The D-Completeness of $T_{\rightarrow}$

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1 Preamble

This paper has an interesting history. In 1986 during a visit by both Bob Meyer and Roger Hindley to Wollongong, Roger raised, in my car, a problem he had worked on. Bob said he had a solution to this, but as we heard nothing more from him on this in the ensuing months we assumed that he had been wrong. Some two years later I mentioned the problem at an AAL conference in ANU. Bob said, "But I have the solution to that, give me five minutes at the end of your talk to explain it". This I did, but after half an hour the conference organisers decided the next speaker should get his chance and Bob could finish his solution after the last conference talk. This he did in another 45 minutes. I wrote all this down, took it back to Wollongong and there shortened Bob's solution to roughly that shown below. Bob was happy with my version, but in his inimitable fashion extended it to a major paper of 78 pages [5], which unfortunately still has only appeared as a preprint.

The joint paper, by now, is probably my most quoted one and I am pleased to publish it here in tribute to Bob, in thanks for his friendship, numerous stimulating discussions, joint work and for bringing relevant logic to Australia.

2 Introduction

A Hilbert-style version of an implicational logic can be represented by a set of axiom schemes and modus ponens or by the corresponding axioms, modus ponens and substitution. Certain logics, for example the intuitionistic implicational logic, can also be represented by axioms and the rule of condensed detachment, which combines modus ponens with a minimal form of substitution. Such logics, for example intuitionistic implicational logic (see Hindley [3]), are said to be D-complete. For certain weaker logics, the version based on condensed detachment and axioms (the CONDENSED version of the logic) is weaker than the original. In this paper we prove that the relevant logic $T_{\rightarrow}$ and any logic of which this is a sublogic, is D-complete.

One feature of condensed detachment is that it is exactly the rule applied to the types of combinators or lambda terms when application is performed. A combinator that has a type, can be interpreted as a Hilbert style proof of the formula that is the type and a lambda term, that has a type, as a natural deduction-style proof of that type (see Hindley [4]). The condensed detachment rule may be viewed as the PUREST FORM of the Resolution Principle of Robinson [7], to which it is closely related.

3 $T_{\rightarrow}$ and Condensed $T_{\rightarrow}$

The logic $T_{\rightarrow}$ (also known as $BB'IW$ logic) is given by:

Definition 1 The logic $T_{\rightarrow}$

Axiom Schemes

$B: \quad (A \rightarrow B) \rightarrow (C \rightarrow A) \rightarrow C \rightarrow B$[†]
$B': \quad (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$
$I: \quad (A \rightarrow A)$
$W: \quad (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$

Rule of Modus Ponens

$$A \rightarrow B, A \quad \Rightarrow \quad B$$

For details on $T_{\rightarrow}$ and other relevant logics such as $T_{\rightarrow} - W$ $(= BB'I$ - logic) see Anderson and Belnap [1]. Condensed logics are given by:

Definition 2 Condensed Logics

The condensed version of a given implicational logic is obtained when its axiom schemes are replaced by axioms and its modus ponens rule by:

[†] We use association to the right for all implicational formulas.

RULE OF CONDENSED DETACHMENT D

$$A \to B, C \;\Rightarrow\; \sigma_1(B)$$

if there are substitutions σ_1 and σ_2 such that

1. $\sigma_1(A) = \sigma_2(C)$
2. Given 1, the total number of variable occurrences in $\sigma_1(B)$ is minimal.
3. Given 1 and 2, the number of distinct variables of $\sigma_1(B)$ is maximal.

NOTES

1. This is a more concise version of RULE D, for details of the original (see also Robinson [7] or Hindley [4]) and a proof of the equivalence see Bunder [2].
2. $\sigma_1(A)$, as defined above, is known as the most general unification (m.g.u) or most general common instance (m.g.c.i) of A and C. Every common substitution instance of A and C is also a substitution instance of $\sigma_1(A)$.

DEFINITION 3 D-COMPLETENESS

An implicational logic is said to be D-complete if it is equivalent to its condensed version.

The names that we have given to our axioms actually are COMBINATORS which have the formulas representing the axioms as their PRINCIPAL TYPES. (For details see Hindley [3]).

If we have combinators X and Y such that

$$X : A \to B$$

$$Y : C$$

and RULE D gives us $\sigma_1(A) = \sigma_2(C)$, we write

$$XY : \sigma_1(B).$$

XY is then a combinator with principal type $\sigma_1(B)$. Thus the combinator indicates the proof, using D, of the formula which is its principal type. For our purposes here, it is sufficient to regard XY as a name for $\sigma_1(B)$.
As examples we have:
$BB' : (p \to q \to r) \to (p \to (r \to s) \to q \to s)$
$BB'W : (p \to p \to q) \to (q \to r) \to p \to r.$

4 The D-completeness of $T_{\rightarrow}$

Before we can prove $T_{\rightarrow}$ D-complete we need a definition and some preliminary results.

Definition 4 Positive and Negative Positions

(*i*) p is in a positive position in p.

