Two-sided Matching with Type-specific Maximal and Minimal Quotas in a Student-Supervisor Assignment†

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1. Introduction

This paper describes a study of the matching problem with type-specific maximal and minimal quotas with the deferred acceptance (DA) mechanism in a student-supervisor assignment. In this problem, both students and supervisors are classified by type according to their affiliations, and supervisors set type-specific maximal and minimal quotas.

In the Japanese university chosen as the case in our study, third-year undergraduate students writing graduation theses typically select their desired thesis supervisors from among university faculty members according to their preferences. This choice is made near the end of the school year. Students and supervisors usually have different majors. Supervisors must set different quotas for each type of student, but they each have a minimal number of students they would like to accept. Thus, this is a matching problem with type-specific maximal and minimal quotas.

The most popular mechanism used in Japanese universities for matching students with supervisors is the *Boston mechanism*. In this mechanism, all students apply to their first-

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choice supervisors, and supervisors accept applicants based on their own priority ordering of students. Once a supervisor's quota is filled, the remaining applicants are rejected. Accepted students' assignment is *final* at this point. Students rejected in this step apply to their second-choice supervisors and the iteration repeats until no more applications.

The other popular matching mechanism is *priority matching* (PM). In this mechanism, given students' submitted preferences for supervisors and supervisors' submitted priorities, the sums of students' ranking for a supervisor and supervisors' ranking for students are calculated, and then student-supervisor pairs are formed in ascending order of the sum (with a tie-breaking rule).

However, neither of these mechanisms is *strategy-proof* (i.e., students can be better off by misrepresenting their true preferences). Therefore, other mechanisms have been proposed to overcome this problem. One such alternative is the *serial dictatorship* mechanism. In this mechanism, given a specified order (based on factors such as GPA score), students apply to their first-choice supervisors and supervisors accept them until they reach their quotas. Accepted students' assignment is *final* at this point. Students rejected in step $k-1$ ($k \geq 2$) apply to their k-th-choice supervisors and the iteration repeats until no more applications. The serial dictatorship mechanism is strategy-proof; unfortunately, the matching outcomes produced by this mechanism—along with the outcomes produced by the Boston mechanism and priority matching—are not *stable*; a student (supervisor) may have *justified envy* of another student (supervisor) in the resulting matching.

Another alternative developed to overcome these problems is the DA mechanism (Gale and Shapley (1962)). This mechanism is strategy-proof, and the matching outcomes produced by DA are always stable. However, although an increasing number of DA applications in resident matching and school choice have been reported, the adoption of DA by Japanese universities is still uncommon.

In 2015, the education committee of Future University Hakodate in Hakodate, Hokkaido, Japan decided to use DA to pair third-year undergraduates with thesis supervisors. One of the authors (Kawagoe) of this paper was a member of the committee. Based on his explanation of the working and desirable properties of DA, the committee soon understood and accepted this mechanism; however, committee members asked for several substantial qualifiers. The most challenging of these concerned imposing *minimal quotas* in the assignment process; the committee wanted every supervisor to be assigned a positive number of students in the resulting matching outcome. This was problematic because of the *rural hospital theorem* in matching theory. In the context of the present study, this theorem would predict that a supervisor not assigned any student in a *stable* matching would never be assigned any student in any other *stable* matchings. Hence, satisfying a minimal quota requires giving up the stability requirement.

However, a significant number of faculty members believed that imposing a minimal quota was necessary. The reasons cited included that if a supervisor was not assigned a certain number of students, it might not be feasible for him/her to manage ongoing research projects. Another reason cited was a request for supervisors to share the educational burden equally; some pointed out that it would be unfair for only some supervisors to be assigned students when others were not. After hearing these arguments, the committee decided to impose a minimal quota, even though imposing minimal quotas generally leads to unstable matching because students' assignment in stable matching must be modified to fulfill such a minimal quota.

Several studies have proposed a mechanism to implement a minimal quota by discarding one of the components in the definition of stability (i.e., no justified envy and non-wastefulness; Ágoston et al. (2018); Fragiadakis et al. (2015); Fragiadakis and Troyan (2017); Tomoeda (2018). After examining the mechanisms, the committee chose the mechanism proposed by Fragiadakis and Troyan (2017) as the basis for the matching mechanism.

The committee was also concerned about handling students' preferences for a supervisor whose affiliation was different from that of the student. In Future University Hakodate, students are segregated into four major courses in the second year.[1] Thereafter, students choose their supervisors from among faculty members who teach their courses. However, some students wish to choose supervisors whose affiliation differs from their own, and such requests are respected. Before 2015, such students were few, and they were assigned supervisors before the assignment process for the other students began. However, this caused problems. For example, if students feel safe being assigned to a supervisor whose affiliation is different (as they need not compete with other students over supervisors whose affiliation is the same), they might misrepresent their preferences as if the supervisor would be their first-choice. Or, if a supervisor accepts a sufficient number of students with different affiliations, that supervisor might be reluctant to accept more students with the same affiliation, in which case students with the same affiliation may experience justified envy with respect to those students whose affiliation was different. Thus, the previous treatment for students whose affiliation is different may not be strategy-proof and/or stable.

This problem is similar to the matching problem under the *affirmative action* policy. Several research papers have focused on the matching mechanisms used with this policy (e.g., Abdulkadiroğlu (2003), (2005); Abdulkadiroğlu and Sönmez (2003); Hafalir et al. (2013); Kawagoe et al. (2018); Kojima (2012); Matsubae (2011)). The committee decided to choose the DA-based mechanism proposed by Kojima (2012) and Matsubae (2011) for handling students on different courses. Before the assignment process begins, each

1) In Future University Hakodate, there are two departments, each with two sub-departments. A sub-department is called a "course" in the university.

supervisor must declare quotas for both students whose affiliation is the same and students whose affiliation is different. It is assumed that all students whose affiliation is the same would be acceptable to supervisors, whereas students whose affiliation is different may not be. In the context of affirmative action, the former students are considered to be the majority and the latter students form the minority.

Thus, the matching problems discussed by the committee were a mixture involving minimal quotas and the affirmative action policy. In the mechanism proposed as a solution, called DAMin, both students and supervisors are classified by type according to their affiliations. Then, supervisors set their maximum and minimal type-specific quotas. The maximal quotas are dynamically adjusted to fulfill the minimal quotas. Unfortunately, the mechanism is not strategy-proof; however, it does eliminate justified envy among students of the same type and achieves feasibility with a certain distributional constraint. Moreover, if the sum of the ranks of students and supervisors in the final assignment is viewed as a measure of welfare, there is no domination relationship between this mechanism and the DA mechanism.

The mechanism was implemented in 2016, and 254 students and 67 supervisors participated. All students were matched, and minimal quotas were fulfilled for every supervisor. The mechanism was not strategy-proof, but most students seemed to submit their true preferences. About 90% of students were assigned to their fifth-choice or better supervisors. For supervisors, about 70% of students matched with them were their fifth choice or better. As for the sum of the ranks of students and supervisors in the student-supervisor pairs in the resulting matching, about 80% were smaller than or equal to 10. Thus, the matching outcome was satisfactory.

The remainder of this paper is structured as follows. Section 2 presents the formal model of student-supervisor assignment with type-specific maximal and minimal quotas, Section 3 presents the results of the 2016 student-supervisor assignment process, and Section 4 concludes. All the proofs for propositions are included in Appendix A. The raw data for the student-supervisor assignment problem are available in an online appendix.

2. Model

In this section, we present a formal model of a student-supervisor assignment problem with type-specific maximal and minimal quotas. Because the algorithm used is a DA-based mechanism, for simplicity, we first describe DA with no distributional constraints.

2.1 The student–supervisor problem in general

The basic setup for a student-supervisor problem is as follows:

(a) A non-empty finite set of students: $S = \{s_1, s_2, \dots, s_n\}$

(b) A non-empty finite set of supervisors: $T = \{t_1, t_2, \dots, t_m\}$

(c) Students' preference profile: $P=(P_{s_1}, P_{s_2}, \ldots, P_{s_n})$, where P_{s_i} is a preference relation over $T \cup \{s_i\}$ for student s_i. We assume in this study that preferences are strict for all students. Herein, $t_k P_{s_i} t_l$ means that student s_i prefers t_k to t_l, and R_{s_i} denotes the weak preference relationship induced by P_{s_i}; that is, $t_k P_{s_i} t_l$ or $t_k = t_l$ if and only if $t_k R_{s_i} t_l$. We also assume that every supervisor is acceptable to every student and there is no constraint on the size of the submitted preference.[2)]

(d) Supervisors' priority profile: $\succsim = (\succsim_{t_1}, \succsim_{t_2}, \ldots, \succsim_{t_m})$, where $\succsim_{t_k}$ is the priority ordering over S for supervisor t_k.[3)] The priority ordering is also assumed to be strict for all supervisors. Here, $\succ_{t_k}$ denotes supervisor t_k's strict priority ordering; thus, $s_i \succ_{t_k} s_j$ if and only if $s_i \succsim_{t_k} s_j$ but not $s_j \succsim_{t_k} s_i$.

(e) Total capacity: Each supervisor $t_k \in T$ has a non-negative integer q_{t_k}, which is his/her capacity t_k (i.e., the total number of students the supervisor can accept); we call this *total capacity*. Here, let $q = \{q_{t_k}\}_{k=1}^{m}$ be the vector of total capacity.

Then, matching μ involves mapping the set $S \cup T$ to the set of all the subsets of $S \cup T$, such that

(M1) $|\mu(s_i)| = 1$ for each *student* s_i, and $\mu(s_i) = s_i$ if $s_i \notin \mu(t_k)$ for any supervisor t_k

(M2) For each $s_i \in S$ and $t_k \in T$, $\mu(s_i) = t_k$ if and only if $s_i \in \mu(t_k)$

(M3) $|\mu(t_k)| \leq q_{t_k}$ and $\mu(t_k) \subseteq S$ for each supervisor t_k

(M1) and (M2) mean that all students will be matched with themselves or, at most, one supervisor. (M3) implies that all supervisors will be matched with up to the number of students allowed by their total capacity.

Mechanism φ is a mapping that produces matching for any preference profile. To determine a matching μ, we used the DA mechanism (Gale and Shapley, 1962), which runs as follows.

