Area and perimeter relationship for different shapes with an inradius

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Abstract: *In this paper, the author shows that a relationship for a triangle, which is Area/Perimeter=r/2, extends to other shapes with an inradius r (i.e., shapes with an inscribed circle). These shapes include squares, circles, rhombuses, regular polygons, and even irregular polygons and more. Students may find it interesting, or perhaps a challenge, to show that this is true.*

Keywords: *geometry, measurement*

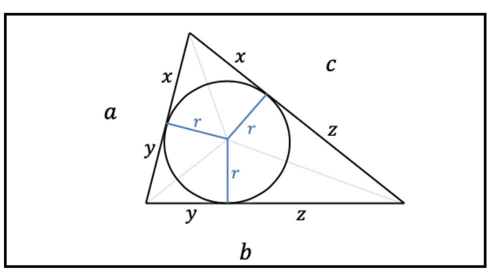
Introduction

We can imagine that a teacher draws a triangle, square, circle, rhombus, and eight-sided polygon ("stop sign" shape) on a whiteboard and says, "Would you believe these shapes all share a relationship?" Of course, a student may correctly state that all the shapes lie on the plane of the board, or that all the shapes have an area. But there is another relationship that they all share. As we show, shapes with an inscribed circle, which therefore have an inradius r, follow the relationship: $Area/Perimeter = r/2$. This includes triangles, squares, circles, rhombuses, regular polygons, and even irregular polygons and more. Students might try to show this relationship for shapes that they already know, or they might try to show the relationship for new shapes that they learn about - reinforcing lessons. We begin by showing the relationship for the triangle, about which much is known and can be learned.

Background: The Triangle

Every triangle, whether acute, right, or obtuse, has an inscribed circle. It can be observed, as illustrated in Figure 1, that line segments which are tangent to a circle and intersect each other are the same length.

Figure 1: *Triangle with inscribed circle (inradius r).*



Therefore, for any triangle, we have that $a = x + y$, $b = y + z$, and $c = x + z$. By adding any two of these sides and subtracting the remaining side, we can obtain each of the following equations (note that each equation shows the addition and subtraction in the numerator – where substitutions from the prior equations can be made):

$$x = \frac{a + c - b}{2} \tag{1}$$

$$y = \frac{a + b - c}{2} \tag{2}$$

$$z = \frac{c + b - a}{2} \tag{3}$$

Although students typically learn that the area of a triangle is half the base times the height, the height of a triangle may not be known (such as in Figure 1). A less familiar equation for a triangle's area is Heron's Formula, which only requires the sides of a triangle:

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{4}$$

where s is the semi-perimeter, $s = (a + b + c)/2$. Simple algebra with s, a, b, and c shows that:

$$Area = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{c+b-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)} \tag{5}$$

Using Eq. (1), (2), and (3), the area of a triangle is:

$$Area = \sqrt{(x+y+z)xyz} \tag{6}$$

The last equation was previously known (Incircle and excircles of a triangle, 2022). Also, using the areas shown in Figure 1, we can show the following result, which was previously known (Area of triangle in terms of inradius, 2022; Chu, 2022).

$$\frac{Area}{Perimeter} = \frac{r}{2} \tag{7}$$

Students can try to develop this last equation on their own. A teacher can suggest that students use the three triangles shown in Figure 1, each with a height of r, and with bases $a = x + y$, $b = y + z$, and $c = x + z$, respectively. The area of each triangle, which is half the base times the height, can be added to give the area of the overall triangle, $Area = \frac{r(x+y)}{2} + \frac{r(y+z)}{2} + \frac{r(x+z)}{2} = \frac{r(x+y+y+z+x+z)}{2} = r(x+y+z) = r(Perimeter)/2$, which establishes Eq. (7).

Using the last two equations, it is easy to show the following, which was previously known (Area of triangle in terms of inradius, 2022; Chu, 2022; Ogilvy & Anderson, 1988):

$$r = \sqrt{\frac{xyz}{(x+y+z)}} \tag{8}$$

From the last two equations, the area A multiplied by the inscribed radius r is: $Ar = xyz$

Area and Perimeter Relationship for Other Shapes with an Inradius

In the previous section, we saw that the relationship $Area/Perimeter = r/2$ holds for any triangle. In this section, we show that this relationship extends to other shapes with an inradius r.

