Side Lengths of a Right Triangle Reproduced as Perimeters of Other Right Triangles: A Sketch and Proof

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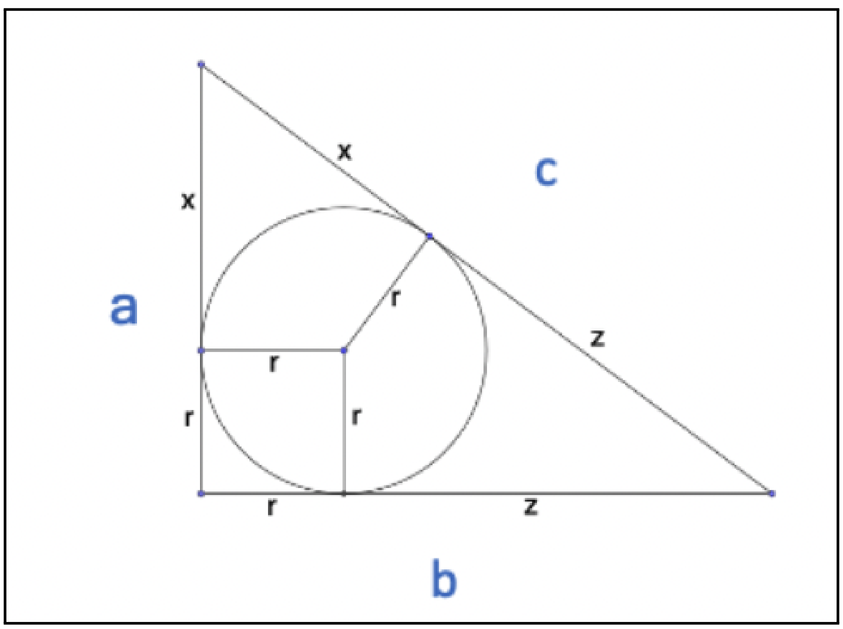
Abstract: *In this paper, the author builds upon the work of DiDomenico (1995) and Alsina and Nelson (2011). Given an arbitrary right triangle with side lengths a, b, and c, the author illustrates a method for constructing three smaller right triangles with perimeters a, b, and c. A link to a dynamic sketch of the construction is provided to help students informally explore geometric relationships related to the proof.*

Keywords: *Proof, argumentation, right triangles, similarity, measurement*

Knowns and Givens

The area of a right triangle is equal to the product of the line segments subdividing the hypotenuse with the triangle's inscribed circle, as demonstrated by DiDomenico (1995) and shown in a proof without words by Alsina and Nelson (2011, see pp. 82–83). Consider, for instance, the arbitrary right triangle depicted in Figure 1 with side lengths a, b, and c.

Figure 1: *The incircle splits the hypotenuse into segments with lengths x and z (DiDomenico, 1995).*



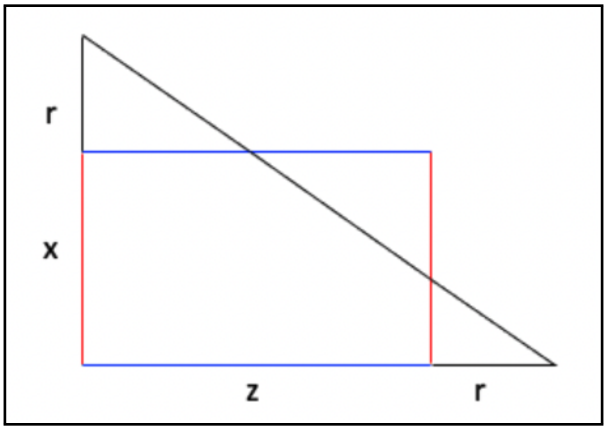
From DiDomenico (1995), we note that the area of the right triangle equals xz, with x and z formed by the intersection of the hypotenuse and incircle. In other words,

