Basic Relevant Theories for Combinators at Levels I and II

Koushik Pal
Department of Mathematics,
Indian Institute of Technology Kanpur
kapal@iitk.ac.in

Robert K. Meyer
Automated Reasoning Group, RSISE,
Australian National University
Bob.Meyer@anu.edu.au

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Abstract: The system B+ is the minimal positive relevant logic. B+ is trivially extended to B+T on adding a greatest truth (Church constant) T. If we leave ∨ out of the formation apparatus, we get the fragment B∧T. It is known that the set of ALL B∧T theories provides a good model for the combinators CL at Level-I, which is the theory level. Restoring ∨ to get back B+T was not previously fruitful at Level-I, because the set of all B+T theories is NOT a model of CL. It was to be expected from semantic completeness arguments for relevant logics that basic combinator laws would hold when restricted to PRIME B+T theories. Overcoming some previous difficulties, we show that this is the case, at Level I. But this does not form a model for CL. This paper also looks for corresponding results at Level-II, where we deal with sets of theories that we call propositions. We adapt work by Ghilezan to note that at Level-II also there is a model of CL in B∧T propositions. However, the corresponding result for B+T propositions extends smoothly to Level-II only in part. Specifically, only some of the basic combinator laws are proved here. We accordingly leave some work for the reader.

1 INTRODUCTION

This paper is an essay at an intersection of Philosophy, Computer Science and Mathematics. That intersection, for present purposes, we take to be Logic—and in particular relevant and other substructural logics. It is also an essay in

Levels—in what happens when we climb a set-theoretic level or two in order to enrich a supply of formal objects with extra ones. It is thus, for example, that the collection of real numbers was placed on a "sound" basis by 19th and early 20th century researchers.

EXAMPLE 1. $\mathfrak{R}$ AT LEVEL-I Consider the problem of locating π among the set $\mathfrak{Q}$ of rational numbers. In $\mathfrak{Q}$, we find 3, 3.1, 3.14 and so forth, but none of these successive "finite approximations" is equal to the infinitely long "decimal expansion of π". *Idea*: Let's extend our usual arithmetic on $\mathfrak{Q}$ to define also arithmetic operations on (properly chosen) subsets of $\mathfrak{Q}$. Such a subset of $\mathfrak{Q}$ will be an *ideal* (number). And among those ideals will be π, with its *usual* arithmetical properties.

Here's the basic idea: We *enrich* a Level-0 collection $\mathfrak{C}$ of formal objects—in this case $\mathfrak{Q}$—by passing to a Level-I collection of (appropriately chosen) *subsets* of $\mathfrak{C}$. Some of these subsets (the *principal ideals*, in the $\mathfrak{Q}$ case) may be taken as the *representatives* at Level-I of the original Level-0 entities. As for the new entities arising at Level-I (in the sample case, the non-principal ideals corresponding to *Dedekind* cuts, which are the other real numbers), the *magic* of set-theoretic ascent gives them the right properties to be the rest of $\mathfrak{R}$.

EXAMPLE 2. BOOLE AT LEVEL-II There is no good reason to stop at Level-I, either. Consider now the case of Boolean algebras, under *Stone representation*. Everybody knows that Boolean algebra is in some appropriate sense the *algebra* of sets. But what that sense is became crystal clear only with the work of Stone [11]. We take (with some redundancy) a Boolean algebra to be a structure $\mathcal{B} = \langle B, \neg, \wedge, \vee, \top, \bot \rangle$, where B is a set closed under complement ($\neg$), meet ($\wedge$) and join ($\vee$), and $\top$ and $\bot$ are respectively the top and the bottom of the algebra under the induced ordering. An *ultrafilter* U in $\mathcal{B}$ will be any subset of B satisfying, for all a, b in B,

[T$\wedge$] $a \wedge b \in U$ iff $a \in U$ and $b \in U$

[T$\neg$] $\neg a \in U$ iff $a \notin U$.

Note that, from the (Level-0) viewpoint of an element b of B, an ultrafilter is at Level-I. But Stone's Representation Theorem, in representing B, takes us already to Level-II, on the following recipe:

- $UB =_{df} \{U : U \text{ is an ultrafilter in } B\}$
- $PUB =_{df} \{W : W \subseteq UB\}$, the power set of UB
- $\mathcal{PUB} = \langle PUB, \neg, \cap, \cup, \emptyset, UB \rangle$ is a Boolean set algebra

where, for any sets α , β of ultrafilters in B, $\alpha \wedge \beta$ and $\alpha \vee \beta$ are respectively the intersection and union, and $\neg\alpha$ the complement relative to UB, of the sets. As for $\bot$ and $\top$, they are respectively the null set and the universal set of ultrafilters in B. The utility of the ascent to PUB is that there is an *isomorphic copy* of B itself two levels up. For let an embedding $h : \mathcal{B} \rightarrow \mathcal{PUB}$ be defined, by assigning to every element b the set of ultrafilters in $\mathcal{B}$ to which it belongs. i.e., for $b \in B$, fix

[Dh] $h(b) = \{U : b \in U \text{ and } U \in UB\}$

Thus every Boolean algebra $\mathcal{B}$ finds a home away from home in a $\mathcal{PUB}$.

EXAMPLE 3. RELEVANT LOGICS Structures may be *enriched* and *completed*, as we have just recalled, by raising Levels. We now turn to relevant logics. They have come, semantically, a long way. Beginning in [10] from the Orlov-Moh-Church-Anderson-Belnap system R of relevant implication, Routley, Meyer and others produced a series of articles on the Semantics of Entailment, based on Kripke-style ternary relational postulates. In [9] such postulates were found, in the $\langle \rightarrow, \wedge, \vee \rangle$ vocabulary, for the Anderson-Belnap systems R+ of positive relevant implication, E+ of positive entailment, and T+ of positive ticket entailment. Among these positive relevant logics there was a new and natural minimal one, which we called B+.

We concentrate, in this paper, on B+, and on the fragments, alternative formulations and conservative extensions that make up what we may call the B+ family of minimal logics. We will be concerned, in our work with the B+ family, with collections of formal objects at Levels I and II. At Level-I our focus is on *theories*—intuitively, sets of sentences that are *logically closed*. We concentrate at Level-II on what we call *propositions*[1]—special collections of theories.

EXAMPLE 4. COMBINATORY LOGIC AND λ-CALCULI Our next topic will be the Combinatory Logic CL of Curry and Feys [3]. There is a delightful coincidence between axiom candidates for various relevant logics and the combinators of which, on the analysis of Curry, these candidate axioms were (in Curry-speak) the "functional characters" (nowadays "types"). In fact the coincidence not only reflected Curry but improved him; some combinators untypeable by Curry correspond anyway to famous theorems of logic, which have been embraced (and repudiated) with vigor.

Twenty years ago, this coincidence was independently rediscovered and deepened by researchers in λ-calculus; specifically, by Dezani and her colleagues in [1] and elsewhere. They added intersection types to Curry's arrow types. (Add with [5] type intersection for logical $\wedge$ to Curry's original function type for logical $\rightarrow$, and behold the improvement.) As relevant semantical analysis predicted, further combinators (sample: W^*, equivalently WI, SII, or $\lambda x.xx$) have

[1]Why do we choose proposition, a philosopher's term for something like the meaning of a sentence? We have in mind the so-called UCLA view, on which a proposition is explicated as the set of possible worlds in which a sentence is true. Reduced to its (Level-II) syntactical residue, this is the set of special theories—namely, the classically consistent and complete ones—to which the sentence belongs. We generalize!

non-trivial types in the enriched setup. And the structure of these types is that conferred by the analysis of conjunction $\wedge$ and implication $\rightarrow$ in the minimal relevant logic B+ of [9].

So exciting are the connections between CL and relevant logics that we have dubbed them The Key to the Universe. There is already good evidence that the key fits because there are the semantical completeness proofs themselves. Here, we shall insert that key a little further into its lock. We look again at how to account for disjunction $\vee$ in modeling CL. For $\vee$ (with its usual truth-functional semantics) has been an ingredient in B+ and other relevant logics from the beginning. Dezani, Meyer and Motohama did offer a B+T model of λ and CL in [5], appealing to (so-called) Harrop theories. A principal result here is a better fit with the semantical and logical intuitions of [9]. We look to prime B+T-theories in general, and not just the Harrop subclass thereof, as the appropriate vehicle with which to make the laws governing primitive CL combinators true. But Dezani and her colleagues, for their part, are not to be denied. For it was (what we call) the Better Bubbling Lemma (henceforth, BBL) of Dezani et al. in [6] that led to our new verification of the primitive combinatory equations in prime B+T theories.

