Paraconsistent Measurement of the Circle

Maarten M^{c}Kubre-Jordens Zach Weber

Abstract

A theorem from Archimedes on the area of a circle is proved in a setting where some inconsistency is permissible, by using paraconsistent reasoning. The new proof emphasizes that the famous method of exhaustion gives approximations of areas closer than any *consistent* quantity. This is equivalent to the classical theorem in a classical context, but not in a context where it is possible that there are inconsistent infinitesimals. The area of the circle is taken 'up to inconsistency'. The fact that the core of Archimedes's proof still works in a weaker logic is evidence that the integral calculus and analysis more generally are still practicable even in the event of inconsistency.

1 Introduction: Reductio without absurdity

"*Reductio ad absurdum*," wrote G.H. Hardy, "is a far finer gambit than any chess gambit; a chess player may offer the sacrifice of a pawn or even a piece, but the mathematician offers the game" [13, p.34]. Hardy means that, if an inconsistency were *per impossible* proved in the process of giving an argument by contradiction, then not only the proof, but the entirety of mathematics would be sacrificed. This may seem a bit dramatic. At least, it raises a straightforward question: how much basic geometry and analysis *would* still be practicable in the event that there were a contradiction somewhere?[1] In the event of inconsistency, which theorems would still be true?

Mathematics as understood in classical logic has an absolute answer. If p and not-p were provable, for some sentence p, then q is provable, for any sentence q whatsoever, by the logical rule of *ex falso quodlibet* or 'explosion'. Any local inconsistency immediately becomes a global catastrophe; in the event of contradiction, *everything* is true—and so pointless. This seems to be what Hardy had in mind. Perhaps relatedly, mathematics is widely *practiced* with a more pragmatic answer: any 'logician's contradiction' cannot matter for *real* mathematics, and is not worth worrying about.[2]

[1] As was the case for the original infinitesimal calculus [4, ch. 5], [6, p.191-219]; [5].

[2] Does the following sentiment sound familiar? "If ever the day comes when the logicians find some inconsistency in arithmetic, our reaction will surely be, 'Oh that's just a trick of the logicians; let them worry about it.' And one can almost hear the inconsistency coming—perhaps there will be a proof of the existence of a contradiction, but that contradiction is too long to even contemplate, so we may quite happily behave as if it did not exist" [14, p.204]. But see also [20].

So the classical answer to our question is either apocalyptic, or indifferent, or both. *Inconsistent mathematics* proposes a middle ground. Paraconsistent (or non-trivially inconsistent) mathematics is the project of seeking a finer-grained answer to the consistency question, by giving proofs that do not rely on 'explosion' [16]. Many have argued that explosion is not in fact a good inference principle, not capturing what is meant by sound mathematical reasoning [1], and have sought to develop mathematics without it [7, 9, 11, 17, 18]. Theorems thereby established are those that hold true independently of consistency. This is an impressive feature of mathematical truth: much as in everyday life, the facts survive even in the face of some level of inconsistency.[3]

In this note, we set ourselves the exercise of returning to the conceptual roots of the integral calculus, and examining an argument by reductio, from Archimedes' *Measurement of the circle* [2, 8]. The basic idea is to approximate the area of a circle by exhausting it with triangles. The point is to see whether the result will still hold in a paraconsistent logic—to imagine that Euclid, Archimedes, et al in the discovery days of basic geometry, happened to be reasoning paraconsistently. Regardless of whether simple geometry would ever be susceptible to contradiction, it is enlightening to see that it could withstand local inconsistency.

Our concerns are practical. This note is not a 'logical foundations of infinitesimal analysis done paraconsistently'; see [15]. As has always been the case historically, the first step is to develop clearer intuitions about what any such foundations will be founding.

2 Archemedes' Original Argument

In the third century BCE, Archimedes wrote a short treatise in which he proves:

> The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle. [2, p. 91]

Archimedes asks us to consider a circle Ω with radius r and circumference C; then he sets out to prove that a right triangle Δ with height r and base C will have the same area. The argument is a reductio, by cases on trichotomy: either the area of Ω is identical to the area of Δ (in which case we are done), or else one is bigger than the other. But, Archimedes argues, either of the latter options lead to contradictions, namely, that some quantity x is

[3]All paraconsistent proofs are also classical proofs: any result established using a paraconsistent logic may also be established classically, by the same argument as the paraconsistent reasoner. (In the event of a contradiction, classically everything follows anyway (by *ex falso*), so in particular the result of the theorem still follows.) In this sense, classical mathematics can be thought of as an *interpretation* of paraconsistent mathematics—an interpretation where there are no local contradictions. (Moretensen [16] calls this the 'special case' thesis.) The mathematical universe, when studied paraconsistently, is at least as rich as the classical one.

both strictly less than *and* strictly greater than some given quantity K:

$$x < K < x$$

If true, then $x = K$, and then $x < x$ and $K < K$, which is absurd, q.e.d. This proof is not paraconsistently valid, for several reasons.

