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Perturbation Methods for Bifurcation Analysis from Multiple Nonresonant Complex Eigenvalues

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Abstract. It is shown that the logical bases of the static perturbation method, which is currently used in static bifurcation analysis, can also be applied to dynamic bifurcations. A two-time version of the Lindstedt–Poincaré Method and the Multiple Scale Method are employed to analyze a bifurcation problem of codimension two. It is found that the Multiple Scale Method furnishes, in a straightforward way, amplitude modulation equations equal to normal form equations available in literature. With a remarkable computational improvement, the description of the central manifold is avoided. The Lindstedt–Poincaré Method can also be employed if only steady-state solutions have to be determined. An application is illustrated for a mechanical system subjected to aerodynamic excitation.

Key words: Perturbation methods, stability, bifurcation, codimension two, periodic and quasi-periodic solutions.

1. Introduction

It is known that, within the framework of the local bifurcation theory [1], analytical techniques make it possible to reduce a multidimensional dynamical system to a lower-dimensional equivalent system that captures all the qualitative aspects of the original system behavior. The dimension of the reduced system is equal to the sum of the algebraic multiplicity of the Jacobian matrix critical eigenvalues [2, 3]. The center manifold reduction [2] is the most commonly followed approach. It calls for two steps: (I) the description of the manifold on which the post-critical steady-state dynamics takes place and (II) the transformation of the bifurcation equations into the simplest form. The first step requires the solution of a functional equation, which is often solved by series expansions; the second step calls for the use of normal form theory. Normal form equations for low codimension bifurcations are well known in literature and have been extensively studied. However, explicit expressions of the coefficients of the reduced system in terms of the coefficients of the original system are not available for general systems. Therefore, the whole procedure described above has to be repeated for each specific problem, thus entailing a larger computational effort.

On the other hand, other methods have also been used in literature to solve static and dynamic bifurcation problems, including the averaging method [4], the harmonic balance method [5] and the multiple scale method [6]. In addition, the Hopf method, used to analyze Hopf bifurcations of codimension one [7], clearly appears as an extension of the Lindstedt–Poincaré Method [8, 9]. Recently, the Lindstedt–Poincaré Method and the Multiple Scale Method have also been extended to discrete-time dynamical systems [10]. All these methods are borrowed from nonlinear dynamics [11] and follow the same logic underlying the theory of static perturbation, where bifurcated paths are approximated by series [8, 12, 13].

Although the methods quoted are perhaps less elegant than the center manifold reduction, they generally require less computational effort. Among them, in the authors' opinion, the multiple scale method is the best tool to study both static and dynamic bifurcations of a general system, regardless of the codimension. The ideas related to the use of the method of multiple scales as a simplification method can be traced to the works by Nayfeh [14], Smith and Morino [15], Maslowe [16], Moroz [17] and Nayfeh and Balachandran [6, 18]. Within bifurcation analysis, the main advantage of the method is the possibility to obtain the reduced equations without describing in advance the central manifold, neither expressing the Jacobian matrix at the critical state in Jordan form. In practice, only the right and left critical eigenvectors need to be evaluated and elementary operations have to be performed. As a result, bifurcation equations are obtained, whose coefficients are expressed in closed form in terms of the derivatives of the original vector field, evaluated at the critical state, similarly to the theory of static bifurcation of conservative systems. As an alternative, the Lindstedt–Poincaré method can be used with identical advantages in order to find steady-state bifurcating solutions. However, if the latter approach is followed, stability analysis is much more complex.

In this paper, both methods are illustrated with reference to a non-resonant double Hopf bifurcation occurring in a two control parameter dynamic system, i.e. for a bifurcation of codimension two. An application of the procedure is presented to analyze the post-critical behavior of a simple mechanical system subjected to aerodynamic excitation. In contrast to the general theory illustrated in the first part of the paper, the applicative analysis is here developed by working on the second order form of the equation of motion.

2. Position of the Problem

The equations of motion of an autonomous dynamical system, reduced in local form [8], are

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \boldsymbol{\mu}), \tag{1}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the state vector and $\boldsymbol{\mu} \in \mathcal{R}^m$ the control parameters vector. Equation (1) admits the trivial equilibrium solution consisting of the set of states $\Gamma := \{(\mathbf{x}, \boldsymbol{\mu}) \mid \mathbf{x} = \mathbf{0}\}$. From Lyapunov's theory, it is well known [19] that the equilibrium position $\mathbf{x} = \mathbf{0}$ is stable (or attracting) if all eigenvalues $\lambda_i(\mu)$ of the Jacobian matrix

$$\mathbf{F}_{\mathbf{x}}(\mathbf{0}, \boldsymbol{\mu}) := \left.\frac{\partial \mathbf{F}(\mathbf{x}, \boldsymbol{\mu})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}} \tag{2}$$

have negative real part, while it is unstable if at least one of the eigenvalues has positive real part. This paper considers systems of type (1) depending on two control parameters, namely $\boldsymbol{\mu} = \{\nu, \eta\}^T$. It is assumed that the trivial solution looses its stability at a bifurcation point $\mathbf{O} \equiv (\mathbf{x} = \mathbf{0}, \nu = \eta = 0)$ in which two pairs of conjugate eigenvalues of the Jacobian matrix simultaneously across the imaginary axis (*double Hopf bifurcation*). The following spectral properties are assumed to hold:

1. At the bifurcation point the Jacobian matrix $\mathbf{F}^0_{\mathbf{x}} := \mathbf{F}_{\mathbf{x}}(\mathbf{x} = \mathbf{0}, \nu = \eta = 0)$ has two pairs of purely imaginary eigenvalues $\lambda_{1,3} = \pm i\omega_{10}, \lambda_{2,4} = \pm i\omega_{20}$. In addition, the frequency ω_{10} and ω_{20} are assumed to be incommensurable (non-resonant eigenvalues). The associated right eigenvectors $\mathbf{u}_j$ $(j = 1, 2)$ are solutions of the following algebraic problems

$$\mathbf{F}^0_{\mathbf{x}}\mathbf{u}_j = i\omega_{j0}\mathbf{u}_j \tag{3}$$

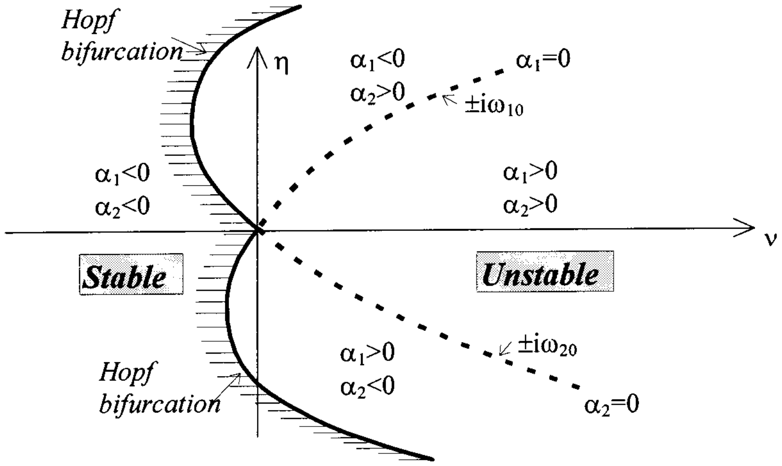

Figure 1. Stability boundary diagram.

being $\mathbf{u}_3 = \bar{\mathbf{u}}_1$ and $\mathbf{u}_4 = \bar{\mathbf{u}}_2$; the associated left eigenvectors satisfy the equations

$$(\mathbf{F}^0_{\mathbf{x}})^T \mathbf{v}_j = -i\omega_{j0}\mathbf{v}_j \tag{4}$$

being $\mathbf{v}_3 = \bar{\mathbf{v}}_1$ and $\mathbf{v}_4 = \bar{\mathbf{v}}_2$. Right and left eigenvectors are orthonormal, i.e. $\mathbf{v}_i^H \mathbf{u}_j = \delta_{ij}$, where H denotes transpose conjugate and δ_{ij} is the Kronecker symbol.

2. At bifurcation, all the remaining eigenvalues λ_h, $h \geq 5$, lie on the left side of the complex plane.
3. The critical eigenvalues

$$\begin{aligned} &\lambda_{1,3} = \alpha_1(\nu,\eta) \pm i\omega_1(\nu,\eta) \\ &\lambda_{2,4} = \alpha_2(\nu,\eta) \pm i\omega_2(\nu,\eta) \\ &\alpha_j(0,0) = 0 \quad (j = 1,2) \end{aligned} \tag{5}$$

satisfy the following *transversality conditions*

$$\det \begin{bmatrix} \alpha_{1\nu} & \alpha_{1\eta} \\ \alpha_{2\nu} & \alpha_{2\eta} \end{bmatrix} \neq 0, \tag{6}$$

where

$$\alpha_{j\nu} := \left.\frac{\partial \alpha_j}{\partial \nu}\right|_{\substack{\nu=0\\ \eta=0}}, \quad \alpha_{j\eta} := \left.\frac{\partial \alpha_j}{\partial \eta}\right|_{\substack{\nu=0\\ \eta=0}} \tag{7}$$

with $(j = 1, 2)$. These properties generalize the simple Hopf bifurcation condition [1].