$$\text{Triangle Area} = \frac{ab}{2} = xz$$

An Interesting Construction

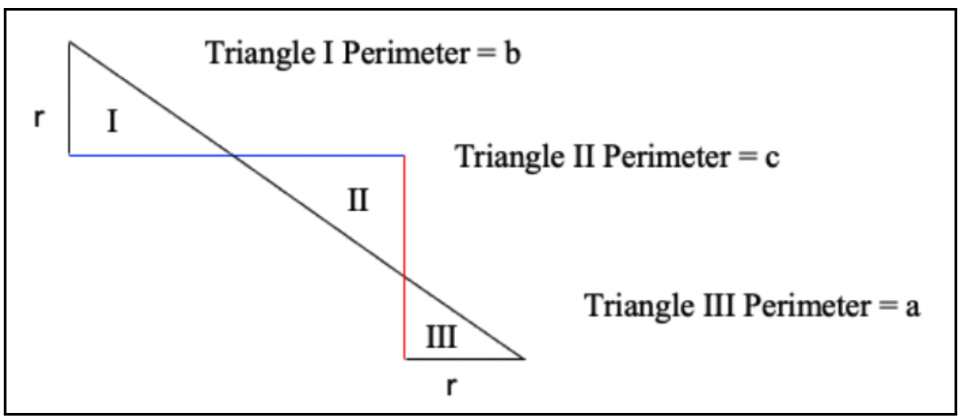
Next, we build a rectangle with side lengths x and z.

Figure 2: *Rectangle with dimensions $x \times z$ constructed using the right triangle illustrated in Figure 1.*



Note that four distinct right triangles are illustrated in Figure 2—namely, the original right triangle and three smaller right triangles formed by the overlap of the original and the rectangle. As Figure 3 suggests, an interesting relationship exists between the three smaller right triangles and side lengths a, b, and c of the original.

Figure 3: *The perimeters of the three right triangles (I, II, and III) have measures equal to the side lengths of the original right triangle (a, b, and c).*



In the following section, we offer a rigorous proof of this result. For readers who are interested in exploring the relationship less formally before delving into the proof, a GeoGebra sketch of the problem context is provided at the following link: https://www.geogebra.org/m/qrhjxvzy.

Formal Proof

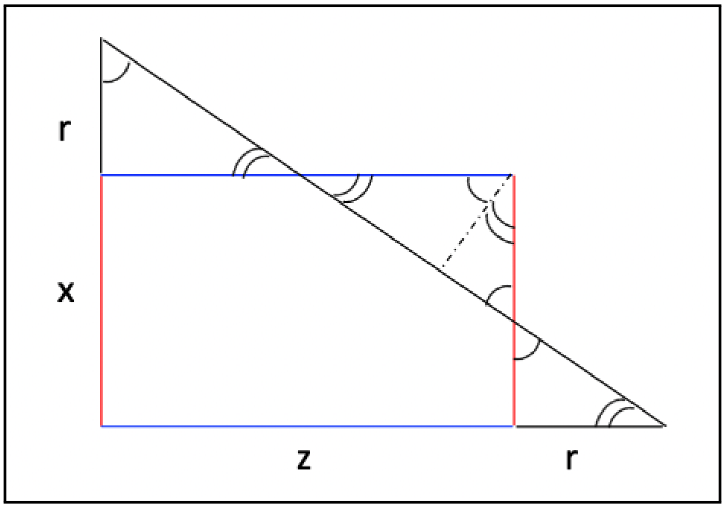
Theorem. *A right triangle ($a = r + x$; $b = r + z$; $c = x + z$) and the rectangle (xz) with the same area reproduces the triangle's side lengths as perimeters of other right triangles.*

Proof. First, refer to Figures 1 and 2. We know the area of the original right triangle equals the area of the rectangle. Therefore, referring to Figures 2 and 3, the area of the region lying within the rectangle but outside the right triangle (i.e. Region II) must equal the area of the region lying within the right triangle but outside the rectangle (i.e., Regions I and III). In other words,

$$\text{Triangle I Area} + \text{Triangle III Area} = \text{Triangle II Area} \tag{1}$$

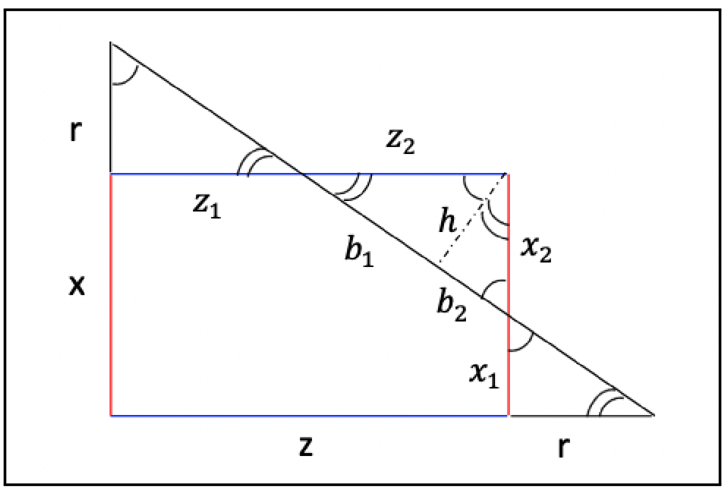
Next, consider the similar right triangles formed by the overlap of the original right triangle and the rectangle shown in Figure 4.