First, Archimedes presumes that when a contradiction is derivable from an assumption, the assumption is proved to be false. But, as with all mathematics, any number of side assumptions may have been used to get to the contradiction. *One* of these assumptions is false, even for a paraconsistent mathematician, but we don't know which one. Perhaps it was the assumption that the areas in question have a non-zero difference; but perhaps it was the 'archimedean' assumption that there are no infinitesimals.

And this leads to the second problem for the proof. Even if we *did* know which assumption is at fault, we would need single it out. Classically, given a disjunction (either assumption p is false or assumption q is false), if p is not false, then it follows that q is the culprit we are looking for. But this pattern of reasoning is exactly what must be avoided if we are to avoid reliance on consistency. For supposing some p and not-p proven; then the argument

$$p, \quad \text{not-}p \text{ or } q \quad \therefore \quad q$$

leads inexorably to explosion, and must be invalid.[4] While this argument form, *disjunctive syllogism*, was one of the Stoic's 'Five Indemonstrables,' denying it is the prerequisite of paraconsistent research; and "we suppose that it is better to deny an Indemonstrable than a Demonstrable" [1, p. 488]. Disjunctive syllogism is not a truth-preserving inference in inconsistent contexts. It leads from a local inconsistency to a global absurdity.

Finally, Archimedes assumes that a quantity ε such that $\varepsilon < \varepsilon$ is simply absurd. To be sure, a quantity being strictly less than (or greater than) itself may well turn out to be absurd. But this begs our general question. Suspending judgement about individual contradictions, and not resorting to ex falso quodlibet, we ought not *presuppose* that this possibility is ruled out.

There are therefore serious obstacles to reconstructing this proof paraconsistently. One might even take these difficulties to vindicate the classical dependence on consistency. This would be premature, however. Below we show that the area of a unit circle is *logic invariant* (cf. [16]), derivable come what may.

[4]The proof is due to CI Lewis: suppose p and not-p. But p implies p or q. Then this together with not-p implies q, where q could be any sentence whatsoever.

3 Axioms and Cancellation Properties

3.1 Logic

An underlying paraconsistent logic is assumed. For concreteness, fix it to be the three valued logic RM3 [16, p. 22], with truth values true (t), false (f), and both (b), and truth tables:

	$\neg$
t	f
b	b
f	t

$\wedge$	t	b	f
t	t	b	f
b	b	b	f
f	f	f	f

$\vee$	t	b	f
t	t	t	t
b	t	b	b
f	t	b	f

$\rightarrow$	t	b	f
t	t	f	f
b	t	b	f
f	t	t	t

The acceptable values are t and b. Then RM3 validity is preservation of t or b forward, or equivalently, falsity preservation backward: if the conclusion of an argument is f, unacceptable, then one of the premises is unacceptable. But all that really matters here is that the logic has a conditional for which modus ponens is valid, and has a De Morgan negation that does everything Boolean negation does—exclude the middle, double negation eliminate, etc.—except it does not explode: in RM3, $p \wedge \neg p \rightarrow q$ can have an acceptable antecedent but unacceptable consequent, making the implication simply fail. Arbitrary contradictions do not entail arbitrary conclusions.[5]

3.2 Order

Assume given a set of *points*, $P, Q, R, \ldots$ and a set of *paths* $PQ, PR, RS, \ldots$. Paths may be represented as single variables $L, M, N \ldots$, and the term 'path' applies equally to magnitude (path length) as it does to a particular path. With Archimedes, we assume that paths are (length-)ordered by $\leqslant$, even in inconsistent settings:

- $L \leqslant L$ (reflexivity)
- If $L \leqslant M$ and $M \leqslant L$ then $L = M$ (antisymmetry)
- If $L \leqslant M$ and $M \leqslant N$ then $L \leqslant M$ (transitivity)
- $L \leqslant M$ or $M \leqslant L$ (linearity)

Define a *strict* order by $L < M := L \leqslant M \wedge M \nleqslant L$. Then $L < M$ implies $L \leqslant M$, but not vice versa. Strict order is irreflexive, asymmetric, transitive, and total (see [15]):

- $L = M$ or $L < M$ or $M < L$ (trichotomy)

[5] This is a *conservative* intuition. If we wanted to maintain the possibility of inconsistency, but didn't mind global absurdity, we would not need a paraconsistent logic!

