Ehrenfeucht-Fraïssé games without identity

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Abstract

Ehrenfeucht-Fraïssé games are usually formulated for a language including identity. In this note, we develop a formulation of the games for languages without identity. The new version is used to show that the identity relation on a structure cannot be characterized if identity is missing in the language.

1 Introduction

Ehrenfeucht-Fraïssé games are used extensively in model theory, especially in finite model theory. Many standard techniques, such as the compactness theorem, fail in the case of finite models, but Ehrenfeucht-Fraïssé games still work. The usual definition of these games is designed for applications to first-order logic with identity. There does not seem to be a formulation of these games in the literature that is adapted to for first-order logic without identity. The object of this note is to present such a formulation; we apply it to show that there is no first-order theory, in which the only atomic predicate is a two-place relation R, that holds of a structure $\langle X, R\rangle$, where $R \subseteq X^2$, if and only if R is the identity relation on X.

2 Games for first order logic without identity

In the original paper [3] introducing the games later known as Ehrenfeucht-Fraïssé games, Ehrenfeucht included identity as a logical predicate. Monographs on finite model theory, such as those by Ebbinghaus and Flum [2] and Libkin [5], or the collection *Finite Model Theory and its Applications* [4], all work in the tradition where the basic language is first-order logic with identity. Here we formulate games where identity is not included in the descriptive language. Definition 2.1 is stated in terms of a language with a single binary predicate, but can be generalized to more extensive languages.

Definition 2.1 *The Ehrenfeucht-Fraïssé game* without identity *is played as follows:*

1. *There are two players, Spoiler and Duplicator, who play k rounds;*
2. *The board on which the game is played consists of two digraphs $G = \langle X, R_1 \rangle$ and $H = \langle Y, R_2 \rangle$;*
3. *The Spoiler moves first by choosing a digraph, and then choosing a vertex in it; the Duplicator replies with a vertex from the other digraph;*
4. *After round j, $j \leq k$, the score of the game is $(a_1, b_1), \ldots, (a_j, b_j)$, where $\{a_i, b_i\}$ are the vertices chosen by the players at round i, $a_i \in X$ and $b_i \in Y$;*
5. *The Spoiler wins if at round $i \leq k$, there are a_p, a_q and b_p, b_q, where $p, q \leq i$, so that $\neg[G \models R_1(a_p, a_q) \Leftrightarrow H \models R_2(b_p, b_q)]$; otherwise the Duplicator wins the k-round game.*

Let L be the first-order language with a two-place relation Rxy as the only atomic predicate, as well as a set of constant symbols. We assume that the sentences of the language are given in prenex normal form; the number of quantifiers in such sentences is the *quantifier rank* of the sentence.

Theorem 2.1 *Let $G = \langle X, R_1 \rangle$ and $H = \langle Y, R_2 \rangle$ be two digraphs. If there is a sentence A in L of quantifier rank k so that $G \models A$ and $H \models \neg A$, then the Spoiler has a winning strategy for the k-round game on G and H.*

Proof. Add to the language L constants $\{\mathbf{x_1}, \ldots, \mathbf{x_p}\}$ and $\{\mathbf{y_1}, \ldots, \mathbf{y_q}\}$ denoting the elements of X and Y. Now assume that the sentence A has the form $\exists w B$. Since $G \models A$, there is an element x_i of G so that $G \models B[\mathbf{x_i}/w]$. The Spoiler chooses x_i in G as the first move. By assumption, $H \models \forall w \neg B$, so if y_j is the element of H chosen by the Duplicator in reply, then $H \models \neg B[\mathbf{y_j}/w]$. If A has the form $\forall w B$, then the Spoiler plays in the digraph H instead.

After k moves in the game, the result is a quantifier-free sentence C so that $G \models C$ and $H \models \neg C$, from which it follows that there are atomic sentences $R_1(a_p, a_q)$ and $R_2(b_p, b_q)$ that differ in their truth-values in G and H; hence, the Spoiler wins the k-round game. $\square$

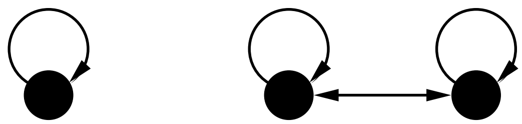

Figure 1: The digraphs G and H

3 Identity is not expressible

In this section, we apply the results of §2 to prove the main result of this note.

Theorem 3.1 *There is no elementary theory (without identity) that is true exactly for those digraphs* $G = \langle X, R \rangle$ *where* $G \models \forall x \forall y (Rxy \Leftrightarrow x = y)$.

Proof. Let G be the digraph $\langle X, R_1 \rangle$, where $|X| = 1$ and R_1 is the identity relation on X, and let H be the digraph $\langle Y, R_2 \rangle$, where $|Y| = 2$ and $R_2 = Y^2$. Thus G is the complete digraph on a unit universe, and H is the complete digraph on a two-element universe. Figure 1 shows a diagram of both structures.

For any k, the Duplicator wins the Ehrenfeucht-Fraïssé game without identity on G and H. Both G and H satisfy the sentence $\forall x \forall y Rxy$, and so no matter what moves the Spoiler and Duplicator make, the Spoiler can never reach a winning position. Hence, the Duplicator can reply randomly, always winning the k-round game.

Assume that there is an elementary theory $\mathcal{T}$ expressed in first-order logic with the predicate R that is true exactly for the models in which R is the identity relation; hence $G \models \mathcal{T}$. We can show in addition that $H \models \mathcal{T}$. If $H \not\models T$, then there is a sentence A of quantifier rank k so that $G \models A$, but $H \models \neg A$. By Theorem 2.1, this contradicts the fact that the Duplicator has a winning strategy for the k-round game. Hence, no such elementary theory as $\mathcal{T}$ can exist. In fact, the proof shows that G and H are elementarily equivalent in the purely relational language. $\square$

The question answered in §3 was posed by Jean-Yves Béziau [1]. My thanks to Jean-Yves for an interesting problem!

References

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