2.1.1 Assignment process of the DA mechanism

Step 1. All students applied to their first-choice supervisor. For each supervisor t_k, when there were fewer than q_{t_k} applicants, all applicants were *tentatively* accepted by t_k; alternatively, when there were more than q_{t_k} applicants, q_{t_k} applicants who had the highest priority for t_k were *tentatively* accepted by t_k, and the others were rejected.

2) Calsamiglia et al. (2010) conducted an experiment to compare school choice problems with and without constraints on the size of the submitted preference list and found that the proportion of (truncated) truth-telling was significantly higher in the unconstrained case than in the constrained case. As a result, efficiency was significantly reduced, and stability was lower in the constrained than in the unconstrained case.

3) This means that every student is acceptable to every supervisor.

Step k ≥ 2. Applicants rejected in step $k-1$ then applied to their next-choice supervisors. For each supervisor t_k, when there were fewer than q_{t_k} applicants, all applicants among the new applicants and those accepted by step $k-1$ were *tentatively* accepted by t_k; alternatively, when there were more than q_{t_k} applicants, q_{t_k} applicants among the new applicants and those accepted by step $k-1$ who had the highest priority for t_k were *tentatively* accepted by t_k, and the others were rejected.

Terminal condition. If either every student was accepted or no more supervisors remained in the submitted preferences for unmatched students, the process terminated.

The mechanism stopped after a finite number of steps, and the resulting matching μ is unique. Moreover, the matching μ is *stable* and *student optimal.*

Stability

A matching μ is *stable* if the following applies:

(S1) $\mu(s_i) P_{s_i} s_i$ for each student $s_i \in \mathrm{S}$

(S2) If $t_k P_{s_i} \mu(s_i)$, then $|\mu(t_k)| = q_{t_k}$ and $s_j \succ_{t_k} s_i$ for any student $s_j \in \mu(t_k)$

Here, (S1) means the condition of *individual rationality* in which all students prefer matching with a supervisor to matching with themselves and (S2) means that no pair (t_k, s_i) could be a *blocking pair* for the matching μ; thus, even if student s_i prefers supervisor t_k to the supervisor $\mu(s_i)$, the total capacity of supervisor t_k is full, and supervisor t_k does not give student s_i higher priority than any other student s_j accepted under μ.

A *stable mechanism* φ is a mapping that produces a stable matching for any preference profile. If the matching outcome is not stable, students would feel justified envy of other students.

Justified envy

For a matching μ, s_i would feel justified envy of another student s_j if $\mu(s_j)\ P_{s_i}\ \mu(s_i)$, $s_i \succ_{\mu}(s_j)\ s_j$.

Student optimality

A matching is student optimal if it is a stable matching that every student weakly prefers to any stable matching.

The mechanism φ is a *student-optimal stable mechanism* if it produces a student-optimal stable matching for any preference profile. Note that in theorem 1 and 2 of Gale and Shapley (1962), they prove that for any (P, $\succsim$), the DA mechanism is a student-optimal stable mechanism.

Strategy–proofness

Let $\varphi(P)$ be a matching induced by a matching mechanism φ under a true preference profile P, and let $\varphi_{s_i}(P)$ be student s_i's matching outcome in $\varphi(P)$. For any student s_i's preference P'_{s_i} and any profile of other students' preferences other than s_i, $\hat{P}_{-s_i}$, then φ is strategy-proof if

$$\varphi_{s_i}(P_{s_i}, \hat{P}_{-s_i}) R_{s_i} \varphi_{s_i}(P'_{s_i}, \hat{P}_{-s_i})$$

Note that in theorem 9 of Dubins and Freedman (1981) or theorem 5 of Roth (1982), they prove that the DA mechanism is strategy-proof for students.

However, the resulting matching with DA in any student-supervisor problem is not always *Pareto-efficient*.[4)]

Pareto efficiency

A matching is *Pareto-efficient* if it is not Pareto-dominated by any other matching.

Here, a matching μ would *Pareto-dominate* another matching ν if $\mu(s_i) R_{s_i} \nu(s_i)$ for every student $s_i \in S$ and $\mu(s_j) P_{s_j} \nu(s_j)$ for at least one $s_j \in S$. Thus, a matching μ would Pareto-dominate another matching ν if every student preferred the supervisor assigned under μ to the supervisor assigned under ν and at least one student strictly preferred the outcome obtained under μ.

2.2 Student–supervisor problem with type–specific maximal and minimal quotas

Next, we consider a student-supervisor problem with type-specific maximal and minimal quotas. Also called the *controlled school choice problem*, this has been studied by Ágoston et al. (2018), Echenique and Yenmez (2015), Ehlers et al. (2014), and Fragiadakis and Troyan (2017). To consider this problem, we add the following settings to the general settings described in the previous subsection.

$\Theta = \{\theta_1, \ldots, \theta_r\}$ is the finite set for types of students, and each student belongs to, at most, one of the following types. We interpret the "type" as the course to which a student belongs. (The subscript indicating type is omitted except in cases where its absence would cause confusion.) The function $\tau: S \cup T \rightarrow \Theta$ assigned one of the types for each student and denoted S_θ is the set of students of type $\theta \in \Theta$, and T_θ is the set of supervisors of type $\theta \in \Theta$. We assume that types are publicly observable (i.e., they cannot be misreported their types.).

For all types $\theta \in \Theta$ and supervisor $t \in T_\theta$, in addition to total quota q_t, each supervisor

4) The outcome of the DA mechanism is not necessarily efficient in the context of school choice (Abdulkadiroğlu, 2003). Ergin (2002) showed that the outcome of the DA mechanism was Pareto-efficient if and only if the school priorities satisfied a certain acyclicity condition. Ehlers and Erdil (2010) generalized the result in the case where school priorities are coarse. This can be interpreted as a negative result for the efficiency of the DA mechanism, since school priorities are not likely to satisfy the acyclicity conditions of Ergin (2002) and Ehlers and Erdil (2010) in applications.

has a type-specific maximal quota, $U_{t,\theta}$, and a type-specific minimal quota, $L_{t,\theta}$. Let $U = (U_{t,\theta})_{t\in T,\theta\in\Theta}$ be the vector of the maximal quota and $L = (L_{t,\theta})_{t\in T,\theta\in\Theta}$ be the vector of the minimal quota with respect to supervisor t's type. Moreover, we denote $U_{t,-\theta}$ as the maximum number of students of type $\theta' \neq \theta$ assigned to supervisor $t \in T_\theta$ and $L_{t,-\theta}$ as the minimal number of students of type $\theta' \neq \theta$ assigned to supervisor $t \in T_\theta$. We assume $0 \leq L_{t,\theta} \leq U_{t,\theta} \leq q_t$ for all (t,θ). We also assume that $0 \leq U_{t,-\theta} \leq q_t$ for all (t,θ) and $L_{t,-\theta} = 0$. Thus, we consider a two-sided matching problem where the minimal quota constraint is binding when student and supervisor types are the same. Moreover, we suppose that $\sum_t q_t = |S|$.

We denote M as the set of matchings. For any $\mu \in M$, let $\mu_\theta(t)$ be the set of students of type θ assigned to supervisor $t \in T_\theta$ under the matching μ.

Feasibility

A matching μ is feasible if $L_{t,\theta} \leq |\mu_\theta(t)| \leq U_{t,\theta}$ for all (t,θ) and $|\mu(t)| \leq q_t$. In other words, a feasible matching satisfies the type-specific minimal and maximal quotas for any type as well as the total capacity for any supervisor.

The following theorem shows that stability and feasibility are not compatible in any market with type-specific maximal and minimal quotas.

Proposition 1. *There is a problem with type-specific maximal and minimal quotas for which feasibility and stability are incompatible.*

We denote $M_f \subset M$ as the set of feasible matchings. We assume that $M_f \neq \emptyset$, which is a requirement for the distributional constraints such as maximal and minimal quotas to be consistent with the number of students of each type actually present in the market. The definition of justified envy given in Section 2.1.1 must be modified after the introduction of maximal and minimal quotas as follows.

Justified envy

For a matching μ, student $s_i \in \mu(t_k)$ would have justified envy for student $s_j \in \mu(t_l)$ if (i) $t_l P_{s_i} t_k$, (ii) $s_i \succ_{t_l} s_j$, and (iii) there exists an alternative matching $\nu \in \mathrm{M}_f$, such that $\nu(s_i) = t_l$, $\nu(s_j) \neq t_l$ and $\nu(s_h) = \mu(s_h)$ for all $h \neq i, j$.

In other words, student s_i would justifiably envy student s_j if (i) student s_i preferred supervisor t_l, to whom student s_j was assigned t_k; if (ii) s_i had a higher rank than s_j on the priority ordering of supervisor t_l; and if (iii) s_i and s_j could be reassigned without violating any distributional constraints (and without altering the allocation of any other student). If no student justifiably envied any other student, then the matching eliminated justified envy.

Before defining the DA mechanism with type-specific maximal and minimal quotas, we modify the DA mechanism in terms of maximal quota vector $U' = (U'_{t,\theta})_{t\in T,\theta\in\Theta}$. We denote

$DA^{(U')}(P)$ as the DA mechanism with maximal quota vector U'.

2.2.1 Assignment process of $DA^{(U')}(P)$

Step 1: All students applied to their first-choice supervisor. Each supervisor tentatively accepted applicants according to their priority ordering, capacity, and maximal quota. Consider a supervisor of type θ and denote him/her as t_θ. Let the number of new applicants for this supervisor in Step 1 be $a(1)$ and the number of new applicants of type θ for this supervisor in Step 1 be $a_\theta(1)$. Then, the following eight-cases should be considered:

Case 1–a: If $a(1)$ was less than q_{t_θ}, $a_\theta(1)$ was less than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was less than $U_{t,-\theta}$, all applicants would be *tentatively* accepted by t_θ.