Figure 4: *Similar right triangles formed by overlapping rectangle and right triangle.*



Next, let $x = x_1 + x_2$ and $z = z_1 + z_2$. Assign h, b_1, and b_2 as shown in Figure 5.

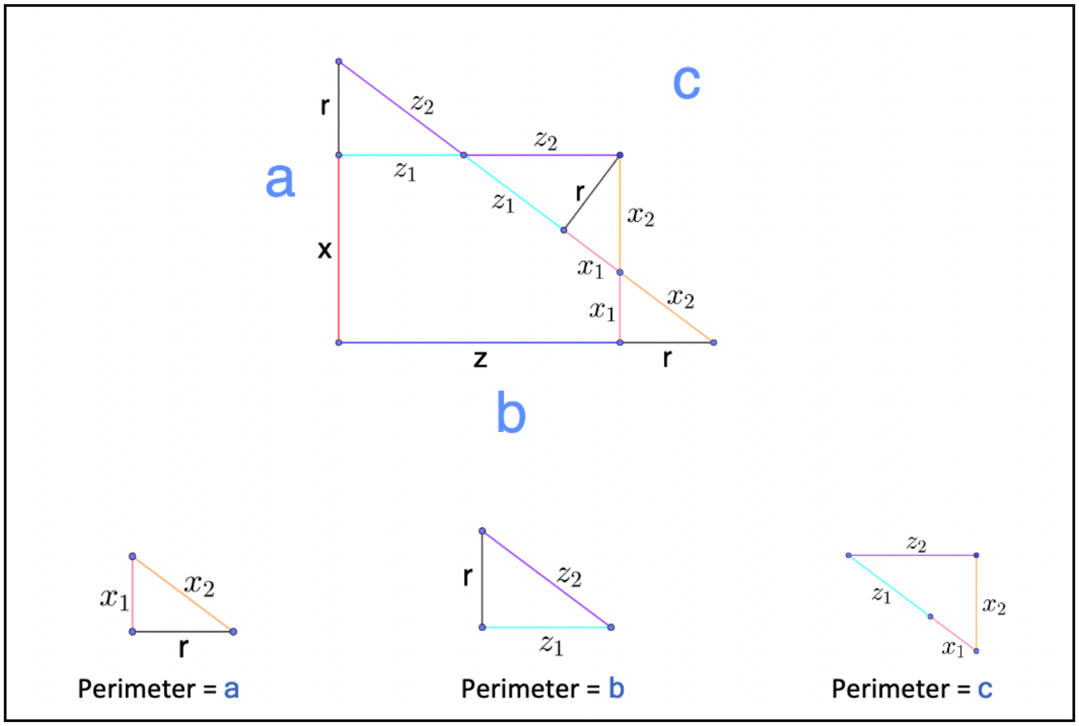
Figure 5: *Assigning variable names within the figure.*



By similar triangles (i.e., Angle-Angle with right triangles), $\frac{b_1}{z_1} = \frac{h}{r}$ and $\frac{b_2}{x_1} = \frac{h}{r}$. Using (1), $\left(\frac{rz_1}{2}\right) + \left(\frac{rx_1}{2}\right) = \left(\frac{(b_1+b_2)h}{2}\right)$. Substituting for $b_1 = \frac{z_1 h}{r}$ and $b_2 = \frac{x_1 h}{r}$ gives $h^2 = r^2$.

We have $h = r$, $b_1 = z_1$, and $b_2 = x_1$. Hence, the desired result holds, as illustrated in Figure 6.

Figure 6: *The desired result holds.*



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Alsina, C. and Nelsen, R. B. (2011). *Icons of mathematics: An exploration of twenty key images*. American Mathematical Society (AMS), 82–83.

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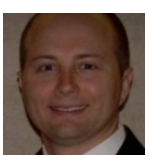

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