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A new method for evaluating the distribution of aggregate claims ☆

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Abstract

In the present paper, we propose a method of practical utility for calculating the aggregate claims distribution in a discrete framework.

It is an approximated method but unlike the other approximated methods proposed in the literature:

- the approximation concerns both the counting distribution and the convolution of the severity distributions;
- the approximation does not consist in truncating the original distribution up to a given number of terms nor in replacing it with another distribution or a more general function (but simply in considering only the significant numerical realizations and in neglecting the others);
- the resulting approximation of the aggregate claims distribution is lower than a prefixed maximum error (10^{-6} in our applications). In particular, the probability distribution and also the first three moments are exact with the prefixed maximum error.

The proposed method does not require special assumptions on the counting distribution nor the identical distribution of the severity random variables and it does not incur in underflow and overflow computational problems.

It proves to be more flexible, easier and cheaper than the (exact and approximated) methods using recursion and Fast Fourier Transform.

We show some applications using both a Poisson distribution and a Generalized Pareto mixture of Poisson distributions as counting distribution.

In addition to the specific application proposed in this paper, the method can be applied in many other (life and non-life) actuarial fields where the sum of discrete random variables and the calculation of compound distributions are involved. Besides, it can be extended in multivariate cases.

☆ The paper is credited to M.G. Bruno.

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1. Introduction

In the present paper, we propose a method of practical utility for calculating the aggregate claims distribution in a discrete framework.

For any non-negative integer s, let P_s denote the class of all discrete probability densities f on the non-negative integers such that $f(s) > 0$ and $f(x) = 0$ for all $x < s$ and let $\overline{P}_s = \bigcup_{j=s}^{\infty} P_j$.

Let N denote the counting variable, i.e. the non-negative, integer-valued random variable counting the number of claims occurring in an insurance portfolio within a given period of time.

Let $p \in \overline{P}_0$ be the discrete density of N.

Further, let U_i ($i = 1, \ldots, N$) denote the severity random variables, i.e. the random variables representing the individual claim amount. We assume that they are discrete (say random integers multiples of some monetary unit), strictly positive, mutually independent (but not necessarily identically distributed) and independent of the counting variable N.

Let $f_i \in \overline{P}_1$ be the discrete density of U_i ($i = 1, \ldots, N$).

Notice that in practical applications we can always determine a finite maximum number of possible claims and, for each claim, a finite maximum amount. This immediately follows by considering that when using a P.C. the precision of the very small probabilities is ensured only up to a given number of digits (up to the eighteenth digit with a high speed execution language like C++). The papers by Sundt [21] and Dhaene and Sundt [7] can be useful to derive the error bounds for the resulting aggregate claims distribution.

Then, let mn be the maximum number of occurring claims and, for each claim i ($i = 1, \ldots, mn$), let mx_i be the maximum amount. Besides, let $mx^{(n)} = \sum_{i=1}^{n} mx_i$ and let $mx = mx^{(mn)}$.

Then, the aggregate claims amount is given by the following random variable:

$$X = \begin{cases} \sum_{i=1}^{N} U_i & \text{if } N \geqslant 1, \\ 0 & \text{if } N = 0 \end{cases}$$

with discrete probability density,

$$g(x) = \sum_{n=0}^{mn} p(n)(*_{i=1}^{n} f_i)(x) \quad \text{for } x = 0, 1, \ldots, mx, \tag{1}$$

where $(*_{i=1}^{n} f_i)$ denote the convolution of the first n discrete densities f_i. Obviously, we have $g(0) = p(0)$ and thus $g \in \overline{P}_0$.

In the present paper, we identify a probability distribution on the integers by its discrete density. We therefore usually mean its discrete density when talking about a distribution. In particular, we mean the evaluation of Eq. (1) when referring to the problem of calculating the aggregate claims distribution.

Eq. (1) involves the calculation of the products $p(n)(*_{i=1}^{n} f_i)(x)$ for all n and x. We can perform this calculation in two different ways:

1. sequentially for each x, that is by letting n vary for each fixed x; or
2. simultaneously for all x, that is by letting x vary for each fixed n.

In the former case, an obvious simplification of Eq. (1) is

$$g(x) = \sum_{n=0}^{x} p(n)(*_{i=1}^{n} f_i)(x) \quad \text{for } x = 0, 1, 2, \ldots, mx \tag{2}$$

since the convolution $(*_{i=1}^{n} f_i)(x)$ and hence the product $p(n)(*_{i=1}^{n} f_i)(x)$ are certainly null for all $n > x$.

In the latter case, a natural reduction of Eq. (1) is instead

$$g(x) = \sum_{n:p(n)>0} p(n)(*_{i=1}^{n} f_i)(x) \quad \text{for } x = 0, 1, 2, \ldots, mx. \tag{3}$$

Despite the above-mentioned simplifications, in most of the practical applications we cannot compute Eq. (1) in a direct way.