The area of a circle is related to its radius by the formula $Area = \pi r^2$. Since the circumference of a circle is $2\pi r$, then students can see that:

A) Circle: $\frac{Area}{Perimeter\ (i.e.,\ circumference)} = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$

A square has an incircle with radius r that is easily shown to be half the length of a side of the square. So, the square's side is of length $2r$. Therefore, the square has an area of $(2r)^2$ and it has a perimeter of $4(2r)$, so students can easily demonstrate that:

B) Square: $\frac{Area}{Perimeter} = \frac{(2r)^2}{4(2r)} = \frac{r}{2}$

The area of a rhombus is related to the inradius r by the formula $Area = 2ar$, where a is a side length of the rhombus (Rhombus, 2022). Since the perimeter of the rhombus is $4a$, we have:

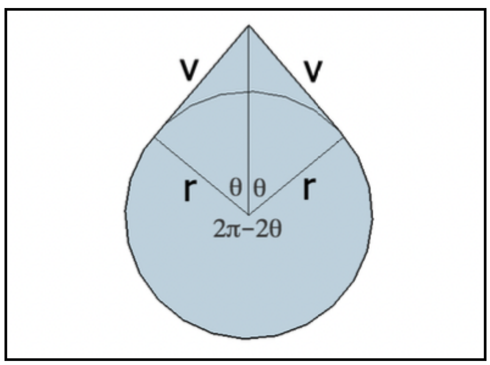
C) Rhombus: $\frac{Area}{Perimeter} = \frac{2ar}{4a} = \frac{r}{2}$

A regular polygon is one that has congruent sides and congruent angles. The area of a regular polygon is $Area = nsa/2$, where a is the apothem, or inradius r, n is the number of sides, and s is the side length (Apothem, 2022). The perimeter of a regular polygon is the product of the number of sides and side lengths. Therefore, we have:

D) Regular Polygon: $\frac{Area}{Perimeter} = \frac{nsr/2}{ns} = \frac{r}{2}$

It can be observed that line segments which are tangent to a circle and intersect each other are the same length (for example, see Figure 1). This is indicated by length v in Figure 2, which shows a shape that we will call Shape E.

Figure 2: *Shape E with Lines Tangent to a Circle.*



The line segments v are perpendicular to the line segment r, which is the radius of the circle. The area of Shape E is:

$$Area = 2\left(\frac{rv}{2}\right) + \left(\frac{2\pi - 2\theta}{2\pi}\right)\pi r^2 = r(v + (\pi - \theta)r)$$

The perimeter of the Shape E is:

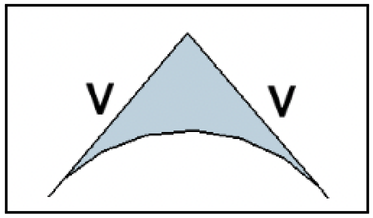
$$Perimeter = 2v + (2\pi - 2\theta)r = 2(v + (\pi - \theta)r)$$

Therefore, for Shape E, we again have the relationship:

E) Shape E: $\frac{Area}{Perimeter} = \frac{r(v+(\pi-\theta)r)}{2(v+(\pi-\theta)r)} = \frac{r}{2}$

Shape E is made up of a circle and a shape that is shown in Figure 3, which we will call Shape F. Although the relationship $Area/Perimeter = r/2$ holds for Shape E and also for the circle, it does not hold for Shape F. The area of Shape E is equal to the area of the circle plus the area of Shape F. However, the perimeter of Shape E is not equal to the circumference of the circle plus the perimeter of Shape F since the arc of Shape F cannot be included (twice at that!) in the overall perimeter of Shape E. This is why Shape F does not follow the relationship $Area/Perimeter = r/2$ itself.

Figure 3: *Shape F (from the top of Shape E).*

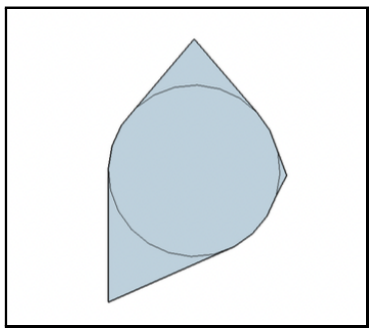


The choice of the tangent lines used in Figure 2 was arbitrary and extends to any number of non-overlapping tangent lines of any length that intersect. Therefore, the following relationship holds for any irregular polygon with an inradius:

F) Irregular Polygon with inradius: $\frac{Area}{Perimeter} = \frac{r}{2}$

And it also holds for unusual shapes such as Figure 4.

Figure 4: *Shape including a circle with different-length tangent lines that intersect.*



In all cases of a triangle, circle, square, rhombus, regular polygon, irregular polygon (with an inradius), and shapes formed with tangent lines to a circle, the following relationship between the area, perimeter, and inradius r holds:

G) $\frac{Area}{Perimeter} = \frac{r}{2}$

Conclusion

Students and teachers may find it interesting how one relationship between the area and perimeter exists in shapes with an inradius. Depending on the shapes learned by students, then a teacher could develop the relationship for some or all of the shapes in this paper. Alternatively, the relationship can provide a challenge for students to demonstrate what they have learned about a shape (or shapes). It is hoped that this paper provides the inspiration for students to explore mathematics and find new relationships and patterns.

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