

Non-representable relation algebras from vector spaces

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Abstract

Extending a construction of Andr eka, Givant, and N emeti (2019), we construct some finite vector spaces and use them to build finite non-representable relation algebras. They are simple, measurable, and persistently finite, and they validate arbitrary finite sets of equations that are valid in the variety RRA of representable relation algebras. It follows that there is no finitely axiomatisable class of relation algebras that contains RRA and validates every equation that is both valid in RRA and preserved by completions of relation algebras. Consequently, the variety generated by the completions of representable relation algebras is not finitely axiomatisable. This answers a question of Maddux (2018).

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1 Introduction

This paper lies in the field of algebraic logic. It is concerned, at least on the face of it, with certain varieties of relation algebras lying between the variety RRA of representable relation algebras and the variety RA of all relation algebras.

The main one is the variety generated by the completions of algebras in RRA . It is an almost brand new variety, introduced by Maddux in [24], in which he denoted it by V . Every relation algebra is a subalgebra of its completion, which is a relation algebra, and it follows that $RRA \subseteq V \subseteq RA$. It was shown in [14] that RRA is not closed under completions, so $RRA \subsetneq V$. In [24, problem 1.1], Maddux asked six questions about V :

*This paper is dedicated to Rob Goldblatt on the occasion of his retirement. Thanks very much to Ed Mares for inviting this contribution and managing the process.

1. Is $V = \text{RA}$?
2. Is V finitely axiomatizable?
3. Is V closed under canonical extensions?
4. Is V closed under completions?
5. Is membership in V decidable for finite algebras?
6. Does V contain any algebras that are not weakly representable (as defined in [17])?

Already in his paper, Maddux proposed a way to answer the first question negatively. He suggested that finite (simple non-representable) relation algebras with no proper simple extensions were candidates for relation algebras that are not in V . It may seem surprising that such algebras could exist, but examples have been known since Frias and Maddux's paper [4].

Maddux's suggestion was soon taken up by Andr eka and N emeti, who showed in [2] that no non-representable simple persistently finite relation algebra can be in V . A relation algebra is *persistently finite* if it is finite and every simple relation algebra extending it is finite. Obviously, a finite relation algebra with no proper simple extensions is persistently finite, so Andr eka and N emeti's result validates and generalises Maddux's suggested approach.

Several kinds of non-representable simple persistently finite relation algebras are now known. As well as the examples from [4] already mentioned, the finite simple non-representable 'coset relation algebras' constructed by Andr eka, Givant, and N emeti in [3] also pass muster, since they were shown to be persistently finite in [2, theorem 3.1] and the remarks following it. There are infinitely many algebras of both kinds, and each of them witnesses that $V \neq \text{RA}$. Indeed, [2] goes on to show that there are continuum-many varieties between V and RA . So the answer to Maddux's first question is resoundingly negative.

On seeing these relation algebras, one wonders if they could also be used to answer Maddux's second question, on whether V is finitely axiomatisable. This could be done if they are *arbitrarily representable* in the sense that while being non-representable, there are algebras among them validating any desired finite subset of the equations defining RRA . Unfortunately, the examples from [4] all violate a particular equation '(L)' [4, theorem 7.1] known to be valid in RRA , and it appears that the examples from [3] do too.

In this paper, we extend the construction in [3]. We exhibit non-representable simple persistently finite relation algebras that are indeed arbitrarily representable. We know from [2] that they are not in V , but a nonprincipal ultraproduct of them will be in RRA and hence in V , from which we can deduce by  os's theorem that V is not finitely axiomatisable (theorem 4.10). So Maddux's second question also receives a negative answer.

In fact, we will take a slightly more general tack. Let $\mathcal{C}(\text{RRA})$ be the variety defined by all equations that are valid in RRA and preserved by completions of relation algebras. Then $\text{RRA} \subsetneq V \subseteq \mathcal{C}(\text{RRA}) \subseteq \text{RA}$. Extending [2, theorem 4.3(iv)], we will show (in corollary 4.8) that no non-representable simple persistently finite relation algebra can lie in

$\mathcal{C}(\text{RRA})$, yielding of course that $\mathcal{C}(\text{RRA}) \subsetneq \text{RA}$. We then conclude by the same ultraproduct argument, using the same algebras, that no variety lying between RRA and $\mathcal{C}(\text{RRA})$ is finitely axiomatisable (theorem 4.9). This includes V as well as $\mathcal{C}(\text{RRA})$ (and RRA).

These results are our main retail items, but our means of production may also be of interest. The simple persistently finite non-representable relation algebras constructed in [3] were in fact *coset relation algebras*. These form a sophisticated generalisation of group relation algebras, and are related to measurable relation algebras (see, e.g., [6]). They have been studied by Andr eka, Givant and N emeti in a long series of papers including [6]. References can be found in [3], together with a helpful introduction to the subject. For each finite abelian group \mathfrak{F} , [3] constructed a finite non-representable coset relation algebra from the group \mathfrak{F}^3 . When \mathfrak{F} is the additive group of a finite field of characteristic 2, this is the ground case of an infinite sequence of relation algebras based on the additive groups of vector spaces constructed in §§2–3 of this paper. They are non-representable simple persistently finite relation algebras, and they are arbitrarily representable.

In summary, we hope that the paper may be of use to those interested in:

- completions of relation algebras,
- non-finite axiomatisability of varieties of relation algebras,
- non-representable but arbitrarily representable simple persistently finite relation algebras,
- coset relation algebras (or measurable relation algebras),
- finite-dimensional vector spaces over finite fields.

We repeat that the work in this article is based on [3], and owes a great debt to it.

2 Vector spaces

Fix, for the rest of the paper, a finite field \mathcal{F} of characteristic 2 — so $\lambda + \lambda = 0$ for every λ in \mathcal{F} . The simplest example is of course \mathbb{Z}_2 , the quotient of the integers \mathbb{Z} by the even integers $2\mathbb{Z}$. We are going to define and study some vector spaces over \mathcal{F} . In the end, we will be interested only in their abelian group structure, but using vector spaces gives us access to useful devices such as bases, dimension, and scalar (or inner) product. Finiteness of \mathcal{F} is actually not needed at all in this section, but it is needed later to construct finite relation algebras from our vector spaces. Some, but perhaps not all, of what we do generalises to fields of other characteristics, and we will discuss this very briefly in the conclusion.

We use some standard notation. For a set S and non-negative integers k, n , we write $S^{[k]}$ for the set $\{T \subseteq S : |T| = k\}$ of k -element subsets of S , and $\binom{n}{k}$ for $|n^{[k]}|$, where $n = \{0, 1, \dots, n-1\}$ as usual. For sets S, T , we write ${}^S T$ for the set of maps $f : S \rightarrow T$.

2.1 The vector space $V_n^{(d)}$

We now come to one of the main definitions of the paper.

DEFINITION 2.1 Let $d \geq 1$ and $n \geq 0$ be integers. We define the following items.

1. Let $K_n^{(d)}$ be a set of cardinality $n + d - 1$. It is empty iff $n = 0$ and $d = 1$. This case is rather degenerate and not very useful, but we will keep an eye on it out of courtesy to the reader.
2. Let $E_n^{(d)} = (K_n^{(d)})^{[d]}$. Thus, $(K_n^{(d)}, E_n^{(d)})$ is the complete d -dimensional hypergraph over the set $K_n^{(d)}$ of nodes, with hyperedges of cardinality d . We will refer to these hyperedges (elements of $E_n^{(d)}$) simply as *edges*.
3. Let $V_n^{(d)}$ be the set $E_n^{(d)}\mathcal{F}$ of all maps from $E_n^{(d)}$ to \mathcal{F} . We regard $V_n^{(d)}$ as a vector space over \mathcal{F} in the usual way by putting $(v + w)(e) = v(e) + w(e)$ and $(\lambda v)(e) = \lambda \cdot v(e)$, for $v, w \in V_n^{(d)}$, $\lambda \in \mathcal{F}$, and $e \in E_n^{(d)}$.

Where d, n are fixed, we will drop the indices $\binom{d}{n}$ and write simply K, E, V for the items defined above. Do not confuse this V with Maddux's variety mentioned in the introduction. We use $a, b, p, q, r, x, y, z, \dots$ to denote elements of K , e, e' to denote elements of E , and v, u, w, g, h, k, \dots to denote elements of (vectors in) V .

It is clear that the dimension $\dim V_n^{(d)}$ of the vector space $V_n^{(d)}$ is equal to $|E_n^{(d)}|$, which is $|(K_n^{(d)})^{[d]}| = \binom{n+d-1}{d}$. When $n \geq 1$, this is equal to $\binom{n+d-1}{n-1}$. So that we can see what we are up against, table 1 lists a few values of this dimension.

$d \setminus n$	0	1	2	3	4	5	n
1	0	1	2	3	4	5	$\binom{n}{1}$
2	0	1	3	6	10	15	$\binom{n+1}{2}$
3	0	1	4	10	20	35	$\binom{n+2}{3}$
d	0	1	$d+1$	$\binom{d+2}{d}$	$\binom{d+3}{d}$	$\binom{d+4}{d}$	$\binom{n+d-1}{d}$

Table 1: Some values of $\dim V_n^{(d)}$

When $n = 0$, we have $|K_0^{(d)}| = d - 1$, so $E_0^{(d)} = \emptyset$. Then $V_0^{(d)} = \emptyset\mathcal{F} = \{\emptyset\}$ since \emptyset is the unique function from \emptyset to \mathcal{F} . This is the degenerate 0-dimensional vector space over \mathcal{F} .

Later, n will mainly be 5, but allowing arbitrary n has its uses, takes no more effort to handle, and in fact makes things clearer.

2.2 Some useful notation

For the rest of §2, we fix integers $d \geq 1$ and $n \geq 0$. Unadorned K, E, V will denote $K_n^{(d)}$, $E_n^{(d)}$, and $V_n^{(d)}$, respectively.

DEFINITION 2.2

1. For $S \subseteq E$, we write $\chi_S : E \rightarrow \mathcal{F}$ for the characteristic function of S :

$$\chi_S(e) = \begin{cases} 1, & \text{if } e \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for $e \in E$. Here, 0 and 1 are taken in the field \mathcal{F} . So $\chi_S \in V$.

2. Plainly, $\mathcal{B} = \{\chi_{\{e\}} : e \in E\}$ is a basis of V , and we call it the *standard basis*.

3. For $W \subseteq V$, we write $\langle W \rangle$ for the subspace of V spanned (generated) by W .

As examples, $v = \sum_{e \in E} v(e) \cdot \chi_{\{e\}} \in \langle \mathcal{B} \rangle$ for $v \in V$, and $\chi_S = \sum_{e \in S} \chi_{\{e\}}$ for $S \subseteq E$. The subspace $\langle W \rangle$ is the set of linear combinations of elements of W .

We now pick out some useful sets of edges.

DEFINITION 2.3 For $X \subseteq K$, we define

- $X\uparrow = \{e \in E : X \subseteq e\}$,
- $X\downarrow = \{e \in E : e \subseteq X\}$.

The latter is of course $X^{[d]}$, but the notation \downarrow is more to the point here. For $a \in K$, we write $a\uparrow$ as short for $\{a\}\uparrow$.

2.3 The subspaces L_a of V ($a \in K$)

Now we define the first of two kinds of subspace of V .

DEFINITION 2.4

1. For $v \in V$, we write $\text{supp}(v) = \{e \in E : v(e) \neq 0\}$ — the support of v .
2. For $a \in K$, we define $L_a = \{v \in V : \text{supp}(v) \subseteq a\uparrow\}$, and $\mathcal{B}_a = \{\chi_{\{e\}} : e \in a\uparrow\}$.