In this paper, we follow [8] to move from Level-I to Level-II to find systems which model λ and hence CL. Levels arise out of the way one looks at systems. We discuss, as anticipated above, three levels. (You can have more, if you like.) Level-0 is that of the elements of a given logical algebra—typically, an Algebra of Formulas, where each element is a Well-Formed-Formula (WFF).[2] For present purposes, at Level-I each element is a theory, which is a collection of WFFS with some nice properties like closure under conjunction and entailment. (For technical reasons involving the relevantly irrelevant combinator K and other *cancellators*, we follow [5] by requiring the theories of *this paper* to be non-empty.) At Level-II we have propositions, which are collections of theories with further nice properties. (We trust that theories and propositions will become more clear as we go along.)

We recall the definition of B$\wedge$T as a relational system from [5]. B$\wedge$T is little more than a fragment of B+. In presenting B$\wedge$T, we had a choice between an *assertional* and a *relational* formulation.[3] This is a distinction without much of a difference, since to assert A $\rightarrow$ B as a theorem of logic comes sensibly to the same thing (when all the "i"s are dotted and "t"s are crossed) as to claim that A logically entails B.[4]

[2] We allow some ambiguity in specifying Level-0 objects. This leaves open the possibility, as in the Boolean algebra example above, that distinct WFFS have been identified via an appropriate quotient construction. Think in this case of Level-0 objects as *congruence classes* of WFFS.

[3] We are even-handed here, formulating the richer B+T below as an assertional system.

[4] More specifically, we may borrow from Curry [4]. The WFFS (Curry's OBS) of a propositional logic are the Level-0 objects built up from atoms (propositional variables and constants) via the primitive operations ($\rightarrow$ and the like). The elementary statements are then formed by attaching predicators to WFFS. To formulate a system assertionally is to choose a 1-place predicator (say $\vdash$) and to insist that the elementary statements (and hence the theorems) are things of the form

$B\wedge T$ is the $\langle \to, \wedge \rangle$ fragment of $B+$, enriched with a greatest truth, given by Church constant T. Upper-case 'A', 'B', etc. are used to denote syntactical variables. $\leqslant$ is a binary predicate in the language, which is read as 'entails'. With these we state the formal definition of the $B\wedge T$ system. The axioms for the $B\wedge T$ system are as follows:

Reflex.	$A \leqslant A$
Top.	$A \leqslant T$
Top→.	$T \leqslant (T \to T)$
Idem∧.	$A \leqslant (A \wedge A)$
∧E.	$(A \wedge B) \leqslant A, (A \wedge B) \leqslant B$
→∧I.	$(A \to B) \wedge (A \to C) \leqslant (A \to (B \wedge C))$

The rules of the system are as follows. (Note that $\Rightarrow$ has been used as a metalogical connective in framing rules.)

Trans∧.	$A \leqslant B \leqslant C \Rightarrow A \leqslant C$
Mon∧.	$A \leqslant A', B \leqslant B' \Rightarrow A \wedge B \leqslant A' \wedge B'$
Mon→.	$A' \leqslant A, B \leqslant B' \Rightarrow A \to B \leqslant A' \to B'$

Our first task now is to prove that the results at Level-I extend smoothly to Level-II for $B\wedge T$ theories. The idea of moving to Level-II is for some obvious reasons which will be made clear soon. But first we start with some definitions and theorems and prove that the results for $B\wedge T$ theories hold at Level-II also.

Definition 1 (theory) A theory or $B\wedge T$-theory is a non-empty set of formulas closed under conjunction and $B\wedge T$-entailment.

Definition 2 (A-theory) An A-theory (denoted "$A\uparrow$") is a theory containing the formula A and all the formulas C that are $B\wedge T$-entailed by A, i.e., $A\uparrow = \{C : A \leqslant C\}$.

Theorem 1 *The intersection of two theories is a theory.*

Proof Let T_1 and T_2 be two theories. So, $T_1 \neq \emptyset \neq T_2$. (i) Suppose $A \in T_1 \cap T_2$ and $A \leqslant B$. It follows that $A \in T_1$ and $A \leqslant B$, so $B \in T_1$ (since T_1 is a theory). Similarly, $B \in T_2$. Hence, $B \in T_1 \cap T_2$. (ii) Suppose $A, B \in T_1 \cap T_2$. So, $A, B \in T_1$ and thus $A \wedge B \in T_1$ (since T_1 is a theory), and similarly, $A \wedge B \in T_2$. Hence, $A \wedge B \in T_1 \cap T_2$. (iii) $T \in T_1$ and $T \in T_2$, so, $T \in T_1 \cap T_2$. So, $T_1 \cap T_2 \neq \emptyset$. Hence, $T_1 \cap T_2$ is a theory. ♯

$\vdash A$. To formulate a system relationally is to choose a 2-place predicator (say $\leqslant$) and to insist that the elementary statements (and hence the theorems) are things of the form $A \leqslant B$.

Definition 3 (proposition) A B∧T-proposition P is a non-empty set of theories closed under sub-theory relation and intersection. That is (i) if T_1 and T_2 are two theories, $T_1 \subseteq T_2$ and $T_1 \in P$, then $T_2 \in P$, and (ii) if $T_1 \in P$ and $T_2 \in P$, then $T_1 \cap T_2 \in P$.

Definition 4 (A-proposition) An A-proposition (written "$\underline{A}$", and also called the principal proposition for the formula A) is the proposition consisting of $A\uparrow$ and all of its its super-theories (it is of course closed under intersection).

In what immediately follows, we will call a B∧T theory simply a theory; B∧T proposition, a proposition; and B∧T entailment, entailment. We will also use T_1, T_2, etc. to denote theories, P, Q, etc. to denote propositions, $\underline{A}$, $\underline{B}$, etc. to denote principal propositions for formulas A, B, etc., and A, B, etc. to denote well formed formulas. Notice that by the axioms of B∧T, T and $T \to T$ belong to every theory of B∧T. The propositions, on the other hand, are by definition non-empty. We take our next definition from [8].

Definition 5 (open sets) A set Θ is said to be open if (i) if $\alpha \in \Theta$ and $\alpha \subseteq \beta$ then $\beta \in \Theta$, and (ii) If $\bigcup_{i \in I} \alpha_i \in \Theta$ then $\exists i_0 \in I$ such that $\alpha_{i_0} \in \Theta$.

Theorem 2 *An* A*-Theory is the minimum theory containing* A.

Proof Suppose T_1 is a theory containing A. Since T_1 is a theory, it is closed under conjunction and entailment relation. Therefore, for all formulas B where $A \leqslant B$, $B \in T_1$. Hence, $A\uparrow \subseteq T_1$. ♯

Theorem 3 *An* A*-proposition is the minimum proposition containing* $A\uparrow$.

Proof Suppose P is a proposition containing $A\uparrow$. Since P is a proposition, it is closed under sub-theory relation and intersection. Therefore, all theories of which $A\uparrow$ is a sub-theory, i.e., all super-theories of $A\uparrow$ belong to P and hence also their intersections. Hence, $\underline{A} \subseteq P$. ♯

Theorem 4 *Any principal proposition is open.*

Proof (i) Let $T_1 \in \underline{A}$ and $T_1 \subseteq T_2$. So $T_2 \in \underline{A}$ (since $\underline{A}$ is closed under sub-theory relation). (ii) Suppose that $\bigcup_{i \in I} \alpha_i \in \underline{A}$ (each α_i is a theory). It follows that $A\uparrow \subseteq \bigcup_{i \in I} \alpha_i$, and hence $A \in \bigcup_{i \in I} \alpha_i$, and it follows that there is some $i_0 \in I$ where $A \in \alpha_{i_0}$. So, $\exists i_0 \in I$ such that $A\uparrow \subseteq \alpha_{i_0}$ (since α_{i_0} is a theory), and it follows that $\exists i_0 \in I$ such that $\alpha_{i_0} \in \underline{A}$. Hence, any principal proposition is open. ♯

Theorem 5 *Any theory* $T_1 = \bigcup\{B\uparrow : B \in T_1\}$.

Proof Let $B \in T_1$. We have $B \in B\uparrow$, so $B \in \bigcup\{B\uparrow : B \in T_1\}$. Conversely, suppose $C \in \bigcup\{B\uparrow : B \in T_1\}$. So, there is some $B \in T_1$ such that $C \in B\uparrow$. Since T_1 is a theory and $B \in T_1$, it follows that $B\uparrow \subseteq T_1$ (by Theorem 2). So, $C \in T_1$, and hence, $T_1 = \bigcup\{B\uparrow : B \in T_1\}$. ♯

Theorem 6 *For principal proposition* $\underline{A}$, $\underline{A} = \bigcup\{\underline{B} : B\uparrow \in \underline{A}\}$.

