A Rejection System for the First-Degree Formulae of some Relevant Logics

Ross T. Brady
Philosophy Program,
La Trobe University
Ross.Brady@latrobe.edu.au

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1 REJECTION

The standard Hilbert-style of axiomatic system yields the assertion of axioms and, via the use of rules, the assertion of theorems. However, there has been little work done on the corresponding axiomatic rejection of non-theorems. Such Hilbert-style rejection would be achieved by the inclusion of certain rejection-axioms (r-axioms) and, by use of rejection-rules (r-rules), the establishment of rejection-theorems (r-theorems). We will call such a proof a rejection-proof (r-proof). The ideal to aim for would be for the theorems and r-theorems to be mutually exclusive and exhaustive. That is, if a formula A is a theorem then it is not an r-theorem, and if A is a non-theorem then it is an r-theorem. The first of these which ensures no overlap between the theorems and the r-theorems, we will call rejection-soundness (r-soundness). The second of these which ensures that all non-theorems are rejected, we will call rejection-completeness (r-completeness).

Both the set of theorems and the set of r-theorems are recursively enumerable, provided there are a finite number of axioms, rules, r-axioms, and r-rules. This can be ensured by use of uniform substitution rules, if necessary. If such a logic L is r-sound and r-complete then, as in [11, p. 307 (see also p. 284)], the class T of all theorems of L is general recursive. Then, the predicate $A \in T$ is general recursive and, by [11, p. 313], L is decidable. So, there are restrictions on the logics for which both r-soundness and r-completeness hold, especially r-completeness whose proof would usually embody a decidability argument. R-soundness, on the other hand, can usually be shown either syntactically, semantically or by metacompleteness. (See Section 2 for metacompleteness.)

Though the idea of rejection goes back to Aristotle, the first rejection system was due to Łukasiewicz, who, in Chapters IV and V of [12], set up a complete axiomatization of sentential logic, coupled with the axiomatic rejection of all and only the non-theorems. His positive system was as follows [12, pp. 80-1]:

PRIMITIVES $\sim, \supset$.

AXIOMS

1. $p \supset q \supset .q \supset r \supset .p \supset r$.
2. $\sim p \supset p \supset p$.
3. $p \supset .\sim p \supset q$.

RULES

1. $A \Rightarrow A^{B}/_{p}$ where B is substituted for each occurrence of p in A. (Rule of Substitution.)
2. $A, A \supset B \Rightarrow B$. (Rule of Detachment.)

For rejection, Łukasiewicz added the following: (We use the rejection sign '$\dashv$' here.)

REJECTION-AXIOM [12, p. 109]

1. $\dashv p$

REJECTION-RULES [12, p. 96]

1. $\vdash A \supset B, \dashv B \Rightarrow \dashv A$ (Rule of rejection by detachment.)
2. $\dashv A^{B}/_{p} \Rightarrow \dashv A$ (Rule of rejection by substitution.)

Łukasiewicz in [12, pp. 109-118], using an implicational normal form, rejected all the non-theorems of his assertion system, thus establishing r-soundness and r-completeness and thereby axiomatically deciding all formulae. Subsequently, Caicedo in [9] established another rejection system for sentential logic, this time without making reference to an assertion system. An induction argument on formula complexity is used to prove r-completeness.

2 SOME PRELIMINARY CONCEPTS

For what follows, we need to introduce the concepts of metacompleteness and degree. Meyer in [13], introduced the concept of a metacompleteness, principally for the purpose of finding an easy proof of 'If $A \vee B$ is a theorem then either A is a theorem or B is a theorem' for a suitable range of logics. In [13], he proved this for a wide range of quantified positive relevant logics, but it was Slaney in [15] who managed to add negation and prove the property for the sentential relevant logics RW and TW.

To define metacompleteness for a sentential logic L, we inductively introduce the following two metavaluations, ν and $\nu^{\star}$, as in Slaney [15, pp. 162-5].

(i) $v(p) = F$, for all sentential variables p.
$v^*(p) = T$, for all sentential variables p.

(ii) $v(A \,\&\, B) = T$ iff $v(A) = T$ and $v(B) = T$.
$v^*(A \,\&\, B) = T$ iff $v^*(A) = T$ and $v^*(B) = T$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.
$v^*(A \vee B) = T$ iff $v^*(A) = T$ or $v^*(B) = T$.

