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Algorithms for the sum of discrete random variables. Actuarial applications.[*]

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Abstract: - In literature, the sum of discrete random variables becomes a problem of heavy (and often impracticable) computation no sooner does the number of convolutions exceed few units (at most in actuarial applications). In this paper, we show how this problem can be easily overcome when using random variables with integer (positive, negative, or null) or referable to integer numerical realizations but not necessarily identically distributed.

Under the above-mentioned condition, we illustrate in particular two exact methods and an approximated one for calculating convolution:

- the first exact method is based on the well-known Fast Fourier Transform (FFT);
- the second exact method is derived from the classical approach using Discrete Fourier Transform (DFT) by means of algebraic manipulations;
- the third method is derived from the definition of convolution and it is approximated by neglecting the probabilities less than a given bound $\varepsilon = 10^{-h}$ $(51 \leq h \leq 100)$[**].

As for the error bounds of the approximated method, it is worth noting that the results obtained by this method differ in relative terms from the corresponding exact values of less than 10^{-9}. This can be tested by comparing the convoluted probability distribution obtained by the approximated method with the one obtained by the other two methods and by also comparing the first four moments with those computed directly on the original random variables. The results (in particular the exact and the approximated probability distribution) are identical in practice. It does not exist therefore the problem of a difference along the tail. As a consequence, although the proposed method is an "approximated method" under a mathematical point of view, it can be considered an "exact method" in the actuarial applications.

As for the efficiency of calculation, we have to distinguish between the simple sum of discrete random variables and the calculation of compound distributions with prefixed counting distributions (i.g. Poisson, Negative Binomial, Pareto):

- in the first case, the approximated method and the second exact method are similar but the approximated method gives further information about the random variables (for instance, the information about the independence using some properties of the characteristic functions);
- in the second case, only the approximated method is applicable in practice.

Finally, in the conditions of interest, the exact method using FFT is less efficient thān the other methods and it has a more limited application field.

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[**] The values of h>100 seldom concern the actuarial applications.

Key-Words: - Fast methods for convolution of discrete random variables, actuarial applications.

1. Theory and practice in computing the sum of random variables

As well-known, the sum of independent but not necessarily identically distributed discrete random variables is generally speaking a problem of heavy (and often impracticable) computation in actuarial applications. The aim of this case study is to show how to perform a fast computation using both exact mathematical procedures and approximated methods based on simple algorithms.

By considering that the insured capitals, the claims amounts and the other quantities useful in actuarial applications can be expressed as integer or referable to integer values, we illustrate two exact methods and an approximated one (but with controlled maximum approximation).

We show that the approximated method can be applied with great flexibility to any random variables with the above-mentioned type of numerical realizations (for instance, it can be easily generalized to the computation of the aggregate claims distribution). Thus, it can be successfully applied in the actuarial field where getting results with the maximum generality and simplicity is necessary. Obviously, it requires an ad-hoc software edited in a fast and powerful language[1].

1.1. The two exact methods and the meaning of "exact" convoluted probabilities

Both the exact methods examined in this paper are based on the Discrete Fourier Transform (DFT).

This approach to the computation of convolution is well-known in literature but for a fast computation one need introduce some simple variants:

- in section 2.1, we make a brief reference to the exact method using the Fast Fourier Transform (FFT);

- in section 2.2, we derive a different exact method by means of algebraic manipulations.

Before proceeding, however, it can be useful to notice that any method, even if exact on the mathematical level, may cause approximation errors on the numerical level. Thus, given the theoretical and practical nature of this case study, we need ask ourselves: when a distribution can be defined exact?

According to a long experience, in the actuarial applications a distribution can be considered exact if the sum of its probabilities is equal to one followed by nine zeros. This means that each probability must have at least nine exact decimal points.

This standard is consistent with the software (such as C++, Delphi and similar languages) and the hardware today commonly available.

1.2. The approximated method and the criteria to obtain a desired level of approximation

The approximated method proposed in section 3 uses the direct definition of convolution.

The approximation consists in neglecting in the calculation the probabilities (both original and convoluted) less than $\varepsilon=10^{-h}$ $(51\leq h\leq 4930)$.

