On the Negative Disjunction Property

C. G. McKay

1 Introduction

An intermediate or superconstructive propositional logic is one obtained from Heyting's formalisation of intuitionist propositional logic by the addition of zero or more classical tautologies as new axioms and the familiar rules of inference – substitution and modus ponens – are retained. In the future I shall refer to these systems merely as logics, and, further, will identify a logic with its set of theorems. They have been an object of intensive study and the reader is referred to the book [2] for an excellent summary of the most important concepts, definitions and results obtain in this field. In particular, I shall assume that the reader is comfortable with the interplay of these logics with the algebraic systems variously known as Heyting algebras/pseudocomplemented lattices, which provide one – if perhaps not the most natural – class of semantic models for the logics.

A logic L is said to have the disjunction property (DP) iff for all formulae A_1 and A_2:

$$\vdash_L A_1 \vee A_2 \;\Leftrightarrow \vdash_L A_1 \textit{ or } \vdash_L A_2.$$

A special case of the DP occurs when the disjuncts which occur are negated. In this case I shall speak of the *negative disjunction property* (NDP). In other words the NDP is the principle:

$$\vdash_L \neg A_1 \vee \neg A_2 \;\Leftrightarrow\; \vdash_L \neg A_1 \textit{ or } \vdash_L \neg A_2.$$

In the case of the DP, we can immediately see that certain generalizations can be trivially established. Let us say that a logic L has the n-term disjunction property iff for all formulae A_i $(1 \leq i \leq n)$:

$$\vdash_L \bigvee_{i=1}^{n} A_i \;\Leftrightarrow\; \vdash_L A_i \textit{ for some } i, 1 \leq i \leq n.$$

Furthermore, a logic will be said to have the *unlimited disjunction property* (UDP) if it has the n-term disjunction property for all n. Then if a logic has the DP, it will also have the UDP by virtue of the fact that an n-term disjunction $\bigvee_{i=1}^{n} A_i$ can always be written as $A_1 \vee \bigvee_{i=2}^{n} A_i$.

The same is no longer true for NDP. Thus the trick of rewriting $\neg A_1 \vee \neg A_2 \vee \neg A_3$ as $\neg A_1 \vee \neg(A_2 \wedge A_3)$ implicitly presupposes that $\vdash_L \neg(A_2 \wedge A_3) \to (\neg A_2 \vee \neg A_3)$, for arbitrary A_2, A_3, which is equivalent to having $\vdash_L \neg A \vee \neg\neg A$ (for arbitrary A). But clearly no L satisfying this last condition will enjoy the NDP. Accordingly if we say that L has the n-term NDP when for all formulae A_i $(1 \leq i \leq n)$:

$$\vdash_L \bigvee_{i-1}^{n} \neg A_i \;\Leftrightarrow\; \vdash_L \neg A_i \textit{ for some } i, 1 \leq i \leq n.$$

and then define L to have the *unrestricted negative disjunction property* (UNDP) when L has the n-term NDP for all n, the NDP (i.e., the 2-term NDP) does not automatically secure the UNDP.

2 Results

In the [4], the Russian mathematician V. A. Jankov studied the logic – called Jankov's logic in what follows[1] – obtained by adding the tautology

$$\big((\neg\neg p \wedge (p \rightarrow q)) \wedge ((q \rightarrow p) \rightarrow p)\big) \rightarrow q$$

to the Heyting axioms and showed that this logic was characterized by the sequence of Heyting algebras J_i $(i \geq 2)$ depicted in Figure 1, in the sense that A is a theorem of Jankov's logic iff $v(A) = 1$ for all J_i and for all valuations v on J_i. The J_i are obtained by taking a Boolean algebra with $i \geq 2$ atoms and adding a new unit element at the top.

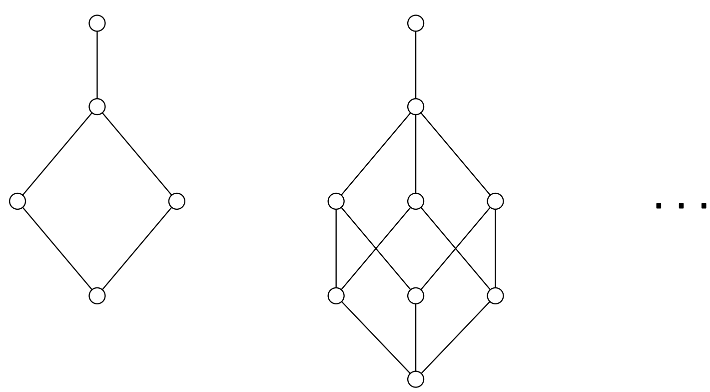

Figure 1: The Jankov Sequence

The following result can be obtained:

Theorem 1 *Jankov's logic has the UNDP.*

Proof. In this proof, H, J, C denote respectively Heyting's logic, Jankov's logic, and classical logic. Clearly, for any logic L, if $\vdash_L \neg A_i$ for some i, $1 \leq i \leq n$ $(n \geq 2)$ then

[1] Warning: in much of the literature this phrase is associated with another logic, namely that mentioned in Section 1 with axiom $\neg p \vee \neg\neg p$, a formula called in Jankov [4] the Weak Law of Excluded middle. Here and in what follows we use $p, q, \ldots$ sometimes with subscripts, as sentence letters or propositional variables.

$\vdash_L \bigvee_{i=1}^{n} \neg A_i$. To prove the converse in the case of $L = J$, suppose that $\nvdash_J \neg A_i$ for all i, $1 \leq u \leq n$. Then since $J \supseteq H$, $\nvdash_H \neg A_i$ for all i. By Glivenko's Theorem, it follows that $\nvdash_C \neg A_i$ for all i. For each n, it can be shown that the disjunction $\bigvee_{i=1}^{n} \neg A_i$ $(n \geq 2)$ fails on the corresponding Heyting algebra J_n in the Jankov series, by the device of taking the direct product of suitable algebras and gluing on a new top element in order to refute a disjunction. This gives us the desired conclusion: that $\nvdash_J \bigvee_{i=1}^{n} \neg A_i$. □

A few further remarks are in order before a corollary is noted.

(*i*) The direct-product-with-added-unit construction invoked at the end of the proof was already put to elegant use by Skolem in [7], p. 199*f.*

(*ii*) Theorem 1 is implicit in McKay's result in [6] on implicationless formulas since in any disjunction composed of negated disjuncts, the negated disjuncts can be transformed into intuitionistically equivalent formulas in which the connectives $\vee$ and $\rightarrow$ do not appear. However, McKay there makes no mention of the concepts of the NDP and UNDP which are central to the present discussion.

(*iii*) The proof given for Theorem 1 shows incidentally that if X is any logic with the n-term NDP for some n, and Y is a logic weaker than X, then Y also has the n-term NDP, using in particular the fact that (by Glivenko) all (intermediate) logics have the same negated formuals derivable in them. Thus suppose that X has the n-term NDP and Y is weaker than X, with $\neg A_1 \vee \ldots \neg A_n$ derivable in Y. Then this disjunction is derivable in the stronger logic X and so at least one of its disjuncts $\neg A_i$ is derivable in X, in which case, since X and Y have the same negated formulae derivable in them, $\neg A_i$ is derivable in Y, showing Y to have the n-term NDP. This is turn implies that if X has the UNDP then so does the weaker Y. This "backwards inheritance" of the UNDP is by no means shared by the DP. Consider Scott's intermediate logic (see for example Anderson [1], p. 126) which extends H by the addition of $F(p) =$

$$\big((\neg\neg p \rightarrow p) \rightarrow (\neg p \vee p)\big) \rightarrow (\neg p \vee \neg\neg p).$$

It is known that this logic has the DP (originally shown by Scott, and also proved in [1]). Let G be the formula

$$\big((F(p) \rightarrow q) \rightarrow q\big) \vee \big((F(q) \rightarrow p) \rightarrow p\big).$$

G is derivable in Scott's logic. (Indeed, G with the main $\vee$ replaced by $\wedge$ is derivable there.) Consider now the logic Φ, say, obtained by adding G to the Heyting axioms, and the Heyting algebra $L7$ illustrated in Figure 2. Note first that if we assign the sentence letter p in Scott's axiom the value 5, the axiom takes the value ω on $L7$ and is thus refutable on it. As a result, the formulae $(F(p) \rightarrow q) \rightarrow q$ and $(F(q) \rightarrow p) \rightarrow p$ both of which is interdeducible with the Scott axiom, are independently refutable on $L7$ by assigning the letter occurring in $F(p)$ and $F(q)$ respectively the value 5 and the other letter occurring in the respective formulas the value ω. Note furthermore that the Scott axiom is *only* refutable on $L7$ when the propositional letter in it is assigned the value 5. As a result $(F(p) \to q) \to q$ is only refutable when p is given the value 5 and q is given the value ω. Similarly $(F(q) \to p) \to p$ is only refutable on $L7$ when q is given the value 5 and p is given the value ω. But the refutation which can be constructed by choosing the appropriate values separately in the case of each individual disjunct in G is no longer available in the case of the disjunction as a whole. As a consequence, G is valid on $L7$ whereas its disjuncts are not. Thus Φ is an example of a logic weaker than Scott's which, unlike it, does not have the DP.