(iv) $v({\sim}A) = T$ iff $v^*(A) = F$.
$v^*({\sim}A) = T$ iff $v(A) = F$.

(v) $v(A \to B) = T$ iff $\vdash A \to B$ *and*, if $v(A) = T$ then $v(B) = T$, *and* if $v^*(A) = T$ then $v^*(B) = T$.
$v^*(A \to B) = T$.

We prove the following two lemmas.

LEMMA 1 *For all formulae* A, *if* $v(A) = T$ *then* $\vdash A$, *and if* $v^*(A) = F$ *then* $\vdash {\sim}A$.

Proof: We prove these together by induction on formulae. □

LEMMA 2 *For all formulae* A, *if* $\vdash A$ *then* $v(A) = T$.

Proof: We use induction on the proof procedure for theorems. □

THEOREM 1 *L is metacomplete, i. e.* $v(A) = T$ *iff* $\vdash A$, *for all formulae* A.

Proof: By Lemmas 1 and 2. □

The class of such metacomplete logics are called M1 logics by Slaney in [16]. Using the above metavaluations, one can also establish a number of metacompleteness properties, such as (I)–(VI) in Section 4 below. In [16], Slaney also defined M2 metacomplete logics by replacing '$v^*(A \to B) = T$' in (v) above by '$v^*(A \to B) = T$ iff, if $v(A) = T$ then $v^*(B) = T$'.

The M1 metacomplete logics include the key contraction-less logics B, DW and TW, and some weaker relevant logics such as DJ and TJ, both with conjunctive syllogism added. The contraction-less logics, EW and RW, are M2 metacomplete, as are the other contraction-less logics with the rule: $A, {\sim}B \Rightarrow {\sim}(A \to B)$, added. Metacomplete logics are important as they encompass contraction-less logics, which have many special properties, the logic DJ, which is conceptualized in Brady [2] and [7] as the logic of meaning containment, and TJ, which can be used to solve the set-theoretic and semantic paradoxes (see Brady [3] and [7]).

Roughly, the degree of a formula is the maximum depth of '$\to$' occurring in it. Inductively:

(i) A sentential variable p has degree 0.

(ii) If A has degree m and B has degree n then $A \,\&\, B$ and $A \vee B$ have degree $max(m, n)$, whilst $A \rightarrow B$ has degree $max(m, n) + 1$.

Formulae of low degree are more likely to occur in practice and indeed, in Anderson and Belnap [1] and in Dunn [10], there has been extensive study of the system E_{fde} of first-degree entailments, common to relevant logics from B through to R. The first-degree formulae of E can also be found in [1], whilst those of the weaker M1 and M2 metacomplete logics can be found in Brady [5]. It is the first degree of these M1 logics that we address in this paper.

3 RECENT REJECTION SYSTEMS

The issue of rejection systems has arisen in three recent contexts. The first was in connection with the metacompleteness of relevant logics including sentential constants some of which are classical. In order to fully represent their classicality, Brady in [8] considered a formal notion of non-derivability to ensure that if a sentential constant p is derivable then $\sim p$ is not derivable. Thus, an axiomatic rejection system was introduced, with the following axioms and rules.

REJECTION-AXIOMS Given that, for each sentential constant p, none, one or both of $\vdash p$ and $\vdash \sim p$ are added, in accordance with some recursive specification, we add $\dashv p$ or $\dashv \sim p$ (or both), whenever the corresponding $\vdash p$ or $\vdash \sim p$ is not included.

REJECTION-RULES

1. $\vdash A \rightarrow B, \dashv B \Rightarrow \dashv A$.
2. $\dashv A, \dashv B \Rightarrow \dashv A \vee B$.
3. $\vdash A \vee B, \dashv A \Rightarrow \vdash B$.

Though metacompleteness and r-soundness was established for appropriate logics, the question of r-completeness was left open.

The second and related recent context is in connection with the formalization of the Law of Non-Contradiction in Brady [6]. There, it is proposed that the Law is most appropriately formalised as $\dashv A \,\&\, \sim A$, or as $\vdash A \Rightarrow \dashv \sim A$, immersed in an axiomatic rejection system.

The third is a rejection system, due to Meyer and Slaney in [14], for Abelian logic A. They show that any formula of A is equivalent to a conjunctive normal form of basic intensional formulae. This theorem can then be used to establish a rejection system for A.