Notice that, even if ε can reach the value of 10^{-4930}, the procedure has sense for ε at the most equal to 10^{-100}. This is for two reasons:

- the computation becomes heavier and heavier, as h increases;

- as above said, today we could not be able to obtain a better approximation in practice.

An important aspect of this method is to assure a desired level of approximation for the sum of the probabilities of each convolution and for the first four moments of the final convolution. In particular, reminding the above-mentioned concept of exact distribution, the software is created in such a way that the sum of the probabilities of each convolution differs from 1 of less than 10^{-9} and, with the same error bound, the first four moments of the final convolution differ in relative terms from the corresponding exact values.

Actually, the software works in the following way:

a. when reading the random variables to be summed up, the software computes the first four moments of their sum exactly;

b. after each convolution performed by neglecting the probabilities (original or convoluted) less than 10^{-h}, supposed h=51, the software performs the sum of the probabilities corresponding to the same numerical realizations and checks that the sum of all the probabilities differs from 1 of less than 10^{-9}. If this condition is not fulfilled, the software signal an error and it re-starts from the point b.

[1] In our applications, we have specifically used the C++ language (just for now, with Builder 5 at 32 bits).

using an higher value of h; on the other case, it proceeds in the same way up to the final convolution;

c. in the end, the software gives the final approximated distribution. It computes the first four moments of this distribution and checks that they simultaneously differ in relative terms from the corresponding exact ones computed at point a. for less than 10^{-9}. In the negative case, the software re-starts from the point b. with an higher value of h; in the positive case, the software ends.

On this subject, we want to discredit a commonplace about the influence of the above-mentioned approximation on the distribution tail.

As a matter of fact, it is easy to verify that the omission of the probabilities smaller than $\varepsilon=10^{-h}$ ($51\leq h\leq 100$) has not a significant weight on the distribution tail, since, according to the point c., the product between the neglected probabilities and the corresponding numerical realizations affects each of the first four moments of the final distribution for less then 10^{-9}.

This has been confirmed in a number of applications by the comparison of the distribution obtained by the approximated method with the distribution resulting from an exact method (see for instance the results in table 3. For further comparisons, we can provide the programs .exe on request).

Finally, notice that a substantial simplification of the approximated method is obtained by summing, after each convolution, the probabilities corresponding to the same numerical realizations. In many applications, this gives a contribution to the calculation efficiency sometimes even greater than the one given by the omission of the very small probabilities.

1.3. The applications in life insurance

In section 4.1, we make a comparison between the exact method based on FFT and the approximated method.

In particular, by means of a numerical exemplification using identically distributed random variables, we compare the number of multiplications required for computing convolution by FFT with those required by the approximated method.

The results show that, in the hypothesis useful in actuarial applications, such as $\varepsilon=10^{-51}$, the number of multiplications required by FFT is greater than the corresponding number required by the approximated method. This is verified even in the case of $\varepsilon=10^{-4930}$.

In section 4.2, we compare the other exact method with the approximated one. The application concerns a portfolio of life insurance policies of different types (that is random variables not necessarily identically distributed) and the comparison is made in terms of processing time.

On this subject, it is useful to notice that the exact method computes only the final distribution while the approximated method computes and gives as output each convoluted distribution up to the final one. The complete calculation of each convolution makes the approximated method very efficient in many other applications in life and non-life insurance as, for instance, for calculating the aggregate claims distribution (see section 1.4).

Besides, the exact method is extremely fast in the case of identically distributed random variables or in the case of few not identically distributed random variables and with a restricted range of numerical realizations. On the contrary, the conditions necessary for running the software of the approximated method do not restrict its concrete applicability (see section 1.5).

In a future work, we will develop the calculation of the ruin probability of an insurance company. This problem is another typical example of actuarial application based on discrete random variables where the exact method and the approximated method proposed in this paper can be easily applied.