Equivalently, $L_a = \{v \in V : \forall e \in E (v(e) \neq 0 \rightarrow a \in e)\}$. When K is empty, no L_a are defined.

LEMMA 2.5 *Let $a \in K$. Then $L_a = \langle \mathcal{B}_a \rangle$.*

Proof. We show first that L_a is a subspace of V . If $v, w \in V$ and $\lambda \in \mathcal{F}$ then $\text{supp}(v+w) \subseteq \text{supp}(v) \cup \text{supp}(w)$ and $\text{supp}(\lambda v) \subseteq \text{supp}(v)$. It follows by definition of L_a that if $v, w \in L_a$ and $\lambda \in \mathcal{F}$ then $v + w, \lambda v \in L_a$, as required.

Plainly, $\mathcal{B}_a \subseteq L_a$, so $\langle \mathcal{B}_a \rangle \subseteq L_a$ since L_a is a subspace of V . Conversely, if $v \in L_a$, then since $\text{supp}(v) \subseteq a\uparrow$, so $v(e) = 0$ whenever $e \in E \setminus a\uparrow$, we see that

$$v = \sum_{e \in E} v(e) \chi_{\{e\}} = \sum_{e \in a\uparrow} v(e) \chi_{\{e\}} \in \langle \mathcal{B}_a \rangle.$$

So $\langle \mathcal{B}_a \rangle = L_a$, proving the lemma. □

Since \mathcal{B}_a is a subset of the standard basis \mathcal{B} of V , it is linearly independent. Hence, it is actually a basis of L_a , giving

$$\dim L_a = |\mathcal{B}_a| = |a\uparrow| = |(K \setminus \{a\})^{[d-1]}| = \binom{n+d-2}{d-1} (= \dim V_n^{(d-1)} \text{ if } d > 1).$$

To see the third equality here, observe that $e \mapsto e \setminus \{a\}$ is a bijection from $a\uparrow$ to $(K \setminus \{a\})^{[d-1]}$.

2.4 The diagonal subspace L_Δ

One more ‘diagonal’ subspace of V is defined as follows. We take Δ to be a special symbol not in any set $K_n^{(d)}$.

DEFINITION 2.6 Let L_Δ be the subspace of V spanned by $\mathcal{X} \stackrel{\text{def}}{=} \{\chi_{(X\uparrow)} : X \in K^{[d-1]}\}$.

The definition of L_Δ is very different from that of the L_a ($a \in K$), but it is really part of the same family. We will not need to show that every permutation of $\{L_x : x \in K \cup \{\Delta\}\}$ is induced by an automorphism of V , but it is true, and a related result is proved in [3, lemma 3.1]. It follows that L_Δ has the same dimension as the L_a ($a \in K$).

If K is empty, so $n = 0$ and $d = 1$, then $\mathcal{X} = \{\chi_{(X\uparrow)} : X \in \emptyset^{[0]}\} = \{\chi_{(\emptyset\uparrow)}\} = \{\chi_E\}$. Here, $E = \emptyset^{[1]} = \emptyset$ too, so $\mathcal{X} = \{\emptyset\}$. But $V = {}^E\mathcal{F} = \{\emptyset\}$ as well. So $L_\Delta = \langle \mathcal{X} \rangle = V$. We see that L_Δ is defined in this case, but both it and V are degenerate.

2.5 Sums of the subspaces

We now examine combinations of these subspaces. Our findings will be useful in §3.

DEFINITION 2.7 For $S \subseteq K \cup \{\Delta\}$, we define $L_S = \sum_{x \in S} L_x \subseteq V$.

Each L_x ($x \in K \cup \{\Delta\}$) is a subspace of V , so L_S is also a subspace of V . Several times we will use the obvious fact that if $S \subseteq S' \subseteq K \cup \{\Delta\}$ then $L_S \subseteq L_{S'}$.

In the case where $S \subseteq K$, it can be checked that for each $v \in V$ we have $v \in L_S$ iff $v(e) = 0$ for all $e \in E$ disjoint from S . We will not need this. However, we will need (in propositions 3.13 and 3.15) a similar characterisation of when an arbitrary $v \in V$ is in $L_{S \cup \{\Delta\}}$, and we give one in proposition 2.12. The two lemmas coming up are needed to prove this proposition. The first one uses that \mathcal{F} has characteristic 2.

As usual, for vectors $v, w \in V$, we define their *scalar product* $v \cdot w = \sum_{e \in E} (v(e) \cdot w(e)) \in \mathcal{F}$, where the \sum and \cdot on the right are taken in the field \mathcal{F} . As examples, for $v \in V$, if $e \in E$ then $v \cdot \chi_{\{e\}} = v(e)$, and if $T \subseteq E$ then $v \cdot \chi_T = \sum_{e \in T} v(e)$. Scalar product is linear in both arguments.

Frequently we will write X for a subset of K of size $d - 1$, and Y for one of size $d + 1$.

LEMMA 2.8 Let $S \subseteq K$ and let $Y \in K^{[d+1]}$ be disjoint from S . Then $v \cdot \chi_{Y\downarrow} = 0$ for every $v \in L_{S \cup \{\Delta\}}$.

Proof. By lemma 2.5 and definitions 2.6 and 2.7, $L_{S \cup \{\Delta\}} = \langle \bigcup_{s \in S} \mathcal{B}_s \cup \mathcal{X} \rangle$. So by linearity of scalar product, it is sufficient to prove that $v \cdot \chi_{Y \downarrow} = 0$ for each $v \in \bigcup_{s \in S} \mathcal{B}_s \cup \mathcal{X}$.

If $v = \chi_{\{e\}} \in \mathcal{B}_s$ for some $s \in S$, then $e \notin Y \downarrow$ since $s \in e \setminus Y$, and so $v \cdot \chi_{Y \downarrow} = \chi_{Y \downarrow}(e) = 0$ as required.

Suppose that $v = \chi_{X \uparrow} \in \mathcal{X}$. Then $|X| = d - 1$, $|Y| = d + 1$, and edges have size d , and it follows that

$$|X \uparrow \cap Y \downarrow| = \begin{cases} 2, & \text{if } X \subseteq Y, \\ 0, & \text{otherwise.} \end{cases}$$

This value is always even. So $v \cdot \chi_{Y \downarrow} = \sum_{e \in X \uparrow \cap Y \downarrow} 1 = 0$, since \mathcal{F} has characteristic 2. \square

The following corollary will be used in definition 3.2.

COROLLARY 2.9 *Suppose that $S \subseteq K$ and $|S| \leq n - 2$. Then $L_{S \cup \{\Delta\}} \subsetneq V$.*

Proof. Since $|S| \leq n - 2$, we have $|K \setminus S| = |K| - |S| \geq (n + d - 1) - (n - 2) = d + 1$, so there exists $Y \in K^{[d+1]}$ disjoint from S . Take any edge $e \in Y \downarrow$. Then $\chi_{\{e\}} \cdot \chi_{Y \downarrow} = \chi_{Y \downarrow}(e) = 1$. So $\chi_{\{e\}} \in V \setminus L_{S \cup \{\Delta\}}$ by lemma 2.8. \square

DEFINITION 2.10 For $S \subseteq K$ and $a \in K \setminus S$, let $\mathcal{E}_S^a = \{\chi_{\{e\}} : e \in E, e \cap (S \cup \{a\}) = \emptyset\}$.

LEMMA 2.11 *Let $S \subseteq K$ and $a \in K \setminus S$. Then $V = L_{S \cup \{\Delta\}} + \langle \mathcal{E}_S^a \rangle$.*

Proof. We write \mathcal{E}_S^a simply as \mathcal{E} . It is enough to show that the standard basis \mathcal{B} of V is contained in $L_{S \cup \{\Delta\}} + \langle \mathcal{E} \rangle$. So let $\chi_{\{e\}} \in \mathcal{B}$. If $e \cap S \neq \emptyset$, then there is $s \in e \cap S$, and so $\chi_{\{e\}} \in L_s \subseteq L_{S \cup \{\Delta\}}$. If $e \cap (S \cup \{a\}) = \emptyset$, then $\chi_{\{e\}} \in \mathcal{E}$.

The remaining case is when e is disjoint from S and contains a . Let $X = e \setminus \{a\} \in K^{[d-1]}$. Now obviously,

$$\begin{aligned} X \uparrow &= \{X \cup \{b\} : b \in K \setminus X\} \\ &= \{e\} \cup \{X \cup \{s\} : s \in S\} \cup \{X \cup \{b\} : b \in K \setminus (S \cup \{a\})\}, \\ \text{so } \chi_{X \uparrow} &= \sum_{e' \in X \uparrow} \chi_{\{e'\}} = \chi_{\{e\}} + \sum_{s \in S} \chi_{\{X \cup \{s\}\}} + \sum_{b \in K \setminus (S \cup \{a\})} \chi_{\{X \cup \{b\}\}}. \end{aligned}$$

Rearranging this (bearing in mind that \mathcal{F} has characteristic 2) gives

$$\chi_{\{e\}} = \underbrace{\chi_{X \uparrow}}_{L_\Delta} + \underbrace{\sum_{s \in S} \chi_{\{X \cup \{s\}\}}}_{L_S} + \underbrace{\sum_{b \in K \setminus (S \cup \{a\})} \chi_{\{X \cup \{b\}\}}}_{\langle \mathcal{E} \rangle}. \quad (1)$$

Now $\chi_{X \uparrow} \in \mathcal{X} \subseteq L_\Delta$ by definition 2.6. For each $s \in S$ we have $\chi_{\{X \cup \{s\}\}} \in L_s$ by definition 2.4, since $X \cup \{s\} \in s \uparrow$. And for each $b \in K \setminus (S \cup \{a\})$ we have $\chi_{\{X \cup \{b\}\}} \in \mathcal{E}$ by definition 2.10. We conclude from (1) that $\chi_{\{e\}} \in L_{S \cup \{\Delta\}} + \langle \mathcal{E} \rangle$, as required. \square

As promised, we now characterise when a vector is in $L_{S \cup \{\Delta\}}$.

PROPOSITION 2.12 *Let $S \subseteq K$. Then for each $v \in V$, the following are equivalent:*

- L1. $v \in L_{S \cup \{\Delta\}}$,
- L2. $v \cdot \chi_{Y \downarrow} = 0$ for every $Y \in K^{[d+1]}$ disjoint from S ,
- L3. $S = K$, or for some $a \in K \setminus S$ we have $v \cdot \chi_{Y \downarrow} = 0$ for every $Y \in K^{[d+1]}$ disjoint from S and containing a .

Proof. Fix any $v \in V$. By lemma 2.8, L1 of the proposition implies L2. If L2 holds, then L3 trivially follows as a special case. Now assume L3; we will prove L1 — that is, $v \in L_{S \cup \{\Delta\}}$.

There are two cases. If $S = K$, then already $L_S = V$ — trivially if $K = \emptyset$, and also if $K \neq \emptyset$ since each member $\chi_{\{e\}}$ of the standard basis \mathcal{B} is in L_a for every $a \in e$, and hence in L_S . So certainly, $v \in L_{S \cup \{\Delta\}}$.