4 THE REJECTION SYSTEM $L1_r$

It would be nice to develop r-sound and r-complete rejection systems for a good range of decidable sentential relevent logics. To start this process, here we examine the first-degree metacomplete logic L_1 of Brady [5]. L_1 is the common first-degree fragment of a range of weak metacomplete logics which are all M1 logics. Thus, L_1 is also B_1, the first-degree fragment of the Routley-Meyer basic logic B, and also DJ_1, the first-degree fragment of the logic DJ of Brady [2] and [7]. The assertion system L_1 is axiomatized as follows in [5]:

AXIOMS

1. $A \to A$.
2. $A \,\&\, B \to A$.
3. $A \,\&\, B \to B$.
4. $A \to A \vee B$.
5. $B \to A \vee B$.
6. $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$.
7. $\sim\sim A \to A$.

RULES

1. $A \to B, B \to C \Rightarrow A \to C$.
2. $A \to B, A \to C \Rightarrow A \to B \,\&\, C$.
3. $A \to C, B \to C \Rightarrow A \vee B \to C$.
4. $A \to \sim B \Rightarrow B \to \sim A$.

NON-ENTAILMENT RULES

5. $A, B \Rightarrow A \,\&\, B$.
6. $A \Rightarrow A \vee B$.
7. $B \Rightarrow A \vee B$.
8. $A \Rightarrow \sim\sim A$.
9. $\sim A \Rightarrow \sim(A \,\&\, B)$.
10. $\sim B \Rightarrow \sim(A \,\&\, B)$.
11. $\sim A, \sim B \Rightarrow \sim(A \vee B)$.

As stated in [5], L_1 consists of all the theorems of the first-degree entailment system E_{fde} of Anderson and Belnap [1, p. 158], obtained by applying rules 1-4 to the axioms 1-7, with all its non-entailment theorems built up from the E_{fde} entailments by applying rules 5-11. The following metacompleteness properties for L_1 should be read in conjunction with it, most of these having allowed its axiomatization to be simplified.

(I) If $\vdash A \vee B$ then $\vdash A$ or $\vdash B$.

(II) NOT $- \vdash {\sim}(A \to B)$.

(III) If $\vdash A \,\&\, B$ then $\vdash A$ and $\vdash B$.

(IV) If $\vdash {\sim\sim}A$ then $\vdash A$.

(V) If $\vdash {\sim}(A \,\&\, B)$ then $\vdash {\sim}A \vee {\sim}B$.

(VI) If $\vdash {\sim}(A \vee B)$ then $\vdash {\sim}A \,\&\, {\sim}B$.

Note that, by (II), the logic L_1 is an M1 logic in the sense of Slaney in [16], in that it has no negated entailment theorems.

We consider the metacompleteness properties (I) and (III)–(VI) as rules for the purposes of framing the rejection-rules in what follows. We add, to the above assertion system L_1, the following rejection-axioms and rejection-rules, yielding L_{1r}.

REJECTION-AXIOMS

1. $\dashv p \,\&\, {\sim}q \,\&\, r \,\&\, {\sim}r \to {\sim}p \vee q \vee s \vee {\sim}s$.
2. $\dashv {\sim}(A \to B)$.

REJECTION-RULES

1. $\vdash A \to B, \dashv A \to C \Rightarrow \dashv B \to C$.
2. $\vdash B \to C, \dashv A \to C \Rightarrow \dashv A \to B$.
3. $\dashv A, \dashv B \Rightarrow \dashv A \vee B$.
4. $\dashv A \Rightarrow \dashv A \,\&\, B$.
5. $\dashv B \Rightarrow \dashv A \,\&\, B$.
6. $\dashv A \Rightarrow \dashv {\sim\sim}A$.
7. $\dashv {\sim}A \Rightarrow \dashv {\sim}(A \vee B)$.
8. $\dashv {\sim}B \Rightarrow \dashv {\sim}(A \vee B)$.
9. $\dashv {\sim}A, \dashv {\sim}B \Rightarrow \dashv {\sim}(A \,\&\, B)$.
10. $\dashv A^{B}/_{p} \Rightarrow \dashv A$, where B is uniformly substituted for p in A.