Proof Consider $\underline{B}$ such that $B\uparrow \in \underline{A}$. Since $\underline{B}$ is the smallest proposition containing $B\uparrow$, $\underline{B}$ and so $\bigcup\{\underline{B} : B\uparrow \in \underline{A}\}$ is a subset of $\underline{A}$. Conversely suppose $T_1 \in \underline{A}$. Clearly, $T_1 \neq \emptyset$. Now $T_1 = \bigcup\{C\uparrow : C \in T_1\}$ and so $\bigcup\{C\uparrow : C \in T_1\} \in \underline{A}$. But $\underline{A}$ is open and $C\uparrow$ is a theory for each C. Hence there is a $C \in T_1$ such that $C\uparrow \in \underline{A}$. Also, $C\uparrow \subseteq T_1$ (by Theorem 2). Therefore $T_1 \in \underline{C}$, and so, $T_1 \in \bigcup\{\underline{C} : C\uparrow \in \underline{A}\}$. Therefore, $\underline{A} \subseteq \bigcup\{\underline{B} : B\uparrow \in \underline{A}\}$, and hence, $\underline{A} = \bigcup\{\underline{B} : B\uparrow \in \underline{A}\}$. ♯

Theorem 7 *The intersection of two propositions is always non-empty.*

Proof Suppose P and Q are two propositions. By definition, $P \neq \emptyset \neq Q$. So, there are T_1, T_2 such that $T_1 \in P$ and $T_2 \in Q$. Now, $T_1 = \bigcup\{C\uparrow : C \in T_1\}$, and $T_2 = \bigcup\{D\uparrow : D \in T_2\}$. Define $T_3 = \{E : \exists C \in T_1 \text{ and } \exists D \in T_2 \text{ where } C \wedge D \leqslant E\}$. Clearly, $T_1 \subseteq T_3$ and $T_2 \subseteq T_3$. Therefore, $T_3 \in P$ and $T_3 \in Q$, i.e., $T_3 \in P \cap Q$. We have shown that $P \cap Q$ is non-empty. ♯

Theorem 8 *The intersection of two propositions is always a proposition.*

Proof Let P and Q be two propositions. By Theorem 6, $P \cap Q \neq \emptyset$. (i) Suppose $T_1 \in P \cap Q$ and $T_1 \subseteq T_2$. Then $T_1 \in P$ and $T_1 \subseteq T_2$ and P is a proposition. Therefore, $T_2 \in P$. Similarly $T_2 \in Q$. And hence $T_2 \in P \cap Q$. (ii) Suppose that T_1 and $T_2 \in P \cap Q$. Then T_1 and $T_2 \in P$ and so $T_1 \cap T_2 \in P$. Similarly, $T_1 \cap T_2 \in Q$. And hence $T_1 \cap T_2 \in P \cap Q$. Therefore, $P \cap Q$ is closed under the sub-theory relation and under intersection, and is non-empty. It follows that it is a propositon. ♯

Theorem 9 *The intersection of two principal propositions is a principal proposition.*

Proof Suppose that $\underline{A}$ and $\underline{B}$ are two principal propositions. Now, $\underline{A} = \{T_1 : A\uparrow \subseteq T_1\}$ and $\underline{B} = \{T_2 : B\uparrow \subseteq T_2\}$. We have already proved that $\underline{A} \cap \underline{B}$ is a proposition. So now we need to show that it is a principal proposition. *Claim*: $\underline{A} \cap \underline{B} = \underline{A \wedge B}$. Let $T_1 \in \underline{A} \cap \underline{B}$ This holds if and only if $A\uparrow \subseteq T_1$ and $B\uparrow \subseteq T_1$ which, in turn, holds if and only if $A \in T_1$ and $B \in T_1$, if and only if $A \wedge B \in T_1$ (since, T_1 is a theory). This is equivalent to $(A \wedge B)\uparrow \subseteq T_1$ (by Theorem 5), and hence $T_1 \in \underline{(A \wedge B)}$. The intersection of two principal propositions is a principal proposition. ♯

2 B∧T is a model for combinators at level-I

The idea of a model for combinators at a Level is that the elements on which the combinators operate are elements of that particular Level. The defining equation for any combinator also depends on the Level at which we work. For example, at Level-I, we assign a theory to each combinator or other combinatorial term. But at Level-II, we assign propositions to combinators. And then we show that, where $t = u$ is a provable equation of CL, both t and u must be assigned the *same* object. The larger part of this task is to show that the defining equations for particular combinators turn out true.[5] But it is meet also to show that we have so fashioned our semantic apparatus that the rules of inference[6] of CL are truth-preserving.

We will look at defining equations for three combinators in particular here, namely, I, K and W^*. For these combinators, we will prove that the defining equations turn out true in B∧T-theories. Proofs for the other combinators are similar and hence are omitted here. Let's start with the definition of the fusion operator at different Levels and a basic lemma.

Definition 6 (fusion operator at level-0) We can introduce the fusion operator '∘' at Level-0 by just including it in our vocabulary as one more sentential connective, together with a "residuation" rule. While [9] shows that this produces conservative extensions of standard relevant logics (including those of this paper), we resist that course here.

Definition 7 (fusion at level-i ($\circ_1$)) Let T_1 and T_2 be two theories. Then $T_1 \circ_1 T_2$ is defined as

$$T_1 \circ_1 T_2 = \{B : \exists A \text{ where } A \to B \in T_1 \text{ and } A \in T_2\}$$

Definition 8 (fusion at level-ii ($\circ_2$)) Let P and Q be two propositions. Then $P \circ_2 Q$ is defined as

$$P \circ_2 Q = \{T_1 : \exists T_2 \in P \text{ and } \exists T_3 \in Q \text{ where } T_2 \circ_1 T_3 \subseteq T_1\}$$

Lemma 10 *The fusion of two theories is also a theory.*

Proof Let T_1 and T_2 be two theories. (i) $T \to T \in T_1$ and $T \in T_2$. Therefore, $T \in T_1 \circ_1 T_2$ and hence, $T_1 \circ_1 T_2$ is non-empty. (ii) Suppose $A \in T_1 \circ_1 T_2$ and $A \leqslant B$. So $\exists C$ such that $C \to A \in T_1$ and $C \in T_2$. But $C \leqslant C$ and $A \leqslant B \Rightarrow C \to A \leqslant C \to B$. Hence, $C \to B \in T_1$ (as T_1 is a theory). Therefore, $B \in T_1 \circ_1 T_2$. (iii) Suppose $A, B \in T_1 \circ_1 T_2$. Therefore ($\exists C$ where $C \to A \in T_1$ and $C \in T_2$) and ($\exists D$ where $D \to B \in T_1$ & $D \in T_2$). Now, $C \wedge D \leqslant C$ and $A \leqslant A \Rightarrow C \to A \leqslant C \wedge D \to A$. Therefore, $C \wedge D \to A \in T_1$. (since T_1 is a theory) Similarly, $C \wedge D \to B \in T_1$. Therefore, $C \wedge D \to A \wedge B \in T_1$. But, $C \wedge D \in T_2$ (since T_2 is closed under conjunction). Hence, $A \wedge B \in T_1 \circ_1 T_2$. Thus, $T_1 \circ_1 T_2$ is non-empty and also closed under conjunction and entailment. Hence it is a theory. ♯

Corollary 11 *The fusion of a finite number of theories is also a theory.*

Proof Use induction and Lemma 10. ♯

[5] These are axioms like $Ix = x$, $Wxy = xyy$, etc.

[6] For example, as found in [3].