Case 1–b: If $a(1)$ was less than q_{t_θ}, $a_\theta(1)$ was less than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was more than $U_{t,-\theta}$, firstly $a_\theta(1)$ applicants would be *tentatively* accepted by t_θ; secondly $U_{t,-\theta}$applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case 1–c: If $a(1)$ was less than q_{t_θ}, $a_\theta(1)$ was more than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was less than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $a(1)-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case 1–d: If $a(1)$ was less than q_{t_θ}, $a_\theta(1)$ was more than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was more than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case 1–e: If $a(1)$ was more than q_{t_θ}, $a_\theta(1)$ was less than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was less than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case 1–f: If $a(1)$ was more than q_{t_θ}, $a_\theta(1)$ was less than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was more than $U_{t,-\theta}$, firstly, $a_\theta(1)$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $U_{t,-\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case 1–g: If $a(1)$ was more than q_{t_θ}, $a_\theta(1)$ was more than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was less than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case 1–h: If $a(1)$ was more than q_{t_θ}, $a_\theta(1)$ was more than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was more than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Step k ≥ 2: Applicants rejected in step $k-1$ then proposed their next-choice supervisors. All supervisors *tentatively* accepted students among the new applicants and those accepted by step $k-1$ according to their priority ordering, capacity, and maximal quota. Consider a supervisor of type θ and denote the supervisor as t_θ. Let the number of new applicants for supervisors in Step k plus ones of the students tentatively accepted in step $k-1$ be $a(k)$. Further, let the number of new applicants for type θ supervisors in Step k plus the ones of the type θ's students tentatively accepted in step $k-1$ be $a_\theta(k)$. Then, the following eight cases should be considered:

Case k–a: If $a(k)$ was less than q_{t_θ}, $a_\theta(k)$ was less than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was less than $U_{t,-\theta}$, all applicants would be *tentatively* accepted by t_θ.

Case k–b: If $a(k)$ was less than q_{t_θ}, $a_\theta(k)$ was less than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was more than $U_{t,-\theta}$, firstly $a_\theta(k)$ applicants would be *tentatively* accepted by t_θ; secondly $U_{t,-\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case k–c: If $a(k)$ was less than q_{t_θ}, $a_\theta(k)$ was more than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was less than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $a(k)-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case k–d: If $a(k)$ was less than q_{t_θ}, $a_\theta(k)$ was more than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was more than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case k–e: If $a(k)$ was more than q_{t_θ}, $a_\theta(k)$ was less than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was less than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case k–f: If $a(k)$ was more than q_{t_θ}, $a_\theta(k)$ was less than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was more than $U_{t,-\theta}$, firstly, a_θ (1) applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $U_{t,-\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case k–g: If $a(1)$ was more than q_{t_θ}, $a_\theta(k)$ was more than $U'_{t,\theta}$, and $a(1)-a_\theta(1)$ was less than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Case k–h: If $a(k)$ was more than q_{t_θ}, $a_\theta(k)$ was more than $U'_{t,\theta}$, and $a(k)-a_\theta(k)$ was more than $U_{t,-\theta}$, firstly, $U'_{t,\theta}$ applicants of type θ who had the highest priority for t_θ would be *tentatively* accepted by t_θ; secondly, $q_{t_\theta}-U'_{t,\theta}$ applicants of type $\theta' \neq \theta$ who had the highest priority for t_θ would be *tentatively* accepted by t_θ, and the other applicants would be rejected.

Then, we defined a DA mechanism with type-specific maximal and minimal quotas (henceforth, DAMin) using $DA^{(U')}(P)$ as follows. Note that the maximal quota for each supervisor would be reduced at each period if the matching outcomes were not feasible.

We also used the following notations in describing DAMin: for a student s and a supervisor t, the mapping $Rank(s\,|\,t)$ assigned a number in $\{1, \ldots, n\}$ for s, which corresponded to s's ranking in t's priority. We define $\bar{T} \coloneqq \{t' \in T : |\mu_\theta(t')| > L_{t',\theta}, \textit{for any } \theta\}$.

For a matching μ, for any $t' \in \bar{T}$, $\overline{Rank}(t') \coloneqq \max_{s \in \mu_\theta(t')} Rank(s\,|\,t'_\theta)$, which would select the worst student rank in $\mu_\theta(t')$ and return its ranking number.

2.2.2 Assignment process of the DAMin mechanism

Step 1: Starting with the maximal quota vector $U^1 = (U^1_{t,\theta})_{t\in T, \theta\in\Theta}$, we determined a matching outcome with $DA^{(U^1)}(P)$ with U^1. If the matching outcome μ was feasible, the algorithm was terminated, and the matching outcome was finalized. If not, the process moved to the next step.

Step $k \geq 2$: If a feasible matching was not obtained in Step $k-1$, $\exists\, t \in T_\theta$ such that $|\mu_\theta(t)| < L_{t,\theta}$, the lowest student rankings were compared among supervisors whose minimal quotas had been fulfilled. Thus, we chose supervisor t', such that $\overline{Rank}(t')$ was the maximum in $t' \in \bar{T}$, and the maximal quota for supervisor t' was adjusted to $U^{k-1}_{t'} - 1$. If two or more supervisors satisfied the above condition, one of them would be randomly chosen. With this updated maximal quota vector U^k, we reran $DA^{(U^k)}(P)$. If the resulting matching μ was feasible, this terminated the algorithm and finalized the matching outcome. If not, the process moved to next Step$(k+1)$.

Terminal condition. The process terminated if (i) every student was accepted, (ii) no more acceptable supervisors remained in the submitted preferences for unmatched students, or (iii) for every supervisor, the number of students of the same type assigned was more than the supervisor's minimal quota, then $L_{t,\theta} \leq |\mu_\theta(t)|$ or (iv) $U^k_{t,\theta} = L_{t,\theta}$ for all (t,θ) (by updating the maximal quota vector, the maximal quota for any supervisor would be less than that supervisor's minimal quota).

The mechanism stopped after a finite number of steps, and the resulting matching μ was unique. The following example shows how DAMin works.

Example 1

There are five students ($S = \{s_1, s_2, s_3, s_4, s_5\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); students are divided into $S_{\theta_1} = \{s_2, s_3\}$ and $S_{\theta_2} = \{s_1, s_4, s_5\}$. There are three supervisors ($T = \{t_1, t_2, t_3\}$) divided into $T_{\theta_1} = \{t_1\}$ and $T_{\theta_2} = \{t_2, t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$
\begin{aligned}
(q, L, U) = \{ & (q_{t_1}, L_{t_1,\theta_1}, U^1_{t_1,\theta_1}) = (3, 1, 2) \\
& (q_{t_2}, L_{t_2,\theta_1}, U^1_{t_2,\theta_1}) = (1, 1, 1) \\
& (q_{t_3}, L_{t_3,\theta_2}, U^1_{t_3,\theta_2})\} = (2, 1, 2)\}
\end{aligned}
$$

The true preference profiles for each student are as follows:[5)]

$P_{s_1} : t_3\ t_1\ t_2$
$P_{s_2} : t_1\ t_3\ t_2$
$P_{s_3} : t_1\ t_2\ t_3$
$P_{s_4} : t_1\ t_2\ t_3$
$P_{s_5} : t_3\ t_1\ t_2$

The priority orderings for each supervisor are as follows:[6)]

$\succ_{t_1} : s_2\ s_4\ s_3\ s_1\ s_5$
$\succ_{t_2} : s_1\ s_2\ s_3\ s_4\ s_5$
$\succ_{t_3} : s_5\ s_2\ s_3\ s_4\ s_1$

Assume that all students submit their true preferences. Then, all students are assigned their first choice, but this matching is not feasible. Indeed, supervisor t_2 has not satisfied his/her minimal quota. Then, since $5 = \overline{Rank}(t_3) > \overline{Rank}(t_1) = 3$, the maximal quota for supervisor t_3 is reduced to $U^2_{t_3} = 1$ and the DA^{U^2} is rerun with this updated quota vector. As a result, student s_1 is rejected by his/her first-choice supervisor t_3. Then, he/she applies to his/her second-choice supervisor t_1 and is again rejected; finally, he/she is accepted by his/her third-choice supervisor t_2. As every student is assigned, DAMin ends and the

5) The notational convention is that supervisors are listed in the order of students' preferences and supervisors not on the preference list are unacceptable: for instance, for student s_1, supervisor t_3 is preferred to supervisor t_1 and supervisor t_1 is preferred to supervisor t_2. Henceforth, the same notation is used in this paper.

6) As in footnote 5), the notational convention here is that students are listed in the order of supervisors' priorities.

resulting matching μ is as follows:

$$\mu = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_2, s_3, s_4 & s_1 & s_5 \end{pmatrix}$$

Remark 1. DAMin starts with DA^{U^1} and then checks if, given the submitted preferences, the resulting matching fulfills the feasibility constraint, especially the minimal quota. If feasibility has been satisfied, the algorithm produces the same matching outcome as DA^{U^1}. If not, one of the supervisors' maximal quotas is reduced by one. This does not necessarily reduce the number of assigned students to the supervisor (i.e., when the number of assigned students is strictly less than the maximal quota, reducing the maximal quota does not change anything). However, repeating this reduction process finally removes a student from the supervisor's quota. If the rejected student then applies to his/her next-choice supervisor and the supervisor accepts him/her, the supervisor may reject his/her lowest-priority student assigned in the previous step. In this way, the rejection chain starts and continues until no student is rejected, according to the DA^{U^k} algorithm. Thus, lowering the maximal quota in sequence, we gradually fulfill the minimal quota for every supervisor.

Remark 2. Fragiadakis and Troyan (2017) proposed that the method for reducing the maximal quota *exogenously* determined one (e.g., a randomly chosen sequence of the supervisors). One of the authors (Kawagoe) strongly recommended this method to the university's education committee, but the chief of the committee finally chose the method for the *endogenously* determined one. Thus, among the supervisors whose number of assigned students was strictly greater than the minimal quota in the previous step, the supervisor assigned the lowest-ranked student with respect to the supervisor's submitted priority order. Changing the reduction sequence from exogenous to endogenous, we lose some good properties of DAMin (e.g., it is no longer strategy-proof).

The reason underlying the committee chief's decision is that with an *exogenously* chosen sequence, a student would not be removed from supervisor t whose worst student's rank was lower (in fact, $\overline{Rank}(t) = 81$) but rather from another supervisor t' whose worst student's rank was relatively high (in fact, $\overline{Rank}(t') = 6$). With an *endogenously* chosen sequence, supervisors were in a better position if they eliminated their lowest-ranked students because those students were virtually unacceptable to them, and students could then have a chance of being assigned to their next-choice supervisors. Hence, changing the reduction sequence from exogenous to endogenous may improve welfare overall. Indeed, in the data on the student-supervisor assignments in 2016 presented in Section 3, the sum of the ranks of students and supervisors in the final assignment with the *endogenously* chosen sequence was slightly lower than the sum with the *exogenously* chosen sequence as well as with DA^U; that is, the *endogenous* one was better.