Sentential variables are used without the use of formula schemes in the r-axioms, except for RA2 which is schematic, and a rejection-substitution rule RR10 is used. Schemes are still used in the assertion system and in the statement of the r-rules. The r-rules RR1-2 are reversals of rule R1 of the assertion system L_1. (The reversals of rules R2-4 of L_1 are not needed for the r-completeness argument for $L1_r$ and so are omitted.) The r-rules RR3-9 embrace all the requirements for the rejection-metacompleteness result of Theorem 3 to follow, some being reversals of appropriate metacompleteness properties of $L1$. Similar to $L1$ in [5], we will add some r-metacompleteness properties to round out the logic $L1_r$.

We need the following rejection-theorems:

R-THEOREMS

1. $\dashv p$. (RA1 $\times$ RR10)
2. $\dashv \sim p$. (RT1 $\times$ RR6 $\times$ RR10)

A rejection system in the style of Caicedo [9], i.e. without reference to the assertion system, can also be created, though rather tediously. This can be done by replacing the assertions $\vdash A \to B$ of RR1 and $\vdash B \to C$ of RR2 by each of the $L1$ theorems actually used in the proof of Theorem 4, i.e. in the proof of r-completeness. We then drop these theorems from the statement of the rules, yielding pure rejection rules. R-soundness and r-metacompleteness will still apply.

5 R-SOUNDNESS AND R-METACOMPLETENESS FOR $L1_r$

R-soundness can be proved immediately by an easy syntactical method.

THEOREM 2 *$L1_r$ is r-sound, i.e. for all formulae A, if $\vdash A$ then* NOT $- \dashv A$.

Proof: We show that if $\dashv A$ then NOT $- \vdash A$, by induction on the r-proof procedure. The r-axioms are non-theorems of $L1$. RA1 is not a tautological entailment, from which it follows that it is not a theorem of any relevant logic from B through to R, and RA2 is a metacompleteness property. The r-rules preserve non-theoremhood of $L1$. RR1-2 preserve non-theoremhood by R1, RR3-9 by metacompleteness properties (I), (III)–(VI), and RR10 by the use of the schematic method in $L1$. □

For rejection-metacompleteness, we inductively introduce the two rejection-metavaluations, v_r and $v_r^\star$, which will relate to the rejection-theorems in a similar way to that of metavaluations and theorems of the assertion logic $L1$ (see Section 2).

(i) $v_r(p) = T$.

$v_r^\star(p) = F$.

(ii) $v_r(A \mathbin{\&} B) = T$ iff $v_r(A) = T$ or $v_r(B) = T$.

$v_r^\star(A \mathbin{\&} B) = T$ iff $v_r^\star(A) = T$ or $v_r^\star(B) = T$.

(iii) $v_r(A \vee B) = T$ iff $v_r(A) = T$ and $v_r(B) = T$.

$v_r^\star(A \vee B) = T$ iff $v_r^\star(A) = T$ and $v_r^\star(B) = T$.

(iv) $v_r(\sim A) = T$ iff $v_r^\star(A) = F$.

$v_r^\star(\sim A) = T$ iff $v_r(A) = F$.

(v) $v_r(A \to B) = T$ iff $\dashv A \to B$.

$v_r^\star(A \to B) = F$.

To establish r-metacompleteness, we prove the following lemmas.

LEMMA 3 *For all formulae* A*, if* $v_r(A) = T$ *then* $\dashv A$*, and if* $v_r^\star(A) = F$ *then* $\dashv \sim A$.

Proof: We prove these together by induction on formulae.

p. By RT1 and RT2, together with RR10 to introduce other variables.
$A \,\&\, B$. By RR4, RR5 and RR9.
$A \vee B$. By RR3, RR7 and RR8.
$\sim A$. By RR6.
$A \to B$. By RA2. □

LEMMA 4 *For all formulae* A*, if* $\dashv A$ *then* $v_r(A) = T$.

Proof: We use induction on the proof procedure for rejection-theorems.

RA1. By (v).
RA2. By (iv), (v).
RR1-2. By (v).
RR3. By (iii).
RR4-5. By (ii).
RR6. By (iv).
RR7-8. By (iii), (iv).
RR9. By (ii), (iv).
RR10. We use induction on formula construction to establish together (I) if $v_r(A^B/_p) = T$ then $v_r(A) = T$, and (II) if $v_r^\star(A^B/_p) = F$ then $v_r^\star(A) = F$. We straight-forwardly use (i)–(v) and RR10. □

THEOREM 3 $L1_r$ *is r-metacomplete, i. e.* $v_r(A) = T$ *iff* $\dashv A$*, for all formulae A.*

Proof: By Lemmas 3 and 4. □

This r-metacompleteness yields a number of properties, which enable the axiomatization of $L1_r$ to be enhanced.