1.4. The applications of the approximated method in non-life insurance

In section 4.3, we show two applications in non-life insurance. The first one concerns the calculation of the aggregate claims distribution using a Poisson distribution as counting distribution; the second one concerns the calculation of the aggregate claims distribution using a Negative Binomial distribution as counting distribution. In both cases, we apply only the approximated method since the exact ones are less efficient and not flexible at all.

1.5. The conditions for running the software of the approximated method and the other utilities

The conditions necessary for running the software[2] of the approximated method are not restrictive.

As a matter of fact:

a) the maximum number of random variables that can be summed up (without considering the

[2] Using a specific software in C++ and hardware PC Pentium 4 HyperThreading 3.4 GHz, 1024 KB.

weight that is the possibility that each random variable can be convoluted more than ones with itself) is 100,000;

b) each random variable can have any number of numerical realizations not lower than two and not greater than 100,000;

c) the maximum numerical realization is 20,000,000.

A limit today existing (but that should be eliminated using a software at 64 bits for the language C++) is that the mean value of the sum of the random variables, each one taken with its own weight, must be not greater than 2,500,000. If not, we have to reduce the amounts of each random variables or the number of numerical realizations.

Before concluding, we mention a further use of the approximated method in addition to the classical applications. This is the possibility to verify the independence among given random variables by the well-known result according to which the random variables are independent if the product of their characteristic functions is equal to the characteristic function of their sum.

1.6. Notation

Let $X_1, X_2, \ldots, X_N$ be the discrete (with integer numerical realizations) and independent (but not necessarily identically distributed) random variables to be summed up and let:

$$(1) \qquad X^{(n)} = \sum_{i=1}^{n} X_i \quad \text{for} \quad n = 1, \ldots, N$$

For each random variable $X^{(n)}$, let:

- $AR^{(n)}_{\max}$ be the greatest numerical realization;
- $AR^{(n)}_{\min}$ be the smallest numerical realization; and let:

$$(2) \qquad M^{(n)} = AR^{(n)}_{\max} - AR^{(n)}_{\min} + 1$$

which is the maximum number of possible numerical realizations.

As for the probability distribution of each random variable $X^{(n)}$ (n=1,...,N), that is the convolution of order n, we use two different notations. In particular:

- in the case of the exact methods, $f_{X^{(n)}}(r)$ denotes the probability (even null) corresponding to the integer number $(AR^{(n)}_{\min} + r)$ for $r=0,1,\ldots,(M^{(n)}-1)$;

- in the case of the approximated method, $f_{X^{(n)}}(x_n)$ denotes the (non null) probability corresponding to the integer numerical realization x_n in the set $[AR^{(n)}_{\min}, AR^{(n)}_{\max}]$.

The reason of a different notation is due to the fact that when using the exact methods we have to consider all the integer values from the minimum to the maximum numerical realization. On the contrary, when using the approximated method, we can consider only the integer numerical realizations with non null probability.

2. The two exact methods

Both the exact methods for calculating convolution mentioned in this paper are based on the classical approach by DFT.

According to this approach, for calculating the final convolution, we proceed in the following way:

a) we calculate the DFT of each random variable X_i (i=1,...,N), that is:

$$(3) \qquad \Phi_{f_{X_i}}(u) = \sum_{r=0}^{(M^{(N)}-1)} f_{X_i}(r) \exp\left(i \frac{2\pi u}{M^{(N)}} r\right)$$

$$\text{for} \quad u = 0, 1, \ldots, (M^{(N)} - 1)$$

where f_{X_i} is the probability distribution of X_i. For $r=0,1,\ldots,(M^{(N)}-1)$, $f_{X_i}(r)$ is in particular the probability corresponding to the numerical realization $(AR^{(N)}_{\min} + r)$ of X_i;

b) then, we calculate the DFT of the convolution of order N. Given the independence of the random variables X_i (i=1,...,N), we have:

$$(4) \qquad \Phi_{f_{X^{(N)}}}(u) = \prod_{i=1}^{N} \Phi_{f_{X_i}}(u)$$

$$\text{for} \quad u = 0, 1, \ldots, (M^{(N)} - 1)$$

c) finally, we obtain the probability distribution of the final convolution by calculating the IDFT, that is:

(5)

$$f_{X^{(N)}}(r)=\frac{1}{M^{(N)}}\sum_{u=0}^{(M^{(N)}-1)}\Phi_{f_{X^{(N)}}}(u)\exp\left(-i\frac{2\pi r}{M^{(N)}}u\right)$$

for $r=0,1,\ldots,(M^{(N)}-1)$

2.1. The exact method using FFT

The FFT is a fast algorithm for the calculus of DFT and its inverse IDFT. Thus, the exact method using FFT just consists in applying this algorithm to Equations (3) and (5).