Note that $N < N$ is an outright contradiction, $N \leqslant N \wedge N \nleqslant N$, but of a sort that could be tolerable if such a path also turned out to have interesting non-trivial properties.

The space of paths comes equipped with the arithmetic operator $+$ such that if PQ and QR are paths, then $PQ + QR$ is a path. The operator $+$ may be seen as *concatenation* of paths. The *direction* of a path can also be indicated: if L is a path, then $-L$ is a path. Two standard properties of inequalities are assumed: when $L + N$ and $M + N$ are defined,

- $L < M$ implies $L + N < M + N$
- $L < M$ implies $-M < -L$

Notably, we do *not* presume that inverses cancel, $L + -L = 0$. It is too close to being the geometric equivalent of the logical ex falso quodlibet, since *any* pairs $L, -L$ give the same 'bottom' particle, 0.

3.3 Archimedes' Axioms

Our target theorem requires little by way of geometric assumptions. We follow Archimedes and take the following as axiomatic.

Axiom 1 (Triangle Inequality)**.** $PQ + QR \geqslant PR$

A *line* is the shortest distance between two points: *the* line L from P to Q is a path such that any other path $(PQ)' \geqslant L$. A line between two points is unique by antisymmetry.

Axiom 2. Let P, Q be points, L the line from P, Q, and M, N two other concave paths from P to Q, both on the same side of L. If M is entirely bounded between L and N, then $M < N$.

Although the goal is eventually to work with infinitesimals, we take the notable step of including the assumption that none exist;[6] therefore any infinitesimals will also just be standard reals, unlike in Robinson's work. We return to this point, about uniformity, in the conclusion §6 below.

Axiom 3 (Archimedes' Axiom)**.** For $0 < L < M$, there is a quantity n such that $M \leqslant Ln$.

The quantity n mentioned in Axiom 3 is traditionally taken to be a natural number. We take the same view here, and accept the intuitive idea of the natural numbers as inexhaustible, in the sense that, given a natural number N, a larger one $(N + 1)$ may always be found. Areas are, likewise, treated intuitively. For the present paper, the only area that

[6]This is, paraconsistently, subtly different from the claim that their existence is *absurd*—the existence of infinitesimals is, under the present assumption, contradictory, but the ambient logic will ideally prevent this from resulting in global incoherence.

matters is that of a triangle and that of a circle. We shall have need only explicitly for the area of a triangle, which, in accordance with normal practice, is taken to be:

$$A\Delta = \frac{1}{2}\text{base} \times \text{perpendicular height}$$

Like Archimedes, we work with Euclid's *Elements* in the background, e.g. that triangles with the same base and height are equal. A large theme of Euclid is that of the common notions and postulates, and we use these here, too: especially the idea that *the whole is greater than the part* (Euclid, Common Notion 5).

4 Inscribing and Circumscribing

The method of Archimedes' proof is approximation by exhaustion. Let Ω be a circle with centre O, radius r and circumference C. Let σ_2 be a square **inscribed** in Ω, and for $n > 2$ let σ_n be the polygon with 2^n sides obtained by dividing arcs of Ω in half along sides of σ_n. Let Σ_2 be a square **circumscribed** around Ω, and let Σ_n be the polygon with 2^n sides ($n > 2$) obtained by dividing arcs of Ω in half along sides of Σ_n; see Fig. 1.

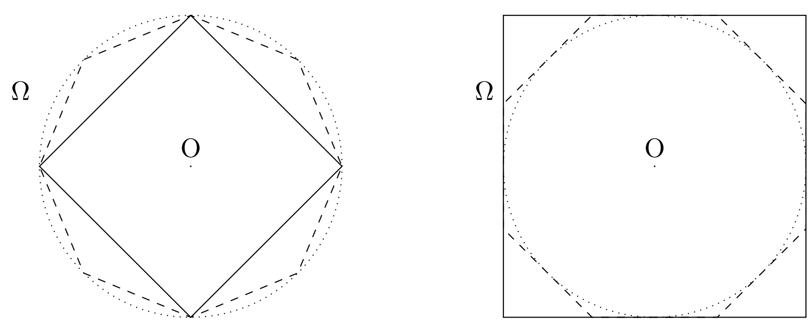

Figure 1: The circle Ω (dotted line), with: polygons σ_2 (solid line) and σ_3 (dashed line) inscribed (left); and polygons Σ_2 (solid line) and Σ_3 (dashed line) circumscribed (right).