The second case is when $S \neq K$. By L3, we can choose $a \in K \setminus S$ with $v \cdot \chi_{Y \downarrow} = 0$ for every $Y \in K^{[d+1]}$ disjoint from S and containing a . Let $\mathcal{E} = \mathcal{E}_a^g$ as in definition 2.10. By lemma 2.11,

$$v = w + \sum_{\gamma \in \mathcal{E}} \lambda_\gamma \gamma \quad (2)$$

for some $w \in L_{S \cup \{\Delta\}}$ and $\lambda_\gamma \in \mathcal{F}$. We show that each λ_γ is zero, so that $v = w \in L_{S \cup \{\Delta\}}$.

So take any $\gamma = \chi_{\{e\}} \in \mathcal{E}$. So $e \in E$ is disjoint from $S \cup \{a\}$. Let $Y = e \cup \{a\}$. Then $Y \in K^{[d+1]}$, and Y is disjoint from S and contains a . We have the following:

- $w \cdot \chi_{Y \downarrow} = 0$ by lemma 2.8.
- $\gamma \cdot \chi_{Y \downarrow} = \chi_{Y \downarrow}(e) = 1$ since $e \in Y \downarrow$.
- For every $\gamma' = \chi_{\{e'\}} \in \mathcal{E} \setminus \{\chi_{\{e\}}\}$, we have $e' \not\subseteq e$ and $a \notin e'$, so $e' \not\subseteq e \cup \{a\} = Y$, giving $e' \notin Y \downarrow$ and hence $\gamma' \cdot \chi_{Y \downarrow} = \chi_{Y \downarrow}(e') = 0$.

So by (2) and linearity of scalar product,

$$v \cdot \chi_{Y \downarrow} = w \cdot \chi_{Y \downarrow} + \sum_{\gamma' \in \mathcal{E}} \lambda_{\gamma'} (\gamma' \cdot \chi_{Y \downarrow}) = \lambda_\gamma (\gamma \cdot \chi_{Y \downarrow}) = \lambda_\gamma.$$

But $v \cdot \chi_{Y \downarrow} = 0$ by choice of a , so $\lambda_\gamma = 0$ as desired. □

Taking $S = \emptyset$ in proposition 2.12 yields $L_\Delta = \{v \in V : v \cdot \chi_{Y \downarrow} = 0 \text{ for every } Y \in K^{[d+1]}\}$, and this is an alternative definition of L_Δ .

We can also see that corollary 2.9 is optimal. For suppose that $S \subseteq K$ and $|S| \geq n - 1$. Then $|K \setminus S| \leq d$, and so no $Y \in K^{[d+1]}$ is disjoint from S . Consequently, for every $v \in V$, L2 of proposition 2.12 holds vacuously, and hence L1 holds. We conclude that $L_{S \cup \{\Delta\}} = V$.

3 Relation algebras from vector spaces

We now fix $n = 5$, but $d \geq 1$ remains arbitrary. Thus, K, E, V denote $K_5^{(d)}, E_5^{(d)}$, and the vector space $V_5^{(d)}$, respectively. So $|K| = d + 4$ and $\dim V = \binom{d+4}{d}$, or equally, $\binom{d+4}{4}$. Since the field \mathcal{F} is finite, so is V . We will still be using the internal structure of V , so the reader will need to bear in mind more of §2 than just the bare results proved there.

In this long section, we will define a finite relation algebra $\mathcal{A}^{(d)}$ over V , and prove some things about it: to wit, it has a $(d + 3)$ -dimensional cylindric basis but no $(d + 4)$ -dimensional relational basis, and it is simple, measurable, and persistently finite. These results may be of interest in their own right, irrespective of the applications they find in §4. We will recall the relevant relation algebra concepts as we go. Readers who would like an introduction to relation algebras are referred to the introduction of [24], which is worth reading in any case, and the survey and books [22, 23, 13], but many other options exist in the literature.

3.1 Non-associative, weakly associative, and relation algebras

A *non-associative algebra* is an algebra of the form

$$\mathcal{A} = (A, +, -, 0, 1, 1', \smile, ;),$$

where $(A, +, -, 0, 1)$ is a boolean algebra, $1'$ is a constant called identity, \smile is a unary function called converse, $;$ is a binary function called composition, $a; 1' = 1'; a = a$ holds for every $a \in A$, and the three conditions $a \cdot (b; c) = 0$, $b \cdot (a; c) = 0$, and $c \cdot (b; a) = 0$ are equivalent for every $a, b, c \in A$. Here, as usual, $a \cdot b$ denotes $-(-a + -b)$. We define the usual partial ordering \leq on A by $a \leq b$ iff $a + b = b$. We say that \mathcal{A} is a *weakly associative algebra* if $(a; 1); 1 = a; (1; 1)$ for every $a \in A$ with $a \leq 1'$, and a *relation algebra* if the operation $;$ is associative. Obviously, every relation algebra is weakly associative.

Whilst these definitions are not equational, they can be replicated by equations, and so the classes of non-associative algebras and weakly associative algebras and the class RA of relation algebras are varieties — classes of algebras defined by equations [19, corollary 1.5].

Examples of relation algebras are algebras of the form

$$\mathfrak{Rc}(U) = (\wp(U \times U), \cup, \sim, \emptyset, U \times U, Id_U, -^{-1}, |),$$

where U is a set, $Id_U = \{(u, u) : u \in U\}$, and for each $a, b \subseteq U \times U$, $a^{-1} = \{(v, u) : (u, v) \in a\}$, and $a | b = \{(u, v) : (u, w) \in a \text{ and } (w, v) \in b \text{ for some } w \in U\}$. So $\mathfrak{Rc}(U)$ is the set of all binary relations on U , endowed with the indicated operations.

The class RRA of *representable relation algebras* is defined to be $\mathbf{SP}\{\mathfrak{Rc}(U) : U \text{ a set}\}$, where \mathbf{P} and \mathbf{S} denote closure under products and isomorphic copies of subalgebras, respectively. It is easy to see that $\mathbf{RRA} \subseteq \mathbf{RA}$. We say that a relation algebra is *representable* if it is in RRA. Non-trivially, RRA is a non-finitely axiomatisable variety [25, 27].

We import some boolean algebra notions to non-associative algebras. Let $\mathcal{A} = (A, +, -, 0, 1, 1', \smile, ;)$ be a non-associative algebra. An *atom* of \mathcal{A} is a \leq -minimal non-zero element

of A , and \mathcal{A} is said to be *atomic* if for every non-zero element $a \in A$, there is an atom b of \mathcal{A} with $b \leq a$. All finite non-associative algebras are atomic.

We say that \mathcal{A} is *complete* if every set S of elements of A has a sum $\sum S$ in \mathcal{A} — the least upper bound of S with respect to \leq . By the De Morgan laws, it is of course equivalent to say that every $S \subseteq A$ has a product $\prod S$ — its greatest lower bound with respect to \leq . As examples, $\mathfrak{Re}(U)$ is of course always complete, and all finite non-associative algebras are complete.

3.2 The algebra $\mathcal{A}^{(d)}$

We now define a finite non-associative algebra $\mathcal{A}^{(d)}$ over V , which we will show later (in corollary 3.14) to be a relation algebra. The algebra $\mathcal{A}^{(d)}$ is what is known as the full complex algebra over a certain atom structure, but we will define it directly. It will help to bear in mind that $|K| = d + 4 \geq 5$. We use the following convention frequently.

NOTATION 3.1 For $x_1, \dots, x_m \in K \cup \{\Delta\}$, we write $L_{x_1 \dots x_m}$ as short for $L_{\{x_1, \dots, x_m\}}$ (see definition 2.7). So, for example, $L_{xy}, L_{yx}, L_{xxy}, \dots$, all denote $L_{\{x, y\}}$.

DEFINITION 3.2

1. Fix pairwise distinct elements $p, q, r \in K$.
2. Fix a vector $c_S \in V$ for each $S \subseteq K$ with $1 \leq |S| \leq 3$, such that:
 - $c_{\{p, q, r\}} \notin L_{pqr\Delta}$. This is possible because $n = 5$, so $|\{p, q, r\}| \leq n - 2$, and hence $L_{pqr\Delta} \neq V$ by corollary 2.9.
 - if $S \neq \{p, q, r\}$ then $c_S = 0$.

Extending notation 3.1, for $x, y, z \in K$ we write c_{xyz} as short for $c_{\{x, y, z\}}$.

3. Let \mathcal{S} be the set of all triples $(x, L_{xy\Delta} + v, y)$, where $x, y \in K$ and $v \in V$. As usual here, $L_{xy\Delta} + v$ is the coset $\{w + v : w \in L_{xy\Delta}\}$ of the subgroup $L_{xy\Delta}$ of the additive group of V . It is an element of the quotient space $V/L_{xy\Delta}$.

For $s = (x, L_{xy\Delta} + v, y) \in \mathcal{S}$, we write $\text{start}(s) = x$ and $\text{end}(s) = y$. We can regard s as ‘going from x to y ’.

For $x, y \in K$ and $v \in V$, we will write (x, \widehat{v}, y) as shorthand for $(x, L_{xy\Delta} + v, y) \in \mathcal{S}$. So \widehat{v} is $L_{xy\Delta} + v$, the x and y here being determined by context. Thus, the meaning of \widehat{v} in (x, \widehat{v}, y) involves all of v, x, y . Of course, $(x, \widehat{v}, y) = (x, \widehat{w}, y)$ iff $v + w \in L_{xy\Delta}$.

So for each $s \in \mathcal{S}$ we have $s = (\text{start}(s), \widehat{v}, \text{end}(s))$, for some (not unique) $v \in V$.

4. The domain of $\mathcal{A}^{(d)}$ is $\wp(\mathcal{S})$.

5. The boolean operations $+$, $-$, 0 , 1 on $\mathcal{A}^{(d)}$ are defined as usual on power sets: $+$ is union, $-$ is unary complement, 0 is \emptyset , and 1 is \mathcal{S} . Under these operations, $\mathcal{A}^{(d)}$ is an atomic boolean algebra and its atoms are the singletons $\{s\}$ for $s \in \mathcal{S}$. We will usually identify the atom $\{s\}$ with the element $s \in \mathcal{S}$.
6. The identity $1'$ of $\mathcal{A}^{(d)}$ is the set $\{(x, \widehat{0}, x) : x \in K\} \subseteq \mathcal{S}$. Here, $\widehat{0}$ is of course the coset $L_{xx\Delta} + 0 = L_{x\Delta}$ of $L_{x\Delta}$ — the zero or identity element of the quotient space $V/L_{x\Delta}$.
7. Converse is defined on atoms by $(x, L_{xy\Delta} + v, y)^\smile = (y, L_{yx\Delta} + v, x)$. This is plainly well defined — independent of the choice of v — because $L_{yx\Delta} = L_{xy\Delta}$ ($= L_{\{x,y,\Delta\}}$). In other words, $(x, \widehat{v}, y)^\smile = (y, \widehat{v}, x)$. So we reverse the order of the entries and don't change the central term at all.

For arbitrary a in $\mathcal{A}^{(d)}$, we then define $\check{a} = \{\check{s} : s \in a\}$.

8. Composition is defined by $a; b = \{s \in \mathcal{S} : s \leq t; u \text{ for some } t \in a, u \in b\}$ for $a, b \subseteq \mathcal{S}$, where the ternary relation ' $s \leq t; u$ ' on \mathcal{S} is defined as follows.

Let $s = (x, \widehat{k}, z)$, $t = (x', \widehat{g}, y')$, and $u = (y'', \widehat{h}, z'')$ be arbitrary elements of \mathcal{S} . We define

$$(x, \widehat{k}, z) \leq (x', \widehat{g}, y'); (y'', \widehat{h}, z'')$$

if the following conditions hold:

- $x = x'$,
- $y' = y'' = y$, say,
- $z = z''$,
- $g + h + k + c_{xyz} \in L_{xyz\Delta}$.