Now we will prove that B∧T-theories model the three combinators mentioned above. Here are defining equations for some combinators.

$$\begin{array}{rcl} \mathsf{I}x & = & x \\ \mathsf{K}xy & = & x \\ \mathsf{C}xyz & = & xzy \\ \mathsf{C}^*xy & = & yx \\ \mathsf{B}xyz & = & x(yz) \\ \mathsf{B}'xyz & = & y(xz) \\ \mathsf{S}xyz & = & xz(yz) \\ \mathsf{W}xy & = & xyy \\ \mathsf{W}^*x & = & xx \end{array}$$

We assign a particular *theory* to each combinator. To indicate the assignment succinctly, we enclose a formula scheme in brackets. The associated theory T is then the smallest set of formulas that contains all conjunctions of one or more instances of the scheme and which is closed under the entailment relation $\leqslant$. Here are the definitions.

$$\begin{array}{rcl} I & = & [A \to A] \\ & = & \{C : \text{for some finite set } M \text{ and formulas } A_m, \\ & & \qquad \bigwedge_{m \in M}(A_m \to A_m) \leqslant C\} \\ K & = & [A \to (B \to A)] \\ C & = & [(A \to (B \to C)) \to (B \to (A \to C))] \\ C^* & = & [A \to ((A \to B) \to B)] \\ B & = & [(A \to B) \to ((C \to A) \to (C \to B))] \\ B' & = & [(A \to B) \to ((B \to C) \to (A \to C))] \\ S & = & [(A \to (B \to C)) \to ((A \to B) \to (A \to C))] \\ W & = & [(A \to (A \to B)) \to (A \to B)] \\ W^* & = & [((A \to B) \wedge A) \to B] \end{array}$$

The definitions for the theories assigned to each of the combinators are similar to that for I. For example, a formula D belongs to the set K iff there is some conjunction C of formulas of the form $A \to (B \to A)$ such that $C \leqslant D$. Here we show that the set I is indeed a theory. Proofs for other combinators are similar.

Claim 12 $[A \to A]$ *is a theory.*

Proof (i) $[A \to A]$ contains all instances of the formulas of the kind $A \to A$. So it is surely non-empty. (ii) Suppose $C, D \in [A \to A]$. Then $\exists M, N$ where $\bigwedge_{m\in M}(A_m \to A_m) \leqslant C$ and $\bigwedge_{n\in N}(A_n \to A_n) \leqslant D$. Thus we have $\bigwedge_{i\in M\cup N}(A_i \to A_i) \leqslant C \wedge D$ and hence $C \wedge D \in [A \to A]$. (iii) Suppose $C \in [A \to A]$ and $C \leqslant D$. Then $\exists M$ where $\bigwedge_{m\in M}(A_m \to A_m) \leqslant C \leqslant D$. Thus $D \in [A \to A]$. Thus $[A \to A]$ is non-empty and closed under conjunction and entailment. Hence, it is a theory. ♯

We come now to the famous Bubbling Lemma (BL), which was a principal weapon invoked in [1] to show that filters on intersection types provide a model of λ. It will play the same role here for us, changing only the vocabulary to say that $B\wedge T$-theories provide a model for combinators.

LEMMA 13 (BUBBLING LEMMA) *Assume it is not the case that* $D \equiv T$. *Assume moreover that* $\bigwedge_{i\in I}(A_i \to B_i) \leqslant C \to D$ *for some finite non-empty index set* I. *Then there is a finite non-empty subset* J *of* I *such that*

$$C \leqslant \bigwedge_{j\in J} A_j \textit{ and } \bigwedge_{j\in J} B_j \leqslant D.$$

PROOF Proof given in [1]. (Note that $\equiv$ is defined by setting $A \equiv B =_{df} A \leqslant B$ and $B \leqslant A$.) ♯

CLAIM 14 $I \circ_1 T_1 = T_1$

PROOF Let $A \in I \circ_1 T_1$. Then $\exists B$ where $B \to A \in I$ and $B \in T_1$. But $B \to A \in I$ means that there is a finite M where $\bigwedge_{k\in M}(A_k \to A_k) \leqslant B \to A$. Therefore, by the Bubbling Lemma, $B \leqslant \bigwedge_{j\in J} A_j$ and $\bigwedge_{j\in J} A_j \leqslant A$ for some $J \subseteq M$. Since $B \in T_1$ and T_1 is a theory, it follows that $\bigwedge_{j\in J} A_j \in T_1$. And hence, $A \in T_1$ (for similar reasons.). So, $I \circ_1 T_1 \subseteq T_1$.

Conversely, suppose $A \in T_1$. Since $A \to A \in I$ (by definition), it follows that $A \in I \circ_1 T_1$. So, $T_1 \subseteq I \circ_1 T_1$. Hence, $I \circ_1 T_1 = T_1$. ♯

CLAIM 15 $K \circ_1 T_1 \circ_1 T_2 = T_1$

PROOF Suppose $A \in K \circ_1 T_1 \circ_1 T_2$. Then $\exists B, C$ such that $B \to (C \to A) \in K, B \in T_1$ and $C \in T_2$. But $B \to (C \to A) \in K$, so $\exists$ finite I such that $\bigwedge_{i\in I}(D_i \to (E_i \to D_i)) \leqslant (B \to (C \to A))$. By the Bubbling Lemma, $\exists J \subseteq I$ such that $B \leqslant \bigwedge_{j\in J} D_j$ and $\bigwedge_{j\in J}(E_j \to D_j) \leqslant C \to A$ Again by Bubbling Lemma, $\exists M \subseteq J$ such that $C \leqslant \bigwedge_{k\in M} E_k$ and $\bigwedge_{k\in M} D_k \leqslant A$. Now, $B \in T_1$ and T_1 is a theory $\Rightarrow \bigwedge_{j\in J} D_j \in T_1$. Therefore, $\forall j \in J,\ D_j \in T_1$. $\Rightarrow \forall k \in M,\ D_k \in T_1$ (as $M \subseteq J$). Hence, $\bigwedge_{k\in M} D_k \in T_1$ (since T_1 is a theory). And so, $A \in T_1$. Therefore, $K \circ_1 T_1 \circ_1 T_2 \subseteq T_1$. Conversely, suppose $A \in T_1$. Since T_2 is not empty, $\exists B$ such that $B \in T_2$. By definition, $(A \to (B \to A)) \in K$. Therefore, $B \to A \in K \circ_1 T_1$. And hence, $A \in K \circ_1 T_1 \circ_1 T_2$. Therefore, $T_1 \subseteq K \circ_1 T_1 \circ_1 T_2$ and so, $K \circ_1 T_1 \circ_1 T_2 = T_1$. ♯

CLAIM 16 $W^* \circ_1 T_1 = T_1 \circ_1 T_1$.

PROOF Suppose $A \in W^* \circ_1 T_1$. Then $\exists B$ such that $B \to A \in W^*$ and $B \in T_1$. But $B \to A \in W^*$ means that $\exists$ finite I such that $\bigwedge_{i\in I}(((D_i \to E_i) \wedge D_i) \to E_i) \leqslant B \to A$. By the Bubbling Lemma, $\exists J \subseteq I$ such that $B \leqslant \bigwedge_{j\in J}((D_j \to E_j) \bigwedge D_j)$ and $\bigwedge_{j\in J} E_j \leqslant A$ Now, $\forall j \in J, \bigwedge_{j\in J}((D_j \to E_j) \wedge D_j) \leqslant ((D_j \to E_j) \wedge D_j)$. Therefore, $\forall j \in J, (D_j \to E_j) \wedge D_j \in T_1$. So, $\forall j \in J, D_j \to E_j \in T_1$ and $D_j \in T_1$. And so, $\forall j \in J, E_j \in T_1 \circ_1 T_1$. Therefore, $\bigwedge_{j\in J} E_j \in T_1$. And hence,

$A \in T_1$. Therefore, $W^* \circ_1 T_1 \subseteq T_1 \circ_1 T_1$. Conversely, suppose $A \in T_1 \circ_1 T_1$. Then $\exists B$ such that $B \to A \in T_1$ and $B \in T_1$. Also $((B \to A) \wedge B) \to A \in W^*$. But, $(B \to A) \wedge B \in T_1$ (since T_1 is closed under conjunction). Therefore, $A \in W^* \circ_1 T_1$. So, $T_1 \circ_1 T_1 \subseteq W^* \circ_1 T_1$. Hence, $W^* \circ_1 T_1 = T_1 \circ_1 T_1$. ♯

We have left it to you, dear reader, to verify the defining axioms for the other combinators. But, to be sure that we have a model of CL in Level-I theories, we must show also that truth of the theorems of this system is preserved under the rules. These are succinctly stated in [2, p. 231f].

- (μ) $X = Y \Rightarrow UX = UY$
- (ν) $X = Y \Rightarrow XU = YU$
- (σ) $X = Y \Rightarrow Y = X$
- (τ) $X = Y$ and $Y = Z \Rightarrow X = Z$

We note simply that, since fusion at Level-I is a single-valued operation on B∧T-theories and since equality is set equality, all of the rules preserve truth, ending the demonstration that there is a model of CL in B∧T-theories, on any assignment of such theories to free variables.