Notation: For any figure Θ, let $A\Theta$ be the **area** of Θ.

Contrary to Archimedes's proof, we proceed by *direct* proof. The following two lemmata are responsible for the main work of the proof:

Lemma 1. *The areas of inscribed polygons are less than the areas of the figures containing them, and circumscribed polygons are greater:*

(i) $A\sigma_n < A\Omega$ *and* $A\sigma_n < A\Delta$*;*

(ii) $A\Sigma_n > A\Omega$ *and* $A\Sigma_n > A\Delta$.

Proof. To prove (i), for each k, take any side s of σ_k and consider a triangle τ_k with base equal to s and height equal to the perpendicular distance d from O to that side. Then the area of the polygon $A\sigma_k$ is

$$A\sigma_k = 2^k A\tau_k = \frac{1}{2}2^k sd.$$

Now the length of the perpendicular d is less than r, and by Axiom 2 the length of the side s is less then length of the segment of the circle it spans (so the perimeter of the polygon, $2^k s$, is less than the circumference of the circle, C), so

$$A\sigma_k = \frac{1}{2}2^k sd < \frac{1}{2}Cr = A\Delta.$$

The argument for (ii) is similar, but with a subtlety. For each k, take any side S of Σ_k and construct a triangle T_k with base equal to S and height equal to the perpendicular distance d from O to that side, which is just r. The area of the polygon $A\Sigma_k$ is $2^k AT_k$. Consider the segment OPQ (see Fig. 2 below) bounded by the circle and two perpendiculars from adjacent sides of Σ_k (OP and OQ). Note that $PR + RQ = \frac{1}{2}S + \frac{1}{2}S = S$. By Axiom 2 the

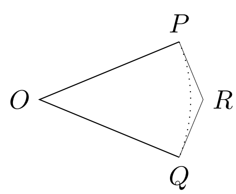

Figure 2: An arc (dotted) of the circle between perpendiculars of adjacent sides of Σ_k.

length $PR + RQ$ is greater than the length of the arc PQ. Thus $2^k S$ is greater than the circumference of the circle, C, and so

$$A\Sigma_k = \frac{1}{2}2^k Sd > \frac{1}{2}Cr = A\Delta$$

as required. □

Lemma 2. *Given any $\varepsilon > 0$, there is a positive integer n such that $A\Sigma_n - A\sigma_n < \varepsilon$.*

Proof. Set $\eta_k = A\Sigma_k - A\sigma_k$. With reference to Fig. 3, observe that

$$\eta_k = 2^k A\Delta_{PRQ}$$
$$\eta_{k+1} = 2^{k+1} A\Delta_{PTS}$$

and that, since Δ_{PRV} is a triangle with the same height but smaller base than Δ_{PRS},

$$\frac{1}{2}A\Delta_{PRQ} = A\Delta_{PRV} > A\Delta_{PTS} + A\Delta_{TRS}. \tag{1}$$

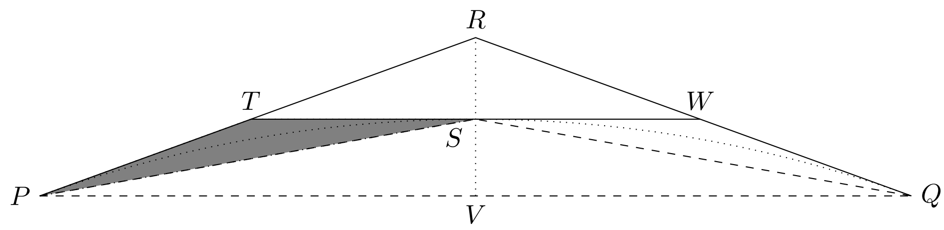

Figure 3: Partial edges of successive Σ_k (solid; edges $PR + RQ$ and $PT + TW + WQ$ for successive k) and σ_k (dashed; edges PQ and $PS + SQ$ for successive k) together with a circle segment (dotted).