This is well defined — independent of the choice of representatives g, h, k of the relevant cosets. For example, if $(x, \widehat{k}, z) = (x, \widehat{k}', z)$, then $L_{xz\Delta} + k = L_{xz\Delta} + k'$, so $k + k' \in L_{xz\Delta} \subseteq L_{xyz\Delta}$. Hence, $g + h + k + c_{xyz} \in L_{xyz\Delta}$ iff $g + h + k' + c_{xyz} \in L_{xyz\Delta}$.

Much of this definition is motivated by [3], including the crucial idea that $c_{pqr} \notin L_{pqr\Delta}$. This gives a 'twist' to composition. Like a wrinkle that stops a carpet being laid flat, it stops $\mathcal{A}^{(d)}$ being representable, but it becomes visible only on fairly large scales, as propositions 3.13 and 3.15 will show.

REMARK 3.3 (on composition)

1. For $s, t, u \in \mathcal{S}$, we have $s \leq t; u$ as defined in part 8 above iff in $\mathcal{A}^{(d)}$ we have $\{s\} \leq \{t\}; \{u\}$, so our use of the symbols \leq and $;$ in the ternary relation on \mathcal{S} in part 8 is not misleading.
2. If $s, t, u \in \mathcal{S}$ and $s \leq t; u$, then $\text{start}(s) = \text{start}(t)$, $\text{end}(s) = \text{end}(u)$, and $\text{end}(t) = \text{start}(u)$.

3. Composition is *additive*: $a;(b+c) = a;b+a;c$ and $(a+b);c = a;c+b;c$ for all a, b, c in $\mathcal{A}^{(d)}$. Hence, it is *monotonic*: if $a \leq a'$ and $b \leq b'$ in $\mathcal{A}^{(d)}$ then $a;b \leq a';b'$.

It is easy to see that $\mathcal{A}^{(d)}$ is a finite, hence complete, non-associative algebra. It is in fact a *coset relation algebra*, as per [3], but we will not need to prove that here. We will show in corollary 3.14 that it is a relation algebra. The proof of that corollary requires $\mathcal{A}^{(d)}$ to be a weakly associative algebra, so we prove this directly as our next step.

3.3 $\mathcal{A}^{(d)}$ is a weakly associative algebra

To clarify the proof of the headline here, we make a quick definition:

DEFINITION 3.4 For $x \in K$ we define the following elements of $\mathcal{A}^{(d)}$:

- ${}_x\mathcal{S} = \{s \in \mathcal{S} : \text{start}(s) = x\}$,
- $\mathcal{S}_x = \{s \in \mathcal{S} : \text{end}(s) = x\}$.

LEMMA 3.5 Let $x \in K$ be arbitrary. Then in $\mathcal{A}^{(d)}$ we have

- $s;1 = {}_x\mathcal{S}$ for each $s \in {}_x\mathcal{S}$,
- $1;s = \mathcal{S}_x$ for each $s \in \mathcal{S}_x$.

Proof. For the first part, let $s = (x, \widehat{g}, y) \in {}_x\mathcal{S}$. We show that $s;1 = {}_x\mathcal{S}$.

So let $(z, \widehat{k}, t) \in \mathcal{S}$ be arbitrary. By definition of composition, $(z, \widehat{k}, t) \in (x, \widehat{g}, y);1$ iff there is $(y, \widehat{h}, t) \in 1 = \mathcal{S}$ with $(z, \widehat{k}, t) \leq (x, \widehat{g}, y);(y, \widehat{h}, t)$. This holds iff $z = x$ and there is $h \in V$ with $g + h + k + c_{xyt} \in L_{xyt\Delta}$. But there always is such an h — for example, $h = g + k + c_{xyt}$. So $(z, \widehat{k}, t) \leq (x, \widehat{g}, y);1$ iff $z = x$, iff $(z, \widehat{k}, t) \in {}_x\mathcal{S}$.

It follows that $s;1 = {}_x\mathcal{S}$, proving the first part. The second part is similar. \square

LEMMA 3.6 $\mathcal{A}^{(d)}$ is a weakly associative algebra.

Proof. In any non-associative algebra, we have $1;1 = 1$ [19, theorem 1.13(5)]. So by additivity of composition, it suffices to show that $(a;1);1 = a;1$ for every atom a of $\mathcal{A}^{(d)}$ with $a \leq 1'$. Take such an atom, say $a = (x, \widehat{0}, x)$. By lemma 3.5, $a;1 = {}_x\mathcal{S}$. By definition of composition and lemma 3.5 again, $(a;1);1 = {}_x\mathcal{S};1 = \{s \in \mathcal{S} : s \leq t;1 \text{ for some } t \in {}_x\mathcal{S}\} = \{s \in \mathcal{S} : s \leq {}_x\mathcal{S}\} = {}_x\mathcal{S} = a;1$. \square

3.4 Networks

In the next three subsections, we mostly recall known material due to Maddux. For the first two of them, let $3 \leq m < \omega$. We take m to be the ordinal $\{0, 1, \dots, m-1\}$ of course. Let \mathcal{B} be an atomic non-associative algebra. We write $\text{At } \mathcal{B}$ for the set of atoms of \mathcal{B} .

DEFINITION 3.7 An m -dimensional (atomic) network over \mathcal{B} is a map $N : m \times m \rightarrow \text{At } \mathcal{B}$ satisfying, for every $x, y, z < m$,

1. $N(x, x) \leq 1'$,
2. $N(x, y) = N(y, x)^\smile$,
3. $N(x, z) \leq N(x, y) ; N(y, z)$.

For m -dimensional atomic networks M, N over \mathcal{B} , and $x, y < m$, we write $M \equiv_x N$ (respectively, $M \equiv_{xy} N$) if $M(z, t) = N(z, t)$ for every $z, t \in m \setminus \{x\}$ (respectively, every $z, t \in m \setminus \{x, y\}$). In Maddux's language, M and N agree off of x (or x, y).

Our algebra $\mathcal{A}^{(d)}$ is an atomic non-associative algebra, so the notion of an m -dimensional atomic network over $\mathcal{A}^{(d)}$ is defined. As we now see, each 'node' in such a network has a 'location' in K which is the start or end of all atoms associated with it.

DEFINITION 3.8 Let N be an m -dimensional atomic network over $\mathcal{A}^{(d)}$. We define a map $\nu_N : m \rightarrow K$ by

$$\nu_N(x) = \text{start}(N(x, x)), \quad \text{for each } x < m.$$

Since $N(x, x) \leq 1'$, we also have $\nu_N(x) = \text{end}(N(x, x))$ and $N(x, x) = (\nu_N(x), \widehat{0}, \nu_N(x))$. The element $\nu_N(x)$ is in some sense the location in K of the node x .

LEMMA 3.9 Let N be an m -dimensional atomic network over $\mathcal{A}^{(d)}$. Then for every $x, y < m$, we have $N(x, y) = (\nu_N(x), \widehat{v}, \nu_N(y))$ for some $v \in V$.

Proof. As N is a network, $N(x, x) \leq N(x, y) ; N(y, x)$. Remark 3.3 now yields that $\text{start}(N(x, y)) = \text{start}(N(x, x)) = \nu_N(x)$. So $\text{end}(N(x, y)) = \text{start}(N(y, x)) = \nu_N(y)$ by this and the remark. \square

3.5 Bases

We now define two kinds of 'basis' of \mathcal{B} . We give references to papers, but the definitions can also be found in [23, §21] and [13, chapter 12]. Let \mathcal{M} be a set of m -dimensional atomic networks over \mathcal{B} .

DEFINITION 3.10 (Maddux, [20, p.78], [21, definition 3]) We say that \mathcal{M} is an m -dimensional relational basis of \mathcal{B} if:

1. for every $a \in \text{At } \mathcal{B}$, there is $N \in \mathcal{M}$ with $N(0, 1) = a$,
2. for every $x, y, z < m$ with $z \neq x, y$, every $N \in \mathcal{M}$, and every $a, b \in \text{At } \mathcal{B}$ with $N(x, y) \leq a; b$, there is $M \in \mathcal{M}$ with $M \equiv_z N$, $M(x, z) = a$, and $M(z, y) = b$. This is called the *triangle addition property*.

DEFINITION 3.11 (Maddux, [21, definition 4]) We say that \mathcal{M} is an *m-dimensional cylindric basis* of \mathcal{B} if:

1. For every $a, b, c \in \text{At } \mathcal{B}$ with $a \leq b; c$, there is $N \in \mathcal{M}$ with $N(0, 1) = a$, $N(0, 2) = b$, and $N(2, 1) = c$.
2. Let $x, y < m$. For each $z < m$, define

$$z' = \begin{cases} y, & \text{if } z = x, \\ z, & \text{otherwise,} \end{cases}$$

Let $N \in \mathcal{M}$ be arbitrary, and let $N' : m^2 \rightarrow \text{At } \mathcal{B}$ be given by $N'(z, t) = N(z', t')$. Then $N' \in \mathcal{M}$. (N' is often written as $N[x/y]$.)

3. If $x, y < m$ are distinct and $N, M \in \mathcal{M}$ satisfy $N \equiv_{xy} M$, then there is some $L \in \mathcal{M}$ with $N \equiv_x L \equiv_y M$. This is called the *amalgamation property*.

Any m -dimensional cylindric basis of an atomic weakly associative algebra is also an m -dimensional relational basis [23, theorem 334(ii)]. Weak associativity is essential here [13, exercise 12.6(5)], which is why we proved that $\mathcal{A}^{(d)}$ is weakly associative, in lemma 3.6.

3.6 Some varieties of relation algebras

We will be using the classes RA_m of relation algebras, for finite $m \geq 3$. RA_m is the closure under subalgebras of the class of complete atomic non-associative algebras that have an m -dimensional relational basis ([21, definition 3(ii–iii)] or [23, §24]). We will need only the following facts.

FACT 3.12 Suppose that $3 \leq m < \omega$.

1. $\text{RA}_{m+1} \subseteq \text{RA}_m$ [20, theorem 3].
2. $\text{RA}_4 = \text{RA}$ [20, theorem 6(2)].
3. $\text{RRA} = \bigcap_{3 \leq m < \omega} \text{RA}_m$ [20, theorems 6(3) and 10].
4. RA_m is a variety [20, theorem 9].
5. If \mathcal{B} is a finite non-associative algebra, then $\mathcal{B} \in \text{RA}_m$ iff \mathcal{B} has an m -dimensional relational basis (\Leftarrow is by definition of RA_m , and \Rightarrow follows from [20, theorem 8]).

The two main results of this section will show that $\mathcal{A}^{(d)}$ has a $(d+3)$ -dimensional cylindric basis but no $(d+4)$ -dimensional relational basis. It will follow that $\mathcal{A}^{(d)} \in \text{RA}_{d+3} \setminus \text{RA}_{d+4}$.

3.7 $\mathcal{A}^{(d)}$ has a $(d + 3)$ -dimensional cylindric basis

PROPOSITION 3.13 *The set \mathcal{M} of all $(d + 3)$ -dimensional atomic networks over $\mathcal{A}^{(d)}$ is a $(d + 3)$ -dimensional cylindric basis of $\mathcal{A}^{(d)}$.*

Proof. The first two properties defining a cylindric basis actually hold for every atomic weakly associative algebra when \mathcal{M} is taken to be the set of *all* atomic networks of appropriate dimension, as it is here. But we will check them anyway.

