Partial Confirmation of a Conjecture on the Boxdot Translation in Modal Logic

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Abstract: The purpose of the present note is to advertise an interesting conjecture concerning a well-known translation in modal logic, by confirming a (highly restricted) special case of the conjecture.

1 The Conjecture

We work with the conventional language of (mono)modal logic, taking $\Box$ and some functionally complete set of boolean connectives as primitive, to generate the set of formulas from the set of propositional variables, $p_1, \ldots, p_n, \ldots$ For p_1 and p_2 we generally write p and q. Given a formula A, $\boxdot A$ abbreviates the formula $\Box A \wedge A$, and a formula in which all occurrences of $\Box$ are in subformulas of the latter form will be called a $\boxdot$-*formula*. The ***boxdot translation***, $\tau_{\boxdot}$, from modal formulas to modal formulas replaces all occurrences of $\Box$ with $\boxdot$. According to a well-known result, this translation embeds KT faithfully in (or 'into') K in the sense that for all formulas A:

$$\vdash_{\mathsf{KT}} A \text{ if and only if} \vdash_{\mathsf{K}} \tau_{\boxdot}(A).$$

Because of its simplicity, the boxdot translation has been a traditional favourite in the philosophical literature on the significance of such embeddings, for example, [7], [4] (and more briefly, by the same authors, in Example 2.9 of [8] and §2 of [5]. Recent technical studies in which it figures prominently include Goris [3] and Litak [6]; see also Zolin [10], Lemma 5.9. ([3] and [10] include references to the literature on modal provability logic, in which again $\boxdot$ and $\tau_{\boxdot}$ have enjoyed considerable currency.)

Now, experience with a considerable number of cases prompts the conjecture that the range of (normal: see below) modal logics in which $\tau_{\boxdot}$ faithfully embeds KT—in the sense that reference to them can replace that to K in the biconditional inset above—comprises precisely the logics S for which $\mathsf{K} \subseteq \mathsf{S} \subseteq \mathsf{KT}$.[1] While it is not hard to see that $\tau_{\boxdot}$ embeds KT faithfully into *all* the logics between K and KT, and also (since $\vdash_{\mathsf{KT}} A \leftrightarrow \tau_{\boxdot}(A)$ for all A) that no proper extension of KT enjoys this status as a candidate 'target' for the embedding (with KT as 'source'), it is not so easy to see that *only* logics between K and KT are suitable candidates. (We have informally circulated this problem amongst several logicians with a special interest in modal translations, but none has so far been able to provide a solution.) What, in particular, is there to rule out the existence of a logic $\subseteq$-incomparable with KT, in which KT is nonetheless embedded by $\tau_{\boxdot}$? According to the conjecture just outlined, no such logic exists: $\tau_{\boxdot}$ embeds KT only into sublogics of KT. "Logic", here, means *normal modal logic*;[2] for these we use the labels of Chellas [1] where they exist, and in general for a set of formulas Γ, $\mathsf{S} \oplus \Gamma$ is the smallest normal extension of S containing all formulas in Γ. When $\Gamma = \{A\}$ for some formula A, we write "$\mathsf{S} \oplus A$" rather then "$\mathsf{S} \oplus \{A\}$". In particular, then, $\mathsf{K} \oplus A$ is the smallest normal modal logic containing the formula A. In view of the facts about $\tau_{\boxdot}$ cited above, the conjecture amounts to the conjecture that for any non-theorem A, of KT (= $\mathsf{K} \oplus \mathsf{T}$, in the notation just introduced), there is a K-unprovable $\boxdot$-formula B with $\vdash_{\mathsf{K} \oplus A} B$. As we may put it, the normal extension of K by any non-theorem

[1] Considerations from the Kripke semantics point in a similar direction, but since these have proved suggestive rather than conclusive for us to date, they will not be in play explicitly in what follows. The idea is that whenever A is not KT-provable, the reflexive closures of frames on which A is valid will validate some formulas—the A^* of the proof of Corollary 4 below—not valid on every reflexive frame. Note that we do not say that A is valid on a frame if and only if A^* is valid on its reflexive closure—by analogy with the well-known fact that $\tau_{\boxdot}(A)$ is valid on a frame iff A is valid on its reflexive closure. In fact, there can be no function $g(\cdot)$ from formulas to formulas with the property that for all formulas A, A is valid on a frame iff $g(A)$ is valid on its reflexive closure, since different frames with the same reflexive closure need not validate the same formulas.