The graphs of the equations $\alpha_j(\nu,\eta) = 0$, $j = 1,2$, in the parameter plane define the diagram of linear stability, also called stability boundary diagram; an example is shown in Figure 1. The critical point associated with a double Hopf bifurcation occupies an isolated position at the intersection of the Hopf boundary lines. Condition (6) requires that the two lines have no common tangent at the intersection point, in such a way that no directions in the (ν, η)-plane exist in which the critical state persists. In Figure 1 it has been assumed that $\alpha_{j\nu} > 0$ for $j = 1, 2$. It implies transversality for both pairs of critical eigenvalues when only ν is increased from zero, η being kept null. To underline the different role of the two parameters, sometimes a parameter like ν is called the *distinguished parameter* [3] while η the *splitting parameter* [20].

Steady-state solutions which bifurcate from point **O** and their stability have to be determined. However, the system behavior around a double Hopf bifurcation is far more complex than around a single Hopf bifurcation [2, 3]. In contrast to the latter, where only periodic motions occur, quasi-periodic motions take place in the former if the eigenvalues ω_{j0} ($j = 1, 2$) are incommensurable [21]. This implies the presence of two different temporal scales and, hence, a resultant motion on a two-dimension torus [19].

In the following, the problem is addressed in two ways. First, a modified version of the classical Lindstedt–Poincaré Method (*LPM*) of determining steady-state solutions is presented; the Multiple Scale Method (*MSM*) is then applied to obtain amplitude-modulation equations.

3. The Two-Times Lindstedt–Poincaré Method

A monoparametric family of double-periodic solutions of the form

$$\begin{cases} \mathbf{x} = \mathbf{x}(\varepsilon, \tau_1, \tau_2) \\ \nu = \nu(\varepsilon) \\ \eta = \eta(\varepsilon) \end{cases} \tag{8}$$

is sought, where

$$\tau_j = \omega_j(\varepsilon)t, \quad j = 1, 2 \tag{9}$$

are independent time-scales, $\omega_j\ (\varepsilon)$ are unknown ε-dependent circular frequencies and ε is a perturbation parameter. Equations (8_2) and (8_3) are the parametric equation of an unknown curve on the (ν, η)-plane on which double-periodic solutions of given amplitudes and unknown periods exist.

By accounting for Equations (8) and for the chain rule $\mathrm{d}/\mathrm{d}t = \omega_1(\varepsilon)\partial/\partial\tau_1 + \omega_2(\varepsilon)\partial/\partial\tau_2$, the following identity follows from Equation (1):

$$\left[\omega_1(\varepsilon)\frac{\partial}{\partial\tau_1} + \omega_2(\varepsilon)\frac{\partial}{\partial\tau_2}\right]\mathbf{x}(\varepsilon, \tau_1, \tau_2) = \mathbf{F}[\mathbf{x}(\varepsilon, \tau_1, \tau_2), \nu(\varepsilon), \eta(\varepsilon)], \quad \forall\varepsilon, \forall(\tau_1, \tau_2) \tag{10}$$

with the periodicity condition

$$\mathbf{x}(\varepsilon, \tau_1 + 2\pi, \tau_2 + 2\pi) = \mathbf{x}(\varepsilon, \tau_1, \tau_2). \tag{11}$$

Under the assumption of regularity, MacLaurin series of the form

$$\begin{Bmatrix} \mathbf{x}(\varepsilon, \tau_1, \tau_2) \\ \nu(\varepsilon) \\ \eta(\varepsilon) \\ \omega_1(\varepsilon) - \omega_{10} \\ \omega_2(\varepsilon) - \omega_{20} \end{Bmatrix} = \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \begin{Bmatrix} \mathbf{x}_k(\tau_1, \tau_2) \\ \nu_k \\ \eta_k \\ \omega_{1k} \\ \omega_{2k} \end{Bmatrix} \tag{12}$$

are introduced, where $\varepsilon = 0$ selects the bifurcation point **O**. By differentiating k times Equations (10) with respect to the parameter ε, and evaluating the derivatives at $\varepsilon = 0$, the perturbative equations of k-th order are obtained; for $k = 1, 2, 3$, by using Equation (12), they read:

$$\begin{aligned}
L_0\mathbf{x}_1 &= \mathbf{0}\\
L_0\mathbf{x}_2 &= 2\left(\omega_{11}\frac{\partial}{\partial\tau_1} + \omega_{21}\frac{\partial}{\partial\tau_2}\right)\mathbf{x}_1 - 2(\nu_1\mathbf{F}^0_{\mathbf{x}\nu} + \eta_1\mathbf{F}^0_{\mathbf{x}\eta})\mathbf{x}_1 - \mathbf{F}^0_{\mathbf{xx}}\mathbf{x}_1^2\\
L_0\mathbf{x}_3 &= 3\left(\omega_{11}\frac{\partial}{\partial\tau_1} + \omega_{21}\frac{\partial}{\partial\tau_2}\right)\mathbf{x}_2 + 3\left(\omega_{12}\frac{\partial}{\partial\tau_1} + \omega_{22}\frac{\partial}{\partial\tau_2}\right)\mathbf{x}_1\\
&\quad - 3(\nu_1\mathbf{F}^0_{\mathbf{x}\nu} + \eta_1\mathbf{F}^0_{\mathbf{x}\eta})\mathbf{x}_2 - 3(\nu_2\mathbf{F}^0_{\mathbf{x}\nu} + \eta_2\mathbf{F}^0_{\mathbf{x}\eta})\mathbf{x}_1 - 3(\nu_1\mathbf{F}^0_{\mathbf{xx}\nu} + \eta_1\mathbf{F}^0_{\mathbf{xx}\eta})\mathbf{x}_1^2\\
&\quad - 3\mathbf{F}^0_{\mathbf{xx}}\mathbf{x}_1\mathbf{x}_2 - 3(\nu_1^2\mathbf{F}^0_{\mathbf{x}\nu\nu} + 2\nu_1\eta_1\mathbf{F}^0_{\mathbf{x}\nu\eta} + \eta_1^2\mathbf{F}^0_{\mathbf{x}\eta\eta})\mathbf{x}_1 - \mathbf{F}^0_{\mathbf{xxx}}\mathbf{x}_1^3,
\end{aligned} \tag{13}$$

where

$$L_0 := -\omega_{10}\frac{\partial}{\partial\tau_1} - \omega_{20}\frac{\partial}{\partial\tau_2} + \mathbf{F}^0_{\mathbf{x}}. \tag{14}$$

Moreover, from Equations (11), the periodicity conditions at the k-th order are similarly obtained

$$\mathbf{x}_k(\tau_1 + 2\pi, \tau_2 + 2\pi) = \mathbf{x}_k(\tau_1, \tau_2). \tag{15}$$

In previous equations the apex 0 indicates that the related quantity is calculated at $\varepsilon = 0$, while the subscripts $\mathbf{x}, \nu, \eta$ denote partial differentiation. All the derivatives of **F** with respect to the parameters ν and η only, have been posed equal to zero so that Equations (13) admits the trivial solution $\mathbf{x}_k = 0, \forall(k, \nu, \eta)$.

The non-decaying solution of (13_1) (i.e., the generating solution of the perturbative process) is

$$\mathbf{x}_1 = A_1\mathbf{u}_1 e^{i\tau_1} + A_2\mathbf{u}_2 e^{i\tau_2} + \text{c.c.}, \tag{16}$$

where $A_j = 1/2 a_j \exp(i\phi_j)$ $(j = 1, 2)$ is a complex constant, with real amplitude a_j and phase ϕ_j, "c.c." stands for the complex conjugate of preceding terms and $\mathbf{u}_j$ $(j = 1, 2)$ is the right eigenvector of $\mathbf{F}^0_{\mathbf{x}}$, associated with the eigenvalue $i\omega_{j0}$ (Equation (3)). Substitution of (16) in (13_2) leads to

$$\begin{aligned}
L_0\mathbf{x}_2 &= 2(i\omega_{11} - \nu_1\mathbf{F}^0_{\mathbf{x}\nu} - \eta_1\mathbf{F}^0_{\mathbf{x}\eta})A_1\mathbf{u}_1 e^{i\tau_1} + 2(i\omega_{21} - \nu_1\mathbf{F}^0_{\mathbf{x}\nu} - \eta_1\mathbf{F}^0_{\mathbf{x}\eta})A_2\mathbf{u}_2 e^{i\tau_2}\\
&\quad - A_1^2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1^2 e^{i2\tau_1} - A_2^2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_2^2 e^{i2\tau_2} - 2A_1A_2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1\mathbf{u}_2 e^{i(\tau_1+\tau_2)} - A_1\bar{A}_1\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1\bar{\mathbf{u}}_1\\
&\quad - A_2\bar{A}_2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_2\bar{\mathbf{u}}_2 - 2A_1\bar{A}_2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1\bar{\mathbf{u}}_2 e^{i(\tau_1-\tau_2)} + \text{c.c.}
\end{aligned} \tag{17}$$

The solvability of Equation (17) requires the coefficients of the resonant terms to be orthogonal to the left eigenvector $\mathbf{v}_j$ of $\mathbf{F}^0_{\mathbf{x}}$ associated with $i\omega_{j0}$ (see Appendix A), thus obtaining,

$$A_j\mathbf{v}_j^H(i\omega_{j1} - \nu_1\mathbf{F}^0_{\mathbf{x}\nu} - \eta_1\mathbf{F}^0_{\mathbf{x}\eta})\mathbf{u}_j = 0, \quad (j = 1, 2). \tag{18}$$