As well known [1], thanks to this algorithm, the number of (real) multiplications necessary to develop the calculus of each transform with $M^{(N)}$ data points reduces to:

(6) $$\text{molt}_{FFT}^{(N)}=M\log_2 M$$

where $M=2^\gamma$ is the smallest integer (expressed as a power of 2) not less than $(M^{(N)}-1)$.

We will use the above mentioned number of multiplications as a measure of efficiency in order to compare this method with the approximated one.

Really, the number of multiplications necessary to develop the calculation by FFT is considerably greater (as one can see by the routine in [6,pp.507-508].

2.2. The exact method derived algebraically

This exact method is derived from the classical approach based on DFT by means of algebraic manipulations.

First of all, we write Equation (3) in the following way:

(7) $$\Phi_{f_{X_i}}(u)=\rho_i(u)\exp(i\alpha_i(u))$$

for $u=0,1,\ldots,(M^{(N)}-1)$

where ρ_i is the module and α_i the argument of the characteristic function of each random variable X_i.

Equation (4) therefore becomes:

(8) $$\Phi_{f_{X^{(N)}}}(u)=\rho(u)\exp(i\alpha(u))$$

for $u=0,1,\ldots,(M^{(N)}-1)$

where:

(9) $$\rho(u)=\prod_{i=1}^{N}\rho_i(u)$$

(10) $$\alpha(u)=\sum_{i=1}^{N}\alpha_i(u)$$

and Equation (5):

(11)

$$f_{X^{(N)}}(r)=\frac{1}{M^{(N)}}\sum_{u=0}^{(M^{(N)}-1)}\rho(u)\exp\left(i\left(\alpha(u)-\frac{2\pi r}{M^{(N)}}u\right)\right)$$

for $r=0,1,\ldots,(M^{(N)}-1)$

Then, we calculate the difference between two successive values of the cumulative distribution of the final convolution.

In particular, for integers m and k (with k>m), we calculate:

(12)

$$\begin{aligned}&F_{X^{(N)}}(k)-F_{X^{(N)}}(m)\\&=\text{prob}\left\{(AR_{\min}^{(N)}+m)\le X^{(N)}<(AR_{\min}^{(N)}+k)\right\}\\&=\sum_{h=m}^{k-1}f_{X^{(N)}}(h)\end{aligned}$$

According to Equation (11), we have[3]:

(13)

$$\begin{aligned}F_{X^{(N)}}(k)-F_{X^{(N)}}(m)=\frac{1}{M^{(N)}}\Bigg[(k-m)\\+\sum_{u=1}^{(M^{(N)}-1)}\rho(u)\frac{\sin\frac{\pi u(k-m)}{M^{(N)}}}{\sin\frac{\pi u}{M^{(N)}}}\\\cdot\cos\left(\alpha(u)-\frac{\pi u(m+k-1)}{M^{(N)}}\right)\Bigg]\end{aligned}$$

For m=r and k=r+1, Equation (13) gives the probability distribution of the final convolution $f_{X^{(N)}}(r)$.

[3] In the program, we further on simplify this formula by cutting the summation interval by half.

3. The approximated method

The principle underlying this method is to apply the definition of convolution in a direct manner by using only real and elementary operations (basically, multiplications and additions) for the random variables (initial or obtained by convolution).