(*ii*) If an occurrence of p is in a positive position in A, this occurrence of p is in a positive position in $B \rightarrow A$ and in a negative position in $A \rightarrow B$.

(*iii*) If an occurrence of p is in a negative position in A, this occurrence of p is in a negative position in $B \rightarrow A$ and in a positive position in $A \rightarrow B$.

lemma 1 *If no variable is repeated in* A *then* $A \rightarrow A$ *is a theorem of condensed* $BB'I$ *logic.*

Proof: By induction on k the number of (distinct) variables in A.
If $k = 1$ by I, $\vdash p \rightarrow p$
If $k > 1$ $\quad A \equiv B \rightarrow C$ and the lemma holds for B and C,
i.e. $\vdash B \rightarrow B$ and $\vdash C \rightarrow C$.
Also $B(B'B)(BBB') : (p \rightarrow q) \rightarrow (r \rightarrow s) \rightarrow (q \rightarrow r) \rightarrow p \rightarrow s$,
so by D, $\vdash (r \rightarrow s) \rightarrow (B \rightarrow r) \rightarrow B \rightarrow s$
and so $\vdash (B \rightarrow C) \rightarrow B \rightarrow C$. □

lemma 2 *If* $A(p)$ *is a formula the variables of which (p among them) appear exactly once each, then if* p *is in a positive position in* $A(p)$ *in condensed* $BB'I$ *logic:*

$$\vdash (p \rightarrow u) \rightarrow A(p) \rightarrow A(u) \tag{1}$$

and if p *is in a negative position in* $A(p)$ *in condensed* $BB'I$ *logic*

$$\vdash (u \rightarrow p) \rightarrow \ A(p) \rightarrow A(u). \tag{2}$$

Proof: We proceed by induction on the length of $A(p)$.
If $A(p)$ is atomic $A(p) = p$ and (1) obviously holds.
Assume now that the result holds for formulas shorter than $A(p)$, then there are 4 cases:

1. $A(p) = B \rightarrow C(p)$ and p is a positive position in $A(p)$
2. $A(p) = C(p) \rightarrow B$ and p is a positive position in $A(p)$
3. $A(p) = B \rightarrow C(p)$ and p is a negative position in $A(p)$
4. $A(p) = C(p) \ \rightarrow B$ and p is a negative position in $A(p)$

(where of course p is not in B).

Case 1 By the induction hypothesis, as p is in a positive position in $C(p)$:

$$\vdash (p \to u) \to C(p) \to C(u) \tag{3}$$

By $BB(B(B'B)(BBB'))$: $(p \to q) \to (t \to r \to s) \to t \to (q \to r) \to p \to s$.

As every variable is in B only once by Lemma 1 :

$$\vdash B \to B$$

so by D :

$$\vdash (t \to r \to s) \to t \to (B \to r) \to B \to s$$

so by D and (3)

$$\vdash (p \to u) \to (B \to C(p)) \to B \to C(u).$$

which is (1).

Case 2 Now p is in a negative position in $C(p)$, so

$$\vdash (u \to p) \to C(p) \to C(u) \tag{4}$$

i.e.

$$\vdash (p \to u) \to C(u) \to C(p) \tag{5}$$

By

$$BB(B(B(B'B')B)B) \ : \ (p \to q) \to [t \to (r \to s). \to: t \to .(s \to p) \to (r \to q)]$$

and by $\vdash B \to B$, we have

$$\vdash (t \to (r \to s)) \to t \to (s \to B) \to r \to B$$

so by (5)

$$\vdash (p \to u) \to (C(p) \to B) \to C(u) \to B,$$

which is (1).

Case 3 Now p is in a negative position in $C(p)$ so (4) holds. The required result (2) is now obtained exactly as in Case 1 with $u \to p$ for $p \to u$ throughout.

Case 4 Now p is in a positive position in $C(p)$ so (3) holds, and hence

$$\vdash (u \to p) \to C(u) \to C(p)$$

the result (2) is now obtained as in Case 2 with $u \to p$ for $p \to u$. □

LEMMA 3 *If* $A(p,q)$ *is a formula, the variables of which* (p *and* q *among them) appear exactly once each, then*

$$\vdash B \to C \to [A(p,q) \to A(u,v)]$$

is a theorem of condensed $BB'I$ *logic, where if* p *is in a positive position in* $A(p,q)$, B *is* $p \to u$ *and if in a negative position,* $u \to p$; *if* q *is in a positive position in* $A(p,q)$, C *is* $q \to v$ *and if in a negative position* $v \to q$.