By separating real and imaginary parts and using $\mathbf{v}_j^H \mathbf{u}_j = 1$, it follows that

$$\begin{bmatrix} a_1\alpha_{1\nu} & a_1\alpha_{1\eta} & 0 & 0 \\ a_2\alpha_{2\nu} & a_2\alpha_{2\eta} & 0 & 0 \\ a_1\omega_{1\nu} & a_1\omega_{1\eta} & -a_1 & 0 \\ a_2\omega_{2\nu} & a_2\omega_{2\eta} & 0 & -a_2 \end{bmatrix} \begin{Bmatrix} \nu_1 \\ \eta_1 \\ \omega_{11} \\ \omega_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{19}$$

where coefficients $\alpha_{j\nu}, \alpha_{j\eta}, \omega_{j\nu}, \omega_{\eta}$ ($j = 1, 2$) are given in Appendix B, where it is shown that they are the partial derivatives of α_j and ω_j evaluated at $\varepsilon = 0$. Equations (19) admit the trivial solution

$$\nu_1 = \eta_1 = \omega_{11} = \omega_{21} = 0. \tag{20}$$

By solving Equation (17), and omitting the complementary function,

$$\begin{aligned} \mathbf{x}_2 \;=\; & A_1^2 \mathbf{z}_{11} e^{i2\tau_1} + A_2^2 \mathbf{z}_{22} e^{i2\tau_2} + A_1 \bar{A}_1 \mathbf{z}_{1\bar{1}} + A_2 \bar{A}_2 \mathbf{z}_{2\bar{2}} \\ & + 2A_1 A_2 \mathbf{z}_{12} e^{i(\tau_1+\tau_2)} + 2A_1 \bar{A}_2 \mathbf{z}_{1\bar{2}} e^{i(\tau_1-\tau_2)} + \text{c.c.} \end{aligned} \tag{21}$$

is found, where the $\mathbf{z}_{rs}$'s and $\mathbf{z}_{r\bar{s}}$'s ($r, s = 1, 2$) are solutions of the non-singular algebraic problems

$$\begin{aligned} &(ip\omega_{10}\mathbf{E} + iq\omega_{20}\mathbf{E} - \mathbf{F}_{\mathbf{x}}^0)\mathbf{z}_{rs} = \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_r \mathbf{u}_s, \\ &(ip\omega_{10}\mathbf{E} - iq\omega_{20}\mathbf{E} - \mathbf{F}_{\mathbf{x}}^0)\mathbf{z}_{r\bar{s}} = \mathbf{F}_{\mathbf{xx}}^0 \mathbf{u}_r \bar{\mathbf{u}}_s, \end{aligned} \tag{22}$$

where p and q are the real coefficients of τ_1 and τ_2, respectively, of the associated exponential functions in Equation (21). Moreover, the following properties hold:

$$\mathbf{z}_{sr} = \mathbf{z}_{rs}, \bar{\mathbf{z}}_{r\bar{s}} = \mathbf{z}_{\bar{r}s}.$$

Taking into account Equations (20), substitution of Equations (16) and (21) in (13_3) leads to

$$\begin{aligned} L_0 \mathbf{x}_3 \;=\; & 3(i\omega_{12} - \nu_2 \mathbf{F}_{\mathbf{x}\nu}^0 - \eta_2 \mathbf{F}_{\mathbf{x}\eta}^0) A_1 \mathbf{u}_1 e^{i\tau_1} \\ & - 3A_1^2 \bar{A}_1 [\mathbf{F}_{\mathbf{xx}}^0 (2\mathbf{u}_1 \mathbf{z}_{1\bar{1}} + \mathbf{z}_{11} \bar{\mathbf{u}}_1) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_1^2 \bar{\mathbf{u}}_1] e^{i\tau_1} \\ & - 6A_1 A_2 \bar{A}_2 [\mathbf{F}_{\mathbf{xx}}^0 (\mathbf{u}_1 \mathbf{z}_{2\bar{2}} + \mathbf{u}_2 \mathbf{z}_{1\bar{2}} + \bar{\mathbf{u}}_2 \mathbf{z}_{12}) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_1 \mathbf{u}_2 \bar{\mathbf{u}}_2] e^{i\tau_1} \\ & + 3(i\omega_{22} - \nu_2 \mathbf{F}_{\mathbf{x}\nu}^0 - \eta_2 \mathbf{F}_{\mathbf{x}\eta}^0) A_2 \mathbf{u}_2 e^{i\tau_2} \\ & - 3A_2^2 \bar{A}_2 [\mathbf{F}_{\mathbf{xx}}^0 (2\mathbf{u}_2 \mathbf{z}_{2\bar{2}} + \mathbf{z}_{22} \bar{\mathbf{u}}_2) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_2^2 \bar{\mathbf{u}}_2] e^{i\tau_2} \\ & - 6A_2 A_1 \bar{A}_1 [\mathbf{F}_{\mathbf{xx}}^0 (\mathbf{u}_2 \mathbf{z}_{1\bar{1}} + \mathbf{u}_1 \mathbf{z}_{2\bar{1}} + \bar{\mathbf{u}}_1 \mathbf{z}_{12}) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_1 \bar{\mathbf{u}}_1 \mathbf{u}_2] e^{i\tau_2} \\ & + \text{c.c.} + \text{NST}, \end{aligned} \tag{23}$$

where "NST" stands for non-secular terms. Solvability conditions read

$$\begin{bmatrix} a_1\alpha_{1\nu} & a_1\alpha_{1\eta} & 0 & 0 \\ a_2\alpha_{2\nu} & a_2\alpha_{2\eta} & 0 & 0 \\ a_1\omega_{1\nu} & a_1\omega_{1\eta} & -a_1 & 0 \\ a_2\omega_{2\nu} & a_2\omega_{2\eta} & 0 & -a_2 \end{bmatrix} \begin{Bmatrix} \nu_2 \\ \eta_2 \\ \omega_{12} \\ \omega_{22} \end{Bmatrix} = -2 \begin{bmatrix} a_1 R_{111} & a_1 R_{122} \\ a_2 R_{112} & a_2 R_{222} \\ a_1 I_{111} & a_1 I_{122} \\ a_2 I_{112} & a_2 I_{222} \end{bmatrix} \begin{Bmatrix} a_1^2 \\ a_2^2 \end{Bmatrix}, \tag{24}$$

where the coefficients R's and I's are given in Appendix C. Under transversality condition (6), Equations (24) make it possible to determine the quantities $\nu_2, \eta_2, \omega_{j2}$ ($j = 1, 2$) as functions of the amplitudes a_j. Two classes of solutions are found.

1. Single-mode (or periodic) solutions, with $(a_1 = a_{1P} \neq 0, a_2 = 0)$ or $(a_1 = 0, a_2 = a_{2P} \neq 0)$:

$$\nu_2 = -\frac{2R_{jjj}}{\alpha_{j\nu}} a_{jP}^2 - \frac{\alpha_{j\eta}}{\alpha_{j\nu}} \eta_2, \quad (j = 1, 2)$$

$$\omega_{j2} = 2\left(I_{jjj} - \frac{\omega_{j\nu}}{\alpha_{j\nu}} R_{jjj}\right) a_{jP}^2 + \left(\omega_{j\eta} - \omega_{j\nu}\frac{\alpha_{j\eta}}{\alpha_{j\nu}}\right)\eta_2, \tag{25}$$

where it has been assumed that $\alpha_{j\nu} \neq 0$. If $\alpha_{j\nu} = 0$, due to condition (6), $\alpha_{j\eta} \neq 0$; therefore, Equations (24_1), (24_2) can be solved with respect to η_2.

2. Mixed-mode (or quasi-periodic) solutions, with $(a_1 = a_{1Q} \neq 0, a_2 = a_{2Q} \neq 0)$:

$$\nu_2 = 2\frac{(\alpha_{1\eta}R_{112} - \alpha_{2\eta}R_{111})a_{1Q}^2 + (\alpha_{1\eta}R_{222} - \alpha_{2\eta}R_{122})a_{2Q}^2}{\alpha_{1\nu}\alpha_{2\eta} - \alpha_{2\nu}\alpha_{1\eta}},$$

$$\eta_2 = 2\frac{(\alpha_{2\nu}R_{111} - \alpha_{1\nu}R_{112})a_{1Q}^2 + (\alpha_{2\nu}R_{122} - \alpha_{1\nu}R_{222})a_{2Q}^2}{\alpha_{1\nu}\alpha_{2\eta} - \alpha_{2\nu}\alpha_{1\eta}}. \tag{26}$$

By substituting Equations (26) into ($24_{3,4}$) the frequency corrections ω_{j2} are obtained as functions of a_{jQ}.

It can be easily checked that, by solving perturbation equations of higher orders, it follows

$$\omega_{1h} = \omega_{2h} = \nu_h = \eta_h = 0, \quad h = 1, 3, 5, \ldots \tag{27}$$

Therefore, the lower-order approximation of the steady-state solutions is

$$\begin{cases} \mathbf{x} = \sum_{j=1}^{2} \varepsilon a_j \,\mathrm{Re}\, \mathbf{u}_j \cos\left(\omega_{j0} + \frac{\varepsilon^2}{2}\omega_{j2} t + \phi_j\right) \\ \qquad - \varepsilon a_j \,\mathrm{Im}\, \mathbf{u}_j \sin\left(\omega_{j0} + \frac{\varepsilon^2}{2}\omega_{j2} t + \phi_j\right) + O(\varepsilon^2) \\ \nu = \frac{\varepsilon^2}{2}\nu_2 + O(\varepsilon^4) \\ \eta = \frac{\varepsilon^2}{2}\eta_2 + O(\varepsilon^4), \end{cases} \tag{28}$$

where ϕ_j are arbitrary initial phases. By fixing a_1 and a_2 and varying ε, a straight line is described on the (ν, η)-plane, on which the true amplitudes εa_j linearly increase.