(I_r) If $\dashv A \,\&\, B$ then $\dashv A$ or $\dashv B$.

(II_r) If $\dashv A \vee B$ then $\dashv A$ and $\dashv B$.

(III_r) If $\dashv \sim\sim A$ then $\dashv A$.

(IV_r) If $\dashv \sim(A \,\&\, B)$ then $\dashv \sim A \vee \sim B$.

(V_r) If $\dashv \sim(A \vee B)$ then $\dashv \sim A \,\&\, \sim B$.

These are admissible rules of $L1_r$, in that r-theoremhood is preserved. Thus, we can use them as such, in addition to the primitive rules RR1-10, and, in particular, we will use (I_r) and (II_r) in determining the r-completeness of $L1_r$ to follow.

6 R-COMPLETENESS FOR $L1_r$

R-completeness does involve some preliminary work. We first need to get some characterization for the non-theorems of L1. However, let us start with the theorems. For each theorem A of L1, using metacompleteness properties (I), (III)–(VI), as well as the rules R5-7, we can establish a disjunctive normal form B consisting of a disjunction of conjunctions of *basic formulae*, i. e. sentential variables, negated sentential variables, entailments and negated entailments. (It is important for the negations to be pushed through first, outer ones before inner ones, as the normal-forming rules do not operate inside the scope of a negation.) Here, we include disjunctions consisting of a single disjunct, which would then be a conjunction of basic formulae. We also include conjunctions consisting of a single conjunct, which would just be a basic formula. As in Slaney [15, p. 166], by metacompleteness property (I), at least one of the disjuncts C of B must be a theorem, and hence each of its conjuncts is a theorem. Since sentential variables, negated sentential variables and negated entailments are non-theorems, the only basic formulae that can be theorems are entailments. So, the theorem disjunct C must consist of a conjunction of first-degree entailments. These first-degree entailments must be theorems of Anderson and Belnap's E_{fde} of [1]. Thus, for any theorem A of L1, there is a disjunctive normal form B, consisting of a disjunction, at least one disjunct C of which consists of a conjunction of E_{fde} theorems.

Non-theorems, on the other hand, are better put into conjunctive normal form. So, each non-theorem A of L1 can be put into a conjunctive normal form B, using (I), (III)–(VI), and R5-7, as above, but with a conjunction of disjunctions of basic formulae, allowing for single conjuncts and disjuncts. By R5, there is at least one conjunct C which is a non-theorem of L1, and, by R6-7, each of its disjuncts is a basic formula which is a non-theorem. Such basic formulae are sentential variables, negated sentential variables, entailments and negated entailments. Each sentential variable, negated sentential variable and negated entailment is a non-theorem, whereas the entailment non-theorems are non-theorems of E_{fde}. So, to prove r-completeness of L1, we have to reject sentential variables, negated sentential variables and negated entailments, as well as the non-theorems of E_{fde}. Then, we have to retrace the conjunctive normal-forming steps, using rejection-rules and r-metacompleteness properties, until we get back to our non-theorem A. We do all this in the following theorem.

However, before we do, we first look at the structure of the non-theorems of E_{fde}. In Anderson and Belnap [1, pp. 151-158], the theorems of E_{fde}, called tautological entailments, can all be put into normal form consisting of disjunctive normal form in the antecedent and conjunctive normal form in the consequent, i. e. $A_1 \vee \cdots \vee A_m \to B_1 \ \& \ \cdots \ \& \ B_n$. Whenever we pair any disjunct A_i from the antecedent with any conjunct B_j from the consequent, i. e. $A_i \to B_j$, we get an entailment between a conjunction of atoms and a disjunction of atoms, with at least one atom in common, where an *atom* is a sentential variable or a negated sentential variable. So, the non-theorems of E_{fde} are those with normal forms $A_1 \vee \cdots \vee A_m \to B_1 \ \& \ \cdots \ \& \ B_n$, with some pair $A_i \to B_j$ having no atom from the conjuncts of A_i in common with an atom from the disjuncts of B_j. Again, this process must be reversed for rejection.

THEOREM 4 *L_{1r} is r-complete, i. e. for all formulae* A, *if* NOT $-\vdash A$ *then* $\dashv A$.