(iv) Similarly, the logic J does not have the DP since, for example, the disjunction

$$\big(((p \to q) \to p) \to p\big) \vee \big(((q \to p) \to q) \to q\big)$$

is valid on all the algebras of the J sequence while the Peirce formula, which may taken as either of the two (interdeducible) disjuncts here, is not.

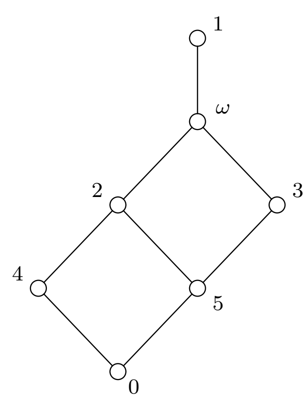

Figure 2: $L7$ (for Remark (iii))

For the record, we state the following corollary of Theorem 1, brought out by remarks (iii) and (iv):

Corollary *UNDP* $\not\Rightarrow$ *DP.*

One of the conspicuous features of NDP is that the number of disjuncts plays an important role. For example in the case of the logic corresponding to the first algebra J_2 in the Jankov sequence we can show the following, in which the logic is itself also denoted by J_2.

Theorem 2 *J_2 has the 2-term NDP but not the 3-term NDP.*

Proof. It is clear that in the case of two formulas $\neg A_1$ and $\neg A_2$, $\vdash_{J_2} \neg A_1 \vee \neg A_2 \Leftrightarrow \vdash_{J_2} \neg A_1$ or $\vdash_{J_2} \neg A_2$. To show that J_2 does not have the 3-term NDP consider the following 3-term disjunction:

$$\neg\neg(p_1 \to (p_2 \vee p_3)) \vee \neg\neg(p_2 \to (p_1 \vee p_3)) \vee \neg\neg(p_3 \to (p_1 \vee p_2)).$$

Each disjunct taken individually is not even a classical tautology. Consider now the whole disjunction, noting the following, in which the labels for the values are as indicated in Figure 3:

If any one of p_1, p_2, p_3 is assigned the value 1, the disjunction takes the value 1.

If any one of p_1, p_2, p_3 is assigned the value 0, the disjunction takes the value 1.

If any two of the values assigned to p_1, p_2, p_3 are the same, the disjunction takes the value 1.

This leaves as the only remaining case the assignment of the values ω, 2, 3 to these sentence letters (in some order). But then the disjunction again takes the value 1. Hence the disjunction takes the value 1 under all valuations. □

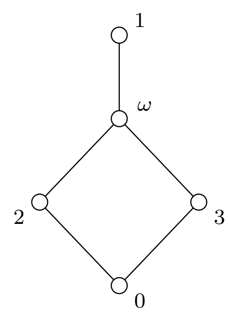

Figure 3: J_2 again, with labels

More generally, it can be shown that each of the logics (corresponding to the algebras) J_i has the i-term NDP but fails to have the j-term NDP for some $j > i$. To see this, using the following variant of the "counting formula" introduced by Gödel (in [3]):

$$\begin{array}{rlll} \neg\neg(p_1 \leftrightarrow p_2) & \vee \neg\neg(p_1 \leftrightarrow p_3) \vee \ldots & \vee \neg\neg(p_1 \leftrightarrow p_k) & \\ & \vee \neg\neg(p_2 \leftrightarrow p_3) \vee \ldots & \vee \neg\neg(p_2 \leftrightarrow p_k) & \\ & & \vdots & (*) \\ & & \vee \neg\neg(p_{k-1} \leftrightarrow p_k) & \end{array}$$

Since each J_i is finite, we can always find a k to ensure the validity of $(*)$ on the algebra concerned. But each individual disjunct is clearly not valid in classical logic. (Note that each formula $(*)$ will eventually be refutable on some J_i in the sequence.)

This general result can no doubt be sharpened to show that for each logic (corresponding to the algebra) J_i has the i-term NDP but not the $(i+1)$-term NDP, $i \geq 2$.

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