Remark 3. Another difference between Fragiadakis and Troyan's (2017) mechanism and ours is the range of students included in the reduction sequence in which the maximal quotas were adjusted to fulfill the minimal quotas. In our mechanism, if a type θ supervisor's minimal quota was not fulfilled, only another type θ supervisor's maximal quota would be reduced, while a supervisor of any type who tentatively accepted type θ students could be a target in the reduction process in Fragiadakis and Troyan's (2017) mechanism. In this manner, feasibility could be attained by the matching process in Fragiadakis and Troyan's (2017) mechanism. In our case, a certain distributional constraint is needed to achieve feasibility, as will be shown in Proposition 2 later. Nevertheless, imposing such a constraint is plausible, and the only difference between Fragiadakis and Troyan's (2017) treatment and ours is that such a constraint is imposed *ex post* or *ex ante*.

Eliminating justified envy among students of the same type

If DAMin satisfies the desired properties described in Section 3.1, it guarantees that the final resulting matching eliminates justified envy among students of the same type.

Proposition 2. *DAMin eliminates justified envy among students of the same type.*

Student–optimal stable matching with a minimal quota

A matching that eliminates justified envy among students of the same type weakly prefers any matching that eliminates justified envy among students of the same type.

The mechanism φ is a *student-optimal stable mechanism* with a minimal quota if it produces student-optimal stable matching with a minimal quota for any preference profile.

Remark 4. The DAMin mechanism is a student-optimal stable mechanism with a minimal quota.

Feasibility

DAMin may not produce feasible matching outcomes as a natural consequence of Theorem 3. The intuitive reasoning is as follows. Any student of type θ can apply to supervisor t, whose type is different from θ. In this case, the maximal quota for t, $U_{t,-\theta}$, would not be reduced with DAMin. If a sufficient number of students of type θ were accepted by supervisors of type $\theta' \neq \theta$, then the algorithm would stop without fulfilling the minimal quotas for type θ supervisors. Then, consider the following distributional constraint for any type $\theta \in \Theta$:

$$\sum_{t\in T_\theta} L_{t,\theta} + \sum_{\theta'\neq\theta} \sum_{t'\in T_{\theta'}} U_{t',\theta'} \leq |S_\theta| \tag{1}$$

The first term is the sum of the minimal quota for type θ students that type θ supervisors must accept; the second term is the sum of the number of students who could be accepted by supervisors of type $\theta' \neq \theta$ and whose type is different from supervisor's type θ'. Thus, this constraint means that even if type θ students are accepted by supervisors of type $\theta' \neq$

θ, type θ supervisors can fulfill their minimal quotas with the remaining students of type θ.

This idea is derived from Fragiadakis and Troyan's (2017) quota adjustment process. This constraint is implicitly assumed in their mechanism (i.e., it is imposed *ex post*). If we explicitly impose this constraint *ex ante* in DAMin, we attain feasibility.

Proposition 3. *For any preference profile, if constraint (1) holds, DAMin is feasible.*

Strategy-proofness

If the method of reducing maximal quotas is *exogenously* determined, as shown by Fragiadakis and Troyan (2017), DAMin is strategy-proof. However, if it is *endogenously* determined, DAMin may not be strategy-proof, as explained in Remark 2.

For example, consider a situation where student s applied to popular supervisor t as his/her first choice. Although the maximal quota of supervisor t has not yet been met, if there is another supervisor t' whose minimal quota has not been filled, the maximal quota of supervisor t may be reduced because he/she is assigned the worst student s among the supervisors who must fulfill minimal quotas. As a result, student s may be assigned to a supervisor ranked worse than t. Anticipating this, student s may hesitate to state his/her true preference.

Proposition 4. *For any preference profile, DAMin is not strategy-proof.*

Efficiency

First, we state that there is no Pareto dominance relationship between DAMin and DA^U. As DAMin includes DA as a special case (e.g., when the final matching outcome is determined in Step 1 with DAMin), no ordinal Pareto domination relationship holds between them.

Proposition 5. *DAMin is not necessarily Pareto-dominated by DA.*

As stated in Remark 2, in the actual matching of student-supervisor data from 2016, the sum of the ranks of students and supervisors in the final assignment with the *endogenously* chosen sequence was slightly lower than that with the *exogenously* chosen sequence and original DA; that is, the endogenous one was better. When we consider the sum of the ranks of students and supervisors to be a measure of welfare, DAMin was cardinally more efficient than DA. We formally define this type of efficiency below.

Cardinal efficiency

A matching outcome μ is cardinally more efficient than μ' if

$$\sum_{s\in S} Rank\ (\mu(s)) + \sum_{t\in T}\sum_{s'\in\mu(t)} Rank(s'|t) < \sum_{s\in S} Rank\ (\mu'(s)) + \sum_{t\in T}\sum_{s'\in\mu'(t)} Rank(s'|t),$$

where the mapping $Rank(s'|t)$ assigns a number in $\{1, \ldots, m\}$ to supervisor t, which corresponds to t's ranking in the preference ordering of student $s' \in \mu(t)$.

Cardinal domination

If a matching μ with a mechanism φ is cardinally more efficient than μ' with another mechanism ψ, then φ cardinally dominates ψ. Then, we have the following proposition.

Proposition 6. *DAMin is not necessarily cardinally dominated by DA.*

However, the following proposition shows that under a certain priority structure, DAMin is cardinally dominated by DA. To prove this, we use the following definition of an essentially homogeneous priority structure introduced by Kojima (2013).

Essentially homogeneous (Kojima (2013))

A priority structure $(\{P_t, q_t\}_{t \in T})$ is essentially homogeneous if no $t, t' \in T$ and $s, s' \in S$ exist, such that (1) $s\ P_t\ s$ and $s'\ P_{t'}\ s$, and (2) sets of students $(S_t\ S_{t'} \subset S \setminus \{s, s'\})$ do exist, such that $|S_t| = q_t - 1, |S_{t'}| = q_{t'} - 1$, $S_t = \{s'' \in S : s''\ P_t s\}$, and $S_{t'} = \{s'' \in S: s''\ P_{t'}\ s'\}$.

Proposition 7. *If a school's priority structures are essentially homogeneous, then DAMin is cardinally dominated by DA.*

The concept of an essentially homogenous priority structure seems to be similar to the *acyclicity* condition introduced by Ergin (2002) and Kesten (2006).[7)] Thus, if a priority structure has a *cycle*, the above negative result for DAMin may not hold. The following example shows that this conjecture may be right.

Example 2

We examine the above statement with the following example. There are five students ($S = \{s_1, s_2, s_3, s_4, s_5\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, the students are divided into $S_{\theta_1} = \{s_3, s_4, s_5\}$ and $S_{\theta_2} = \{s_1, s_2\}$. There are three supervisors $T = \{t_1, t_2, t_3\}$, divided into $T_{\theta_1} = \{t_1, t_2\}$ and $T_{\theta_2} = \{t_3\}$. The total quotas as well as type-specific minimal and maximal quotas are as follows:

$$(q, L, U) = \{(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (2, 1, 2)$$

7) A strong cycle of priority structure P holds if the following two conditions are met: (1) there are $t, t' \in T$ and $s_i, s_j, s_k \in S$, such that $s_i\ P_t\ s_j\ P_t\ s_k, s_k\ P'_t\ s_i$, and (2) there exist (possibly empty) disjoint sets of students $S_t, S_{t'} \subset S \setminus \{s_i, s_j, s_k\}$, such that $S_t \subset Upper_t(s_j) := \{s \in S : s\ P_t\ s_j\}$, $|S_t| = q_t - 1$, $S_{t'} \subset Upper_{t'}(s_i)$, $|S_{t'}| = q_{t'} - 1$. Priority structure P is weakly acyclic if it has no strong cycle (Ergin (2002)). A cycle of priority structure P holds if the following two conditions are met: (1) there are $t, t' \in T$ and $s_i, s_j, s_k \in S$, such that
$s_i\ P_t\ s_j\ P_t\ s_k$ *and* $s_k\ P_{t'}\ s_i, s_j$, and (2) there is a (possibly empty) set $S_t \subset S \setminus \{s_i, s_j, s_k\}$, such that $S_t \subset Upper_t(s_i) \cup (Upper_t(s_j) \setminus Upper_{t'}(s_k))$ and $|S_t| = q_t - 1$. Priority structure P is acyclic if it has no cycle (Kesten (2006))

$$(q_{t_2}, L_{t_2,\theta 1}, U_{t_2,\theta_1}) = (3, 1, 3)$$
$$(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (2, 1, 2)\}$$

Under the true preference profiles for each student:

$$P_{s_1} : t_3\ t_1\ t_2$$
$$P_{s_2} : t_3\ t_1\ t_2$$
$$P_{s_3} : t_2\ t_1\ t_3$$
$$P_{s_4} : t_2\ t_1\ t_3$$
$$P_{s_5} : t_2\ t_1\ t_3$$

and the priority orderings for each supervisor:

$$\succ_{t_1} : s_3\ s_1\ s_2\ s_4\ s_5$$
$$\succ_{t_2} : s_4\ s_5\ s_3\ s_2\ s_1$$
$$\succ_{t_3} : s_1\ s_2\ s_3\ s_4\ s_5$$

The resulting final matching with DA is as follows (note that the priority structures have a *cycle* structure):

$$\mu = \begin{pmatrix} t_1 & t_2 & t_3 \\ \varnothing & s_3, s_4, s_5 & s_1, s_2 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 14. The result of DAMin is μ^{Min}:

$$\mu^{Min} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_3 & s_4, s_5 & s_1, s_2 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 12. This example shows that if a priority structure has a *cycle*, DAMin cardinally dominates DA. However, this property does not hold for any matching market, as the following example shows.