Proof: By RT1, RT2 and RR10, any sentential variable q and any negated sentential variable $\sim s$ is an r-theorem. By RA2, any negated entailment is an r-theorem as RA2 is schematic. We proceed in five stages. In the first three stages, we reject the non-theorems of E_{fde}.

(1) We start with entailments between conjunctions of atoms and disjunctions of atoms. By RA1, $\dashv p \ \& \sim q \ \& \ r \ \& \sim r \to \sim p \vee q \vee s \vee \sim s$ and, by applying RR1 to the antecedent and RR2 to the consequent, $\dashv p \ \& \ p \ \& \ \cdots \ \& \ p \ \& \sim q \ \& \sim q \ \& \ \cdots \ \& \sim q \ \& \ r \ \& \ r \ \& \ \cdots \ \& \ r \ \& \sim r \ \& \sim r \ \& \ \cdots \ \& \sim r \to \sim p \vee \sim p \vee \cdots \sim p \vee q \vee q \vee \cdots \vee q \vee s \vee s \vee \cdots \vee s \vee \sim s \vee \sim s \vee \cdots \vee \sim s$, for any number of repetitions of the variables, and with the occurrences of the variables arranged in any order and bracketed in any way within the antecedent and within the consequent. Applying RR10 to expand the range of variables in their respective positions, we get $\dashv p_1 \ \& \ p_2 \ \& \ \cdots \ \& \ p_i \ \& \sim q_1 \ \& \sim q_2 \ \& \ \cdots \ \& \sim q_m \ \& \ r_1 \ \& \ r_2 \ \& \ \cdots \ \& \ r_p \ \& \sim r_1 \ \& \sim r_2 \ \& \ \cdots \ \& \sim r_p \to \sim p_1 \vee \sim p_2 \vee \cdots \sim p_j \vee q_1 \vee q_2 \vee \cdots \vee q_n \vee s_1 \vee s_2 \vee \cdots \vee s_q \vee \sim s_1 \vee \sim s_2 \vee \cdots \vee \sim s_q$. We can still allow repetitions of variables within each of these four categories, represented by the four types of variables and also rearrangements of variables within the antecedent and within the consequent. Note that the p's, and also the q's, can properly overlap, i. e. some p's can occur in the antecedent side and not in the consequent side or vice versa, and similarly with the q's. We confine the r's and $\sim r$'s to the same variable range on the antecedent side and the s's and $\sim s$'s to the same variable range on the consequent side. This then exhausts all possibilities for non-theorems of E_{fde}, which are entailments between conjunctions of atoms and disjunctions of atoms, provided the four distinct classes of variables, represented by the p's, q's, r's and s's are all non-empty. In order to

reject such non-theorems of E_{fde} where at least one of these classes of variables are empty (leaving at least one of the p's, q's or r's non-empty and at least one of the p's, q's or s's non-empty), we take sub-conjunctions of the antecedent and sub-disjunctions of the consequent, this being justified by RR1 and RR2. Thus, we are able to reject all entailments between conjunctions of atoms and disjunctions of atoms, where there is no atom in common.

(II) We now build up the rejection of entailments between disjunctive normal forms and conjunctive normal forms, $A_1 \vee \cdots \vee A_m \to B_1 \ \& \ \cdots \ \& \ B_n$, where at least one pair $A_i \to B_j$ has no atom from the conjuncts of A_i in common with an atom from the disjuncts of B_j. These rejections of form, $\dashv A_1 \vee \cdots \vee A_m \to B_1 \ \& \ \cdots \ \& \ B_n$, follow from $A_i \to B_j$ by RR1 and RR2, since $\vdash A_i \to A_1 \vee \cdots \vee A_m$ and $\vdash B_1 \ \& \ \cdots \ \& \ B_n \to B_j$, for any i and j.

(III) All the normal-forming equivalences, $C \leftrightarrow D$, and their contextual forms, $\mathcal{C}(C) \leftrightarrow \mathcal{C}(D)$, are provable in E_{fde}, and hence in L_1. So, by RR1 and RR2, any non-theorem of E_{fde} can now be rejected.

(IV) We now reject the appropriate conjunctive normal forms, consisting of a conjunction of disjunctions of basic formulae. We have already rejected all basic formulae that are non-theorems. By repeated uses of RR3, any disjunction of rejected basic formulae is also rejected, and, by repeated RR4 and RR5, any conjunction with at least one conjunct being a rejected disjunction of basic formulae is rejected. So, we have rejected all non-theorem conjunctive normal forms of basic formulae.