3 B∧T IS A MODEL FOR COMBINATORS AT LEVEL-II

Having proved that B∧T-theories provide a good model for the combinators at Level-I, we will now prove the same results at Level-II, i. e., B∧T-propositions also provide a good model for the combinators. Let's start with the definitions of the combinators at Level-II. (i) $\iota = \{T_1 : I \subseteq T_1\}$. (ii) $\kappa = \{T_1 : K \subseteq T_1\}$. (iii) $\omega = \{T_1 : W^* \subseteq T_1\}$. Here ι, κ and ω are the Level-II correspondance of I, K and W^* respectively. The definitions of the other combinators follow on similar lines. The idea behind using Greek notations for the combinators at Level-II is just to remove ambiguity with those at Level-I.

LEMMA 17 *Suppose* T_1, T_2, T_3 *and* T_4 *are four theories such that* $T_1 \subseteq T_2$ *and* $T_3 \subseteq T_4$. *Then* $T_1 \circ_1 T_3 \subseteq T_2 \circ_1 T_4$.

PROOF Suppose $A \in T_1 \circ_1 T_3$. Then there is some B such that $B \to A \in T_1$ and $B \in T_3$. Therefore, $B \to A \in T_2$ and $B \in T_4$ (since $T_1 \subseteq T_2$ and $T_3 \subseteq T_4$). Therefore, $A \in T_2 \circ_1 T_4$. Hence, $T_1 \circ_1 T_3 \subseteq T_2 \circ_1 T_4$. ♯

LEMMA 18 *Suppose* P, Q, R *and* S *are four propositions such that* $P \subseteq R$ *and* $Q \subseteq S$. *Then* $P \circ_2 Q \subseteq R \circ_2 S$.

PROOF Suppose $T_1 \in P \circ_2 Q$. $\Rightarrow \exists\, T_2 \in P$ and $T_3 \in Q$ such that $T_2 \circ_1 T_3 \subseteq T_1$. But then, $T_2 \in R$ and $T_3 \in S$ (since $P \subseteq R$ and $Q \subseteq S$). Therefore, $T_1 \in R \circ_2 S$. Hence, $P \circ_2 Q \subseteq R \circ_2 S$. ♯

Lemma 19 *The fusion of two propositions is also a proposition.*

Proof Let P, Q be two propositions. (i) Since $P \neq \emptyset \neq Q$, it follows that $\exists T_1$ and T_2 such that $T_1 \in P$ and $T_2 \in Q$. Now, $T_1 \circ_1 T_2$ is a theory (by Lemma 10). And by definition, $T_1 \circ_1 T_2 \in P \circ_2 Q$. Hence, $P \circ_2 Q$ is non-empty. (ii) Suppose $T_1 \in P \circ_2 Q$ and $T_1 \subseteq T_2$ and T_2 is a theory. So, $\exists T_3 \in P$ and $T_4 \in Q$ where $T_3 \circ_1 T_4 \subseteq T_1$. Therefore $T_3 \circ_1 T_4 \subseteq T_2$ (since $T_1 \subseteq T_2$). Therefore, $T_2 \in P \circ_2 Q$. (iii) Suppose $T_1, T_2 \in P \circ_2 Q$. It follows that $(\exists\, T_3 \in P$ and $T_4 \in Q$ where $(T_3 \circ_1 T_4 \subseteq T_1))$ and $(\exists\, T_5 \in P$ and $T_6 \in Q$ where $(T_5 \circ_1 T_6 \subseteq T_2))$. Hence, $(T_3 \cap T_5) \in P$ and $(T_4 \cap T_6) \in Q$. Also, $(T_3 \cap T_5) \circ_1 (T_4 \cap T_6) \subseteq (T_3 \circ_1 T_4) \subseteq T_1)$ and $(T_3 \cap T_5) \circ_1 (T_4 \cap T_6) \subseteq (T_5 \circ_1 T_6) \subseteq T_2)$ (by Lemma 17). Therefore, $(T_3 \cap T_5) \circ_1 (T_4 \cap T_6) \subseteq T_1 \cap T_2$. Hence, $T_1 \cap T_2 \in P \circ_2 Q$. Thus, $P \circ_2 Q$ is non-empty and closed under intersection and sub-theory relation. And hence is a proposition. ♯

Theorem 20 *The fusion of finite number of propositions is a proposition.*

Proof By induction on the previous result. ♯

Claim 21 $\iota \circ_2 P = P$.

Proof Suppose $T_1 \in \iota \circ_2 P$. It follows that there is some $T_2 \in \iota$ and $T_3 \in P$ such that $T_2 \circ_1 T_3 \subseteq T_1$. But $T_2 \in \iota \Rightarrow I \subseteq T_2$ (by definition). Therefore, $I \circ_1 T_3 \subseteq T_2 \circ_1 T_3$ (by Lemma 17). Hence, $I \circ_1 T_3 \subseteq T_1$. But, $I \circ_1 T_3 = T_3$ (by Claim 14). $\Rightarrow T_3 \subseteq T_1$ and hence, $T_1 \in P$. Therefore, $\iota \circ_2 P \subseteq P$. Conversely, suppose $T_1 \in P$. It follows that $I \circ_1 T_1$ $(= T_1) \subseteq T_1$. Hence, $T_1 \in \iota \circ_2 P$. Therefore, $P \subseteq \iota \circ_2 P$. And so, $\iota \circ_2 P = P$. ♯

Claim 22 $\kappa \circ_2 P \circ_2 Q = P$.

Proof Suppose $T_1 \in \kappa \circ_2 P \circ_2 Q$. So, there is some $T_2 \in \kappa$, and a $T_3 \in P$ and $T_4 \in Q$ such that $T_2 \circ_1 T_3 \circ_1 T_4 \subseteq T_1$. But $T_2 \in \kappa \Rightarrow K \subseteq T_2$. Therefore, $K \circ_1 T_3 \circ_1 T_4 \subseteq T_2 \circ_1 T_3 \circ_1 T_4 \subseteq T_1$. But $K \circ_1 T_3 \circ_1 T_4 = T_3$ (by Claim 15). $\Rightarrow T_3 \subseteq T_1$ and hence $T_1 \in P$. Therefore, $\kappa \circ_2 P \circ_2 Q \subseteq P$. Conversely, suppose $T_1 \in P$. Since Q is not empty by definition, there is a $T_2 \in Q$. K of course belongs to κ. Also, $K \circ_1 T_1 \circ_1 T_2$ $(= T_1) \subseteq T_1$. Hence, $T_1 \in \kappa \circ_2 P \circ_2 Q$. and thus $P \subseteq \kappa \circ_2 P \circ_2 Q$. Therefore, $\kappa \circ_2 P \circ_2 Q = P$. ♯

Claim 23 $\omega \circ_2 P = P \circ_2 P$.

Proof Suppose $T_1 \in \omega \circ_2 P$. It follows that there is some $T_2 \in \omega$ and $T_3 \in P$ such that $T_2 \circ_1 T_3 \subseteq T_1$. But $T_2 \in \omega \Rightarrow W^* \subseteq T_2$. Therefore, $W^* \circ_1 T_3 \subseteq T_2 \circ_1 T_3 \subseteq T_1$. But $W^* \circ_1 T_3 = T_3 \circ_1 T_3$ (by Claim 16). So $T_3 \circ_1 T_3 \subseteq T_1$ and hence $T_1 \in P \circ_2 P$. Therefore, $\omega \circ_2 P \subseteq P \circ_2 P$. Conversely, suppose $T_1 \in P \circ_2 P$. $\Rightarrow T_2 \in P$ and $T_3 \in P$ such that $T_2 \circ_1 T_3 \subseteq T_1$. But $T_2 \in P$ and $T_3 \in P \Rightarrow T_2 \cap T3 \in P$. W^* of course belongs to ω. Also, $W^* \circ_1 (T_2 \cap T_3) = (T_2 \cap T_3) \circ_1 (T_2 \cap T_3) \subseteq (T_2 \circ_1 T_3) \subseteq T_1$. Hence, $T_1 \in \omega \circ_2 P$. So $P \circ_2 P \subseteq \omega \circ_2 P$. Therefore, $\omega \circ_2 P = P \circ_2 P$. ♯

We conclude this section as we did the last, leaving the verification of other primitive combinator equalities to readers; note that the rules of [2] preserve equality of propositions at Level-II as they did of theories at Level-I.