For having the same results of the exact methods (but with a greater efficiency), we apply the following conditions:

a) we proceed step by step, that is we consider the first random variable (convolution=1), then we add the second (convolution=2), then we add the third (convolution=3) and so on until convolution=N;

b) at each step, we neglect the initial or convolved probabilities less than 10^{-h} with $51 \leq h \leq 4930$ (in the actuarial applications $51 \leq h \leq 100$ is sufficient);

c) for each convolution, we verify that the sum of the probabilities is equal to 1 with a prefixed error (in the actuarial applications an error less than 10^{-9} is sufficient);

d) we verify that the first four moments of the final convolution differ from the corresponding exact values by an error of the same order (less than 10^{-9}).

Formally, the method consists in considering, for each n, only the numerical realizations $y_n : f_{X^{(n)}}(y_n) < 10^{-h}$ (with $51 \leq h \leq 4930$) fulfilling the following conditions:

- after the computation of the convolution of order n:

$$(14) \qquad 1 - \sum_{y_n} f_{X^{(n)}}(y_n) < 10^{-9}$$

- after the computation of the final convolution:

$$(15) \qquad 1 - \frac{E_{y_n}\left((X^{(N)})^k\right)}{E\left((X^{(N)})^k\right)} < 10^{-9} \quad \text{for} \quad k = 1,2,3,4$$

where $E\left((X^{(N)})^k\right)$ is the k-th exact moment of the final convolution and $E_{y_n}\left((X^{(N)})^k\right)$ the corresponding value obtained by only considering the realizations y_n and their probabilities.

Given the simplicity of this method, it is easy to compute the number of multiplications necessary to develop each successive order of convolution.

Let NR_i (i=1,2,...N) be the number of the numerical realizations (with non null probability) of the i-th random variable X_i and $NR^{(n)}$ the number of the numerical realizations of the random variable $X^{(n)}$ obtained after n convolutions. Notice that, generally speaking, $NR^{(n)} \leq M^{(n)}$ for two reasons:

- the random variables to sum up do not necessarily have as many numerical realizations as all the integers from the minimum to the maximum;

- as above said at point b), in the calculation of each convolution the approximated method neglects the numerical realizations with probability (initial and/or convoluted) less than a prefixed error.

In the case of identically distributed random variables, $X_i = X_1$ (i=1,...,N), the minimum and the maximum numerical realization of $X^{(n)}$ (n=1,...,N) are respectively:

$$(16) \qquad AR^{(n)}_{min} = nAR^{(1)}_{min}$$

$$(17) \qquad AR^{(n)}_{max} = nAR^{(1)}_{max}$$

and the number of numerical realizations is at most:

$$(18) \qquad NR^{(n)} = n(AR^{(1)}_{max} - AR^{(1)}_{min}) + 1$$

In this case, the number of multiplications required for calculating the n-th (n=1,...,N) convolution is at most:

$$(19) \qquad molt^{(n)}_{AM} = NR^{(n-1)} NR_1$$

As a matter of fact:

a) the random variable $X^{(1)}$ relative to the first convolution is equal to the original random variable X_1. Thus, it has:

$$(20) \qquad NR^{(1)} = NR_1$$

possible values and the corresponding probabilities can be obtained without any multiplications;

b) the random variable $X^{(2)}$ obtained after the second convolution of the original random variable has at most:

$$(21) \qquad NR^{(2)} = 2(AR^{(1)}_{max} - AR^{(1)}_{min}) + 1$$

possible values. The probabilities are calculated by multiplying each probability of $X^{(1)} = X_1$ (for a total of $NR^{(1)}=NR_1$) for the probabilities of the same X_1. Then the multiplications are:

$$molt_{AM}^{(2)} = NR^{(1)} NR_1 \tag{22}$$

c) the random variable $X^{(3)}$ obtained after the third convolution of the original random variable has at most:

$$NR^{(3)} = 3(AR_{max}^{(1)} - AR_{min}^{(1)}) + 1 \tag{23}$$

possible realizations. The probabilities are calculated by multiplying each probability of $X^{(2)}$ (for a total of $NR^{(2)}$) for the probabilities of X_1 (that is NR_1). Then the multiplications are:

$$molt_{AM}^{(3)} = NR^{(2)} NR_1 \tag{24}$$

Generally speaking, the random variable $X^{(n)}$ obtained after the n-th convolution of X_1 has at most $NR^{(n)}$ possible realizations given by Equation (18) and the probabilities are calculated by multiplication of each probability of $X^{(n-1)}$ (that is $NR^{(n-1)}$) by the probabilities of X_1 (that is NR_1). Then the multiplications are given by Equation (19).