Example 3

There are five students ($S = \{s_1, s_2, s_3, s_4, s_5\}$) of two types ($\Theta = \{\theta_1, \theta_2\}$); hence, the students are divided into $S_{\theta_1} = \{s_3, s_4, s_5\}$ and $S_{\theta_2} = \{s_1, s_2\}$. There are three supervisors $T = \{t_1, t_2, t_3\}$, divided into $T_{\theta_1} = \{t_1, t_2\}$ and $T_{\theta_2} = \{t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$(q, L, U) = \{(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (2, 1, 2)$$
$$(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (3, 1, 3)$$
$$(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (2, 1, 2)\})$$

Under the true preferences, the profiles for each student are as follows:

$$P_{s_1} : t_3\ t_1\ t_2$$
$$P_{s_2} : t_3\ t_1\ t_2$$
$$P_{s_3} : t_2\ t_3\ t_1$$

$P_{s_4}: t_2\ t_1\ t_3$
$P_{s_5}: t_2\ t_1\ t_3$

and the priority orderings for each supervisor are as follows:

$\succ_{t_1}: s_4\ s_1\ s_2\ s_3\ s_5$
$\succ_{t_2}: s_4\ s_5\ s_3\ s_2\ s_1$
$\succ_{t_3}: s_1\ s_2\ s_3\ s_4\ s_5$

The resulting final matching with DA is as follows (note that the priority structure has a *cycle*):

$$\mu = \begin{pmatrix} t_1 & t_2 & t_3 \\ \varnothing & s_3, s_4, s_5 & s_1, s_2 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 14. The result of DAMin is μ^{Min}:

$$\mu^{Min} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_3 & s_4, s_5 & s_1, s_2 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 17.

Next, we address the question about whether or not DAMin with an *endogenous* sequence Pareto-dominate ones with an *exogenous* sequence in general. Proposition 7 shows that there is no Pareto domination relationship among these mechanisms.

Proposition 8. *There is no Pareto domination relationship between DAMin with an endogenous sequence and one with an exogenous sequence for any matching market.*

Then, does DAMin with an *endogenous* sequence cardinally dominate one with an *exogenous* sequence? In the following proposition, we show that there is no cardinal domination relationship between both types of DAMin.

Proposition 9. *There is no cardinal domination relationship between both types of DAMin for any matching market.*

Our findings suggest that the only limitation of DAMin with an *endogenous* sequence rather than one with an *exogenous* sequence is that the former is not strategy-proof. However, experimental findings in the laboratory (Chen and Sönmez (2006); Chen et al.(2016); Featherstone and Niederle (2008); Kawagoe et al.(2018); Pais and Pintér (2008)) and in the field data (e.g., Chen and Kesten (2017); Echenique et al.(2016))) show that strategy-proofness is not satisfied even for the DA mechanism. Therefore, the lack of strategy-proofness may not be a major problem in practice.

Comparison with related mechanisms

Fragiadakis et al. (2015) also studied a model of school choice with minimal quotas (they called minimal quotas "constraints" such as hard-floor constraints).[8] The model they proposed was restricted to the case in which each school had an aggregate floor constraint (i.e., they did not allow a school to have separate constraints for different types of students). Unlike the model proposed by Fragiadakis et al. (2015) and Fragiadakis and Troyan (2017), DAMin allows a supervisor to have separate constraints for different types of students.

The extended-seat DA (ESDA) mechanism proposed by Fragiadakis et al. (2015) fills an aggregate minimal quota by separating each school into virtually two smaller schools according to their minimal quotas: total quota for one school (they called it the standard school) corresponds to minimal quota and the one for another school (they called it the extended school) corresponds to the remaining seats. Then, as ESDA is designed so that the number of students assigned for the extended schools is no more than the number of total students minus the sum of minimal quotas, all the standard schools fill its capacities, thereby all the minimum quotas are filled in the original assignment problem. Specifically, when students apply to their first-choice schools, the schools accept them up to their *minimal quotas* based on their priority orderings. Then, students rejected by their first-choice schools apply for the *remaining seats* (total quotas minus minimal quotas) at their first-choice schools. If they are again rejected, they then apply to their second-choice schools and so on. The ESDA mechanism fulfills each school's seats *from the bottom*, while the model proposed by Fragiadakis and Troyan (2017) and DAMin reduces each school's seats *from the top* to fulfill the minimal quotas.[9] Fragiadakis et al. (2015) proved that ESDA mechanism is strategy-proof and satisfies a modified sense of stability (Theorem 3.1).

In the DAMin, the maximal quota is *reduced* one by one. This causes students to have unwarranted and unfair feelings for supervisors who removed students assigned in the previous steps of the algorithm. On the contrary, ESDA fulfills the minimal quotas unless it causes justified envy. In that case, as students are not *removed* but *added* for each supervisor, unfair feelings do not seem to occur. There is no cardinal domination between ESDA and DAMin. Indeed, in the environment used in Proposition 2, the resulting matching with ESDA is feasible:

$$\mu^{ESDA} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_2, s_3 & s_1, s_4 & s_5 \end{pmatrix}.$$

Proposition 10. *There is no cardinally domination relationship between DAMin and*

8) See also Goto et al. (2016), (2017) for a generalization.

9) Ágoston et al. (2018) also studied two-sided matching with type-specific maximal and minimal quotas as an integer programming problem. They showed that feasible matching with a type-specifically modified version of stability exists. Although it was not explicitly described, they also used a quota adjustment process like that used by Fragiadakis and Troyan (2017) and by us. However, Ágoston et al. (2018) started *from the bottom*—that is, they stated that the maximal quotas were equal to the minimal quotas—and then gradually raised the maximal quotas by keeping the minimal quotas fulfilled.

ESDA for any matching market.

Tomoeda (2018) also studied a model of school choice with minimal quotas. He proposed a mechanism called the *DA mechanism with precedence lists*. He also considered a dynamic DA-based mechanism. In his model, to fulfill the minimal quotas, the *rankings* over students of the same type were dynamically changed, while the *maximal quotas* were changed in DAMin. The DA mechanism with precedence lists is strategy-proof and satisfies a modified sense of stability (Tomoeda (2018), Proposition 2). Tomoeda's proposed model is like our model, but it does not seem applicable to our situation because the interpretations of types in Tomoeda's (2018) model and ours are different. Tomoeda (2018) interprets as "type" a given student's individually characteristics, but we interpret a type as a *student's as well as supervisor's* affiliation. In other words, in Tomoeda (2018), "type" is defined only for students.

The inherent weakness of DAMin is its lack of strategy-proofness when a reduction sequence is *endogenously* determined. However, some studies have shown that mechanisms with dynamic adjustment processes do not always satisfy strategy-proofness. For example, Haeringer and Iehlé (2019), who studied a dynamic DA-based mechanism in which students could resubmit their preferences to obtain a better match in the later stages of the admission process for a French college, demonstrated that the mechanism was not strategy-proof. Okumura (2017) studied a certain kind of school choice problem for resolving shortages in childcare in nursery schools in Japan, where different quotas are set for different age groups in each nursery school, and these quotas are dynamically adjusted in the school to resolve the coexistence of excess demand and supply for different age groups. The proposed mechanism satisfied a modified sense of stability but was not strategy-proof.

However, if we put DAMin in the context of a large economy, we could show that it is strategy-proof.[10)]

3. Empirical data

In this section, we present a case study in which a Japanese university implemented a student-supervisor matching problem with type-specific maximal and minimal quotas to evaluate the performance of the DAMin mechanism in a practical environment. The DAMin mechanism was implemented in 2016 at Future University Hakodate with 254 students and 67 supervisors. There were four courses, and each student belonged to, at most, one course. In the following analyses, we refer to these courses as A, B, C, and D.[11)]

10) For example, Kojima and Pathak (2009) studied a large economy and in the large economy DAMin can be satisfied strategy-proofness.

Supervisors belonged either to one of the four courses or to the Communication Media Laboratory (CML).

Students were asked to apply to supervisors who belonged to their own course, but they could apply to any supervisor on any course. Indeed, 21.2% of students (54 of 254) were assigned to supervisors on different courses. Thus, although the minimal quota was fulfilled for all supervisors, the maximal quota for students on the same course was not fulfilled for 7.5% supervisors (5 of 67).

The maximal quotas for students on the same course were equal among all the supervisors on that course. The maximal quotas were set to ensure that if every student on the same course applied to a supervisor on that course, students would not be unmatched. Therefore, the maximal quotas on each course were equal to the number of students on the course divided by the number of supervisors on the same course (if the calculated number contained decimal points, it was rounded up to the closest integer).

The minimal quotas for students on the same course were equal among supervisors belonging to that course. However, they were different for the four courses, reflecting the educational objectives of each course. Supervisors had to set strictly positive minimal quotas for students on the same course, but could set their minimal quotas equal to zero for students on different courses. For supervisors who belonged to the CML, the maximal quotas were four, and they could set minimal quotas to zero for students on any course. Table 1 summarizes the number of students and supervisors as well as the maximal and minimal quotas for students on the same course.

Before starting the matching process, the chief of the education committee of the university explained the matching process and basic properties of the DA mechanism for students, including the fact that truth-telling was the best choice for them. For two weeks, students engaged in interviews with the supervisors to which they wanted to be assigned. After that, they submitted paper forms listing their supervisor preferences. They ranked every supervisor on the same course to avoid no match being found. While they could also rank supervisors on different courses, that number was restricted to two.

Then, each supervisor was informed by an electronic file delivery system about

Table 1 Number of Students, Number of Supervisors, and Maximal and Minimal Quotas for Students on the Same Course

	A	B	C	D	CML	Total
Number of students	62	62	86	44	0	254
Number of supervisors	13	12	19	13	10	67
Maximal quota	6	6	4	4	4	–
Minimal quota	2	2	2	3	0	–

11) A, B, C, and D correspond to the Complex Systems, Intelligent Systems, Information Systems, and Information Design courses, respectively.

students' preferences along with additional information about students such as their GPA, the number of compulsory courses completed or remaining, and the total number of courses for which students had already secured credits. Supervisors could only see the preferences of students who had ranked them. After reviewing this information, they submitted their priority orderings of students using the electronic file delivery system. Supervisors had to rank all students on the same course, but could eliminate students on different courses if they did not want to accept them.

The matching outcome was determined by DAMin using the preferences, priority orderings, and maximal and minimal quotas. No student was left unmatched, but eight supervisors had not fulfilled their minimal quotas after the first step of the DAMin algorithm. This triggered the quota-adjusting process until, finally, every supervisor had a fulfilled minimal quota after the 47th iteration.