4 EXTENSION OF THE $B \wedge T$ RESULTS TO $B+T$ AT LEVEL-I

Having shown that $B \wedge T$-theories are a model for the combinators at Level-I and $B \wedge T$-propositions are a model for the combinators at Level-II, we now try to extend the results to $B+T$-theories and propositions. But we are stuck at the very beginning because it is shown in [5] that there is no model for λ in the set of *all* $B+T$-theories. For the same reason, there is no model for the combinators there. But we can model the basic combinator laws in *prime* $B+T$ theories, as we proceed to show. Recall that T_1 is a prime theory if $A \vee B \in T_1 \Rightarrow$ either $A \in T_1$ or $B \in T_1$. Again we will talk of only the three combinators I, K and W^*. The proofs for other primitive combinators from among C, C^*, B, B', S and W are similar. For the proofs to go through, we will need what we call the *Better Bubbling Lemma* (BBL). This is a very important generalization of the Bubbling Lemma BL above.[7] For proofs of BBL, see [7] and [6]. We only state it here.

LEMMA 24 (BETTER BUBBLING LEMMA) *For any finite sets* $\{\sigma_i \to \tau_i\}_{i \in M}$ *and* $\{\sigma'_j \to \tau'_j\}_{j \in J}$ *of arrow types, the following equivalence holds:* $\mathfrak{A} \Leftrightarrow \mathfrak{F}$ *where*

$$\mathfrak{A} =_{df} \bigwedge_{i \in M}(\sigma_i \to \tau_i) \leqslant \bigvee_{j \in J}(\sigma'_j \to \tau'_j)$$
$$\mathfrak{F} =_{df} \exists j \in J(\sigma'_j \leqslant \bigvee_{i \in M} \sigma_i) \textit{ and } \forall M' \subsetneq M(\sigma'_j \leqslant \bigvee_{i \in M'} \sigma_i) \textit{ or } (\bigwedge_{i \in M \setminus M'} \tau_i \leqslant \tau'_j)$$

PROOF Given in [6]. ♯

From this point on, we will call a $B+T$ theory simply a theory; a $B+T$ proposition, a proposition; and $B+T$ entailment, entailment. We will also use T_1, T_2, etc. to denote theories, P, Q, etc. to denote propositions, $\underline{A}$, $\underline{B}$, etc. to denote principal propositions for formulas A, B, etc., and A, B, etc. to denote well formed formulas. Let us first look at the set of axioms and rules of $B+T$. The axioms are as follows.[8]

Reflex.	$A \to A$
$\wedge$E.	$(A \wedge B) \to A, (A \wedge B) \to B$
$\to \wedge$I.	$(A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$
$\to \vee$E.	$(A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$
$\vee$I.	$A \to (A \vee B), B \to (A \vee B)$
Dist$\wedge\vee$.	$A \wedge (B \vee C) \to ((A \wedge B) \vee (A \wedge C))$

[7] It is our impression, based on conversations with Dezani, that BL is mainly due to her and BBL to Castagna.

[8] in contrast to $B \wedge T$ above, we formulate $B+T$ as an *assertional* system. But we suppress $\vdash$. To restore it in approved Curry [4] fashion, preface each formula asserted as an axiom with $\vdash$. Make a similar adjustment in the rules.

The rules for the system B+T are as follows. Note that $\Rightarrow$ is again a metalogical connective used to express rules.

$$\begin{array}{rl} \to\text{E.} & A \to B \text{ and } A \Rightarrow B \\ \wedge\text{I.} & A \text{ and } B \Rightarrow A \wedge B \\ \text{RulB.} & B \to C \Rightarrow (A \to B) \to (A \to C) \\ \text{RulB}'. & A \to B \Rightarrow (B \to C) \to (A \to C) \end{array}$$

In what follows, we will show how the prime theories satisfy the combinators. We will show this for the combinators I, K and W*. For the others, the proofs follow on similar lines. *Caution*: It is necessary to be delicate at this point. What we are proving here (or leaving for you to prove) is that the defining equations for particular primitive combinators are true, when their arguments are assigned *prime* B+T theories. It is also the case that the primitive combinators will be assigned prime theories. Not only that, but any theory all of whose members are entailed by conjunctions of arrow statements will also be prime.[9] This means that prime theories are rather thick on the ground. Nonetheless, as we warn again below, neither the fusion nor the intersection of two prime theories is in general prime. Beware.

Claim 25 *Suppose* T_1 *is a prime theory. Then* $I \circ_1 T_1 = T_1$.

Proof Suppose $A \in I \circ_1 T_1$. So there is a B where $B \to A \in I$ and $B \in T_1$. Now, if $B \to A \in I$ then there is a finite index set K such that $\bigwedge_{k \in K}(C_k \to C_k) \leqslant B \to A$. By Lemma 24, $B \leqslant \bigvee_{k \in K} C_k$. But $B \in T_1$. Hence, $\bigvee_{k \in K} C_k \in T_1$. Since T_1 is prime, there is some $k_0 \in K$ such that $C_{k_0} \in T_1$. Define $S = \{k \in K : C_k \in T_1\}$. ($S \neq \emptyset$ as $k_0 \in S$.) Define $K' = K \setminus S$. ($K' \subsetneq K$ as $S \neq \emptyset$.) Clearly, $B \nleqslant \bigvee_{j \in K'} C_j$. Therefore, $\bigwedge_{j \in S} C_j \leqslant A$. Now, $\forall j \in S, C_j \in T_1$ (by definition of S). Hence, $\bigwedge_{j \in S} C_j \in T_1$ (since T_1 is a theory). Therefore, $A \in T_1$. And so, $I \circ_1 T_1 \subseteq T_1$. Conversely, suppose $A \in T_1$. Obviously, $A \to A \in I$ (by definition). Hence, $A \in I \circ_1 T_1$. Therefore, $T_1 \subseteq I \circ_1 T_1$. And so, $I \circ_1 T_1 = T_1$. ♯

Claim 26 *Suppose* T_1 *and* T_2 *are prime theories. Then* $K \circ_1 T_1 \circ_1 T_2 = T_1$.

Proof Suppose $A \in K \circ_1 T_1 \circ_1 T_2$. So there are B, C such that $B \to (C \to A) \in K$, $B \in T_1$ and $C \in T_2$. Now if $B \to (C \to A) \in K$ then there is a finite index set M such that $\bigwedge_{i \in M}(E_i \to (D_i \to E_i)) \leqslant (B \to (C \to A))$. By Lemma 24, $B \leqslant \bigvee_{i \in M} E_i$. Since, $B \in T_1$ and T_1 is a theory, therefore, $\bigvee_{i \in M} E_i \in T_1$. Since, T_1 is prime, there is a $i_0 \in M$ such that $E_{i_0} \in T_1$. Define $S = \{i \in M : E_i \in T_1\}$ ($S \neq \emptyset$ as $i_0 \in S$). Define $M' = M \setminus S$ ($M' \subsetneq M$ as $S \neq \emptyset$). Clearly, $B \nleqslant \bigvee_{j \in M'} E_j$. Therefore, $\bigwedge_{j \in S}(D_j \to E_j) \leqslant (C \to A)$. Again by Lemma 24, $C \leqslant \bigvee_{j \in S} D_j$. Since, $C \in T_2$ and T_2 is a theory, therefore, $\bigvee_{j \in S} D_j \in T_2$. Since, T_2 is prime, there is a $j_0 \in S$ such that $D_{j_0} \in T_2$. Define $S' = \{j \in S : D_j \in T_2\}$ ($S' \neq \emptyset$ as $j_0 \in S'$.) Define $M'' = S \setminus S'$ ($M'' \subseteq S$ as $S' \neq \emptyset$). Clearly, $C \nleqslant \bigvee_{j \in M''} D_j$. Therefore, $\bigwedge_{k \in S'} E_k \leqslant A$. Now $\forall k \in S'$, $E_k \in T_1$ (by definition and since

[9] Dezani showed this in [6].

$S' \subsetneq S$). Hence, $\bigwedge_{k \in S'} E_k \in T_1$ (since T_1 is a theory) . Therefore, $A \in T_1$. And so, $K \circ_1 T_1 \circ_1 T_2 \subseteq T_1$.

Conversely, suppose $A \in T_1$. Since T_2 is not empty, there is some B such that $B \in T_2$. By definition, $(A \to (B \to A)) \in K$. Therefore, $B \to A \in K \circ_1 T_1$. And hence, $A \in K \circ_1 T_1 \circ_1 T_2$. Therefore, $T_1 \subseteq K \circ_1 T_1 \circ_1 T_2$. And so, $K \circ_1 T_1 \circ_1 T_2 = T_1$. ♯

Claim 27 *Suppose* T_1 *is a prime theory. Then* $W^* \circ_1 T_1 = T_1 \circ_1 T_1$.