For the final convolution, the random variable $X^{(N)}$ has at most:

$$NR^{(N)} = N(AR_{max}^{(1)} - AR_{min}^{(1)}) + 1 \tag{25}$$

possible realizations. The probabilities are calculated by multiplying each probability of $X^{(N-1)}$ (that is $NR^{(N-1)}$) for the probabilities of X_1 (that is NR_1). Then the multiplications are:

$$molt_{AM}^{(N)} = NR^{(N-1)} NR_1 \tag{26}$$

In the more general case of not identically distributed random variables, the reasoning for computing the number of multiplications is similar but with some merely formal complications.

4. Applications

In this section, we show some applications in life and non-life insurance.

4.1. A single random variable convoluted more times. Comparison between the exact method based on FFT and the approximated method

Let us compute the convolution of order 1000 of a random variable with 14 realizations with minimum value 14 and maximum 60. Let therefore N=1000, $NR_1=14$, $AR_{max}^{(1)} = 60$ and $AR_{min}^{(1)} = 14$.

Using FFT, the number of multiplications is according to equation (6):

$$molt_{FFT}^{(1000)} = M \log_2 M = 65536 \cdot 16 = 1048576 \tag{27}$$

where:

$$M \geq 1000(AR_{max}^{(1)} - AR_{min}^{(1)}) = 1000(60-14) = 46000$$

and exactly $M=2^{16}=65536$ as it must be a power of 2.

In the approximated method, if we neglect the probabilities less than $\varepsilon=10^{-51}$, the number of realizations after 999 convolutions is (according to the program) equal to $NR^{(999)}=17111$ (instead of 45955 according to the theoretical formula, that is 999(60-14)+1).

In these conditions, according to equation (26) the number of multiplications is:

$$molt_{AM}^{(1000)} = NR^{(999)} NR_1 = 17111 \cdot 14 = 239554 \tag{28}$$

If we take $\varepsilon=10^{-4930}$ (yet this is of no help in the applications since the value is excessively little for actuarial purposes), the realizations after 999 convolutions become $NR^{(999)}=38252$ and then the number of multiplications is:

$$molt_{AM}^{(1000)} = NR^{(999)} NR_1 = 38252 \cdot 14 = 535528 \tag{29}$$

In conclusion, as for the number of multiplications (results being equal), the more efficient method for convolution is the approximated method.

In Table 1, we show some further results as the number of convolutions vary.

Table 1. Number of real multiplications

Num. Conv.	FFT	Approx. Meth.*
2	896	196
10	10240	5446
100	106496	57204

1000	1048576	239554
10000	20971520	765940

* *The calculation is performed by neglecting the probabilities lower than $\varepsilon=10^{-51}$. The results concerning the single probabilities and the first four moments of the final convolution are identical (with a spread of order 10^{-9}) to those obtained by FFT.*

4.2. Many random variables convoluted more times. Comparison between the exact method of section 2.2 and the approximated method.

Let us compute the convolution of the 13,500 random variables not necessarily identically distributed used in [3] .

In Table 2, we show some results in terms of processing times by varying the weight of each random variable.

Table 2. Processing times (in seconds)

Num. Conv.	Exact Meth. Sec. 2.2	Appr. Meth.*
1350	1	1
13500	11	11
27000	25	29

* *The calculation is performed by neglecting the probabilities lower than $\varepsilon=10^{-51}$. The results concerning the cumulative distribution functions and the first four moments of the final convolution are identical (with a spread of order 10^{-9}) to those obtained by the exact method.*

Notice that in the programs there are fixed components having a great weight for a relatively small number of convolution.

Besides, consider that, as anticipated in section 1.3, the exact method takes the indicated processing times to compute only the final convolution while the approximated method takes the indicated processing times to compute all the convolution up to the final one (it computes also the product of the characteristic functions of each original random variables and the characteristic function of their sum).