Now since Δ_{TRS} is right-angled, and Δ_{PTS} isosceles, we have

$$TR > TS = PT.$$

Consequently,

$$A\Delta_{PTS} < A\Delta_{TRS}. \tag{2}$$

Combining (1) and (2) gives

$$\frac{1}{2}A\Delta_{PRQ} > 2A\Delta_{PTS}.$$

Multiplication by 2^k yields

$$\frac{1}{2}\eta_k > \eta_{k+1}.$$

Given $\varepsilon > 0$, use Axiom 3 to choose n such that $\frac{1}{2^n}(A\Sigma_1 - A\sigma_1) < \varepsilon$; then

$$A\Sigma_n - A\sigma_n = \eta_n < \frac{1}{2}\eta_{n-1} < \frac{1}{2^2}\eta_{n-2} < \cdots < \frac{1}{2^n}\eta_1 = \frac{1}{2^n}(A\Sigma_1 - A\sigma_1) < \varepsilon$$

as required. $\square$

5 Measuring the circle

In the sequel, by not taking consistency for granted, we do not lose any information, but we do gain insight into what the theorem really says about approximations: that without the assumption of consistency, they are only approximate.

A final bit of notation: for any quantities a, b,

$$|a - b| = \begin{cases} a - b & \text{if } a > b \text{ or } b = a \\ b - a & \text{if } b > a \end{cases}$$

The notation is unambiguous, even given the possibility of inconsistency, since if $b < a$ and $a < b$ then $b = a$, so $b - a = a - b$.

First let us argue the proof in the spirit of Archimedes, as a 'reductio', before giving the official proof. There are three cases:

$$A\Omega < A\Delta \quad \text{or} \quad A\Omega = A\Delta \quad \text{or} \quad A\Omega > A\Delta$$

Suppose $A\Omega > A\Delta$. Inscribe polygons σ_n. Set $\delta_n = A\Omega - A\sigma_n$. Find N_1 such that $\delta_{N_1} < A\Omega - A\Delta$, using Lemma 2. Then for each $N \geqslant N_1$,

$$A\Omega - A\sigma_N < A\Omega - A\Delta$$

Let $\eta := A\Omega - A\Omega$. Since $A\Omega - A\sigma_N < A\Omega - A\Delta$,

$$A\Delta + \eta < A\sigma_N + \eta \tag{3}$$

But by Lemma 1, $A\sigma_N < A\Delta$, so

$$A\sigma_N + \eta < A\Delta + \eta \tag{4}$$

Combining (3) and (4) we have

$$A\sigma_N + \eta = A\Delta + \eta \tag{5}$$

for each $N \geqslant N_1$, and a fortiori

$$A\Delta + \eta < A\Delta + \eta \tag{6}$$

A contradiction several times over.

Well. In the other case, supposing that $A\Omega < A\Delta$, then carrying out same procedure with circumscribed Σ_n, obtain N_2 such that

- $A\Delta - A\Omega < A\Sigma_N - A\Omega$ and
- $A\Delta - A\Omega > A\Sigma_N - A\Omega$

i.e. $A\Delta + \eta = A\Sigma_N + \eta$, and a fortiori

$$A\Delta + \eta < A\Delta + \eta$$

Therefore in this case, and a fortiori for all cases,

$$A\Omega = A\Delta$$

or else

$$|A\Omega - (A\Delta + \eta)| < |A\Omega - (A\Delta + \eta)|$$

as required.

Theorem 3. *Let Ω be a circle with radius r and circumference C, and Δ a right triangle with height r and base C. Then for any $\varepsilon > 0$, $|A\Delta - A\Omega| < \varepsilon$.*

Proof. For any k, Lemma 1 gives

$$A\sigma_k < A\Omega < A\Sigma_k$$
$$A\sigma_k < A\Delta < A\Sigma_k$$

Subtracting the two,

$$A\sigma_k - A\Sigma_k < A\Omega - A\Delta < A\Sigma_k - A\sigma_k.$$

Lemma 2 gives n such that

$$-\varepsilon < A\sigma_n - A\Sigma_n < A\Omega - A\Delta < A\Sigma_n - A\sigma_n < \varepsilon,$$

so that

$$|A\Omega - A\Delta| < \varepsilon$$

as required. □

The areas of Ω and Δ are arbitrarily—infinitesimally—close. They are within inconsistency of each other:

Proposition 4. *If $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then either $a = b$ or $|a - b| < |a - b|$.*