Each student had to rank 14 to 23 supervisors depending on the number of supervisors on the course and his/her preference for supervisors on different courses.

Assignments in the final matching outcome

In the resulting matching, 71.7% of students were matched with their first-choice supervisor. In total, 90% were matched with their fifth-choice or better supervisor. In the worst case, a student was matched with his 19th choice. In the next-worst case, a student was matched with his 14th choice. Table 2 shows the number of students assigned up to their 10th choice, the percentages, and the cumulative percentages in the population.

Except for supervisors who belonged to the CML, each supervisor had to rank 47 to 96 students depending on the number of students on the course and supervisor's preference for students on different courses. Up to four students applied to supervisors in the CML.

In the resulting matching, 21.7% of the students matched with supervisors were those supervisors' first-choice students. In total, 90.2% were matched with supervisors who had ranked that student as their 13th choice or better. In the worst case, a supervisor was matched with his 81st choice. In the next worst case, a supervisor was matched with his 73rd choice. Although this may suggest that the matching outcomes for supervisors were worse than the outcomes for students, supervisors had to accept up to their maximal quota. Even in the best case, each supervisor had to accept four or more students if the maximal quota was fulfilled. Then, as 60.2% (73.6%) of the students with whom supervisors were matched were up to their fourth (sixth) choice, the relative performance

Table 2 Matching Results for Students

Rank	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Frequency	182	22	13	9	5	6	2	4	4	0
%	71.7	8.7	5.1	3.5	2.0	2.4	0.8	1.6	1.6	0.0
Cumulative %	71.7	80.3	85.4	89.0	90.0	93.3	94.1	95.7	97.2	97.2

of the mechanism for supervisors was not worse than that for students. In addition, remember that DAMin produces a student-optimal matching.

Table 3 shows the number of supervisors assigned up to their 10th choice, the percentages, and the cumulative percentages in the population.

Table 3 Matching Results for Supervisors

Rank	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Frequency	55	42	30	26	22	12	12	11	5	4
%	21.7	16.5	11.8	10.2	8.7	4.7	4.7	4.3	2.0	1.6
Cumulative %	21.7	38.2	50.0	60.2	68.9	73.6	78.3	82.7	84.6	86.2

Welfare

Table 4 shows the sums of the ranks of students and supervisors in the student-supervisor pairs. If both a student and a supervisor were matched with their first-choice, the sum of the ranks was 2. Such cases occurred in 20.9% of the matching cases. As already pointed out, each supervisor had to accept four or more students; if the maximal quota was fulfilled, the sum of the ranks would be 5 to 8 in the best case. Then, 74% of the matchings are in such cases.

Example 2 in Section 3 showed that if we consider the sum of the ranks of students and supervisors in the final assignment to be a measure of welfare, DAMin can dominate DA. That is also the case in this matching. Indeed, the sum of the ranks of students and supervisors in the final assignment with the original DA was 2,460 and the sum with DAMin with an endogenous sequence was 2,398. Thus, if we consider the sum of the ranks of students and supervisors in the final assignment to be a measure of welfare, DAMin improved welfare more than DA did.

Table 4 Distribution of the Sum of the Ranks of Students and Supervisors

Rank	2	3	4	5	6	7	8	9	10	11 −
Frequency	53	38	33	25	20	13	6	8	6	52
%	20.9	15.0	13.0	9.8	7.9	5.1	2.4	3.1	2.4	20.5
Cumulative %	20.9	35.8	48.8	58.7	66.5	71.7	74.0	77.2	79.5	100.0

Strategy-proofness

In this student-supervisor matching case study, it was not possible to determine if students had submitted their true preferences. Therefore, after the matching process was complete, we asked students about this using an anonymous online questionnaire. About 45.3% of students (115 of 254) responded to the questionnaire. Nearly 87.8% of respondents (101 of 115) answered that they were aware that the matching process was based on the DA mechanism; 57.4% (58 of 101) had read books or searched the Internet for information about the DA mechanism; 65.5% (38 of 58) answered that they understood the workings and properties of the DA mechanism; nearly 54.8% (63 of 115) answered that the idea of strategy-proofness was attractive; 50.4% (58 of 115) said that eliminating justified envy was desirable.

Then, about 41.7% (48 of 115) said that they had submitted their true preferences in full length (i.e., they had ranked all the supervisors according to their true preferences); and 55.7% (64 of 115) said they had submitted their true preferences for those supervisors whom they had ranked relatively high. Notably, only 2.6% of respondents (3 of 115) said that they had not submitted their true preferences intentionally. Indeed, they had avoided their first-choice supervisors and may have avoided the most popular supervisors by misrepresenting their second choice as their first choice.

Other strategic problems

In this matching process, before submitting their preferences, students had the opportunity to interview all supervisors during a two-week period. For all students, their preferences for supervisors were usually based on incomplete information; this is partly because not all had attended lectures taught by all supervisors. Thus, during this interview period, many students had to form their "true" preferences based on what they discerned in the interviews about supervisors' personality and educational and research objectives as well as what they learned from other students. In this sense, their preferences were *endogenous* (Antler, 2015).

Nearly 97.4% of questionnaire respondents (112 of 115) said that they had interviewed at least one supervisor, whereas only 8% (9 of 112) said they had interviewed five or more. About 75% of those who had interviewed supervisors (84 of 112) had interviewed two to four supervisors, but 17% (19 of 112) had interviewed only one. Although students were all forced to submit their preferences for all supervisors on the same course to avoid being unmatched, this result suggests that many students considered a significant number of supervisors to be unacceptable. In other words, students' preferences were virtually *truncated*; only supervisors they ranked as their first (or a relatively high) choice were reliable. Truncated preferences can cause undesirable outcomes. However, since some students and supervisors already knew each other, it may be that those students did not need to search out other supervisors during the interview period to achieve a better match.

Another concern is that students' preference may have been affected by supervisors' persuasion during the interview period. We realized through informal talks with students after the matching process was complete that some supervisors had made *credible threats* to students, telling students that if they did not rank the supervisor as their first-choice, he/she would not accept them. Some supervisors seemed to be concerned about fulfilling their maximal quotas; for example, some might have felt pressured to gather sufficient number of students to run ongoing research projects. These threats were credible because supervisors could submit their priority orderings *after* knowing students' preferences.[12)] This might explain why a large number of students were matched with their first choice.

Nonetheless, it is not clear if such truncated preferences and/or credible threats distorted the matching outcome. According to the follow-up questionnaire for students, 80% of respondents (92 of 115) said that they had decided on the rankings they submitted according to their own preferences. About 15.7% (18 of 115) said that they had been worried about competition for popular supervisors and 13.9% (16 of 115) said that a factor in their choice was whether they had known senior students (or their mutual friends) who had already been assigned to or had applied for the same supervisor, similar to resident matching with couples (Kamada and Kojima (2015)). Only 6.1% of respondents (7 of 115) said that they had randomly decided. These findings seem to suggest that most students had submitted their true preferences.

4. Discussion and conclusions

This paper discussed two-sided matching with type-specific maximal and minimal quotas in a student-supervisor matching in a Japanese university. We showed that the proposed mechanism, DAMin, could eliminate justified envy among students of the same type and that the process attained feasibility with a certain distributional constraint, although it was not strategy-proof. With respect to strategy-proofness, experimental evidence from laboratory experiments (Chen and Sönmez (2006); Chen et al. (2016); Featherstone and Niederle (2008); Kawagoe et al. (2018); Pais and Pintér (2008)) and field data (e.g., Chen and Kesten (2017); Echenique et al. (2016)) suggests that a significant number of students mispresent their preferences. Thus, it is still debatable whether a lack of strategy-proofness is detrimental.

During our discussion with the university education committee, one member questioned whether submitting *cardinal* preferences might be better than submitting ordinal preferences to measure the strength of preferences. To date, studies of the school choice problem with cardinal preferences have used the context of resolving problems caused by tie-breaking.[13)] It is not yet clear what happens in a matching process with

12) The chief of the education committee and one of the authors (Kawagoe) strongly recommended that supervisors submit their priority orderings *without knowing* students' preferences to avoid any strategic effect. However, a significant number of supervisors claimed that they could not rank all students on the same course *without* knowing their preferences. In personal communication with Fuhito Kojima, we were informed that he had ever heard of Alvin Roth facing a similar situation when he consulted the matching process for freshers in a certain university.

13) See Abdulkadiroğlu et al. (2015).

type-specific quotas when cardinal preferences are used. Another concern relates to the *lattice structure* of stable matching. Because DAMin is based on student-proposed DA, the resulting matching is student-optimal. However, a significant number of supervisors seemed to feel uneasy about this. In the context of two-sided matching (e.g., a college admission problem), addressing such a claim seems to be important for designing a mechanism. One reason why an *endogenous* reduction sequence was adopted for DAMin was to improve supervisors' welfare (see Remark 2 in Section 2.2). If reflecting both supervisors' and students' welfare is desirable, then *median matching* or fractional matching might be a useful method to achieve this, although neither is strategy-proof. Reflecting supervisors' welfare is an important objective in a matching mechanism between students and supervisors because supervisors are long-run players and therefore have more invested in the matching outcome than students. We welcome future research answering these questions.

Appendix A.