Proof Suppose $A \in W^* \circ_1 T_1$. It follows that there is a B such that $B \to A \in W^*$ and $B \in T_1$. Now, $B \to A \in W^* \Rightarrow \exists$ a finite index set M such that $\bigwedge_{i \in M}(((C_i \to D_i) \wedge C_i) \to D_i) \leqslant B \to A$. By Lemma 24, $B \leqslant \bigvee_{i \in M}((C_i \to D_i) \wedge C_i)$. Since, $B \in T_1$ and T_1 is a theory, it follows that, $\bigvee_{i \in M}((C_i \to D_i) \wedge C_i) \in T_1$. Since, T_1 is prime, there is some $i_0 \in M$ such that $(C_{i_0} \to D_{i_0}) \wedge C_{i_0} \in T_1$. Define $S = \{i \in M : (C_i \to D_i) \wedge C_i \in T_1\}$ ($S \neq \emptyset$ as $i_0 \in S$).

Define $M' = M \backslash S$ ($M' \subsetneq M$ as $S \neq \emptyset$). Clearly, $B \not\leqslant \bigvee_{j \in M'}((C_j \to D_j) \wedge C_j)$. Therefore, $\bigwedge_{j \in S} D_j \leqslant A$. Now, $\forall j \in S$, $(C_j \to D_j) \wedge C_j \in T_1$ (by definition of S). Therefore, $\forall j \in S$, $C_j \to D_j \in T_1$ and $C_j \in T_1$ (since T_1 is a theory). It follows that $\forall j \in S$, $D_j \in T_1 \circ_1 T_1$ (by definition). Hence, $\bigwedge_{j \in S} D_j \in T_1 \circ_1 T_1$ (since $T_1 \circ_1 T_1$ is a theory). Therefore, $A \in T_1 \circ_1 T_1$. And so, $W^* \circ_1 T_1 \subseteq T_1 \circ_1 T_1$.

Conversely, suppose $A \in T_1 \circ_1 T_1$. Then there is a B such that $B \to A \in T_1$ and $B \in T_1$. Also $((B \to A) \wedge B) \to A \in W^*$. But, $(B \to A) \wedge B \in T_1$ (since T_1 is closed under conjunction). Therefore, $A \in W^* \circ_1 T_1$. So, $T_1 \circ_1 T_1 \subseteq W^* \circ_1 T_1$. Hence, $W^* \circ_1 T_1 = T_1 \circ_1 T_1$. ♯

5 THE REAL PROBLEM

In the above section, we have seen that the prime theories of B+T do a really good job, satisfying all the primitive combinator equalities. So the question arises: Is the set of prime theories of B+T a model for the combinators? The answer is "No." We must face the real problem, because prime theories are neither closed under fusion nor closed under intersection.[10]

So what do we do? The most logical thing to do is to expand a non-prime theory T_1 to a prime theory. But this can be done in more than one way. So

[10]Specifically, consider the following counterexample, adapted from Dezani. Recall that we interpret $W = [(A \to (A \to B)) \to (A \to B)]$, now in the B+T vocabulary. Where p, q, r are propositional variables, define theories T_1 and T_2 thus: T_1 is the principal theory $(p \to (p \to r)) \wedge (q \to (q \to r))\uparrow$. Similarly let T_2 be the principal theory $p \vee q\uparrow$. T_1 is prime, though T_2 clearly is not prime. Computing, $r \in W \circ_1 T_1 \circ_1 T_2$. But r sadly *fails* to belong to $T_1 \circ_1 T_2 \circ_1 T_2$. This refutes on interpretation in *arbitrary* B+T theories the W law. Worse, this counterexample can be massaged so that all theories involved are prime. For, where s is another propositional variable, we may simply let $T_2 = T_3 \circ_1 T_4$, where T_3 is the principal theory $s \to p \vee q\uparrow$ and T_4 is just $s\uparrow$. The verification that we have left for the reader now fails for W, our appeal to BBL being blocked because T_2, though now defined as a fusion of prime theories, remains resolutely non-prime.

which theory should we choose as our prime extension of T_1? A natural solution is to look at *all* the prime theories which are super theories of T_1. And that is how propositions originate. But in the B+T case we will try to define something more precise, namely, prime propositions. We supply these with a corresponding fusion operator (say, prime fusion) in the hope that the propositions will then have nice properties. So here we go.

Definition 9 (prime proposition) A non-empty collection of theories P is said to be a prime proposition if (i) for each $T_1 \in P$, T_1 is a prime theory. (ii) If $T_1 \in P$, $T_1 \subseteq T_2$ and T_2 is a prime theory, then $T_2 \in P$.

Definition 10 (prime fusion operator ($\circ_{2'}$)) Suppose P and Q are two prime propositions. Then their prime fusion is defined as follows

$$P \circ_{2'} Q = \{T_1 : (\exists T_2 \in P)(\exists T_3 \in Q)(T_2 \circ_1 T_3) \subseteq T_1 \text{ and } T_1 \text{ is prime.}\}$$

Theorem 28 *The intersection of two prime propositions is always non-empty.*

Proof Suppose P and Q are two prime propositions. By definition, $P \neq \emptyset \neq Q$. It follows that there are $T_1 \in P$, $T_2 \in Q$. Now, $T_1 = \bigcup\{C\uparrow : C \in T_1\}$ And, $T_2 = \bigcup\{D\uparrow : D \in T_2\}$. Define $T_3 = \{E : \exists C \in T_1 \text{ and } \exists D \in T_2 \text{ such that } C \wedge D \leqslant E\}$. Clearly, $T_1 \subseteq T_3$ and $T_2 \subseteq T_3$. However, T_3 need not be prime. But we can always extend it to some prime theory T_4 such that $T_3 \subseteq T_4$. Therefore, $T_4 \in P$ and $T_4 \in Q$, i. e., $T_4 \in P \cap Q$. And hence, $P \cap Q$ is non-empty. ♯

Theorem 29 *The intersection of two prime propositions is always a prime proposition.*

Proof Let P and Q be two prime propositions. By Theorem 27, $P \cap Q \neq \emptyset$. (i) Suppose $T_1 \in P \cap Q$. Then $T_1 \in P$ and hence, T_1 is prime. (ii) Suppose $T_1 \in P \cap Q$ and $T_1 \subseteq T_2$ and T_2 is prime. Then $T_1 \in P$, $T_1 \subseteq T_2$, T_2 is prime and P is a prime proposition. Therefore, $T_2 \in P$. Similarly $T_2 \in Q$. And hence $T_2 \in P \cap Q$. Therefore $P \cap Q$ is non-empty, contains only prime theories and is closed under sub-theory relation. And hence is a prime proposition. ♯

Theorem 30 *The prime fusion of two prime propositions is a prime proposition.*

Proof Suppose P and Q are two prime propositions. (i) Let $T_1 \in P \circ_{2'} Q$. By definition of $\circ_{2'}$, T_1 is prime. (ii) Since $P \neq \emptyset \neq Q$, there are $T_1 \in P$ and $T_2 \in Q$. Now, $T_1 \circ_1 T_2$ is a theory (by Lemma 10). But it need not be prime. However we can extend it to a prime theory T_3 such that $T_1 \circ_1 T_2 \subseteq T_3$. Then by definition, $T_3 \in P \circ_{2'} Q$. Hence, $P \circ_{2'} Q$ is non-empty. (iii) Suppose $T_1 \in P \circ_{2'} Q$ and $T_1 \subseteq T_2$ and T_2 is a prime theory. It follows that there are $T_3 \in P$ and $T_4 \in Q$ such that $T_3 \circ_1 T_4 \subseteq T_1$. So, $T_3 \circ_1 T_4 \subseteq T_2$. (since $T_1 \subseteq T_2$) Therefore, $T_2 \in P \circ_{2'} Q$. Thus, $P \circ_{2'} Q$ is non-empty, contains only prime theories and is closed under sub-theory relation. And hence is a prime proposition. ♯

THEOREM 31 *The prime fusion of a finite number of propositions is a prime proposition.*

PROOF By induction on the previous result. ♯

Prime propositions and the prime fusion operator satisfy the combinator laws for K, B, B′, C and C*. But they do not work so nicely for the combinators W, W* and S. We will carry out the arguments for I and K. Proofs for B, B′, C and C* are similar. We will indicate where a similar argument breaks down for W*. The same difficulties afflict arguments for W and S. We begin anew with interpretations of the combinators ι, κ and ω at Level-II of B+T. $\iota = \{T_1 : I \subseteq T_1$ and T_1 is prime$\}$. $\kappa = \{T_1 : K \subseteq T_1$ and T_1 is prime$\}$. $\omega = \{T_1 : W^* \subseteq T_1$ and T_1 is prime$\}$.