First let $s, t, u \in \mathcal{S}$ with $s \leq t; u$. We will define $N \in \mathcal{M}$ with $N(0, 1) = s$, $N(0, 2) = t$, and $N(2, 1) = u$.

We start out by defining the part of N on the indices $0, 1, 2$. Of course we define $N(0, 1) = s$, $N(0, 2) = t$, and $N(2, 1) = u$. We put $N(1, 0) = \check{s}$, $N(2, 0) = \check{t}$, and $N(1, 2) = \check{u}$. We also put $N(0, 0) = (\text{start}(s), \widehat{0}, \text{start}(s))$, $N(1, 1) = (\text{end}(s), \widehat{0}, \text{end}(s))$, and $N(2, 2) = (\text{end}(t), \widehat{0}, \text{end}(t))$ — note that $\text{end}(t) = \text{start}(u)$, by remark 3.3. Extend the definition of N to all indices $x, y < d + 3$ by putting $N(x, y) = N(\min(x, 2), \min(y, 2))$, which we have already defined. It is easily checked that $N \in \mathcal{M}$ and is as required.

It is extremely easy to check that if $N \in \mathcal{M}$ then the map N' in definition 3.11 is an atomic network, so is in \mathcal{M} as well.

We come to the amalgamation property now. So let $x, y < d + 3$ be distinct and let $N, M \in \mathcal{M}$ satisfy $N \equiv_{xy} M$. We will define $L \in \mathcal{M}$ with $N \equiv_x L \equiv_y M$. See figure 1.

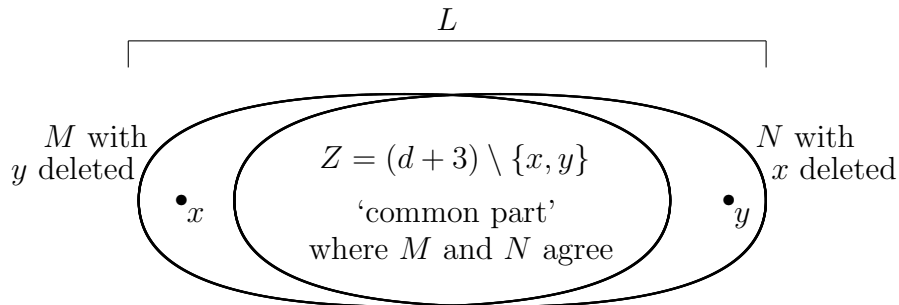


Figure 1: Amalgamation property

We need to define the atom $L(z, t) \in \mathcal{S}$ for every $z, t < d + 3$. Now $L(z, t)$ is already determined whenever $\{z, t\} \subseteq (d + 3) \setminus \{x\}$, because we need $L \equiv_x N$, so we have to put $L(z, t) = N(z, t)$ in this case. Similarly, $L(z, t)$ is determined whenever $\{z, t\} \subseteq (d + 3) \setminus \{y\}$: we need $L \equiv_y M$, so we have to put $L(z, t) = M(z, t)$. This is well defined when $\{z, t\} \subseteq (d + 3) \setminus \{x, y\}$, since $M \equiv_{xy} N$.

The remaining case is when $x, y \in \{z, t\}$, so since $x \neq y$, we have $\{z, t\} = \{x, y\}$. We will define $L(x, y)$, and then let $L(y, x)$ be its converse.

Recall from definition 3.8 that $\nu_N(t) = \text{start}(N(t, t)) \in K$, and similarly for M . For each $t < d + 3$, write

$$\check{t} = \begin{cases} \nu_N(t), & \text{if } t \neq x, \\ \nu_M(t), & \text{if } t \neq y. \end{cases}$$

Since $N \equiv_{xy} M$, if $t \neq x, y$ then $\nu_M(t) = \nu_N(t)$, so this is well defined (we will have $\check{t} = \nu_L(t)$).

Let $Z = (d+3) \setminus \{x, y\}$, the ‘common part’ in figure 1. Since M and N are networks, it follows from lemma 3.9 that for each $z \in Z$ we can choose $g_z, h_z \in V$ such that

$$\begin{aligned} L(x, z) &= M(x, z) = (\check{x}, \widehat{g}_z, \check{z}), \\ L(z, y) &= N(z, y) = (\check{z}, \widehat{h}_z, \check{y}). \end{aligned} \quad (3)$$

We need to define $L(x, y) = (\check{x}, \widehat{k}, \check{y})$ for some $k \in V$ with $(\check{x}, \widehat{k}, \check{y}) \leq (\check{x}, \widehat{g}_z, \check{z}); (\check{z}, \widehat{h}_z, \check{y})$ for each $z \in Z$, so that $L(x, y) \leq L(x, z); L(z, y)$. So by definition of composition in $\mathcal{A}^{(d)}$, we need to find $k \in V$ such that

$$k + g_z + h_z + c_{\check{x}\check{y}\check{z}} \in L_{\check{x}\check{y}\check{z}\Delta} \quad \text{for each } z \in Z. \quad (4)$$

The way we do it is one of the two critical points of the paper.

First, we define a field element $\lambda_Y \in \mathcal{F}$ for each $Y \in K^{[d+1]}$ with $\check{x}, \check{y} \notin Y$, as follows.

1. If there is $z \in Z$ with $\check{z} \notin Y$, then we put $\lambda_Y = (g_z + h_z + c_{\check{x}\check{y}\check{z}}) \cdot \chi_{Y\downarrow}$.
2. If there is no such z , we put $\lambda_Y = 0$ (actually, any value will do).

Claim. λ_Y is well defined.

Proof of claim. Take $Y \in K^{[d+1]}$ with $\check{x}, \check{y} \notin Y$. Suppose that $z, z' \in Z$, $z \neq z'$, and $\check{z}, \check{z}' \notin Y$. We need to show that

$$(g_z + h_z + c_{\check{x}\check{y}\check{z}}) \cdot \chi_{Y\downarrow} = (g_{z'} + h_{z'} + c_{\check{x}\check{y}\check{z}'}) \cdot \chi_{Y\downarrow}. \quad (5)$$

It will be useful to define

$$S = \{\check{x}, \check{y}, \check{z}, \check{z}'\} \subseteq K \setminus Y.$$

Let $M(z, z') = N(z, z') = (\check{z}, \widehat{w}, \check{z}')$, say, where $w \in V$. The value is the same in M and N because $M \equiv_{xy} N$ and $z, z' \notin \{x, y\}$. Since M is a network, we have $M(z, z') \leq M(z, x); M(x, z')$, and from this and (3) we obtain $g_z + g_{z'} + w + c_{\check{x}\check{z}\check{z}'} \in L_{\check{x}\check{z}\check{z}'\Delta} \subseteq L_{S \cup \{\Delta\}}$. On the other side, since N is a network, $N(z, z') \leq N(z, y); N(y, z')$, which with (3) gives $h_z + h_{z'} + w + c_{\check{y}\check{z}\check{z}'} \in L_{\check{y}\check{z}\check{z}'\Delta} \subseteq L_{S \cup \{\Delta\}}$. Adding these yields

$$g_z + g_{z'} + w + c_{\check{x}\check{z}\check{z}'} + h_z + h_{z'} + w + c_{\check{y}\check{z}\check{z}'} \in L_{S \cup \{\Delta\}}.$$

Bear in mind that \mathcal{F} has characteristic 2. So the two w here cancel, and by adding $c_{\check{x}\check{y}\check{z}} + c_{\check{x}\check{y}\check{z}'} + c_{\check{x}\check{y}\check{z}} + c_{\check{x}\check{y}\check{z}'} = 0$ and rearranging, we get

$$(g_z + h_z + c_{\check{x}\check{y}\check{z}}) + (g_{z'} + h_{z'} + c_{\check{x}\check{y}\check{z}'}) + \underbrace{(c_{\check{x}\check{z}\check{z}'} + c_{\check{y}\check{z}\check{z}'} + c_{\check{x}\check{y}\check{z}} + c_{\check{x}\check{y}\check{z}'})}_{v, \text{ say}} \in L_{S \cup \{\Delta\}}. \quad (6)$$

We will show that the vector v indicated in (6) is zero. Put $I = \{x, y, z, z'\}$, so that $S = \{\check{i} : i \in I\}$. Define a map $f : I \rightarrow \wp(S)$ by $f(i) = \{\check{j} : j \in I \setminus \{i\}\}$. So

$$v \stackrel{\text{def}}{=} c_{\check{x}\check{z}\check{z}'} + c_{\check{y}\check{z}\check{z}'} + c_{\check{x}\check{y}\check{z}} + c_{\check{x}\check{y}\check{z}'} = c_{f(y)} + c_{f(x)} + c_{f(z')} + c_{f(z)} = \sum_{i \in I} c_{f(i)}.$$

Now observe that $S \subseteq K \setminus Y$ and $|K \setminus Y| = |K| - |Y| = (d+4) - (d+1) = 3$. So, crucially, $|S| \leq 3$, while $|I| = 4$ since we assumed $z \neq z'$. So by the pigeonhole principle, there are distinct $i, j \in I$ with $\check{i} = \check{j}$. It follows that $f(i) = f(j) = S$, and that $|f(k)| \leq 2$ for each $k \in I \setminus \{i, j\}$, so $c_{f(k)} = 0$ by definition 3.2(2). So indeed,

$$v = \sum_{i' \in I} c_{f(i')} = c_S + c_S + 0 + 0 = 0.$$

As a result, (6) reduces to

$$(g_z + h_z + c_{\check{x}\check{y}\check{z}}) + (g_{z'} + h_{z'} + c_{\check{x}\check{y}\check{z}'}) \in L_{S \cup \{\Delta\}}.$$

As Y is disjoint from S , by proposition 2.12(L1 \Rightarrow L2) we obtain

$$((g_z + h_z + c_{\check{x}\check{y}\check{z}}) + (g_{z'} + h_{z'} + c_{\check{x}\check{y}\check{z}'})) \cdot \chi_{Y\downarrow} = 0.$$

By linearity of scalar product, this yields

$$(g_z + h_z + c_{\check{x}\check{y}\check{z}}) \cdot \chi_{Y\downarrow} = (g_{z'} + h_{z'} + c_{\check{x}\check{y}\check{z}'}) \cdot \chi_{Y\downarrow},$$

proving (5), and with it, the claim.

We now have a clear run to the finish. Since $|\{\check{t} : t < d+3\}| < d+4 = |K|$, we can select some $a \in K \setminus \{\check{t} : t < d+3\}$. We now define our $k \in V$ by

$$k(e) = \begin{cases} \lambda_{e \cup \{a\}}, & \text{if } \check{x}, \check{y}, a \notin e, \\ 0, & \text{otherwise,} \end{cases}$$

for $e \in E$. We check that (4) holds. So fix arbitrary $z \in Z$; we use proposition 2.12 to show that $k + g_z + h_z + c_{\check{x}\check{y}\check{z}} \in L_{\check{x}\check{y}\check{z}\Delta}$.

