On Lundberg's estimate for ruin probability under reinsurance

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We consider the risk process $(X^x(t))$ defined by

$$X^x(t) = x + pt - S(t)$$

where $x > 0$ is the initial capital, $p > 0$ is the (constant) premium rate and the aggregate claims process $(S(t))$ is a compound Poisson process (classical case) or a Markov modulated compound Poisson process (Markov modulated case). More precisely we have $S(t) = \sum_{k=1}^{N(t)} U_k$ and $N(t) = \sum_{k \geq 1} 1_{T_k \leq t}$, where (U_k) is a sequence of positive random variables and $(N(t))$ is a counting process with points (T_k). Roughly speaking, when we deal with the Markov modulated case, claims intensity and claims size distribution depend on the evolution of a finite state space Markov chain; from the actuarial point of view the Markov chain describes the environmental conditions that influence the phenomena, such as weather conditions in car insurance.
We analyze the small claim case, so that $\frac{S(t)}{t}$ converges to some limit value ℓ as $t \to \infty$. The (infinite horizon) ruin probabilities $(\psi(x))_{x>0}$ are defined by

$$\psi(x) = P(\tau^x < \infty), \text{ where } \tau^x = \inf\{t \geq 0 : X^x(t) < 0\}$$

and, in order to avoid the trivial case $\psi(x) = 1$ for all $x > 0$, the so called *net profit condition* is required, i.e. $p = (1 + \kappa)\ell$ for some relative safety loading $\kappa > 0$.

In our model reinsurance is allowed, i.e. the insurance company may insure part of the risk at another company (the reinsurance company) in return for a part of the premium pt. We consider a general family of reinsurance contracts including the usual *proportional* and *excess-of-loss* forms. A reinsurance policy is described by a measurable function $\mathcal{R} : [0, \infty) \times [0, \infty) \to [0, \infty)$, for which we use the notation $\mathcal{R}(t, \alpha) = R_t(\alpha)$; for such a function we require the condition $0 \leq R_t(\alpha) \leq \alpha$ for all $t, \alpha \geq 0$. This means that $R_t(\alpha)$ is the part of the claim that the company pays when a claim of size α occurs at time t. Since the reinsurance policy is chosen dynamically, the premium rate for the reinsurer is in general not constant in time, as it happen in the classical risk model. We denote with $q_{\mathcal{R}}(t)$ the premium up to time t paid by the insurer to the reinsurer. We assume that reinsurer uses the *expected value principle* with relative safety loading $\eta > 0$ for premium calculation. We point out that it is interesting to consider $\eta > \kappa$, i.e. the case in which reinsurance is more expensive than insurance, otherwise the insurer would reinsure the whole portfolio. Then the reserve process $(X^x_{\mathcal{R}}(t))$ under the reinsurance policy $\mathcal{R}$ is defined by

$$\begin{cases} X^x_{\mathcal{R}}(t) = x + p_{\mathcal{R}}(t) - S_{\mathcal{R}}(t), \text{ where} \\ S_{\mathcal{R}}(t) = \sum_{k=1}^{N(t)} R_{T_k}(U_k) \text{ and } p_{\mathcal{R}}(t) = pt - q_{\mathcal{R}}(t). \end{cases}$$

The aim of this paper is to derive the so called Lundberg's estimate for the infinite time ruin probability

$$\psi_{\mathcal{R}}(x) = P(\tau^x_{\mathcal{R}} < \infty), \text{ where } \tau^x_{\mathcal{R}} = \inf\{t \geq 0 : X^x_{\mathcal{R}}(t) < 0\}.$$

This estimate shows that, in a sense related to large deviations, $\psi_{\mathcal{R}}(x)$ decays exponentially as $x \to \infty$. The result is obtained as a consequence of some large deviation results (see [2]). We point out that the Markov modulated case is a generalization of the classical case; on the other hand, the large deviation result for the classical case is more sophisticated. Moreover, we discuss the existence of the asymptotically optimal reinsurance strategy, that is the reinsurance strategy $\mathcal{R}$ that maximizes the adjustment coefficient.

A number of papers focus their analysis on giving asymptotic results for the ruin probability for risk processes with reinsurance. Most of them study the classical case. Waters [5] considers constant reinsurance strategies. He finds that in the case of *proportional* reinsurance there exists a unique constant strategy that maximizes the adjustment coefficient. In the case of *excess-of-loss* reinsurance strategies, he argues that the same result holds if the premium is calculated according to the *expected value principle*. Schmidli studies the asymptotics for risk processes under optimal *proportional* reinsurance in the small claim case (see [3]) and large claim case (see [4]). In both case, he provides the Cramér-Lundberg approximation as well as the convergence of the optimal strategies. In particular, in the small claim case he proves that the optimal reinsurance strategy converges to the asymptotically optimal strategy as the initial capital increases to infinity. For Markov modulated risk processes, Hald and Schmidli [1] (section 4.2) treats the question of how to calculate the *proportional* reinsurance strategy maximizing the adjustment coefficient. As far as we know, there are no asymptotic results for Markov modulated risk processes with *excess-of-loss* reinsurance.

References

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