The theories determined by the combinators are known to be prime [6]. In particular, I, K and W^* are prime. Therefore, $I \in \iota$, $K \in \kappa$ and $W^* \in \omega$.

CLAIM 32 *Suppose* P *is a prime proposition. Then* $\iota \circ_{2'} P = P$.

PROOF Suppose $T_1 \in \iota \circ_{2'} P$. So T_1 is prime. It follows that $T_2 \in \iota$ and $T_3 \in P$ such that $T_2 \circ_1 T_3 \subseteq T_1$. But $T_2 \in \iota \Rightarrow I \subseteq T_2$ (by definition). Therefore, $I \circ_1 T_3 \subseteq T_2 \circ_1 T_3$ (by Lemma 17). Hence, $I \circ_1 T_3 \subseteq T_1$. But, T_3 is prime and so $I \circ_1 T_3 = T_3$ (by Claim 25). So $T_3 \subseteq T_1$ and hence, $T_1 \in P$ (since T_1 is prime). Therefore, $\iota \circ_{2'} P \subseteq P$. Conversely, suppose $T_1 \in P$. Therefore T_1 is prime, and $I \circ_1 T_1$ (= T_1) $\subseteq T_1$. Hence, $T_1 \in \iota \circ_{2'} P$. Therefore, $P \subseteq \iota \circ_{2'} P$. And so, $\iota \circ_{2'} P = P$. ♯

CLAIM 33 *Suppose* P *and* Q *are prime propositions. Then* $\kappa \circ_{2'} P \circ_{2'} Q = P$.

PROOF Suppose $T_1 \in \kappa \circ_{2'} P \circ_{2'} Q$. Therefore T_1 is prime. So, there are $T_2 \in \kappa$, $T_3 \in P$ and $T_4 \in Q$ such that $T_2 \circ_1 T_3 \circ_1 T_4 \subseteq T_1$. But if $T_2 \in \kappa$ then $K \subseteq T_2$. Therefore, $K \circ_1 T_3 \circ_1 T_4 \subseteq T_2 \circ_1 T_3 \circ_1 T_4 \subseteq T_1$. But T_3 and T_4 are prime. Hence, $K \circ_1 T_3 \circ_1 T_4 = T_3$ (by Claim 26), and $T_3 \subseteq T_1$ which gives $T_1 \in P$ (since T_1 is prime). Therefore, $\kappa \circ_{2'} P \circ_{2'} Q \subseteq P$.

Conversely, suppose $T_1 \in P$. Therefore, T_1 is prime (as P is a prime proposition). Since Q is not empty by definition, there is a $T_2 \in Q$. K of course belongs to κ. Also, $K \circ_1 T_1 \circ_1 T_2$ $(= T_1) \subseteq T_1$ and T_1 is prime. Hence, $T_1 \in \kappa \circ_{2'} P \circ_{2'} Q$. So, $P \subseteq \kappa \circ_{2'} P \circ_{2'} Q$. Therefore, $\kappa \circ_{2'} P \circ_{2'} Q = P$. ♯

So we have proved that prime propositions and the prime fusion operator work fine for the combinators I and K. Proofs for the combinators B, B′, C and C* are similar. Now we show where we get stuck with these definitions for the combinator W*. Similar difficulties afflict W and S. We will make the following claim and try to prove it. Keep an eye out for *Trouble*.

CONJECTURE 34 *Suppose* P *is a prime proposition. Then* $\omega \circ_{2'} P = P \circ_{2'} P$.

ATTEMPTED PROOF Suppose $T_1 \in \omega \circ_{2'} P$. Hence T_1 is prime. There are $T_2 \in \omega$ and $T_3 \in P$ such that $T_2 \circ_1 T_3 \subseteq T_1$. But $T_2 \in \omega \Rightarrow W^* \subseteq T_2$. Therefore,

$W^* \circ_1 T_3 \subseteq T_2 \circ_1 T_3 \subseteq T_1$. But $W^* \circ_1 T_3 = T_3 \circ_1 T_3$ (by Claim 27). It follows that $T_3 \circ_1 T_3 \subseteq T_1$ and hence $T_1 \in P \circ_{2'} P$ (as T_1 is prime). Therefore, $\omega \circ_{2'} P \subseteq P \circ_{2'} P$.

This part was trivial. It is the other part where we actually get stuck. Conversely, suppose $T_1 \in P \circ_{2'} P$. So, there are $T_2 \in P$ and $T_3 \in P$ such that $T_2 \circ_1 T_3 \subseteq T_1$. Suppose there is a $T_4 \in P$ where $T_4 \subseteq T_2$ and $T_4 \subseteq T_3$ [!!!]. W^* of course belongs to ω. Also, $W^* \circ_1 T_4 = (T_4 \circ_1 T_4) \subseteq (T_2 \circ_1 T_3) \subseteq T_1$. Hence, $T_1 \in \omega \circ_{2'} P$. It follows that $P \circ_{2'} P \subseteq \omega \circ_{2'} P$. Therefore, $\omega \circ_{2'} P = P \circ_{2'} P$. ♭

Now that the attempted proof is over, look back at the part marked "[!!!]". This line causes a problem. It is not always possible to find a prime theory T_4 which is a subset of two given arbitrary prime theories T_2 and T_3. Consider in this context the boolean algebra on the base set $\{a, b\}$. This lattice illustrates our B+T problems. We have $\{\{a\}, \{a, b\}\}$ and $\{\{b\}, \{a, b\}\}$ as two prime theories. (Prime *filters*, as an algebraist would say.) But the only non-empty theory (filter) that is a subset of both of these is $\{\{a, b\}\}$, which unfortunately is not prime. And that is exactly where we get stuck in all the three cases of W, W^* and S. Perhaps we can modify the definitions of prime propositions and/or the prime fusion operator so that the existence of such a T_4 is always guaranteed for arbitrary prime theories T_2 and T_3 while satisfying also nice properties like closure under intersection and fusion. If so, we can hope to get a model for the combinators in B+T-prime propositions. Also note that because of what we have claimed in this paper, any subset of the prime theories of B+T, which is closed under intersection and fusion, is definitely a model for CL. But finding one such subset is not an easy goal. We entreat *your* help.

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References

[1] Henk Barendregt, Mario Coppo and Mariangiola Dezani-Ciancaglini, "A Filter Model and the Completeness of Type Assignment", *The Journal of Symbolic Logic*, Volume 48, Issue 4 (Dec. 1983), 931–940.

[2] M. W. Bunder, "Combinators, Proofs and Implicational Logics", in D. M. Gabbay and F. Guenther, eds., *Handbook of Philosophical Logic*, 2nd Edition, Kluwer, Dordrecht (2002), 229–286.

[3] H. B. Curry, and R. Feys, *Combinatory Logic* vol. I, N. Holland, Amsterdam, (1958).

[4] H. B. Curry, *Foundations of Mathematical Logic*, McGraw-Hill, N. Y. (1963).

[5] Mariangiola Dezani-Ciancaglini, Robert K. Meyer and Yoko Motohama "The Semantics of Entailment Omega", *Notre Dame Journal of Formal Logic*, vol. 43, no. 3 (2002), 129–145.

[6] Mariangiola Dezani-Ciancaglini, Alain Frisch, Elio Giovannetti and Yoko Motohama, "The Relevance of Semantic Subtyping", *Electronic Notes in Theoretical Computer Science* 70, No. 1 (2002).

[7] A. Frisch, G. Castagna and V. Benzaken, "Semantic subtyping", in *17th IEEE Symposium on Logic in Computer Science*, 2002, IEEE Computer Society Press, to appear.

[8] Silvia Ghilezan, "Full Intersection Types and Topologies in Lambda Calculus", *Journal of Computer and Science Systems* 62 (2001), 1–14.

[9] Richard Routley and Robert K. Meyer, "The Semantics of Entailment: III", *Journal of Philosophical Logic* 1 (1972), 192–208.

[10] R. Routley, and R. K. Meyer, "The Semantics of Entailment I", in H. Leblanc, ed., *Truth, Syntax and Modality*, N. Holland, Amsterdam (1973), 199–243.

[11] M. H. Stone, "The representation theorem for Boolean algebra", *Trans. Am. Math. Soc.* 40 (1936), 37–111.

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