To this end, let $Y \in K^{[d+1]}$ be disjoint from $\{\check{x}, \check{y}, \check{z}\}$ and contain a . The edges in $Y\downarrow$ are $Y \setminus \{a\}$, on which k takes the value λ_Y , and a bunch of edges containing a , on which k takes the value 0. The sum of these values is λ_Y . So by definition of scalar product and of λ_Y ,

$$k \cdot \chi_{Y\downarrow} = \sum_{e \in Y\downarrow} k(e) = \lambda_Y = (g_z + h_z + c_{\check{x}\check{y}\check{z}}) \cdot \chi_{Y\downarrow}.$$

So by linearity of scalar product and the characteristic of \mathcal{F} as usual,

$$(k + g_z + h_z + c_{\check{x}\check{y}\check{z}}) \cdot \chi_{Y\downarrow} = k \cdot \chi_{Y\downarrow} + (g_z + h_z + c_{\check{x}\check{y}\check{z}}) \cdot \chi_{Y\downarrow} = k \cdot \chi_{Y\downarrow} + k \cdot \chi_{Y\downarrow} = 0.$$

This holds for every such Y , so since $a \notin \{\check{x}, \check{y}, \check{z}\}$, by proposition 2.12(L3 \Rightarrow L1) we obtain $k + g_z + h_z + c_{\check{x}\check{y}\check{z}} \in L_{\check{x}\check{y}\check{z}\Delta}$, proving (4).

Now, defining $L(x, y) = (\check{x}, \widehat{k}, \check{y})$ and $L(y, x) = (\check{y}, \widehat{k}, \check{x})$ as intimated earlier, it is straightforward to check that $L \in \mathcal{M}$, and plainly $N \equiv_x L \equiv_y M$. \square

COROLLARY 3.14 $\mathcal{A}^{(d)} \in \text{RA}_{d+3}$. Hence, $\mathcal{A}^{(d)}$ is a relation algebra.

Proof. By proposition 3.13, $\mathcal{A}^{(d)}$ has a $(d+3)$ -dimensional cylindric basis, \mathcal{M} . By lemma 3.6, $\mathcal{A}^{(d)}$ is a weakly associative algebra, so by [23, theorem 334(ii)], already mentioned, \mathcal{M} is also a $(d+3)$ -dimensional relational basis of $\mathcal{A}^{(d)}$. By fact 3.12 and since $d+3 \geq 4$, we obtain $\mathcal{A}^{(d)} \in \text{RA}_{d+3} \subseteq \text{RA}_4 = \text{RA}$. \square

We can show a little more for those interested. Since $\mathcal{A}^{(d)}$ is finite, it is complete. So [21, theorem 10(i,iii)] applies, showing that there is an atomic $(d+3)$ -dimensional cylindric algebra \mathcal{C} such that $\mathcal{A}^{(d)}$ is isomorphic to the relation algebra reduct of \mathcal{C} . (Actually, \mathcal{C} is the standard complex algebra over the cylindric basis \mathcal{M} , and is therefore finite.) So in fact, $\mathcal{A}^{(d)} \in \mathfrak{RacA}_{d+3}$, a proper subclass of RA_{d+3} [12].

3.8 $\mathcal{A}^{(d)}$ has no $(d+4)$ -dimensional relational basis

We now come to the second critical point of the paper. It is where the wrinkle mentioned before shows up.

PROPOSITION 3.15 $\mathcal{A}^{(d)}$ has no $(d+4)$ -dimensional relational basis.

Proof. Suppose for contradiction that \mathcal{M} is a $(d+4)$ -dimensional relational basis for $\mathcal{A}^{(d)}$. We will find a network $M \in \mathcal{M}$ that is ‘full’ in the sense that $\nu_M : (d+4) \rightarrow K$ is surjective — every possible ‘location’ appears in M . We will then obtain our contradiction by showing that no such ‘full’ network can exist.

Enumerate K as $\{x_i : i < d+4\}$. We will define networks $M_j \in \mathcal{M}$ for $j = 1, \dots, d+3$ by induction, such that

$$\nu_{M_j}(i) = x_i \quad \text{for each } i = 0, \dots, j. \quad (7)$$

As \mathcal{M} is a relational basis, there is $M_1 \in \mathcal{M}$ with $M_1(0, 1) = (x_0, \widehat{0}, x_1)$. By lemma 3.9, $\nu_{M_1}(0) = x_0$ and $\nu_{M_1}(1) = x_1$, so M_1 satisfies (7).

Let $1 \leq j < d+3$ and assume inductively that M_j has been defined. So from (7) and because $M_j(0, 0) \leq 1$, we have $M_j(0, 0) = (\nu_{M_j}(0), \widehat{0}, \nu_{M_j}(0)) = (x_0, \widehat{0}, x_0)$. It follows by definition of composition in $\mathcal{A}^{(d)}$ that

$$M_j(0, 0) = (x_0, \widehat{0}, x_0) \leq (x_0, \widehat{0}, x_{j+1}); (x_{j+1}, \widehat{0}, x_0),$$

since $c_{x_0 x_{j+1} x_0} = 0$ by definition 3.2(2), so $0 + 0 + 0 + c_{x_0 x_{j+1} x_0} = 0 \in L_{x_0 x_{j+1} \Delta}$. So by the triangle addition property of \mathcal{M} , there is $M_{j+1} \in \mathcal{M}$ with $M_{j+1} \equiv_{j+1} M_j$ and $M_{j+1}(0, j+1) = (x_0, \widehat{0}, x_{j+1})$. So $\nu_{M_{j+1}}(j+1) = x_{j+1}$; and since $M_{j+1} \equiv_{j+1} M_j$, which inductively satisfies (7), for each $i \leq j$ we have $\nu_{M_{j+1}}(i) = \nu_{M_j}(i) = x_i$. It follows that M_{j+1} satisfies (7) as well. This completes the induction.

From now on, we write M for $M_{d+3} \in \mathcal{M}$. By (7), $\nu_M(i) = x_i$ for each $i < d+4$. So M is ‘full’ in the sense described above, and indeed, $\nu_M : (d+4) \rightarrow K$ is a bijection.

Using that ν_M is bijective, we can define a map $N : K^{[1]} \cup K^{[2]} \rightarrow V$ such that

$$M(\nu_M^{-1}(x), \nu_M^{-1}(y)) = (x, N(\widehat{\{x, y\}}), y) \quad \text{for every } x, y \in K. \quad (8)$$

It can be done as follows. For each $i \leq j < d + 4$, we have $M(i, j) = (\nu_M(i), \widehat{v}, \nu_M(j))$ for some (not unique) $v \in V$. Choose such a v , and define $N(\{\nu_M(i), \nu_M(j)\}) = v$. Since $M(j, i) = M(i, j)^\smile = (\nu_M(j), \widehat{v}, \nu_M(i))$ for the same v , we can see that (8) holds. For brevity, for $x, y \in K$ we will often write $N(\{x, y\})$ simply as $N(x, y)$. Obviously, $N(x, y) = N(y, x)$.

Since M is a network, it follows from definition 3.2(8) that

$$N(x, y) + N(y, z) + N(z, x) + c_{xyz} \in L_{xyz\Delta} \quad \text{for every } x, y, z \in K. \quad (9)$$

Take any $S = \{x, y, z\} \in K^{[3]}$. Define

$$T_S \stackrel{\text{def}}{=} \sum \{N(x, y)(e) + N(y, z)(e) + N(z, x)(e) : e \in E, x, y, z \notin e\} \in \mathcal{F}.$$

This is independent of the choice of enumeration of S as $\{x, y, z\}$, so is well defined. Since $|K| = d + 4$, the complement $K \setminus S$ of S has size $d + 1$, and hence $K \setminus S$ is the unique $Y \in K^{[d+1]}$ disjoint from S . So for each $v \in V$ we have

$$\begin{aligned} v \in L_{xyz\Delta} &\iff v \cdot \chi_{(K \setminus S)\downarrow} = 0 && \text{by proposition 2.12(L1} \iff \text{L2)} \\ &\iff \sum \{v(e) : e \in E, x, y, z \notin e\} = 0 && \text{by evaluating.} \end{aligned}$$

Applying this with $v = N(x, y) + N(y, z) + N(z, x)$ gives

$$N(x, y) + N(y, z) + N(z, x) \in L_{xyz\Delta} \iff T_S = 0. \quad (10)$$

Put $S_0 = \{p, q, r\}$ (see definition 3.2(1) for p, q, r). Now if $S \neq S_0$ then $c_S = 0$ by definition 3.2(2), so by (9), the left-hand side of (10) holds. We deduce from (10) that

$$T_S = 0 \text{ for every } S \in K^{[3]} \setminus \{S_0\}.$$

Hence,

$$\mathcal{T} \stackrel{\text{def}}{=} \sum_{S \in K^{[3]}} T_S = T_{S_0}. \quad (11)$$

But let us inspect the sum \mathcal{T} of all T_S in its own right. Consider a term $N(\{x, y\})(e)$ in this sum. It comes from one or more of the T_S whose sum is \mathcal{T} . In fact, a little thought shows that this term occurs once in each T_S with $S \in K^{[3]}$ disjoint from e and containing x, y , and not at all in any other T_S . Now $|K| = d + 4$ and $|e| = d$. So $|K \setminus e| = 4$, and hence there are distinct $t, u \in K$ such that $K = e \cup \{x, y, t, u\}$. Then $\{x, y, t\}$ and $\{x, y, u\}$ are the only sets in $K^{[3]}$ containing x and y and disjoint from e . So $N(\{x, y\})(e)$ occurs precisely twice in \mathcal{T} : once in $T_{\{x, y, t\}}$ and once in $T_{\{x, y, u\}}$. Since \mathcal{F} has characteristic 2, the sum of these two occurrences is zero. This holds for all terms in \mathcal{T} — they all occur in pairs and cancel. So we obtain $\mathcal{T} = 0$.

We conclude from this and (11) that $T_{S_0} = 0$. So by (10) again, $N(p, q) + N(q, r) + N(r, p) \in L_{pqr\Delta}$. Now (9) yields that $c_{pqr} \in L_{pqr\Delta}$ as well, contradicting the original (wrinkly) choice of $c_{pqr} \notin L_{pqr\Delta}$ in definition 3.2(2). \square

COROLLARY 3.16 $\mathcal{A}^{(d)} \notin \text{RA}_{d+4}$. Hence, $\mathcal{A}^{(d)}$ is not representable.

Proof. By proposition 3.15, $\mathcal{A}^{(d)}$ has no $(d+4)$ -dimensional relational basis. We saw in fact 3.12(5) that every *finite* algebra in RA_{d+4} actually has a $(d+4)$ -dimensional relational basis of its own. Since $\mathcal{A}^{(d)}$ is finite, it follows that $\mathcal{A}^{(d)} \notin \text{RA}_{d+4}$. Since $\text{RRA} \subseteq \text{RA}_{d+4}$, we obtain $\mathcal{A}^{(d)} \notin \text{RRA}$, so $\mathcal{A}^{(d)}$ is not representable. \square

3.9 $\mathcal{A}^{(d)}$ is simple, measurable, and persistently finite

Now that we know (from corollary 3.14) that $\mathcal{A}^{(d)}$ is a relation algebra, we can establish three more of its basic properties.

As is standard in algebra, a relation algebra \mathcal{A} is said to be *simple* if it is nontrivial and, up to isomorphism, its only homomorphic images are itself and the trivial one-element algebra. A nontrivial relation algebra \mathcal{A} is simple iff $1; a; 1 = 1$ for every non-zero element a of \mathcal{A} [23, theorem 379]. So RA is a *discriminator variety*: it has the unary discriminator term $1; x; 1$. For relation algebras, simplicity is defined by the universal first-order sentence $0 \neq 1 \wedge \forall x(x \neq 0 \rightarrow 1; x; 1 = 1)$, and it follows that every subalgebra of a simple relation algebra is also simple.

LEMMA 3.17 $\mathcal{A}^{(d)}$ is simple.