Proposition 1. *In any student-supervisor problem with type-specific maximal and minimal quotas, there is a problem for which feasibility and stability are incompatible.*

Proof. We prove the statement with the following example. There are two students ($S = \{s_1, s_2\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1\}$ and $S_{\theta_2} = \{s_2\}$. There are two supervisors ($T = \{t_1, t_2\}$), divided into $T_{\theta_1} = \{t_1\}$ and $T_{\theta_2} = \{t_2\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$(q, L, U) = \{(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (1, 1, 1),$$
$$(q_{t_2}, L_{t_2,\theta_2}, U_{t_2,\theta_2}) = (1, 1, 1)\}.$$

Under the preference profiles for each student,

$P_{s_1} : t_2\ t_1$
$P_{s_2} : t_1\ t_2$

and priority orderings for each supervisor,

$\succ_{t_1} : s_2\ s_1$
$\succ_{t_2} : s_1\ s_2$

a unique stable matching μ based on the preferences profile follows:

$$\mu = \begin{pmatrix} t_1 & t_2 \\ s_2 & s_1 \end{pmatrix}.$$

However, μ is not feasible because no type θ_1 (or θ_2) student is assigned to a type θ_1 (or θ_2) supervisor t_1 (or t_2). *Q.E.D.*

Proposition 2. *DAMin eliminates justified envy among students of the same type.*

Proof. Without a loss of generality, consider DA with U^k ($k = 1, \ldots, r$). Denote the matching produced by

DA with U^k as μ. Assume that student s_i envies student s_j, who is of the same type: $\mu(s_j)\ P_{s_i}\ \mu(s_i)$ and $\tau(s_i) = \tau(s_j) = \theta$. Let step r be the step in the DA algorithm in which student s_i is rejected from $\mu(s_j)$. In step r, s_i is rejected because the type θ-specific minimal quota is filled with $L_{\mu(s_j),\theta}$ students of type θ ranked higher than s_i, according to $\succ_\mu(s_j)$; the remaining seats are also filled with $q^{k}_{\mu(s_j)} - \sum_{\theta\in\Theta} L_{\mu(s_j),\theta}$ students of any type ranked higher than s_i according to $\succ_{\mu(s_j)}$. In future steps, a student accepted in step r can be rejected from the type θ-specific minimal quota only if a higher-ranked student of type θ applies, and the same is true for students in the remaining seats. Thus, at the end of the algorithm, all students assigned to $\mu(s_j)$ in either the type θ-specific minimal quota or the remaining seats must be ranked higher than s_i. Since $\tau(s_j) = \theta$, this implies that $s_j \succ_\mu(s_j)\ s_i$; that is, s_i does not have any justified envy against s_j. *Q.E.D.*

Proposition 3. *For any preference profile, if constraint (1) holds, DAMin is feasible.*
Proof. Suppose that DAMin is not feasible. Then, there are some type θ supervisors whose minimal quotas are not fulfilled; that is, there are some type θ students not assigned to a type θ supervisor. As the quotas for type θ students for supervisors of type $\theta' \neq \theta$ have already been fulfilled in this case because of constraint (1), those students are unassigned. However, because all students are acceptable for all supervisors and the quotas are set so that all students are assigned to either supervisor by assumption, this is a contradiction. *Q.E.D.*

Proposition 4. *For any preference profile , DAMin is not strategy-proof.*
Proof. We prove the statement with the following example. There are five students ($S = \{s_1, s_2, s_3, s_4, s_5\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$). Thus, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3, s_4\}$ and $S_{\theta_2} = \{s_5\}$. There are three supervisors ($T = \{t_1, t_2, t_3\}$), divided into $T_{\theta_1} = \{t_1, t_2\}$ and $T_{\theta_2} = \{t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$(q, L, U) = \{(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (2, 1, 2),$$
$$(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (1, 1, 1),$$
$$(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (2, 1, 2)\}.$$

Under the true preference profiles for each student,

$P_{s_1}: t_1\ t_2\ t_3$
$P_{s_2}: t_1\ t_2\ t_3$
$P_{s_3}: t_3\ t_1\ t_2$
$P_{s_4}: t_1\ t_3\ t_2$
$P_{s_5}: t_3\ t_1\ t_2$

and the priority orderings for each supervisor,

$\succ_{t_1}: s_2\ s_1\ s_3\ s_4\ s_5$
$\succ_{t_2}: s_1\ s_3\ s_2\ s_4\ s_5$
$\succ_{t_3}: s_4\ s_5\ s_3\ s_2\ s_1$

the resulting final matching with DAMin is as follows:

$$\mu(P) = \begin{pmatrix} t_1 & t_2 & t_3 & s_3 \\ s_2 & s_1 & s_4, s_5 & s_3 \end{pmatrix}.$$

Consider that student s_3 states the following preference instead (any other student reveals his/her true preference):

$P'_{s_3}: t_2\ t_1\ t_3$.

Then, the resulting final matching with DAMin is as follows:

$$\mu(P_{-s_3}, P'_{s_3}) = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1, s_2 & s_3 & s_4, s_5 \end{pmatrix}$$

In this matching outcome, student s_3 is strictly better off than in $\mu(P)$. Thus, student s_3 has an incentive to misrepresent his/her true preferences. *Q.E.D.*

Proposition 5. *DAMin is not necessarily Pareto-dominated by DA.*

Proof. We prove the statement with the following example. There are five students ($S = \{s_1, s_2, s_3, s_4\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3\}$ and $S_{\theta_2} = \{s_4\}$. There are three supervisors ($T = \{t_1, t_2, t_3\}$), divided into $T_{\theta_1} = \{t_1, t_2\}$ and $T_{\theta_2} = \{t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$\begin{aligned}(q, L, U) = \{&(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (2, 1, 2)\\ &(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (1, 1, 1)\\ &(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (2, 1, 2)\}\end{aligned}$$

Under the true preference profiles for each student,

$P_{s_1}: t_1\ t_2\ t_3$
$P_{s_2}: t_1\ t_2\ t_3$
$P_{s_3}: t_2\ t_1\ t_3$
$P_{s_4}: t_3\ t_1\ t_2$

and the priority orderings for each supervisor,

$\succ_{t_1}: s_2\ s_1\ s_3\ s_4$
$\succ_{t_2}: s_1\ s_3\ s_2\ s_4$
$\succ_{t_3}: s_4\ s_3\ s_2\ s_1$

the resulting final matching with DAMin is as follows:

$$\mu^{DAMin}(P) = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1, s_2 & s_3 & s_4 \end{pmatrix}$$

and the resulting final matching with DA is as follows:

$$\mu^{DA}(P) = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1, s_2 & s_3 & s_4 \end{pmatrix}$$

Thus, the matching outcomes are the same. Therefore, we conclude that DAMin is not Pareto-dominated by DA. *Q.E.D.*

Proposition 6. *The DAMin is not necessarily cardinally dominated by DA.*

Proof. We prove this statement using an example. There are four students ($S = \{s_1, s_2, s_3, s_4\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3\}$ and $S_{\theta_2} = \{s_4\}$. There are three supervisors ($T = \{t_1, t_2, t_3\}$), divided into $T_{\theta_1} = \{t_1, t_2\}$ and $T_{\theta_2} = \{t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$\begin{aligned}(q, L, U) = \{&(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (3, 1, 3)\\ &(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (2, 1, 2)\end{aligned}$$

$(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (1, 1, 1)\}$

Under the preference profiles for each student,

$P_{s_1} : t_1\ t_2\ t_3$
$P_{s_2} : t_1\ t_2\ t_3$
$P_{s_3} : t_1\ t_2\ t_3$
$P_{s_4} : t_3\ t_1\ t_2$

and the priority orderings for each supervisor,

$\succ_{t_1} : s_1\ s_2\ s_3\ s_4$
$\succ_{t_2} : s_3\ s_1\ s_2\ s_4$
$\succ_{t_3} : s_4\ s_1\ s_2\ s_3$

the resulting final matching with DA is as follows:

$$\mu(P) = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1,s_2,s_3 & \varnothing & s_4 \end{pmatrix}.$$

The sum of the ranks of students and supervisors after the final assignment in this case is 9. The resulting final matching with DAMin with either an *endogenous* or an *exogenous* sequence is as follows:

$$\mu'(\mathrm{P}) = \begin{pmatrix} \mathrm{t}_1 & \mathrm{t}_2 \\ \mathrm{s}_1,\mathrm{s}_2 & \mathrm{s}_3 \end{pmatrix}$$

The sum of the ranks of students and supervisors after the final assignment in this case is 8. *Q.E.D.*

Proposition 7. *If a school's priority structures are essentially homogeneous, then DAMin is cardinally dominated by DA.*

Proof. Let a be the sum of the ranks of students and supervisors under the original DA matching outcome. If the matching outcome with DA satisfied the minimal quotas for all supervisors, then the sum of the ranks of students and supervisors with the DAMin matching outcome would be a.

Next, we consider the case in which the matching outcome with DA does not satisfy the minimal quotas for some supervisors. We divide a into two: $a = a^S + a^T$. Let a^S be the sum of the ranks of students under the DA matching outcome and a^T be the sum of the ranks of supervisors under the DA matching outcome. Let a'^S be the sum of the ranks of the same students under the DAMin matching outcome and a'^T be the sum of the ranks of the same supervisors under the DAMin matching outcome. As the minimal quota is not satisfied with the DA, the maximal quotas for some supervisors will be reduced until every supervisor's minimal quota is satisfied by the DAMin. In this process, some students will be reallocated to other supervisors that students ranked lower. Thus, a^S is larger than a'^S. To prove that a^T is larger than a'^T, we assume that $a'^T < a^T$. Thus, some supervisors will be matched with better students with the DAMin than with the DA. Suppose supervisor t_1 is one of them; he/she is matched with student s_1. Hence, s_1 is matched with a better supervisor under the DA process than under the DAMin process. As for the assumption $a'^T < a^T$, we must consider the following two cases:

(**Case 1**) t_1's minimal quota is not satisfied with the DA
(**Case 2**) t_1 is matched with a student worse than s_1 with the DA

In these cases, even allocations for some students, including s_1, will be changed with the DAMin because the priority structure is essentially homogeneous; $a'^T = a^T$ holds in both cases (e.g., s_1's affiliation is changed, but his/her rankings are the same as between supervisors.) This is a contradiction. Thus, we have the following inequality:

$$a = a^S + a^T < a'^S + a'^T = a' \qquad Q.E.D.$$

Proposition 8. *There is no Pareto domination relationship between DAMin with an endogenous sequence and one with an exogenous sequence for any matching market.*

Proof. We prove the statement by the following example. There are five students ($S = \{s_1, s_2, s_3, s_4, s_5\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3\}$ and $S_{\theta_2} = \{s_4, s_5\}$. There are three supervisors ($T = \{t_1, t_2, t_3\}$), divided into $T_{\theta_1} = \{t_1, t_2\}$ and $T_{\theta_2} = \{t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$\begin{aligned}(q, L, U) = \{&(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (2, 1, 2)\\ &(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (2, 1, 1)\\ &(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (2, 2, 2)\}\,)\end{aligned}$$