It may also be interesting to make a specific comparison between the final convolution obtained by the exact method and the one obtained by the approximated method. The results are identical (up to the tenth decimal digit).

In table 3, we show in particular the "tails" of the cumulative distribution for 13,500 convolutions.

Table 3. Tails of the cumulative distribution function

Realizations*	Cumulative distribution
$m-5.00\sigma$	0.0000000017
$m-4.00\sigma$	0.0000033001
$m-3.50\sigma$	0.0000494136
$m-3.00\sigma$	0.0005507423
$m+3.00\sigma$	0.9976084881
$m+3.50\sigma$	0.9994448125
$m+4.00\sigma$	0.9998758140
$m+5.00\sigma$	0.9999966871
$m+8.00\sigma$	1.0000000000

* *Where $m=128575.052$ is the mean value and $\sigma=736.935$ is the standard deviation.*

In addition to the above mentioned results, the approximated method provides also both the characteristic function of the sum of the random variables and the product of their characteristic functions. These values are identical and, in particular, the real part is equal to – 0.000532462 and the imaginary part is equal to – 0.000351474.

4.3. Computation of the aggregate claims distribution by means of the approximated method

In table 4, we show some results concerning the calculation of the aggregate claims cumulative distribution using a Poisson distribution with parameter $\lambda=4.841423259$ as counting distribution. In Table 5, we show some results concerning the calculation of the aggregate claims cumulative distribution using a Negative Binomial with parameters $r=23.43937917$ and $\alpha=0.2065508316$ as counting distribution.

In both cases, we use the distribution in [5,pp.178-229] as severity distribution and we apply only the approximated method since the exact ones are less efficient and not flexible at all (results being equal). On this subject, see also [2].

Table 4. Aggregate claims cumulative distribution by the approximated method using a Poisson counting distribution*

Realizations**	Cumulative distribution
$m-20.00\sigma$	0.007895808
$m-1.50\sigma$	0.041573460
$m-1.00\sigma$	0.158407992
$m-0.75\sigma$	0.239006941
$m-0.50\sigma$	0.335345410
$m-0.25\sigma$	0.443531561

m	0.543173551
m + 0.25σ	0.628233889
m + 0.50σ	0.713022866
m + 0.60σ	0.745325000
m + 0.75σ	0.781696707
m + 1.00σ	0.839987301
m + 1.50σ	0.921775029
m + 2.00σ	0.965070471
m + 2.50σ	0.985514660
m + 3.00σ	0.994508388
m + 4.00σ	0.999385086
m + 5.00σ	0.999947238
m + 6.00σ	0.999996345
m + 8.00σ	0.999999991
m + 10.00σ	1.000000000

** The calculation is performed by neglecting the probabilities lower than $\varepsilon=10^{-51}$.*

*** Where m=207.025055 is the mean value and σ=102.602026 is the standard deviation.*

Table 5. Aggregate claims cumulative distribution by the approximated method using a Negative Binomial counting distribution*

Realizations**	Cumulative distribution
m – 20.00σ	0.012263691
m – 1.50σ	0.039570081
m – 1.00σ	0.157682964
m – 0.75σ	0.248705182
m – 0.50σ	0.346776542
m – 0.25σ	0.439798971
m	0.550177095
m + 0.25σ	0.632109689
m + 0.50σ	0.718709110
m + 0.60σ	0.746800061
m + 0.75σ	0.788076093
m + 1.00σ	0.843476297
m + 1.50σ	0.919629987
m + 2.00σ	0.962628379
m + 2.50σ	0.983628885
m + 3.00σ	0.993394096
m + 4.00σ	0.999069226
m + 5.00σ	0.999891334
m + 6.00σ	0.999989109
m + 8.00σ	0.999999728
m + 16.00σ	0.999999760

** The calculation is performed by neglecting the probabilities lower than $\varepsilon=10^{-51}$.*

*** Where m=207.025055 is the mean value and σ=111.156177 is the standard deviation.*

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