Proof. By the above, it is enough to show that $1; a; 1 = 1$ for each nonzero element a of $\mathcal{A}^{(d)}$, and by monotonicity of composition, we can take a to be an atom. By lemma 3.5, $1; a = \mathcal{S}_{\text{end}(a)}$, and so $1; a; 1 = \mathcal{S}_{\text{end}(a)}; 1 = \bigcup_{b \in \mathcal{S}_{\text{end}(a)}} (b; 1) = \bigcup_{\text{end}(b) = \text{end}(a)} \text{start}(b) \mathcal{S} = \bigcup_{z \in K} z \mathcal{S} = \mathcal{S} = 1$. \square

An element a of a relation algebra \mathcal{A} is said to be *functional* if $\check{a}; a \leq 1'$, and an *identity atom* if it is an atom of \mathcal{A} and $a \leq 1'$. We say that \mathcal{A} is *measurable* if $1'$ is a sum of atoms, and for each identity atom a , the element $a; 1; a$ is a sum of functional elements.

LEMMA 3.18 $\mathcal{A}^{(d)}$ is measurable.

Proof. Certainly, $1'$ is a sum of atoms since $\mathcal{A}^{(d)}$ is atomic. Take an identity atom $a = (x, \widehat{0}, x)$ of $\mathcal{A}^{(d)}$. Plainly,

$$\begin{aligned} a; 1; a &= \{(u, \widehat{g}, v) : (u, \widehat{g}, v) \leq (x, \widehat{0}, x); (y, \widehat{h}, z); (x, \widehat{0}, x) \text{ for some } (y, \widehat{h}, z) \in \mathcal{S}\} \\ &= \{(x, \widehat{h}, x) : h \in V\}. \end{aligned}$$

This is a sum of atoms of the form (x, \widehat{h}, x) . Each such atom is functional, because it is self-converse, and for any $(y, \widehat{g}, z) \in \mathcal{S}$ we have

$$\begin{aligned} (y, \widehat{g}, z) \leq (x, \widehat{h}, x); (x, \widehat{h}, x) &\iff y = z = x \text{ and } g + h + h + c_{xxx} = g \in L_{x\Delta} \\ &\iff (y, \widehat{g}, z) = (x, \widehat{0}, x). \end{aligned}$$

So $(x, \widehat{h}, x)^\smile; (x, \widehat{h}, x) = (x, \widehat{0}, x) \leq 1'$. \square

A relation algebra \mathcal{A} is said to be *persistently finite (in RA)* if \mathcal{A} is finite and every simple relation algebra extending \mathcal{A} is finite as well [2, p. 3]. Under this definition, every finite *non-simple* relation algebra \mathcal{A} is vacuously persistently finite, since as we said above, every subalgebra of a simple relation algebra is simple, and so \mathcal{A} has no simple relation algebra extensions at all. So we are mainly interested in *simple* persistently finite relation algebras. As we said in the introduction, non-representable examples were constructed in [3]. Additional examples are the simple relation algebras with *no* proper simple extensions. The representable ones are precisely the algebras isomorphic to $\mathfrak{Rc}(U)$ for a finite set U — for \Leftarrow see [18, lemma 7.4], and \Rightarrow is an exercise — and non-representable examples were given in [4], as mentioned in the introduction. We remark that a relation algebra is persistently finite iff it is finite and, up to isomorphism, it has only finitely many simple extensions [2, p. 4].

LEMMA 3.19 $\mathcal{A}^{(d)}$ is persistently finite.

Proof. By the non-trivial [2, theorem 3.1], if \mathcal{A} is a finite measurable relation algebra, \mathcal{B} is a simple relation algebra, and $\mathcal{A} \subseteq \mathcal{B}$, then \mathcal{B} is finite (and measurable). So every finite measurable relation algebra is persistently finite. Since $\mathcal{A}^{(d)}$ is finite, and by lemma 3.18, measurable, it is persistently finite. \square

4 Main results

Having defined our relation algebras $\mathcal{A}^{(d)}$ and established some of their properties, we will soon be able to prove our main results. They involve completions, which we now recall.

4.1 Completions of relation algebras; the variety $\mathcal{C}(\text{RRA})$

For relation algebras \mathcal{A}, \mathcal{B} , we say that \mathcal{A} is a *dense subalgebra* of \mathcal{B} , and that \mathcal{B} is a *dense extension* of \mathcal{A} , if \mathcal{A} is a subalgebra of \mathcal{B} and for every non-zero element b of \mathcal{B} , there is a non-zero element a of \mathcal{A} with $a \leq b$.

A *completion* of a relation algebra \mathcal{A} is a complete relation algebra (see §3.1) that is a dense extension of \mathcal{A} . Monk proved in [26] that every relation algebra \mathcal{A} has a completion, which is unique up to isomorphism over \mathcal{A} . So we are justified in speaking of ‘the’ completion of a relation algebra \mathcal{A} , and we write it as \mathcal{A}^c .

For a class \mathbf{K} of relation algebras, we write $\mathbf{K}^c = \{\mathcal{A}^c : \mathcal{A} \in \mathbf{K}\}$, and $\text{Var } \mathbf{K}$ for the variety generated by (i.e., the smallest variety containing) \mathbf{K} .

DEFINITION 4.1 Let $\mathcal{C}(\text{RRA})$ be the variety defined by all equations ε in the signature of relation algebras that are valid in RRA and preserved by completions of relation algebras — that is, if \mathcal{A} is a relation algebra and $\mathcal{A} \models \varepsilon$, then $\mathcal{A}^c \models \varepsilon$.

Here are some observations about $\mathcal{C}(\text{RRA})$. They are easy to prove.

1. $\text{RRA} \subseteq \mathcal{C}(\text{RRA})$.

2. Each of the equations defining the class RA of relation algebras is valid in RRA, and obviously preserved by completions of relation algebras since the completion of a relation algebra is a relation algebra. So $\mathcal{C}(\text{RRA}) \subseteq \text{RA}$.
3. $\mathcal{C}(\text{RRA})$ is closed under completions: $\mathcal{C}(\text{RRA})^c \subseteq \mathcal{C}(\text{RRA})$.

Our first main result (theorem 4.9 below) will be that no class \mathbf{K} with $\text{RRA} \subseteq \mathbf{K} \subseteq \mathcal{C}(\text{RRA})$ is finitely axiomatisable.

REMARK 4.2 (for those interested in canonicity) By a result of Gehrke et al. [5, theorem 3.6], any universal class of monotone lattice expansions that is closed under completions (of a given type) is canonical — closed under canonical extensions. Since $\mathcal{C}(\text{RRA})$ is such a class, $\mathcal{C}(\text{RRA})$ is *canonical*.

Furthermore, $\mathcal{C}(\text{RRA})$ is *elementarily generated*. This follows from Goldblatt [8, theorem 5.8]. Alternatively, Goldblatt showed in [9, corollary 3.4] that a canonical singleton-persistent variety is elementarily generated, and since $\mathcal{C}(\text{RRA})$ is closed under completions, by [9, theorem 5.1] it is singleton-persistent.

For simplicity, let us take an equation of the signature of relation algebras to be *canonical* if whenever it is valid in a *relation algebra*, it is valid in its canonical extension. That is to say, we restrict attention to RA. Each equation ε that is preserved by completions of relation algebras is canonical — apply [5, theorem 3.6] to the class $\{\mathcal{A} \in \text{RA} : \mathcal{A} \models \varepsilon\}$. So $\mathcal{C}(\text{RRA})$ has a *canonical equational axiomatisation*.

Now by [16], RRA is *barely canonical*: it is a canonical variety, but every first-order axiomatisation of it has infinitely many non-canonical formulas. $\mathcal{C}(\text{RRA})$, on the other hand, is a possibly rare example of a variety of relation algebras that is non-finitely axiomatisable (as we will see in theorem 4.9) and canonical, but not barely canonical, because as we just saw, it has a canonical axiomatisation. So whilst we do need infinitely many non-canonical formulas to define RRA, it seems that we can still get a long way inside RA using only canonical ones. See [11] for work along similar lines.

4.2 Diagrams

To prove our first main result, we will use model-theoretic diagrams. Let $\mathcal{A} = (A, +, -, 0, 1, \cdot, \smile, ;)$ be a finite relation algebra. The (*basic*) *diagram* $\Delta\mathcal{A}(\bar{v})$ of \mathcal{A} is the conjunction of the following formulas of the signature of relation algebras, involving variables v_a ($a \in A$): $v_a \neq v_b$ for each distinct $a, b \in A$, $v_0 = 0$, $v_1 = 1$, $v_{1'} = 1'$, $v_{-a} = -v_a$ and $v_{\bar{a}} = (v_a)^\smile$ for each $a \in A$, and $v_{a+b} = v_a + v_b$ and $v_{a;b} = v_a ; v_b$ for each $a, b \in A$. Given any relation algebra \mathcal{B} , a map $f : \mathcal{A} \rightarrow \mathcal{B}$ is a relation algebra embedding iff $\mathcal{B} \models \Delta\mathcal{A}(\langle f(a) : a \in A \rangle)$. So

$$\mathcal{B} \models \exists \bar{v} \Delta\mathcal{A}(\bar{v}) \iff \mathcal{A} \text{ embeds into } \mathcal{B}. \quad (12)$$

We will use diagrams in conjunction with a well-known fact about relation algebras (it holds for all discriminator varieties): for every quantifier-free formula $\varphi(\bar{x})$ of the signature of relation algebras, there is a relation algebra term $\tau_\varphi(\bar{x})$ such that $\varphi(\bar{x})$ is equivalent to $\tau_\varphi(\bar{x}) = 0$ in every simple relation algebra. See, e.g., [23, theorem 381].

DEFINITION 4.3 For a finite relation algebra \mathcal{A} , let $\varepsilon_{\mathcal{A}}$ be the equation $\tau_{\neg\Delta\mathcal{A}}(\bar{v}) = 0$.

LEMMA 4.4 For every simple relation algebra \mathcal{B} , we have $\mathcal{B} \models \varepsilon_{\mathcal{A}}$ iff \mathcal{A} does not embed into \mathcal{B} .

Proof. By definition, $\mathcal{B} \models \varepsilon_{\mathcal{A}}$ iff $\mathcal{B} \models \forall\bar{v}(\tau_{\neg\Delta\mathcal{A}}(\bar{v}) = 0)$. As \mathcal{B} is simple, this is iff $\mathcal{B} \models \forall\bar{v}\neg\Delta\mathcal{A}(\bar{v})$. By (12), this is iff \mathcal{A} does not embed into \mathcal{B} . \square

Now let \mathcal{Q} be any non-representable simple persistently finite relation algebra.

LEMMA 4.5 $\varepsilon_{\mathcal{Q}}$ is not valid in \mathcal{Q} .

Proof. \mathcal{Q} is simple and $\mathcal{Q} \subseteq \mathcal{Q}$, so $\mathcal{Q} \not\models \varepsilon_{\mathcal{Q}}$ by lemma 4.4. \square

LEMMA 4.6 $\varepsilon_{\mathcal{Q}}$ is valid in RRA.

Proof. Let \mathcal{A} be a simple representable relation algebra. Since \mathcal{Q} is not representable and RRA is closed under subalgebras, \mathcal{Q} does not embed into \mathcal{A} . By lemma 4.4, $\mathcal{A} \models \varepsilon_{\mathcal{Q}}$. Since RRA is generated by its simple algebras, we deduce that RRA $\models \varepsilon_{\mathcal{Q}}$. \square

LEMMA 4.7 $\varepsilon_{\mathcal{Q}}$ is preserved by completions of relation algebras.