[2] We are taking a normal modal logic to be a set of formulas in the language described above, containing all the theorems of K and closed under Necessitation, Modus Ponens, and Uniform Substitution. For such a logic S we write "$\vdash_{\mathsf{S}} A$" for "$A \in \mathsf{S}$" and read this as: "A is provable in (or: is a theorem of) S". Because of the Uniform Substitution condition, it doesn't matter whether we think of KT as the smallest normal modal logic containing the formula $\Box p \to p$ or as the smallest such logic containing all instances of the schema $\Box A \to A$. Whereas Chellas [1] uses T as the name of the schema, we use it as the name for the formula just mentioned.

of KT is $\boxdot$-nonconservative. (Note that the $\boxdot$-formulas comprise precisely the image of the translation $\tau_{\boxdot}$.)

Massaged into the form just given, the conjecture clearly admits of no counterexample in which A is of modal degree 0, since for such an $A \notin \mathsf{KT}$, $\mathsf{K} \oplus A$ is the inconsistent logic (as A is then a $\Box$-free formula which is not a two-valued tautology). In the following section, we shall extend this to cover the case of formulas of modal degree 1—formulas, that is, in which no occurrence $\Box$ lies within the scope of another occurrence of $\Box$. (On this convention a formula of modal degree n is also of modal degree $n+1$.) Evidently this is a considerable limitation. The normal proper extensions of K axiomatizable by one-variable first degree formulas are precisely KD, $\mathsf{KD_c}$, KD!, KT, $\mathsf{KT_c}$, KT!, $\mathsf{K} \oplus \Box\bot$ and the inconsistent logic. Amongst first-degree formulas considered as axioms by which to extend K without restriction as to the number of variables, the most famous are perhaps those from the infinite sequence of formulas Alt_n ($n \in \mathbb{N}$) from Segerberg [9], p. 52. This section concludes with some remarks on the general case, without any restriction as to modal degree.

Sometimes when $A \notin \mathsf{KT}$, we can see that $\mathsf{K} \oplus A$ is not $\boxdot$-conservative because its theorems include $\tau_{\boxdot}(A)$, and $\nvdash_{\mathsf{K}} \tau_{\boxdot}(A)$. Examples include the cases in which A is 4 ($= \Box p \to \Box\Box p$) or B ($= p \to \Box\Diamond p$, where $\Diamond p$ is $\neg\Box\neg p$), or $\mathsf{T_c}$ ($= p \to \Box p$). In other cases, $\nvdash_{\mathsf{K}\oplus A} \tau_{\boxdot}(A)$, such as when A is $\Box\bot$, $\Box\bot \vee \Diamond\Box\bot$, or $\mathsf{D_c}$ ($= \Diamond p \to \Box p$, a minor variant of Segerberg's Alt_1); but here it is not hard to find $\boxdot$-formulas other than $\tau_{\boxdot}(A)$ which are not K-provable but which are of the form $\tau_{\boxdot}(B)$ for KT-unprovable B. (In the cases just mentioned, take B as $\mathsf{T_c}$, $\Diamond(\Box p \vee \Box\neg p)$, and $p \vee \Box(p \to q) \vee \Box(p \to \neg q)$, respectively.) The function f described in the following section applies to any first degree A to provide a B with the desired behaviour.

2 RESOLVING THE FIRST DEGREE CASE

In any normal modal logic, a formula A is provably equivalent to a conjunction of disjunctions of formulas of the form $\neg B \vee \neg\Box C \vee \Box D_1 \vee \cdots \vee \Box D_n$ in which B is $\Box$-free, and for the case of A of modal degree 1, the formulas $C, D_1, \ldots, D_n$ are also $\Box$-free;[3] we will write such 'basic disjunctions' in the implicational form:

$$(B \wedge \Box C) \to (\Box D_1 \vee \cdots \vee \Box D_n),$$

in which the consequent is understood as $\bot$ when $n = 0$ and either or both of the conjuncts in the antecedent may be absent.[4] (If both are, we may think

[3]Fine [2] is the *locus classicus* for normal forms in (normal) modal logic. Fine takes $\Diamond$ as primitive rather than $\Box$, and (essentially) works with disjunctive rather than (as here) conjunctive normal forms.