4. The Multiple Scale Method

A monoparametric family of solutions of the type

$$\begin{cases} \mathbf{x} = \mathbf{x}(\varepsilon, t_0, t_1, \ldots) \\ \nu = \nu(\varepsilon) \\ \eta = \eta(\varepsilon), \end{cases} \tag{29}$$

where $t_0 = t$, $t_1 = \varepsilon t, \ldots, t_k = (\varepsilon^k/k!)t$ are independent temporal scales, is sought. Under hypotheses of regularity, Equations (29) are expressed in MacLaurin series as

$$\begin{Bmatrix} \mathbf{x} \\ \nu \\ \eta \end{Bmatrix} = \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \begin{Bmatrix} \mathbf{x}_k \\ \nu_k \\ \eta_k \end{Bmatrix}, \tag{30}$$

where $\mathbf{x}_k = \mathbf{x}_k(t_0, t_1, \ldots)$ and $\varepsilon = 0$ selects the bifurcation point $\mathbf{O}$. The time derivative is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} = d_0 + \varepsilon d_1 + \frac{\varepsilon^2}{2!}\, d_2 + \cdots + \frac{\varepsilon^k}{k!}\, d_k + \cdots, \tag{31}$$

where $d_k = \partial/\partial t_k$. Guided by the results obtained in the previous section, the odd terms of the control parameter expansion are assumed to be zero and only temporal scales of even order are considered. The following perturbation equations up to order three are obtained:

$$\begin{aligned}
(d_0\mathbf{E} - \mathbf{F}^0_{\mathbf{x}})\mathbf{x}_1 &= 0\\
(d_0\mathbf{E} - \mathbf{F}^0_{\mathbf{x}})\mathbf{x}_2 &= \mathbf{F}^0_{\mathbf{xx}}\mathbf{x}_1^2\\
(d_0\mathbf{E} - \mathbf{F}^0_{\mathbf{x}})\mathbf{x}_3 &= 3(\nu_2\mathbf{F}^0_{\mathbf{x}\nu} + \eta_2\mathbf{F}^0_{\mathbf{x}\eta})\mathbf{x}_1 + 3\mathbf{F}^0_{\mathbf{xx}}\mathbf{x}_1\mathbf{x}_2 + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{x}_1^3 - 3\, d_2\mathbf{x}_1
\end{aligned} \tag{32}$$

The non-decaying solution of (32_1) is

$$\mathbf{x}_1 = A_1(t_2, t_4, \ldots)\, \mathbf{u}_1 e^{i\omega_{10}t_0} + A_2(t_2, t_4, \ldots)\mathbf{u}_2 e^{i\omega_{20}t_0} + \text{c.c.}, \tag{33}$$

where $A_j(t_2, t_4, \ldots) = 1/2a_j(t_2, t_4, \ldots)\exp[i\phi_j(t_2, t_4, \ldots)]$ $(j = 1, 2)$ is a function of the slow time scales, with real amplitude a_j and phase ϕ_j. Substitution of Equation (33) in (32_2) leads to

$$\begin{aligned}
(d_0\, \mathbf{E} - \mathbf{F}^0_{\mathbf{x}})\mathbf{x}_2 = \; & A_1^2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1^2 e^{i2\omega_{10}t_0} + A_2^2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_2^2 e^{i2\omega_{20}t_0} + 2A_1A_2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1\mathbf{u}_2 e^{i(\omega_{10}+\omega_{20})t_0}\\
& + A_1\bar{A}_1\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1\bar{\mathbf{u}}_1 + A_2\bar{A}_2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_2\bar{\mathbf{u}}_2\\
& + 2A_1\bar{A}_2\mathbf{F}^0_{\mathbf{xx}}\mathbf{u}_1\bar{\mathbf{u}}_2 e^{i(\omega_{10}-\omega_{20})t_0} + \text{c.c.}
\end{aligned} \tag{34}$$

Equation (34) does not contain resonant terms; for this reason, temporal scales and control parameter derivatives of odd order are unnecessary. By solving Equation (34)

$$\begin{aligned}
\mathbf{x}_2 = \; & A_1^2\mathbf{z}_{11}e^{i2\omega_{10}t_0} + A_2^2\mathbf{z}_{22}e^{i2\omega_{20}t_0} + A_1\bar{A}_1\mathbf{z}_{1\bar{1}} + A_2\bar{A}_2\mathbf{z}_{2\bar{2}}\\
& + 2A_1A_2\mathbf{z}_{12}e^{i(\omega_{10}+\omega_{20})t_0} + 2A_1\bar{A}_2\mathbf{z}_{1\bar{2}}e^{i(\omega_{10}-\omega_{20})t_0} + \text{c.c.}
\end{aligned} \tag{35}$$

is obtained, where $\mathbf{z}_{rs}$ has the same meaning as in the previous section. By substituting Equations (33) and (35) in (32_3) it follows that

$$\begin{aligned}
(d_0 - \mathbf{F}^0_{\mathbf{x}})\mathbf{x}_3 = \; & 3(-d_2 + \nu_2\mathbf{F}^0_{\mathbf{x}\nu} + \eta_2\mathbf{F}^0_{\mathbf{x}\eta})A_1\mathbf{u}_1 e^{i\omega_{10}t_0}\\
& + 3A_1^2\bar{A}_1[\mathbf{F}^0_{\mathbf{xx}}(2\mathbf{u}_1\mathbf{z}_{1\bar{1}} + \mathbf{z}_{11}\bar{\mathbf{u}}_1) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_1^2\bar{\mathbf{u}}_1]e^{i\omega_{10}t_0}\\
& + 6A_1A_2\bar{A}_2[\mathbf{F}^0_{\mathbf{xx}}(\mathbf{u}_1\mathbf{z}_{2\bar{2}} + \mathbf{u}_2\mathbf{z}_{1\bar{2}} + \bar{\mathbf{u}}_2\mathbf{z}_{12}) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_1\mathbf{u}_2\bar{\mathbf{u}}_2]e^{i\omega_{10}t_0}\\
& + 3(-d_2 + \nu_2\mathbf{F}^0_{\mathbf{x}\nu} + \eta_2\mathbf{F}^0_{\mathbf{x}\eta})A_2\mathbf{u}_2 e^{i\omega_{20}t_0}\\
& + 3A_2^2\bar{A}_2[\mathbf{F}^0_{\mathbf{xx}}(2\mathbf{u}_2\mathbf{z}_{2\bar{2}} + \mathbf{z}_{22}\bar{\mathbf{u}}_2) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_2^2\bar{\mathbf{u}}_2]e^{i\omega_{20}t_0}\\
& + 6A_2A_1\bar{A}_1[\mathbf{F}^0_{\mathbf{xx}}(\mathbf{u}_2\mathbf{z}_{1\bar{1}} + \mathbf{u}_1\mathbf{z}_{2\bar{1}} + \bar{\mathbf{u}}_1\mathbf{z}_{12}) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_1\bar{\mathbf{u}}_1\mathbf{u}_2]e^{i\omega_{20}t_0}\\
& + \text{c.c.} + \text{NST}.
\end{aligned} \tag{36}$$

This equation contains terms that would lead to secular terms; to eliminate them, the orthogonality of the coefficients of the resonant terms to the critical left eigenvectors $\mathbf{v}_j$ has to be imposed, leading to

$$\begin{aligned} d_2A_1 \;=&\; \mathbf{v}_1^H\{(\nu_2\mathbf{F}^0_{\mathbf{x}\nu} + \eta_2\mathbf{F}^0_{\mathbf{x}\eta})A_1\mathbf{u}_1 + A_1^2\bar{A}_1[\mathbf{F}^0_{\mathbf{xx}}(2\mathbf{u}_1\mathbf{z}_{1\bar{1}} + \mathbf{z}_{11}\bar{\mathbf{u}}_1) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_1^2\bar{\mathbf{u}}_1] \\ &+ 2A_1A_2\bar{A}_2[\mathbf{F}^0_{\mathbf{xx}}(\mathbf{u}_1\mathbf{z}_{2\bar{2}} + \mathbf{u}_2\mathbf{z}_{1\bar{2}} + \bar{\mathbf{u}}_2\mathbf{z}_{12}) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_1\mathbf{u}_2\bar{\mathbf{u}}_2]\} \\ d_2A_2 \;=&\; \mathbf{v}_2^H\{(\nu_2\mathbf{F}^0_{\mathbf{x}\nu} + \eta_2\mathbf{F}^0_{\mathbf{x}\eta})A_2\mathbf{u}_2 + A_2^2\bar{A}_2[\mathbf{F}^0_{\mathbf{xx}}(2\mathbf{u}_2\mathbf{z}_{2\bar{2}} + \mathbf{z}_{22}\bar{\mathbf{u}}_2) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_2^2\bar{\mathbf{u}}_2] \\ &+ 2A_2A_1\bar{A}_1[\mathbf{F}^0_{\mathbf{xx}}(\mathbf{u}_2\mathbf{z}_{1\bar{1}} + \mathbf{u}_1\mathbf{z}_{2\bar{1}} + \bar{\mathbf{u}}_1\mathbf{z}_{12}) + \mathbf{F}^0_{\mathbf{xxx}}\mathbf{u}_1\bar{\mathbf{u}}_1\mathbf{u}_2]\}. \end{aligned} \tag{37}$$