Proof. Either $|a - b| = 0$ (in which case we are done) or else $|a - b| > 0$. But then since $|a - b| < \varepsilon$ for *all* $\varepsilon > 0$, pick $|a - b|$ for ε. □

An approximation up to infinitesimal closeness is therefore only an *identity* if the possibility of inconsistency is disregarded, that is, if the case where $|a - b| < |a - b|$ is eliminated by disjunctive syllogism. This corollary to theorem 3 is stated thus:

Theorem 5. *Let Ω be a circle with radius r and circumference C, and Δ a right triangle with height r and base C. If $a = b$ whenever $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then*

$$A\Omega = A\Delta.$$

6 Identical up to inconsistency

Theorems 3 and 5 are classically equivalent restatements of the classical theorem. In fact, Theorem 5 is a version of the theorem with *paraconsistently* the same conclusion as the classical one—but it requires hypotheses which are stronger than the classical hypotheses in this setting. The attraction of this version of the theorem is that it clearly demonstrates a so-called "consistency requirement" that is classically trivial and so unmentioned, brought out explicitly here. Moreover, this version of the theorem does little violence to the intuition of the classical mathematician, who regards the additional hypothesis as inviolably true.[7]

We hasten to point out that making consistency assumptions explicit (e.g. in the distinction between theorems 3 and 5), is not being offered as a 'solution' to possible inconsistency. Adding a consistency assumption in the premise does not lead to consistency. From [1, p503, vol 2]: "One thing is clear is that adding premises cannot possibly reduce threat". Nor are we suggesting that, if presented with an inconsistency, mathematicians should generally accept it and carry on, rather than looking for revised consistent theories. Rather the point is that consistency can be considered to be a substantive assumption, made explicit, and in cases when it cannot be assured (like in the original naive theory of sets, or even in arithmetic due to Gödel's second theorem) mathematicians can still work and reason through.

The area of Ω is identical to the area of Δ *up to inconsistency*, in the same vein as two structures are said to be identical up to isomorphism. One could understand classicality to be the thesis that identity up to inconsistency just is identity. Classically, 'the fact that p-up-to-inconsistency' is just 'the fact that p'. The only classical possibility is the 'consistentized' version of Proposition 4,

$$\text{if } \forall \varepsilon > 0(|x - y| < \varepsilon) \text{ then } x = y. \tag{7}$$

This is precisely the property of real numbers that is responsible for producing Archimedes's conclusion. A paraconsistent reconstruction isolates the impact of various classical assumptions without altering the fundamental facts.[8]

It is well-known that (7) fails in the hyperreals [12], though its "transfer" holds (that

[7] This is a well-known feature of constructive mathematics, also(see e.g. [3]). For example, the *classical* intermediate value theorem (IVT),

> [IVT] *For any continuous function f, if $f(0) < 0$ and $f(1) > 0$ then there exists $x \in (0,1)$ such that $f(x) = 0$.*

is not constructively provable, whereas two versions, classically equivalent to the IVT, *are*:

> [Constructive IVT] *For any continuous* locally non-constant *function f, if $f(0) < 0$ and $f(1) > 0$ then there exists $x \in (0,1)$ such that $f(x) = 0$.*
>
> [Approximate IVT] *Given $\varepsilon > 0$ and any continuous function f, if $f(0) < 0$ and $f(1) > 0$ then there exists $x \in (0,1)$ such that $|f(x)| < \varepsilon$.*

[8] Anderson and Belnap called this situation 'p or Funny Bussiness'.

is, if ε is allowed to range over infinitesimals). This points to the difference between our approach and Robinson's famous work [19]. What we present is a *uniform practice*. In contrast, the traditional theory of the nonstandard numbers involves: a 'special' class of infinitesimal numbers; two separate formal languages (one for the 'standard' numbers, and one for infinitesimals, etc.) with some strict rules about what can and cannot be formulated in these languages; use of Zorn's Lemma to show that free ultrafilters exist; a transfer principle to translate statements between these languages (only if they are "appropriately formulated" [12, p. 11]); a distinction between 'internal' and 'external' objects; etc. Without diminishing Robinson's significant achievements, the work here begins to show how it is possible to recapture the flexibility of 'infinitesimal' arguments without needing to call on advanced machinery from model theory.

There is comfort to be had knowing that the area of the circle is indifferent to changes in logic. The core of Archimede's insight is derivable even in the event of inconsistency. Truth is not so fragile.[9]

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