Under the true preference profiles for each student,

$$\begin{aligned}&P_{s_1}: t_1\ t_2\ t_3\\ &P_{s_2}: t_2\ t_1\ t_3\\ &P_{s_3}: t_1\ t_1\ t_1\\ &P_{s_4}: t_3\ t_1\ t_2\\ &P_{s_5}: t_3\ t_1\ t_2\end{aligned}$$

and the priority orderings for each supervisor,

$$\begin{aligned}&\succ_{t_1}: s_2\ s_1\ s_3\ s_4\ s_5\\ &\succ_{t_2}: s_2\ s_3\ s_1\ s_4\ s_5\\ &\succ_{t_3}: s_4\ s_5\ s_3\ s_2\ s_1\end{aligned}$$

the resulting final matching μ^{en} with DAMin with an *endogenous* reduction sequence is as follows:

$$\mu^{en} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1\ s_3 & s_2 & s_4,\ s_5 \end{pmatrix}$$

For DAMin with an *exogenous* reduction sequence, suppose that the sequence is given as $\{U^1 = (2, 1, 2), U^2 = (1, 1, 1)\}$. Then, the resulting final matching μ^{ex} is as follows:

$$\mu^{ex} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1\ s_3 & s_2 & s_4,\ s_5 \end{pmatrix}$$

Both matchings are the same. Thus, there is no Pareto domination relationship among both types of DAMin. *Q.E.D.*

Proposition 9. *There is no cardinal domination relationship between both types of DAMin.*

Proof. We show the statement with the following example. There are six students ($S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3, s_6\}$ and $S_{\theta_2} = \{s_4, s_5\}$. There are three supervisors ($T = \{t_1, t_2, t_3\}$), divided into $T_{\theta_1} = \{t_1\}$ and $T_{\theta_2} = \{t_2, t_3\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$\begin{aligned}(q, L, U) = \{&(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (4, 1, 3)\\ &(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (2, 1, 2)\end{aligned}$$

$(q_{t_3}, L_{t_3,\theta_2}, U_{t_3,\theta_2}) = (2, 1, 2)\}$

Under the true preference profiles for each student,

$P_{s_1}: t_1\ t_2\ t_3$
$P_{s_2}: t_1\ t_3\ t_2$
$P_{s_3}: t_1\ t_2\ t_3$
$P_{s_4}: t_3\ t_2\ t_1$
$P_{s_5}: t_3\ t_2\ t_1$
$P_{s_6}: t_1\ t_2\ t_3$

and the priority orderings for each supervisor,

$\succ_{t_1}: s_6\ s_1\ s_3\ s_2\ s_4\ s_5$
$\succ_{t_2}: s_4\ s_3\ s_2\ s_1\ s_6\ s_5$
$\succ_{t_3}: s_4\ s_2\ s_5\ s_1\ s_3\ s_6$

the resulting final matching with DAMin with the *endogenous* sequence μ^{en} is as follows (note that the priority structures have a cyclic structure):

$$\mu^{en} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1, s_3, s_6 & s_5 & s_2, s_4 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 18. For DAMin with an *exogenous* reduction sequence, suppose that the sequence is given as $\{U^1 = (3, 2, 2), U^2 = (3, 2, 1)\}$. Then, the resulting final matching with DAMin with an *endogenous* sequence μ^{ex} is

$$\mu^{ex} = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_1, s_2, s_3, s_6 & s_5 & s_4 \end{pmatrix}.$$

The sum of the ranks of students and supervisors is 24.

On the contrary, the following example shows that the sum of the ranks of students and supervisors under DAMin with an *exogenous* reduction sequence is higher than those under DAMin with an *endogenous* one.

There are seven students ($S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $S_{\theta_2} = \{s_7\}$. There are four supervisors ($T = \{t_1, t_2, t_3, t_4\}$), divided into $T_{\theta_1} = \{t_1, t_2, t_3\}$ and $T_{\theta_2} = \{t_4\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$(q, L, U) = \{(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (3, 1, 3)$
$(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (2, 1, 2)$
$(q_{t_3}, L_{t_3,\theta_1}, U_{t_3,\theta_1}) = (3, 1, 3)\}$
$(q_{t_4}, L_{t_4,\theta_4}, U_{t_4,\theta_2}) = (1, 1, 1)\}$

Under the true preference profiles for each student,

$P_{s_1}: t_1\ t_2\ t_3\ t_4$
$P_{s_2}: t_3\ t_1\ t_2\ t_4$
$P_{s_3}: t_1\ t_2\ t_3\ t_4$
$P_{s_4}: t_3\ t_2\ t_1\ t_4$
$P_{s_5}: t_3\ t_2\ t_1\ t_4$

$P_{s_6}: t_1\ t_2\ t_3\ t_4$
$P_{s_6}: t_4\ t_2\ t_3\ t_1$

and the priority orderings for each supervisor,

$\succ_{t_1}: s_6\ s_1\ s_4\ s_2\ s_3\ s_5\ s_7$
$\succ_{t_2}: s_5\ s_4\ s_2\ s_1\ s_6\ s_3\ s_7$
$\succ_{t_3}: s_4\ s_2\ s_5\ s_1\ s_3\ s_6\ s_7$
$\succ_{t_3}: s_7\ s_2\ s_5\ s_1\ s_3\ s_6\ s_4$

the resulting final matching with DAMin with an *endogenous* sequence μ^{en} is as follows (note that the priority structures have a cyclic structure):

$$\mu^{en} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_1, s_6 & s_3 & s_2, s_4, s_5 & s_7 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 25. For DAMin with an *exogenous* reduction sequence, suppose that the sequence is given as $\{U^1 = (3, 2, 3,1), U^2 = (3, 2, 2,1)\}$. Then, the resulting final matching with DAMin with an *endogenous* sequence μ^{ex} is as follows:

$$\mu^{ex} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_1, s_3, s_6 & s_5 & s_2, s_4 & s_7 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 21. *Q.E.D.*

Proposition 10. *There are no cardinally domination relationships between DAMin and ESDA for any matching market.*

Proof. We show the statement with the following example. There are six students ($S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3, s_4, s_5\}$ and $S_{\theta_2} = \{s_6\}$. There are four supervisors ($T = \{t_1, t_2, t_3, t_4\}$), divided into $T_{\theta_1} = \{t_1, t_2, t_3\}$ and $T_{\theta_2} = \{t_4\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$\begin{aligned}(q, L, U) = \{&(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (3, 1, 3)\\ &(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (2, 1, 2)\\ &(q_{t_3}, L_{t_3,\theta_1}, U_{t_3,\theta_1}) = (2, 1, 2)\\ &(q_{t_4}, L_{t_4,\theta_2}, U_{t_4,\theta_2}) = (1, 1, 1)\}\end{aligned}$$

Under the true preference profiles for each student,

$P_{s_1}: t_1\ t_2\ t_3\ t_4$
$P_{s_2}: t_1\ t_2\ t_3\ t_4$
$P_{s_3}: t_1\ t_2\ t_3\ t_4$
$P_{s_4}: t_2\ t_3\ t_1\ t_4$
$P_{s_5}: t_2\ t_1\ t_3\ t_4$
$P_{s_6}: t_4\ t_1\ t_3\ t_2$

and the priority orderings for each supervisor,

$\succ_{t_1}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6$
$\succ_{t_2}: s_5\ s_4\ s_3\ s_2\ s_1\ s_6$
$\succ_{t_3}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6$

$\succ_{t_3}: s_6\ s_1\ s_2\ s_3\ s_4\ s_5$

the resulting final matching with DAMin μ^{Min} is as follows:

$$\mu^{Min} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_1, s_2 & s_4, s_5 & s_3 & s_6 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 18. The resulting final matching with ESDA μ^{ESDA} is as follows:

$$\mu^{ESDA} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_1, s_2, s_3 & s_4 & s_4 & s_6 \end{pmatrix}.$$

The sum of the ranks of students and supervisors is 19. On the contrary, in the following example, ESDA cardinally dominates DAMin.

There are six students ($S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$) and two types ($\Theta = \{\theta_1, \theta_2\}$); hence, students are divided into $S_{\theta_1} = \{s_1, s_2, s_3, s_4, s_5\}$ and $S_{\theta_2} = \{s_6\}$. There are four supervisors ($T = \{t_1, t_2, t_3, t_4\}$), divided into $T_{\theta_1} = \{t_1, t_2, t_3\}$ and $T_{\theta_2} = \{t_4\}$. The total quotas and type-specific minimal and maximal quotas are as follows:

$$\begin{aligned}(q, L, U) = \{&(q_{t_1}, L_{t_1,\theta_1}, U_{t_1,\theta_1}) = (3, 1, 3)\\ &(q_{t_2}, L_{t_2,\theta_1}, U_{t_2,\theta_1}) = (2, 1, 2)\\ &(q_{t_3}, L_{t_3,\theta_1}, U_{t_3,\theta_1}) = (2, 1, 2)\\ &(q_{t_4}, L_{t_4,\theta_2}, U_{t_4,\theta_2}) = (1, 1, 1)\})\end{aligned}$$

Under the true preference profiles for each student,

$P_{s_1}: t_1\ t_2\ t_3\ t_4$
$P_{s_2}: t_1\ t_2\ t_3\ t_4$
$P_{s_3}: t_1\ t_2\ t_3\ t_4$
$P_{s_4}: t_2\ t_3\ t_1\ t_4$
$P_{s_5}: t_2\ t_1\ t_3\ t_4$
$P_{s_6}: t_4\ t_1\ t_3\ t_2$

and the priority orderings for each supervisor,

$\succ_{t_1}: s_1\ s_2\ s_3\ s_4\ s_5\ s_6$
$\succ_{t_2}: s_5\ s_4\ s_3\ s_2\ s_1\ s_6$
$\succ_{t_3}: s_4\ s_1\ s_2\ s_3\ s_5\ s_6$
$\succ_{t_4}: s_6\ s_1\ s_2\ s_3\ s_5\ s_4$

the resulting final matching with DAMin μ^{Min} is as follows:

$$\mu^{Min} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_1, s_2 & s_4, s_5 & s_3 & s_6 \end{pmatrix}$$

The sum of the ranks of students and supervisors is 19. The resulting final matching with ESDA μ^{ESDA} is as follows:

$$\mu^{ESDA} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_1, s_2, s_3 & s_5 & s_4 & s_6 \end{pmatrix}.$$

The sum of the ranks of students and supervisors is 16. *Q.E.D.*

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