By separating real and imaginary parts of the solvability conditions, the amplitudes and phases modulation equations on the t_2-scale are obtained. By coming back to the t scale and reabsorbing the parameter ε they read:

$$\begin{cases} \dot{a}_1 = (\alpha_{1\nu}\nu + \alpha_{1\eta}\eta)a_1 + R_{111}\, a_1^3 + R_{122}\, a_1a_2^2 + O(|a_1|^5 + |a_2|^5) \\ \dot{a}_2 = (\alpha_{2\nu}\nu + \alpha_{2\eta}\eta)a_2 + R_{112}\, a_1^2a_2 + R_{222}\, a_2^3 + O(|a_1|^5 + |a_2|^5), \end{cases} \tag{38}$$

$$\begin{cases} \dot{\phi}_1 = (\omega_{1\nu}\nu + \omega_{1\eta}\eta) + I_{111}a_1^2 + I_{122}\, a_2^2 + O(|a_1|^4 + |a_2|^4) \\ \dot{\phi}_2 = (\omega_{2\nu}\nu + \omega_{2\eta}\eta) + I_{112}\, a_1^2 + I_{222}\, a_2^2 + O(|a_1|^4 + |a_2|^4). \end{cases} \tag{39}$$

The amplitude modulation equations (38) are uncoupled from the phase modulation equations (39) and can be studied, for example, by phase techniques. Since they are invariant under the transformations $a_1 \to -a_1$ and $a_2 \to -a_2$, is sufficient to consider positive a_1 and a_2 only. Equations (38) constitute the bifurcation equations in standard normal form for a non-resonant double Hopf bifurcation [3]. Their complete classification, containing twelve different cases, is shown in Guckenheimer and Holmes [2]. Constant solutions of Equations (38) are determined by setting $\dot{a}_1 = \dot{a}_2 = 0$. These solutions correspond to one-frequency periodic motion or two-frequency quasi-periodic motions of the original system, Equation (1). They coincide with the steady-state solutions (Equations (25, 26)) found with the *LPM*, as it can be seen by multiplying Equation (24) by ε^3 and reabsorbing ε in the amplitudes and in the control parameters. Equations (38) make it possible to detect the stability of periodic and quasi-periodic motions by analyzing the stability of equilibrium points.

5. Two-Rod System under Aerodynamic Excitation

In this section, the procedure described above is applied to the structure illustrated in Figure 2a. The structure consists of two vertical rigid rods of length l, constrained by two visco-elastic hinges of torsional rigidity k_H and damping coefficient $c_H > 0$. The rods are joined at their ends by a visco-elastic device, that either dissipates or puts energy into the system, whose rigidity is k_D and damping coefficient $c_D \neq 0$. The structure is loaded by a fluid flow of uniform velocity U in a direction orthogonal to the plane of the motion. The exciting mechanism is such that an aerodynamic force, depending on U and on the shape of the cross-section, arises in the plane of motion leading to possible Hopf bifurcations (galloping instability). By assuming the rotations q_1 and q_2 as Lagrangian parameters (Figure 2b), applying the quasi-static theory for aerodynamic forces and expanding nonlinearities up to third order (see, e.g., [22]), the following non-dimensional equations of motion are found:

$$\begin{cases} \ddot{x} + (\xi_x - \xi_a u)\dot{x} + x = 2\beta xy^2 + 4\xi_a\eta xy\dot{y} + c_2(\dot{x}^2 + \dot{y}^2) + \frac{c_3}{u}(\dot{x}^3 + 3\dot{x}\dot{y}^2) \\ \ddot{y} + (\xi_y - \xi_a u)\dot{y} + \omega_0^2 y = 2\beta x^2y + \frac{4}{3}\beta y^3 + 4\xi_a\eta(x^2\dot{y} + xy\dot{x} + y^2\dot{y}) + c_2\dot{x}\dot{y} \\ \qquad + \frac{c_3}{u}(\dot{y}^3 + 3\dot{x}^2\dot{y}). \end{cases} \tag{40}$$

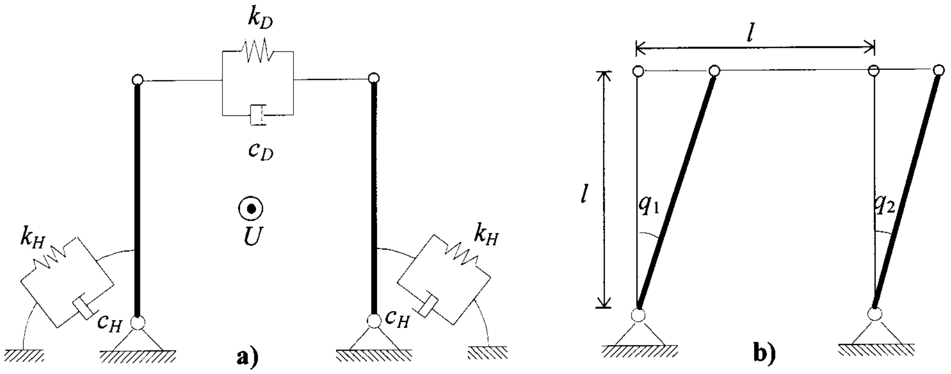

Figure 2. (a) System of two rigid rods under aerodynamic excitation; (b) Lagrangian parameters.

In Equations (40), modal co-ordinates have been used to uncouple the linear part; namely

$$x = \frac{1}{2}(q_1 + q_2); \quad y = \frac{1}{2}(q_1 - q_2) \tag{41}$$

are the amplitude of the antisymmetric ($q_1 = 1, q_2 = 1$) and the symmetric ($q_1 = -1, q_2 = 1$) modes, respectively. Moreover, the following positions have been made:

$$\begin{aligned} u &= \frac{\rho b}{m\omega_x}U; \quad \omega_x^2 = \frac{3k_H}{ml^3}; \quad \beta = \frac{2k_D l^2}{k_H}; \quad \omega_0^2 = 1 + \beta; \\ \xi_x &= \frac{3c_H}{ml^3\omega_x}; \quad \xi_y = \xi_x + \xi_a\eta; \quad \xi_a = \frac{|c_d + c_l'|}{2}; \quad \eta = \frac{3c_D}{ml\omega_x}\frac{1}{\xi_a} \\ c_2 &= \frac{3}{16}\left(\frac{\rho bl}{m}\right)(c_l'' + c_l + 2c_d'); \quad c_3 = -\frac{1}{20}\left(\frac{\rho bl}{m}\right)^2 (c_l''' + c_l' + 3c_d'' + 3c_d), \end{aligned} \tag{42}$$

where ρ is the air density; b is an appropriate characteristic length of the cross-section of the rods; m is the mass per unit length of the rods; ξ_x and ξ_y are the modal structural dampings; ξ_a is the aerodynamic modal damping; c_d and c_l are the drag and lift non-dimensional coefficients, respectively; c_d', c_d'', c_l', c_l'' and c_l''' are their derivatives with respect to the attack angle; $\omega_0 = \omega_y/\omega_x$ is the ratio between the two undamped frequencies, assumed to be incommensurable; u is the non-dimensional wind velocity, which is assumed as a control parameter; η is the non-dimensional damping of the visco-elastic device, assumed as a control splitting parameter. In Equations (40) the dot denotes differentiation with respect to non-dimensional time $\tau = \omega_x t$.

Equations (40) are in second-order local form. To apply the theory developed above, they should be expressed in form (1); however, as an example, perturbative methods will be applied directly to them.

5.1. Stability Analysis of the Trivial Path

The trivial equilibrium position $x = y = 0$ loses its stability through a Hopf bifurcation when the coefficient of the velocities $\dot{x}$ and $\dot{y}$ in Equations (40) vanish. This occurs for two critical wind velocities, $u_{c1} = \xi_x/\xi_a$ and $u_{c2} = \xi_x/\xi_a + \eta$, which trigger an antisymmetrical and a symmetrical galloping mode, respectively. By posing $\nu := u - u_{c1}$, and assuming

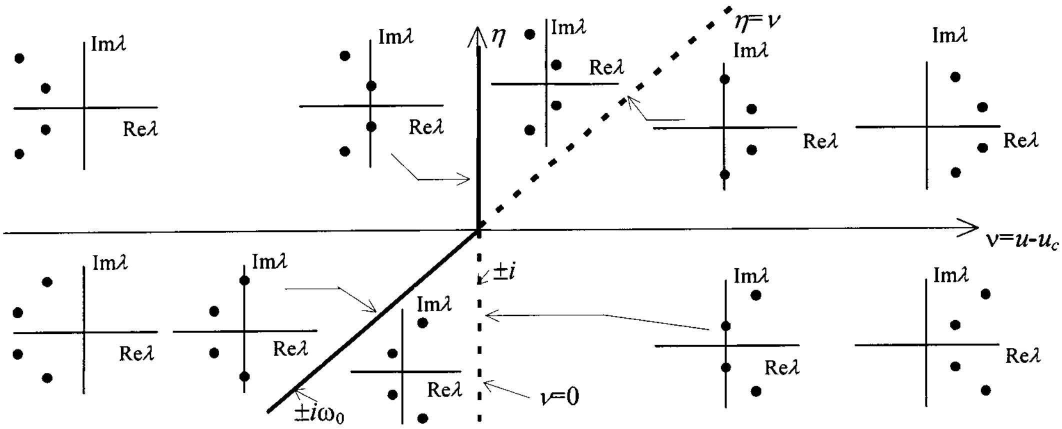

Figure 3. Stability boundaries in the parameter space for the two-rod system.