[4]We don't need to write "$\Box C_1 \wedge \cdots \wedge \Box C_m$" in the antecedent, since by normality this simplifies to $\Box(C_1 \wedge \cdots \wedge C_m)$. In terms of the disjunctive formulation, this is why we have just $\neg\Box C$ rather than $\neg\Box C_1 \vee \cdots \vee \neg\Box C_m$.

of the antecedent as $\top$ or, equivalently, identify the conditional with its consequent.)

Whether or not a given A is a first degree modal formula, if A is written as a conjunction $A_1 \wedge \cdots \wedge A_k$ with each A_i $(1 \leqslant i \leqslant k)$ a basic formula (as inset above), we define $f(A)$ to be the conjunction of the formulas $f(A_i)$ where f maps A_i, to the formula:

$$(s \wedge B \wedge \boxdot C) \to (\boxdot(D_1 \vee s) \vee \cdots \vee \boxdot(D_n \vee s)),$$

in which s is a propositional variable not occurring in A_i (i.e. not occurring in the given $B, C, D_1, \ldots, D_n$), and the $\boxdot$-notation is as explained in Section 1.

So defined, $f(A)$ is not unique, since various different conjunctive normal forms $\bigwedge_{i=1}^{k} A_i$ of A can make a difference, as well as the choice of s from the countable list of propositional variables (see the opening sentence of Section 1), though the first difference is inconsequential and the second could be ironed out by choosing s for A_i as the first p_j in that enumeration which does not occur in A_i. Rather than making any such moves, however, we can simply take $f(A)$ to denote an arbitrarily selected formula satisfying the defining conditions.

LEMMA 1 *For any formula* A: $\vdash_{\mathsf{K}} A \to f(A)$.

Proof: It suffices to observe that where $\bigwedge_{i=1}^{k} A_i$ is a normal form for A, we have $\vdash_{\mathsf{K}} A_i \to f(A_i)$, for $1 \leqslant i \leqslant k$. □

LEMMA 2 *For any formula* A *of modal degree 1, if* $\vdash_{\mathsf{K}} f(A)$ *then* $\vdash_{\mathsf{KT}} A$.

Proof: As in the proof of Lemma 1, it suffices to check this for each A_i. Take A_i as above, i.e., as $(B \wedge \Box C) \to (\Box D_1 \vee \cdots \vee \Box D_n)$, so that $f(A)$ is $(s \wedge B \wedge \boxdot C) \to (\boxdot(D_1 \vee s) \vee \cdots \vee \boxdot(D_n \vee s))$. Our hypothesis is that this is provable in K, so weakening the "$\boxdot$"s in the consequent to simple "$\Box$"s, and unpacking the "$\boxdot$" in the antecedent (together with some re-arranging), we have:

$$\vdash_{\mathsf{K}} ((s \wedge B \wedge C) \wedge \Box C) \to (\Box(D_1 \vee s) \vee \cdots \vee \Box(D_n \vee s)).$$

Since A, and therefore A_i, is of modal degree (at most) 1, $s \wedge B \wedge C$ is a $\Box$-free formula, so, using the fact that K is a (fully) modalized logic in the sense of [10], we can infer that either (a) $\vdash_{\mathsf{K}} ((s \wedge B \wedge C) \to \bot$ or (b) $\vdash_{\mathsf{K}} \Box C \to (\Box(D_1 \vee s) \vee \cdots \vee \Box(D_n \vee s))$. In case (a) we have $\vdash_{\mathsf{K}} B \to \neg C$ (after substituting B or $\top$ for s), and therefore (since $\vdash_{\mathsf{KT}} \Box C \to C$) $\vdash_{\mathsf{KT}} B \to \neg \Box C$, from which we conclude that $\vdash_{\mathsf{KT}} A_i$. In case (b) we have, substituting $\bigwedge_{i=1}^{n} D_i$ for s and simplifying: $\vdash_{\mathsf{K}} \Box C \to (\Box D_1 \vee \cdots \vee \Box D_n)$, so $\vdash_{\mathsf{K}} A_i$ and therefore again $\vdash_{\mathsf{KT}} A_i$. □

Putting these ingredients together settles the conjecture of Section 1 insofar as it bears on first degree formulas:

THEOREM 3 *Suppose* A *is a first degree modal formula not provable in* KT. *Then* $\mathsf{K} \oplus A$ *proves some* $\boxdot$*-formula not provable in* K.