Proof. Let \mathcal{Q} be the variety of relation algebras that validate the equation $\varepsilon_{\mathcal{Q}}$. We need to show that $\mathcal{Q}^c \subseteq \mathcal{Q}$.

Suppose for contradiction that there is some $\mathcal{A} \in \mathcal{Q}^c \setminus \mathcal{Q}$. We start by showing that $\mathcal{Q} \in \text{Var}(\mathcal{Q}^c)$. As \mathcal{A} is clearly a relation algebra, it is a subdirect product of simple relation algebras \mathcal{S}_i ($i \in I$). That is, $\mathcal{A} \subseteq \prod_{i \in I} \mathcal{S}_i$ up to isomorphism, and the projection of \mathcal{A} onto each \mathcal{S}_i is surjective. Each \mathcal{S}_i is therefore a homomorphic image of \mathcal{A} , and so $\mathcal{S}_i \in \text{Var}(\mathcal{Q}^c)$ since $\mathcal{A} \in \mathcal{Q}^c$. If all \mathcal{S}_i validated $\varepsilon_{\mathcal{Q}}$, then so would \mathcal{A} , contradicting $\mathcal{A} \notin \mathcal{Q}$. So some \mathcal{S}_i does not validate $\varepsilon_{\mathcal{Q}}$, and as \mathcal{S}_i is simple, we obtain $\mathcal{Q} \subseteq \mathcal{S}_i$ (up to isomorphism) by lemma 4.4. Thus, indeed, $\mathcal{Q} \in \text{Var}(\mathcal{Q}^c)$.

We now cite a special case of [2, theorem 2.2], by taking ‘V’ in that theorem to be RA:

Assume that \mathfrak{A} is simple and persistently finite in RA. Then $\mathfrak{A} \in \text{Var}(\mathcal{K}^d \cap \text{RA})$ implies that $\mathfrak{A} \in \text{Var } \mathcal{K}$ for any subclass \mathcal{K} of RA.

Here, \mathcal{K}^d denotes the class of dense extensions of members of \mathcal{K} . Take $\mathcal{K} = \mathcal{Q}$, observe that $\mathcal{Q}^c \subseteq \mathcal{Q}^d \cap \text{RA}$, so that $\mathcal{Q} \in \text{Var}(\mathcal{Q}^d \cap \text{RA})$, and deduce from the theorem that $\mathcal{Q} \in \text{Var } \mathcal{Q} = \mathcal{Q}$. So $\mathcal{Q} \models \varepsilon_{\mathcal{Q}}$, contradicting lemma 4.5. \square

COROLLARY 4.8 There are no non-representable simple persistently finite relation algebras in $\mathcal{C}(\text{RRA})$.

Proof. It is immediate from lemmas 4.6–4.7 and definition 4.1 of $\mathcal{C}(\text{RRA})$ that $\varepsilon_{\mathcal{Q}}$ is valid in $\mathcal{C}(\text{RRA})$. But by lemma 4.5, it is not valid in \mathcal{Q} . Since \mathcal{Q} was arbitrary, we are done. \square

4.3 Non-finite axiomatisability

We now prove our first main result. The previous meaning of n ($n = 5$) is no longer needed.

THEOREM 4.9 *No class \mathbf{K} with $\mathbf{RRA} \subseteq \mathbf{K} \subseteq \mathcal{C}(\mathbf{RRA})$ is finitely axiomatisable (i.e., definable by a single first-order sentence).*

Proof. For each $d \geq 1$, the algebra $\mathcal{A}^{(d)}$ is a simple persistently finite non-representable relation algebra (lemmas 3.17 and 3.19, and corollaries 3.16 and 3.14). So by corollary 4.8, $\mathcal{A}^{(d)} \notin \mathcal{C}(\mathbf{RRA})$, and hence $\mathcal{A}^{(d)} \notin \mathbf{K}$.

Now let \mathcal{B} be a nonprincipal ultraproduct of the $\mathcal{A}^{(d)}$ for $d \geq 1$. Let $n \geq 4$. Since \mathbf{RA}_n is a variety (fact 3.12(4)), it is elementary, and by Łoś's theorem, elementary classes are closed under ultraproducts. By corollary 3.14 and fact 3.12(1), $\mathcal{A}^{(d)} \in \mathbf{RA}_{d+3} \subseteq \mathbf{RA}_n$ for all $d \geq n$. It follows that $\mathcal{B} \in \mathbf{RA}_n$.

This holds for all n , so $\mathcal{B} \in \bigcap_{n \geq 4} \mathbf{RA}_n = \mathbf{RRA}$ (fact 3.12(3)). The theorem now follows by Łoś's theorem in the usual way: if \mathbf{K} is finitely axiomatisable then so is its complement, which is therefore elementary and closed under ultraproducts, so it cannot be that $\mathcal{A}^{(d)} \notin \mathbf{K}$ for all d , and yet an ultraproduct of the $\mathcal{A}^{(d)}$ is in \mathbf{RRA} and hence in \mathbf{K} . \square

As shown in the proof above, for each $n \geq 4$ we have $\mathcal{A}^{(n)} \in \mathbf{RA}_n \setminus \mathcal{C}(\mathbf{RRA})$, and hence $\mathbf{RA}_n \not\subseteq \mathcal{C}(\mathbf{RRA})$ and $\mathcal{C}(\mathbf{RRA}) \subsetneq \mathbf{RA}$. For those interested, because $\mathcal{A}^{(d)} \in \mathfrak{RaCA}_{d+3}$ (see the comments after corollary 3.14), we also have $\mathfrak{RaCA}_n \not\subseteq \mathcal{C}(\mathbf{RRA})$ for every finite n . In some sense, $\mathcal{C}(\mathbf{RRA})$ is going off in a different direction from these classes.

Theorem 4.9 still holds if we replace $\mathcal{C}(\mathbf{RRA})$ by the variety \mathbf{NSPF} defined by $\{\varepsilon_{\mathcal{Q}} : \mathcal{Q} \text{ a non-representable simple persistently finite relation algebra}\}$, or even the variety \mathbf{AD} defined by $\{\varepsilon_{\mathcal{A}^{(d)}} : d \geq 1\}$, plus in both cases the equations defining relation algebras. The proof is the same. Since $\mathcal{C}(\mathbf{RRA}) \subseteq \mathbf{NSPF} \subseteq \mathbf{AD}$, these are formally more general results, but the definitions of \mathbf{NSPF} and \mathbf{AD} are perhaps less attractive than that of $\mathcal{C}(\mathbf{RRA})$.

4.4 Maddux's variety is not finitely axiomatisable

In [24], Maddux defined the variety V generated by the completions of representable relation algebras. In symbols, $V = \mathbf{Var}(\mathbf{RRA}^c)$. We discussed it in the introduction, and we can now prove our second main result, solving Maddux's problem 1.1(2) from [24].

THEOREM 4.10 *V is not finitely axiomatisable.*

Proof. As is easily seen, $\mathbf{RRA} \subseteq V \subseteq \mathcal{C}(\mathbf{RRA})$, so the result follows from theorem 4.9. \square

5 Conclusion

We have constructed non-representable simple persistently finite measurable relation algebras $\mathcal{A}^{(d)}$, for finite $d \geq 1$, that are 'arbitrarily representable', and in fact, $\mathcal{A}^{(d)} \in$

$\mathbf{RA}_{d+3} \setminus \mathbf{RA}_{d+4}$. We used them to show that various varieties of relation algebras are not finitely axiomatisable, including Maddux's variety $V = \text{Var}(\mathbf{RRA}^c)$ — this solves [24, problem 1.1(2)] negatively.

We end with some comments.

When $d = 1$, the construction that we have given is a special case of the one in [3], which essentially replaces our field \mathcal{F} by any finite abelian group \mathfrak{F} . The machinery of coset relation algebras is used in [3], and our construction (for all $d \geq 1$) can be shown directly to fit into that framework — for example, the required automorphisms can be shown to exist. [3] constructed a relation algebra from the cube \mathfrak{F}^3 of \mathfrak{F} . We too could have worked with $V_3^{(d)}$, which for $d = 1$ is isomorphic to \mathcal{F}^3 , but using $V_5^{(d)}$ seems to allow a simpler presentation. For example, we did not need automorphisms.

We did not use $V_n^{(d)}$ for $n > 5$ here, but perhaps they might be useful for other things.

We assumed that our field \mathcal{F} has characteristic 2. This assumption was not used often, and sometimes, such as in the proof of lemma 2.11, the use was only cosmetic. It is unclear whether the assumption can be relaxed, and what is to be gained by doing so, but some parts of the argument, such as the proof of lemma 2.8, do not seem straightforward to replicate in other characteristics.

We also assumed that \mathcal{F} is finite. This ensured that each relation algebra $\mathcal{A}^{(d)}$ is finite, hence not in \mathbf{RA}_{d+4} , and persistently finite, which was needed in the proof of the first main theorem (4.9).

A relation algebra is *completely representable* if it has a complete representation — an embedding of it into a product of relation algebras of the form $\mathfrak{Rc}(U)$ that preserves all existing sums. Givant and Andréka [6, problem 7.11] asked whether there is an atomic, complete, and representable measurable relation algebra that is not completely representable. Could the completion \mathcal{B}^c of a nonprincipal ultraproduct \mathcal{B} of the algebras $\mathcal{A}^{(d)}$ be an example? Unfortunately, assuming the continuum hypothesis, it appears that \mathcal{B}^c is completely representable. So this line of attack does not look promising.

Finally, we list some open questions, for specialists.

PROBLEM 5.1 Here, V denotes Maddux's variety $\text{Var}(\mathbf{RRA}^c)$, and $\mathcal{C}(\mathbf{RRA})$ is as defined in definition 4.1.

1. Do we have $V = \mathcal{C}(\mathbf{RRA})$?
2. Is V elementarily generated? The question is motivated by the fact that V is *canonical*. This is a simple consequence of [5, theorem 3.8], and is recorded in [15], so answering Maddux's [24, problem 1.1(3)] quoted in the introduction. All elementarily generated varieties of boolean algebras with operators are canonical [7, theorem 3.6.7], but not conversely [10].
3. We saw in remark 4.2 that $\mathcal{C}(\mathbf{RRA})$ is elementarily generated. Is there a 'nice' elementary class that generates it — for example, a finitely axiomatisable class?
4. Does V have a canonical axiomatisation?

5. Do V or $\mathcal{C}(\text{RRA})$ have Sahlqvist axiomatisations (cf. [11])?
6. Are the relation algebras $\mathcal{A}^{(d)}$ weakly representable (as defined in [17])? Most likely not, since [2, p.8] states that the algebras from [3], on which the $\mathcal{A}^{(d)}$ are based, are not weakly representable.

Problems 4–6 of [24, problem 1.1], quoted in the introduction, also remain open as far as we know. A positive answer to no. 4 would give a positive answer to problem 5.1(2) above, using [8, 9].

In case the reader is wondering, we mention that RRA is not finitely axiomatisable over V [15, theorem 2.2].

I dedicate this paper to Rob Goldblatt. Perhaps he may find some of it interesting, since he has worked on completions (e.g., [9]) and algebras of relations (e.g., [1]), he is familiar with coset relation algebras, and he is an authority on canonicity, which is involved in this article, though rather glancingly. The collaboration I have had with him, leading to nine joint papers from 2003 to the present day, is one of the two most fruitful of my life, and we have spent some wonderful times together. His generosity, wisdom, and magnificent New Zealand sense of humour are inspiring. I wish him the very happiest and most rewarding retirement.

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