ν and η as control parameters, the stability diagram of Figure 3 is obtained. In this figure the two boundary stability curves are determined by the equations $\nu = 0$ and $\nu = \eta$. For positive damping η and increasing ν, two successive Hopf bifurcations, associated with the antisymmetric and the symmetric modes, occur; for negative damping η the two bifurcations occur in the reverse order.

5.2. Perturbation Analysis

When the *MSM* is applied to Equations (40), the following set of perturbation equations is drawn:

$$\begin{cases} (d_0^2+1)x_1 = 0 \\ (d_0^2+\omega_0^2)y_1 = 0 \end{cases} \tag{43}$$

$$\begin{cases} (d_0^2+1)x_2 = 2c_2((d_0x_1)^2+(d_0y_1)^2) \\ (d_0^2+\omega_0^2)y_2 = 2c_2\, d_0x_1\, d_0y_1 \end{cases} \tag{44}$$

$$\begin{cases} (d_0^2+1)x_3 = 3\nu_2\xi_a\, d_0x_1 - 6\, d_0\, d_2x_1 + 6\frac{c_3}{u_c}\, d_0x_1(d_0y_1)^2 + 6\frac{c_3}{u_c}(d_0x_1)^3 \\ \qquad + 6c_2(d_0x_1\, d_0x_2 + d_0y_1\, d_0y_2) + 12\beta x_1y_1^2 \\ (d_0^2+\omega_0^2)y_3 = 3(\nu_2-\eta_2)\xi_a\, d_0y_1 - 6\, d_0\, d_2y_1 + 6\frac{c_3}{u_c}(d_0x_1)^2\, d_0y_1 \\ \qquad + 3c_2(d_0x_1\, d_0y_2 + d_0x_2\, d_0y_1) + 6\frac{c_3}{u_c}(d_0y_1)^3 \\ \qquad + 12\beta\, x_1^2y_1 + 8\beta\, y_1^3 \end{cases} \tag{45}$$

in which $d_h := \partial/\partial\, \tau_h$ and $d_0^2 := \partial^2/\partial^2\, \tau_0$ with $h = 0, 2$. The general solution of (43) is

$$\begin{cases} x_1 = A_1(\tau_2)e^{it_0} + \text{c.c.} \\ y_1 = A_2(\tau_2)e^{i\omega_0t_0} + \text{c.c.} \end{cases} \tag{46}$$

By substituting Equations (46) in Equations (44) and solving it,

$$\begin{cases} x_2 = \frac{2}{3}c_2A_1^2e^{i2t_0} - 2c_2\bar{A}_1A_1 + \frac{2}{3}c_2A_2^2e^{i2\omega_0t_0} + c_2\omega_0^2\bar{A}_2A_2 + \text{c.c.} \\ y_2 = 2c_2\frac{\omega_0}{2\omega_0-1}A_1\bar{A}_2e^{i(1-\omega_0)t_0} + 2c_2\frac{\omega_0}{2\omega_0+1}A_1A_2e^{i(1+\omega_0)t_0} + \text{c.c.} \end{cases} \tag{47}$$

are found. Elimination of the resonant terms from Equations (45) leads to two ordinary differential equations in the amplitudes $A_j(\tau_2)$ $(j = 1, 2)$. After expressing A_j in polar form (i.e. $A_j = (1/2)\exp(i\phi_j)$), separating real and imaginary parts and absorbing the parameter ε, the following equations are obtained

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\xi_a \nu a_1 + \frac{1}{4}\frac{c_3}{u_c}\omega_0^2 a_1 a_2^2 + \frac{3}{8}\frac{c_3}{u_c} a_1^3 \\ \dot{a}_2 = \frac{1}{2}\xi_a(\nu - \eta)a_2 + \frac{1}{4}\frac{c_3}{u_c}a_1^2 a_2 + \frac{3}{8}\frac{c_3}{u_c}\omega_0^2 a_2^3, \end{cases} \tag{48}$$

$$\begin{cases} \dot{\phi}_1 = -\frac{1}{6}c_2^2 a_1^2 - \left(\frac{2\omega_0^2-1}{4\omega_0^2-1}\frac{\omega_0^2 c_2^2}{2} + \frac{\beta}{2}\right) a_2^2 \\ \dot{\phi}_2 = -\frac{1}{4}\left(c_2^2 \frac{\omega_0}{4\omega_0^2-1} + 2\frac{\beta}{\omega_0}\right) a_1^2 - \frac{1}{4}\left(\frac{1}{3}\omega_0 c_2^2 + 2\frac{\beta}{\omega_0}\right) a_2^2. \end{cases} \tag{49}$$

The amplitude Equations (48) are uncoupled from the phases ϕ_1 and ϕ_2, which can be evaluated successively. In order to draw the bifurcation diagram it is necessary to determine the steady-state solutions of the dynamical system (48) and to perform stability analysis.

5.3. Existence and Classification of Steady-State Solutions

Equations (48) admit the trivial solution $a_{1T} = a_{2T} = 0$. Non-trivial steady-state solutions with one or two non-vanishing components are sought. If $a_2 = 0$, Equation (47_2) is identically satisfied, while (47_1) and (48_1) yield to

$$a_{1P}^2 = -\frac{4}{3}\frac{u_c}{c_3}\xi_a \nu; \quad \phi_{1P} = \frac{2}{9}\frac{c_2^2 u_c}{c_3}\xi_a \nu\tau + \phi_{10}, \tag{50}$$

respectively. Similarly, if $a_1 = 0$, Equation (47_1) is identically satisfied, while Equations (47_2) and (48_2) yield to

$$a_{2P}^2 = \frac{4}{3}\frac{u_c}{c_3}\frac{\xi_a}{\omega_0^2}(\eta - \nu); \quad \phi_{2P} = \left(-\frac{1}{3}\left(\frac{1}{3}\omega_0 c_2^2 + 2\frac{\beta}{\omega_0}\right)\frac{u_c \xi_a}{c_3 \omega_0^2}(\eta - \nu)\right)\tau + \phi_{20}, \tag{51}$$

respectively. Both solutions (50) and (51) correspond to periodic responses of the original system (40), whose first-order approximations are given by Equations (46). It is apparent that, since a_j is real, solutions (50) and (51) exist only for certain ranges of the control parameters, depending on the sign of c_3. For example, if $c_3 < 0$ (as is the case with a square cross-section) solution (50) exists for $\nu > 0$ and solution (51) for $\eta - \nu < 0$.

Finally, if both a_1 and a_2 are different from zero, solution (50) gives

$$a_{1Q}^2 = -\frac{8}{5}\frac{\xi_a u_c}{c_3}\left(\frac{\nu}{2} + \eta\right); \quad a_{2Q}^2 = -\frac{12}{5}\frac{\xi_a u_c}{c_3 \omega_0^2}\left(\frac{\nu}{3} - \eta\right); \tag{52}$$

while the corresponding ϕ_{jQ} $(j = 1, 2)$ are obtained by direct substitution of (51) in (49). With reference again to a square cross-section, the domain of definition of solution (52) is $\nu/2 + \eta > 0$ and $\nu/3 - \eta > 0$. Since the frequencies of the two interacting modes are incommensurable, the resultant motion is quasi-periodic.

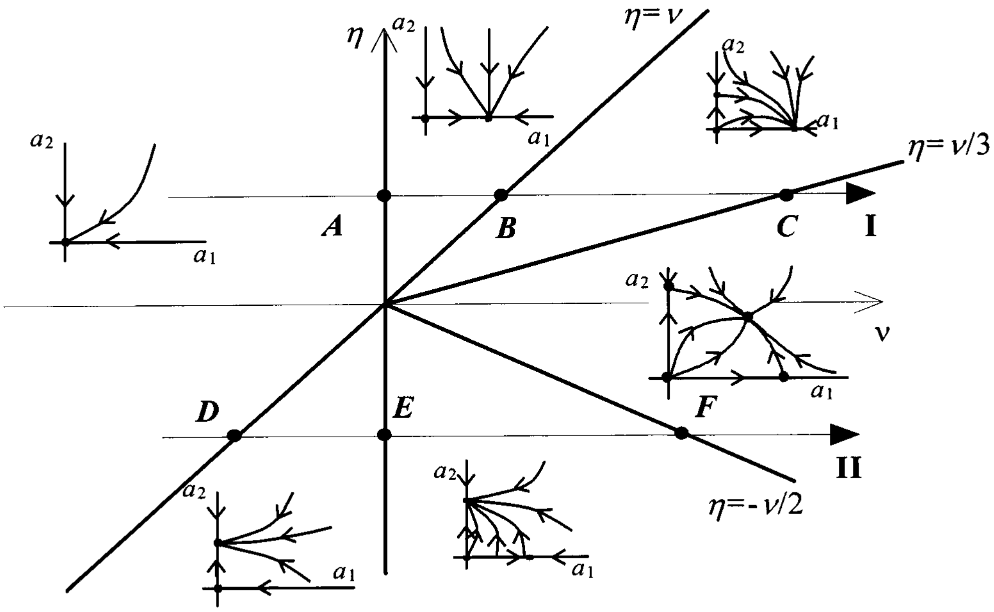

Figure 4. Bifurcation diagram in the (ν, η) parameter plane and phase portraits for the double rod system ($c_3 < 0$).