Proof: Let A be as described. We may choose $f(A)$ as the desired $\boxdot$-formula, since it is provable in $K \oplus A$, by Lemma 1 (and Modus Ponens), but not provable in K, by Lemma 2 (contraposed). $\square$

For the first degree KT-unprovable formulas A listed at the end of Section 1, the formulas B given there as having B provable in $K \oplus A$ without $\tau_{\boxdot}(B)$ provable in K are minor variations on what the definition of $f(A)$ would deliver. For example, for the case of A as $\Box\bot$ we gave as a candidate B the formula $p \to \Box p$. Since this A contains no propositional variables, taking s as the first variable not occurring in A means that s is p (alias p_1), and $f(A)$ is therefore $p \to \boxdot(\bot \vee p)$, which is provably equivalent in K to $p \to \Box p$. More generally, since we may have to make substitutions, we can say for the proffered examples of B that B and $f(A)$ are K-interducible in the sense that $K \oplus B = K \oplus f(A)$.[5] This happens in the case of $A = p \to \Box p$ in which again we gave B as $p \to \Box p$, while $f(A)$ is $(p \wedge q) \to \boxdot(p \vee q)$ and we need to substitute p for q before proceeding further.

Returning to the conjecture as first formulated in Section 1, we have the following:

COROLLARY 4 If $S = K \oplus \Gamma$ for a set Γ of first degree formulas and $S \nsubseteq KT$, then $\tau_{\boxdot}$ does not embed KT faithfully in S.

Proof: Given that $S \nsubseteq KT$, pick $A \in \Gamma \smallsetminus KT$, and the proof of Theorem 3 shows that $f(A)$ is a $\boxdot$-formula in $K \oplus A \smallsetminus K$. Thus we have (1) $\nvdash_K f(A)$ while (2) $\vdash_S f(A)$. Since $f(A)$ is a $\boxdot$-formula, replacing every $\boxdot$ in $f(A)$ with $\Box$ gives a formula A^* for which $f(A) = \tau_{\boxdot}(A^*)$, and (1) and (2) can be reformulated as $(1')$: $\nvdash_K \tau_{\boxdot}(A^*)$, $(2')$: $\vdash_S \tau_{\boxdot}(A^*)$. From the fact (recalled in Section 1) that $\tau_{\boxdot}$ embeds KT in K, we infer from $(1')$ that $\nvdash_{KT} A^*$. If it were also the case that $\tau_{\boxdot}$ faithfully embedded KT in S, $(2')$ would imply that $\vdash_{KT} A^*$: a contradiction. $\square$

What is wanted, then, is a way of lifting the restriction to first degree formulas from Theorem 3 (and Corollary 4)—or else of showing, with a counterexample, that it cannot be lifted. Should the former turn out to be possible, perhaps the proof of Theorem 3 can be turned into the inductive step of a proof of the desired general result (by induction on the modal degree of formulas); however, we do not currently see how the details of such an argument would go.

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[5] Indeed, since the rule of necessitation is not required here, we could say more informatively that $K + B = K + f(A)$, where $S + \Gamma$ is the smallest (not necessarily normal) modal logic extending S and containing all formulas in Γ. In fact, everything said in Lemma 2 about KT applies to the quasi-normal modal logic $K + T$ (= $K[T]^0$ from [9], p. 179), suggesting a variant on Corollary 4 below in which the references to $K \oplus \Gamma$ and KT are replaced by references to $K + \Gamma$ and $K + T$, the qualification "first degree" is dropped, and reference to $\tau_{\boxdot}$ is replaced by a one to the translation τ_0, where $\tau_0(A)$ is the result of replacing only the outermost (i.e., not modally embedded) occurrences of $\Box$ in A by $\boxdot$. We do not go into this in any detail because of the marginal interest of τ_0 by comparison with $\tau_{\boxdot}$.

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