Proof:

$$B[B(B'B')B'](BB') : (s \to r \to p) \to (u \to p \to q) \to s \to u \to r \to q$$

By Lemma 2,

$$\vdash B \to A(p,q) \to A(u,q)$$

and

$$\vdash C \to A(u,q) \to A(u,v)$$

so by two applications of D

$$\vdash B \to C \to A(p,q) \to A(u,v) \qquad \Box$$

LEMMA 4 *Any proof in an implicational logic can be rewritten as a proof in the corresponding condensed logic followed by zero or more applications of the rule of substitution.*

Proof: We prove this by induction on the length of the proof. Any one step proof, i.e. an axiom scheme, is a substitution instance of an axiom of the corresponding condensed logic.
If $A \to B = [A_1/p_1, \ldots, A_n/p_n](C \to D)$
and $A = [B_k/p_k, \ldots, B_m/p_m]E$ where $C \to D$ and E and theorems of the condensed logic, it is clear that C and E have an m.g.u. F, and that A is a substitution instance of F. G the result of applying D to $C \to D$ and E, will than have B as a substitution instance. Thus the lemma holds. $\Box$

THEOREM 1 *If* $A(p_1)$ *is a theorem of any condensed logic that includes* B, B' *and* I *and* r *and* s *are variables not in* $A(p_1)$ *then* $A(r \to s)$ *is a theorem of that logic.*

Proof: We will write $A(p_1)$ as $A'(p_1, p_1, \ldots, p_2, p_2, \ldots, p_n)$ where all the occurrences of the variables of $A(p_1)$ (namely $p_1, p_2, \ldots, p_n$) are represented.

In A' we now replace the ith occurrence of p_j by p_{ji} for all appropriate i and $1 \leqslant j \leqslant n$). $A'(p_{11}, \ldots, p_{1k_1}, p_{21}, \ldots, p_{n1}, \ldots, p_{nk_n})$ is then a formula in which every variable appears exactly once.
By Lemma 1,

$$\vdash A'(r \to s, p_{12}, \ldots, p_{1k_1}, \ldots, p_{nk_n}) \to A'(r \to s, p_{12}, \ldots, p_{1k_1}, \ldots, p_{nk_n})$$

Also $\vdash A'(p_1, p_1, \ldots, p_1, p_2, \ldots, p_n)$,
so by D $\vdash A'(r \to s, r \to s, \ldots r \to s, p_2, \ldots, p_n)$
i.e. $\vdash A(r \to s)$. $\Box$

COROLLARY *Any substitution that does not identify variables is admissible in any condensed logic that has B,B′ and I.*

THEOREM 2 *If* $A(p_1, p_2)$ *is a theorem of any condensed logic that includes* B, B', I *and* W *then* $A(p_1, p_1)$ *is a theorem of that logic.*

Proof: We write $A(p_1, p_2)$ as $A'(p_1, p_1, \ldots, p_2, \ldots, p_n, \ldots, p_n)$, as in the proof of Theorem 1 and choose distinct variables $p_{11}, \ldots, p_{nk_n}$ to replace these as before.
We then have by Lemma 3:

$$\begin{aligned}\vdash B \to C \to\ & A'(p_{11}, \ldots, p_{1k_1}, p_{21}, \ldots, p_{nk_n}) \\ & \to A'(u, p_{12}, \ldots, p_{1k_1} v, p_{22}, \ldots, p_{nk_n})\end{aligned}$$

where B is $p_{11} \to u$ or $u \to p_{11}$ and C is $p_{21} \to v$ or $v \to p_{21}$.
If $B = p_{11} \to u$ and $C = p_{21} \to v$ or $B \equiv u \to p_{11}$ and $C \equiv v \to p_{21}$, we have by D and W:

$$\begin{aligned}\vdash B \to\ & A'(p_{11}, p_{12}, \ldots, p_{1k_1}, p_{11}, p_{22}, \ldots, p_{nk_n}) \\ & \to A'(u, p_{12}, \ldots, p_{1k_1}, u, p_{22}, \ldots, p_{nk_n})\end{aligned}$$

and by D and I

$$\begin{aligned}\vdash\ & A'(p_{11}, p_{12}, \ldots, p_{1_{k_1}}, p_{11}, p_{22}, \ldots, p_{n_{k_n}}) \\ & \to A'(p_{11}, p_{12}, \ldots, p_{1_{k_1}}, p_{11}, p_{22}, \ldots, p_{n_{k_n}})\end{aligned}$$

We have

$$\vdash A'(p_1, \ldots, p_1, p_2, p_2, p_3, \ldots, p_n)$$

so by D

$$\vdash A'(p_1, \ldots, p_1, p_1, \ldots, p_1, p_3, \ldots, p_n)$$

i.e. $A(p_1, p_1)$.
If $B \equiv p_{11} \to u$ and $C \equiv v \to p_{21}$
or $B \equiv u \to p_{11}$ and $C \equiv p_u \to v$,
we have by D and W

$$\begin{aligned}\vdash B \to\ & A'(p_{11}, p_{12}, \ldots, p_{1k_1}, u, \ldots, p_{nk_n}) \\ & \to A'(u, p_{12}, \ldots, p_{1k_1}, p_{11}, \ldots, p_{nk_n})\end{aligned}$$

and by D and I the same as above. □

THEOREM 3 *Any logic that has* $B, B',\ I$ *and* W *is D-complete.*

Proof: By Lemma 4 and Theorems 1 and 2. □

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