5.4. Stability and Bifurcation Analysis

Let a_{j0} be a steady-state solution to Equation (48) and δa_j a perturbation. The stability of a_{j0} depends on the evolution of the perturbed motion, governed by the variational equation

$$\left\{ \begin{array}{c} \delta\dot{a}_1 \\ \delta\dot{a}_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{\xi_a \nu}{2} + \frac{c_3}{4u_c}\left(\omega_0^2 a_{20}^2 + \frac{9}{2}a_{10}^2\right) & \frac{c_3\omega_0^2}{2u_c} a_{10}a_{20} \\ \frac{c_3}{2u_c} a_{10}a_{20} & \frac{\xi_a(\nu-\eta)}{2} + \frac{c_3}{4u_c}\left(\frac{9}{2}\omega_0^2 a_{20}^2 + a_{10}^2\right) \end{array} \right] \left\{ \begin{array}{c} \delta a_1 \\ \delta a_2 \end{array} \right\}, \quad (53)$$

where the matrix **J** of the coefficients is the Jacobian of the vectorial field defined by the right hand side of (48) and calculated on a_{j0}. In particular, **J** has a diagonal form for trivial solutions ($a_{10} = 0$, $a_{20} = 0$, $\mathbf{J} = \mathbf{J}_T$),

$$\mathbf{J}_T = \left[\begin{array}{cc} \frac{\xi_a}{2}\nu & 0 \\ 0 & \frac{\xi_a}{2}(\nu - \eta) \end{array} \right] \quad (54)$$

for antisymmetric periodic modes ($a_{10} = a_{1P}$, $a_{20} = 0$, $\mathbf{J} = \mathbf{J}_{AP}$) and for symmetric periodic modes ($a_{10} = 0$, $a_{20} = a_{2P}$, $\mathbf{J} = \mathbf{J}_{SP}$)

$$\mathbf{J}_{AP} = \left[\begin{array}{cc} -\xi_a \nu & 0 \\ 0 & \frac{\xi_a}{2}\left(\frac{\nu}{3} - \eta\right) \end{array} \right]; \quad \mathbf{J}_{SP} = \left[\begin{array}{cc} \frac{\xi_a}{3}\left(\eta + \frac{\nu}{2}\right) & 0 \\ 0 & \xi_a(\eta - \nu) \end{array} \right]. \quad (55)$$

On the other hand, for mixed modes, $\mathbf{J} = \mathbf{J}_Q$ is a full matrix; which trace and determinant are

$$\mathrm{tr}\,\mathbf{J}_Q = -\frac{3}{5}\xi_a(2\nu - \eta), \quad \det \mathbf{J}_Q = \frac{\xi_a^2}{5}(\nu + 2\eta)(\nu - 3\eta). \quad (56)$$

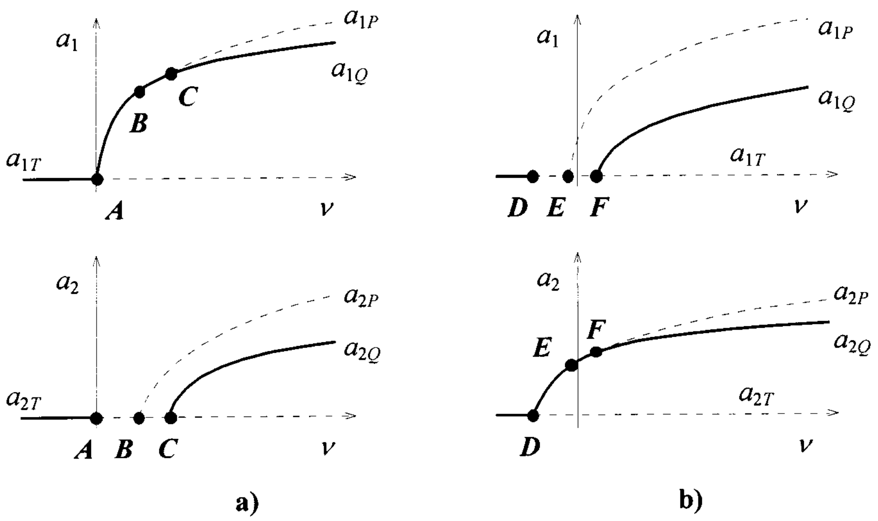

Figure 5. Bifurcated steady-state amplitudes vs. ν along: (a) arrow I; (b) arrow II in Figure 4.

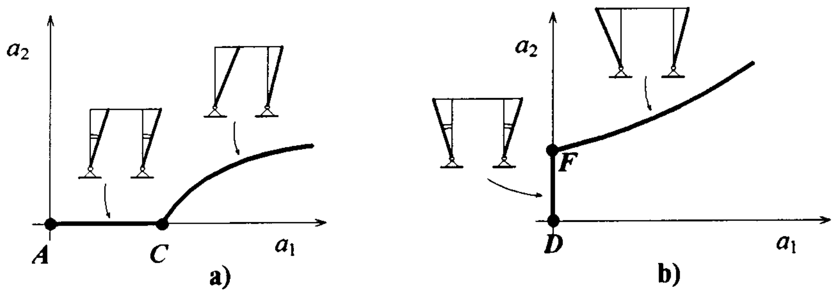

Figure 6. Stable equilibrium paths as ν is varied along: (a) arrow I; (b) arrow II in Figure 4.

Previous results (Equations (50) to (56)) have an effective geometric representation in the bifurcation diagram of Figure 4, obtained for $c_3 < 0$, in which phase-portraits are sketched for different regions of the control parameters plane. To illustrate the phenomenology, two representative paths, I and II, have been selected on the diagram. By moving along line I ($\eta > 0$) from the negative half-plane, the trivial equilibrium position looses its stability at point A after a pitchfork bifurcation and a stable antisymmetric solution arises. For increasing values of ν, a second static bifurcation occurs at point B and an unstable symmetric mode appears. Finally, if ν is further increased, the antisymmetric periodic solution bifurcates at point C in a quasiperiodic stable solution. An analogous discussion can be applied for the line II ($\eta < 0$).

The steady-state paths (Equations (50) to (51)) are plotted on the (a_j, ν)-plane in Figure 5. The stable paths are represented by solid lines, the unstable paths by dashed lines. At the marked points, bifurcations occur, as already described with reference to Figure 4. In particular, if the splitting parameter η vanishes, all the bifurcation points A to F coalesce and a unique stable steady-state quasi-periodic motion exists, directly bifurcating from the trivial state.

Stable steady-state paths are also represented on the (a_1, a_2)-plane (Figure 6) together with the associated deflection shapes. However, while in periodic motions the shapes are preserved, in quasi-periodic motions they change continuously.

If the aerodynamic coefficient $c_3 > 0$ (as is the case with rectangular sections with a large aspect ratio (e.g., 1:3), with wind acting on the short side [23]), the post-critical behavior is of a subcritical type and stable postcritical solutions do not exist.

6. Concluding Remarks

The Lindstedt–Poincaré Method (*LPM*) and the Multiple Scale perturbation Method (*MSM*) have been arranged to analyze non-resonant double (codimension two) Hopf bifurcations of a general two control parameter dynamical system. The following remarks can be made.

1. In the two-time version of the *LPM*, by expanding the two frequencies $\omega_j(\varepsilon)$ and the two control parameters $\mu_j(\varepsilon)$ in a series of a perturbation parameter ε, *linear* equations in the ε-derivatives of ω_j and μ_j are obtained at each step, in terms of the amplitudes a_i of the two interacting modes. Therefore, bifurcated paths are described in the form $\mu_j = \mu_j(a_i)$. In order ascertain stability of these paths, Floquet theory should be employed to study the variational equation.
2. In the *MSM*, amplitude and phase modulation equations identical to normal form equations are obtained. As they depend on parameters μ_j, to obtain bifurcated paths in the form $a_i = a_i(\mu_j)$, *nonlinear* equations in the amplitudes have to be solved. Stability analysis is then easily accomplished using the same modulation equations.
3. Both methods are straightforward and can be applied to any order to improve approximation. In addition, they furnish closed-form expressions for coefficients that could be used directly in applications, thus avoiding the need of developing the procedure each time.
4. An example has been developed in details, using the equations of motion in second order form. This approach requires a lighter computational burden with slightly formal differences in comparison with the general theory. Around the bifurcation point, the structure under analysis is a quasi-Hamiltonian linear system. Therefore, the critical eigenvectors are real and remarkable simplifications in the procedure are obtained. Similar problems have been addressed in technical literature in the analysis of free motions of two weakly coupled nonlinear oscillators [6, 24].
5. A comparative study of the *MSM* and the center manifold method (*CMM*) has been performed by Moroz [17] for a particular two-parameter system undergoing a Bogdanov–Takens bifurcation. In that analysis it is concluded that *CMM* requires less computational efforts than *MSM*, and there is less likelihood of omitting important nonlinear terms. However, such a conclusion is strongly related to the particular system under analysis (i.e., a nilpotent Jacobian matrix) and to the steps that are followed (the two methods are not applied to the same form of the original equation). The conclusion drawn here, which completely agrees with Nayfeh and Balachandran [6], refers instead to *general* systems, especially if they are characterized by large dimensions. In addition, the method presented here, is systematic and consistent, so that no possibilities arise of omitting any term.
6. The *LPM* and the *MSM* can be easily extended to analyze higher codimension bifurcation problems, whether static, dynamic or mixed. In addition, resonances can be accounted for, practically without any additional effort. Forthcoming papers will deal with this subject.