(V) It remains to reject all the non-theorems of L_1. We use rule versions of the normal-forming equivalences to reject arbitrary non-theorems of L_{1r}, given the rejection of the above conjunctive normal forms. We need to use the r-rules RR3-9, together with the help of r-metacompleteness properties (I_r) and (II_r). We build up the inner negations first as we can only apply the rejection rules and properties (I_r) and (II_r) in conjunctive and disjunctive contexts. The use of metacompleteness properties are justified here since they essentially backtrack down a given r-proof to a certain step or steps, yielding an r-proof of this formula (or formulae) or allowing a rebuilding of the proof of a new r-theorem from it (or them). □

7 A SIMPLE SEMANTICS FOR L_1

One of the side benefits of an r-sound and r-complete rejection system for a logic is that one can test a potential semantics for such a logic by just proving soundness twice, once for the assertion system and once for the rejection system, both with respect to the chosen semantics. The result of this would be that the usual soundness would show that all theorems are valid in the semantics and that the other soundness would show that all r-theorems, i. e. non-theorems, are invalid in the semantics. We will appropriately call this other soundness, converse soundness or c-soundness.

One would expect this process to be easier than proving completeness, together with soundness, thus providing an opportunity to experiment with the semantics. And, as stated in Brady [5], one of the hopes in studying degree fragments of logics was to try to establish simple semantics for them. This leaves us with the immediate task of putting up a suitable semantics for the first-degree fragment L_1. The semantics given in [5] is a standard Routley-Meyer semantics over two levels, with just T and T^* at level 1 and with the usual (possibly infinite) set of worlds at level 0, the levels representing the maximum degree of formulae to be evaluated there.

To get some idea of how to get a simple semantics for L_1, we should first look at the semantics, given in Anderson and Belnap [1, pp. 206-7], for the first-degree fragment E_{fdf} for the logic E. It is a 2-stage semantics, first assessing first-degree entailments as true or false using an intensional De Morgan lattice, and then, using these assignments of truth or falsity to sentential variables, evaluating first-degree formulae in terms of these entailments and sentential variables, using classical truth-functions.

Let Q be a model, $\langle L, \nu \rangle$, where L is an intensional De Morgan lattice and v is an assignment of sentential variables to elements of L. v extends to an interpretation I for all zero-degree formulae, evaluating them as elements of L, in the usual De Morgan manner. The valuation conditions of the semantics for E_{fdf} are then as follows:

Let A and B be zero-degree formulae, positioned inside an '→', i. e. of depth 1 in a first-degree formula under consideration for validity. Alternatively, let C and D be of depth 0 in such a formula.

(i) $I(A) = T$ iff $\nu(A) \in T$, where T is the truth filter of L.

(ii) $I(A \to B) = T$ iff $I(A) \leqslant I(B)$ in L.

(iii) $I({\sim}C) = T$ iff $I(C) = F$.

(iv) $I(C \,\&\, D) = T$ iff $I(C) = T$ and $I(D) = T$.

(v) $I(C \vee D) = T$ iff $I(C) = T$ or $I(D) = T$.

So, the mode of evaluation of a zero-degree subformula depends on where in the overall formula it lies.

A first-degree formula C is valid in the semantics iff $I(C) = T$, for all interpretations I.

For our semantics S_1 of L_1, we should and can rectify this disparity by maintaining the same valuation conditions throughout a first-degree formula. In fact, we will project the valuation conditions for Dunn's intuitive semantics of E_{fde} (see [10]) over formulae of depth 0 as well. However, what we do is to evaluate the entailment $A \rightarrow B$ by quantifying over all interpretations, which is quite appropriate for a true entailment and it is what Dunn does for validity of $A \rightarrow B$. If we do not do this, the entailment would be evaluated like an implication, i. e. just 'if $T \in I(A)$ then $T \in I(B)$' in Dunn's semantics, which would not only be inappropriate for the logics L of [5], of which L_1 is their first-degree fragment, but also would let in formulae such as $A \vee (A \rightarrow B)$ and $A \vee {\sim}B \vee (A \rightarrow B)$ as valid.

As in Dunn [10], valuation in S_1 is over the 4 subsets of $\{T, F\}$ and interpretation conditions are stated in terms of the membership of T and F in such subsets. The valuation v assigns a subset of $\{T, F\}$ to each sentential variable and is extended to an interpretation I over all first-degree formulae, using the following conditions:

(i) $I(p) = v(p)$, for all sentential variables p.