Appendix A: Solvability Conditions for Perturbation Equations (13)

At the k-th step of the perturbative process, the perturbation equation appears as follows

$$L_0\mathbf{x}_k(\tau_1,\tau_2) = \sum_{l,m} \mathbf{f}_{lm} e^{i(l\tau_1+m\tau_2)} + \text{c.c.}, \quad \mathbf{f}_{lm} \in C^n; \quad l,m \in Z; \quad l,m \leq k. \tag{A1}$$

By taking into account periodicity conditions (15), Equation (A1) admits the solution

$$\mathbf{x}_k(\tau_1,\tau_2) = \sum_{l,m} \mathbf{b}_{lm} e^{i(l\tau_1+m\tau_2)} + \text{c.c.} \quad \mathbf{b}_{lm} \in \mathcal{R}^n; \quad l,m \in Z; \quad l,m \leq k, \tag{A2}$$

where $\mathbf{b}_{lm}$ is the solution to

$$(-il\omega_{10} - im\omega_{20} + \mathbf{F}^0_{\mathbf{x}})\mathbf{b}_{lm} = \mathbf{f}_{lm}. \tag{A3}$$

Since the algebraic operator on the left hand side of Equation (A3) is singular for

$$l = \pm 1, m = 0; \quad l = 0, m = \pm 1, \tag{A4}$$

Equation (A3) admits solutions if and only if (solvability conditions)

$$\mathbf{v}_l^H \mathbf{f}_{10} = 0; \quad \mathbf{v}_2^H \mathbf{f}_{01} = 0. \tag{A5}$$

Since the vectors $\mathbf{f}_{-l-m}$ are complex conjugate of $\mathbf{f}_{lm}$, conditions relevant to $l = -1$ and $m = -1$ are automatically satisfied.

Appendix B: First-Order Derivatives of the Critical Eigenvalues

Coefficients appearing in solvability conditions (Equations (19), (24), (38), and (39)) are defined as follows:

$$\begin{aligned} \alpha_{j\nu} &= \text{Re}(\mathbf{v}_j^H \mathbf{F}^0_{\mathbf{x}\nu}\mathbf{u}_j), \qquad \omega_{j\nu} = \text{Im}(\mathbf{v}_j^H \mathbf{F}^0_{\mathbf{x}\nu}\mathbf{u}_j) \\ \alpha_{j\eta} &= \text{Re}(\mathbf{v}_j^H \mathbf{F}^0_{\mathbf{x}\eta}\mathbf{u}_j), \qquad \omega_{j\eta} = \text{Im}(\mathbf{v}_j^H \mathbf{F}^0_{\mathbf{x}\eta}\mathbf{u}_j) \end{aligned} \tag{B1}$$

Their meaning as first-order derivatives of the critical eigenvalues emerges from the stability analysis of the trivial path Γ. It is governed by the variational equation

$$\delta\dot{\mathbf{x}} = \mathbf{F}_{\mathbf{x}}(\mathbf{0},\nu,\eta)\delta\mathbf{x} \tag{B2}$$

from which the eigenvalue problem

$$\mathbf{F}_{\mathbf{x}}(\mathbf{0},\nu,\eta)\mathbf{w}(\nu,\eta) = \lambda(\nu,\eta)\mathbf{w}(\nu,\eta) \tag{B3}$$

follows. Under the spectral hypotheses assumed for the operator $\mathbf{F}_{\mathbf{x}}(\mathbf{0},\nu,\eta)$, (Section 2), the equilibrium path Γ is asymptotically stable if $\nu < 0$, and $\eta = 0$ and unstable if $\nu > 0$ and $\eta = 0$. In order to analyze the stability of Γ for arbitrary variations of both parameters, consider Equation (B2) calculated in a point of Γ near the double bifurcation point $\mathbf{O}$ (i.e. $\nu \ll 1, \eta \ll 1$). Under these hypotheses, Equation (B3) can be considered as a perturbation of the eigenvalue problem

$$\mathbf{F}^0_{\mathbf{x}}\mathbf{w}_0 = \lambda_0\mathbf{w}_0 \tag{B4}$$

in which $\mathbf{F}_{\mathbf{x}}^0 := \mathbf{F}_{\mathbf{x}}(\mathbf{0}, 0, 0)$, $\lambda_0 = \lambda(0,0)$ and $\mathbf{w}_0 = \mathbf{w}(0,0)$. Therefore, the eigenvalues $\lambda(\nu,\eta)$ and the eigenvectors $\mathbf{w}(\nu,\eta)$ of problem (B3) can be determined using a perturbative procedure. By assuming for $\lambda, \mathbf{w}, \nu$ and η the expressions

$$\begin{Bmatrix} \lambda \\ \mathbf{w} \\ \nu \\ \eta \end{Bmatrix} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \begin{Bmatrix} \lambda_k \\ \mathbf{w}_k \\ \nu_k \\ \eta_k \end{Bmatrix} \tag{B5}$$

the following perturbation equations are obtained

$$\begin{aligned} &(\mathbf{F}_{\mathbf{x}}^0 - \lambda_0 \mathbf{E})\mathbf{w}_0 = \mathbf{0}, \\ &(\mathbf{F}_{\mathbf{x}}^0 - \lambda_0 \mathbf{E})\mathbf{w}_1 = \lambda_1 \mathbf{w}_0 - (\nu_1 \mathbf{F}_{\mathbf{x}\nu}^0 + \eta_1 \mathbf{F}_{\mathbf{x}\eta}^0)\mathbf{w}_0. \end{aligned} \tag{B6}$$

To study the stability of Γ, only the critical eigenvalues $\lambda_{j0} = \omega_{j0}$ ($j = 1, 2$) are of interest, since the remaining ones are a great distance from the imaginary axis. From Equation (B6$_1$)

$$\mathbf{w}_0 = \mathbf{u}_j \tag{B7}$$

follows and (B6$_2$) reads

$$(\mathbf{F}_{\mathbf{x}}^0 - i\omega_{j0}\mathbf{E})\mathbf{w}_1 = [\lambda_{j1} - (\nu_1 \mathbf{F}_{\mathbf{x}\nu}^0 + \eta_1 \mathbf{F}_{\mathbf{x}\eta}^0)]\mathbf{u}_j. \tag{B8}$$

Since, by hypothesis, matrix $(\mathbf{F}_{\mathbf{x}}^0 - i\omega_{j0}\mathbf{E})$ is singular, Equation (B8) admits a solution if and only if the right hand side is orthogonal to the left eigenvector $\mathbf{v}_j$, defined by Equation (4). Hence,

$$\lambda_{j1} = \mathbf{v}_j^H [(\nu_1 \mathbf{F}_{\mathbf{x}\nu}^0 + \eta_1 \mathbf{F}_{\mathbf{x}\eta}^0)\mathbf{u}_j] \tag{B9}$$

follows, where the normalization condition $\mathbf{v}_j^H \mathbf{u}_j = 1$ has been used. By truncating (B5$_1$) at the ε-order and absorbing the parameter ε, the following expression is obtained for the critical eigenvalues

$$\lambda_j = i\omega_{j0} + \mathbf{v}_j^H [(\nu \mathbf{F}_{\mathbf{x}\nu}^0 + \eta \mathbf{F}_{\mathbf{x}\eta}^0)\mathbf{u}_j] + O(\nu^2 + \eta^2). \tag{B10}$$

From Equation (B10), by putting $\lambda_j = \alpha_j + i\omega_j$, it follows that coefficients (B1) are the control parameter derivatives at the critical state of the real and imaginary parts of λ_j (i.e. the so-called first-order sensitivities of the critical eigenvalues).

Appendix C: Coefficients in Equations (38) and (39)

By defining

$$c_{jjj} := \mathbf{v}_j^H [\mathbf{F}_{\mathbf{xx}}^0 (2\mathbf{u}_j \mathbf{z}_{j\bar{j}} + \mathbf{z}_{jj}\bar{\mathbf{u}}_j) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_j^2 \bar{\mathbf{u}}_j], \quad (j = 1, 2), \tag{C1}$$

$$c_{122} := 2\mathbf{v}_1^H [\mathbf{F}_{\mathbf{xx}}^0 (\mathbf{u}_1 \mathbf{z}_{2\bar{2}} + \mathbf{u}_2 \mathbf{z}_{1\bar{2}} + \bar{\mathbf{u}}_2 \mathbf{z}_{12}) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_1 \mathbf{u}_2 \bar{\mathbf{u}}_2], \tag{C2}$$

$$c_{112} := 2\mathbf{v}_2^H [\mathbf{F}_{\mathbf{xx}}^0 (\mathbf{u}_2 \mathbf{z}_{1\bar{1}} + \mathbf{u}_1 \mathbf{z}_{2\bar{1}} + \bar{\mathbf{u}}_1 \mathbf{z}_{12}) + \mathbf{F}_{\mathbf{xxx}}^0 \mathbf{u}_1 \bar{\mathbf{u}}_1 \mathbf{u}_2], \tag{C3}$$

the following positions hold in Equations (38) and (39)

$$R_{jjj} = \frac{1}{8}\,\mathrm{Re}(c_{jjj}), \quad I_{jjj} = \frac{1}{8}\,\mathrm{Im}(c_{jjj}); \quad R_{1j2} = \frac{1}{8}\,\mathrm{Re}(c_{1j2}), \quad I_{1j2} = \frac{1}{8}\,\mathrm{Im}(c_{1j2}). \quad \text{(C4)}$$

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