(ii) $T \in I({\sim}A)$ iff $F \in I(A)$.

$F \in I({\sim}A)$ iff $T \in I(A)$.

(iii) $T \in I(A \,\&\, B)$ iff $T \in I(A)$ and $T \in I(B)$.

$F \in I(A \,\&\, B)$ iff $F \in I(A)$ or $F \in I(B)$.

(iv) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$.

$F \in I(A \vee B)$ iff $F \in I(A)$ and $F \in I(B)$.

(v) $T \in I(A \rightarrow B)$ iff, for all interpretations I', if $T \in I'(A)$ then $T \in I'(B)$, and if $F \in I'(B)$ then $F \in I'(A)$.

$F \notin I(A \rightarrow B)$.

We add 'if $F \in I'(B)$ then $F \in I'(A)$' to the T-valuation for $A \rightarrow B$ to directly show that the contraposition rule R4 preserves truth. R4 can still be shown to hold without it, as in Dunn [10], but the duality between T and F here is nice. Also, the choice of F-valuation for $A \rightarrow B$ is appropriate for the logics L of [5], which are all M1 metacomplete with no negated entailment theorems (see Slaney [16]).

A first-degree formula A is *valid* in the semantics S_1 iff $T \in I(A)$, for all interpretations I. S_1 will also apply to zero-degree formulae, i. e. those without '$\rightarrow$', but as with metacomplete logics there are no valid zero-degree formulae. As indicated above, if we prove soundness and c-soundness for L_1 with respect to S_1 we will indeed have established soundness and completeness.

THEOREM 5 (Soundness Theorem) *For all first-degree formulae A, if A is a theorem of L_1 then A is valid in the semantics S_1.*

Proof: Clearly, the axioms of L_1 are all valid in S_1 and the rules of L_1 preserve the membership of T in interpretations in S_1. □

THEOREM 6 (C-Soundness Theorem) *For all first-degree formulae* A, *if* A *is an r-theorem of* L_{1r} *then* A *is invalid in the semantics* S_1.

Proof: We show that each axiom is invalid and that each rule preserves invalidity.

RA1. Put $v(p) = T$, $v(q) = F$, $v(r) = \{T, F\}$ and $v(s) = \varnothing$.

RA2. By the F-valuation for $A \to B$.

RR1-2, 4-8, 10. By the corresponding validity preservation or by re-use of the interpretation yielding the invalidity of the premise to yield the invalidity of the conclusion.

RR3, 9. Because of the two premises in these rules, different interpretations can yield their respective invalidities. We need to construct an interpretation which suffices to make the conclusion invalid. For RR3, let $T \notin I_1(A)$ and $T \notin I_2(B)$, for some interpretations I_1 and I_2. We construct I_3 by assigning values to the variables occurring in $A \vee B$. If the variable p occurs in $A \vee B$ at depth 0, then put $I_3(p) = I_1(p) \cap I_2(p)$. If p does not occur at depth 0 in $A \vee B$, then its value in I_3 is immaterial. By the interpretation conditions, $T \notin I_1(A)$ iff some mix of conjunctions and disjunctions of $T \notin I_1(p)$, $F \notin I_1(q)$, $T \notin I_1(B \to C)$ and $F \notin I_1(D \to E)$, for some variables p, q, ... (or none), and for some entailments $B \to C$, $D \to E$, ... holds. These variables and entailments are all the expessions of such type that occur at depth 0 in A. Now, $I_3(B \to C) = I_1(B \to C)$ and $I_3(p) \subseteq I_1(p)$, for any p and $B \to C$ at depth 0, and so the same mix of conjunctions and disjunctions of expressions, $T \notin I_3(p)$, $F \notin I_3(q)$, $T \notin I_3(B \to C)$ and $F \notin I_3(D \to E)$, holds for I_3. Hence, $T \notin I_3(A)$. Similarly, by $I_3(B \to C) = I_2(B \to C)$ and $I_3(p) \subseteq I_2(p)$, for any p and $B \to C$ at depth 0, we also conclude $T \notin I_3(B)$. Thus, $T \notin I_3(A \vee B)$, as required.

RR9. This is similar. Given $F \notin I_1(A)$ and $F \notin I_2(B)$, for some I_1 and I_2, we construct I_3 such that $F \notin I_3(A \,\&\, B)$. □

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