As a consequence, various methods have been proposed in the literature for evaluating it. In Section 2, we make a brief reference to Sundt's generalization of Panjer's recursion and to the recursive methods based on De Pril Transform while in Section 3, we discuss the methods actively used in practice based on Fast Fourier Transform (FFT).

However, by putting together the ideas underlying Eqs. (2) and (3), we can furtherly simplify Eq. (1) and then be able to compute it directly.

Virtually, we have to determine those $n : p(n) > 0$ and, for each n, those $x : (*_{i=1}^{n} f_i)(x) > 0$. In this way, we can reduce the number of operations to be done since the products $p(n)(*_{i=1}^{n} f_i)(x)$ are certainly null otherwise. In Section 4, we propose a general and flexible method to do this.

It is an approximated method but unlike the other approximated methods proposed in the literature ([2, pp. 385–392,9, pp. 55–65,18]; in the continuous case, also [1]):

- the approximation concerns both the counting distribution and the convolution of the severity distributions;
- the approximation does not consist in truncating the original distribution up to a given number of terms nor in replacing it with another distribution or a more general function;
- the resulting approximation of the aggregate claims distribution can be as accurate as we desire.

According to this method, we consider non-null the probabilities not lower than a suitably determined value.

For the counting distribution, this value is the maximum number not greater than 10^{-21} such that the sum of the probabilities not lower than it is equal to one with a given maximum error and simultaneously the first three moments of the aggregate claims distribution obtained by neglecting the others are exact with the same maximum error (lower than 10^{-10} in our applications).

For each convolution, instead the above-mentioned value is the maximum number not greater than 10^{-21} such that the sum of the probabilities not lower than it is equal to one with a given maximum error and simultaneously the first three moments of the distribution obtained by neglecting the others are exact with the same maximum error (lower than 10^{-6} in our applications).

We show that the resulting aggregate claims distribution and its first three moments are exact with a maximum error equal to the greater among those chosen in the approximation to the counting distribution and to the convolutions (say, an error lower than 10^{-6} in our applications).

This method can be applied in any practical circumstances since it does not require special assumptions on the counting distribution nor the identical distribution of the severity random variables and it does not incur in underflow and overflow computational problems.

It therefore proves to be more flexible, easier and cheaper than the (exact and approximated) methods using recursion and FFT.

In Section 5, we show some applications of the proposed method using both a Poisson distribution and a Generalized Pareto mixture of Poisson distributions as counting distribution. In the former case, we also compare the obtained results with those of the other methods existing in literature quoted in the paper.

It may be useful to notice that the main purpose of the paper is to illustrate a new calculation methodology and not the specific software elaborated for implementing it (even though it is important especially for reducing the execution time). This methodology can also be applied in many other (life and non-life) actuarial fields where the sum of discrete random variables (with integer or referable to integer – positive, negative and also null – numerical realizations) and the calculation of compound distributions are involved and it can be extended in multivariate cases.

2. Recursive methods

In this section, we make a brief reference to some recursive methods existing in the literature useful for evaluating the aggregate claims distribution. We can group them into two general classes: methods based on Panjer's formula and its generalizations and methods based on De Pril Transform.

The methods of the former class assume that the severity random variables are identically distributed with common probability density $h \in \overline{P}_1$.

In this case, by also considering (2), Eq. (1) becomes

$$g(x) = \sum_{n=0}^{x} p(n)h^{*n}(x) \quad \text{for } x = 0, 1, 2, \ldots, mx, \tag{4}$$

with $g(0) = p(0)$, where h^{*n} denote the n-fold convolution of the severity distribution.

They also assume that the counting distribution has a positive probability in zero.

On the contrary, the methods using De Pril Transform can also be applied when the severity random variables are not identically distributed and $p \in \overline{P}_0$.

Unfortunately, these methods as well as the previous ones may be rather time-consuming. Thus, they can be applied only in some practical circumstances or in an approximated way.

Besides, as well as all recursive methods, they incur in underflow problems which are not always easy to overcome and which therefore furtherly restrict their concrete applicability.

2.1. Sundt's generalization of Panjer's formula

Sundt [22] studied the class of counting distributions $p \in P_0$ satisfying:

$$p(n) = \sum_{j=1}^{k} \left(a_j + \frac{b_j}{n}\right) p(n-j) \quad \text{for } n = \omega+1, \omega+2, \ldots, \tag{5}$$

for some integer $k \leqslant \infty$ and constants a_j, b_j ($j = 1, \ldots, k$).

He derived the following recursion for the corresponding compound distribution with $h \in \overline{P}_1$:

$$g(x) = \sum_{y=1}^{x} \sum_{j=1}^{k} \left(a_j + \frac{b_j}{j}\frac{y}{x}\right) h^{*j}(y) g(x-y) + \sum_{n=1}^{\omega} \left[p(n) - \sum_{j=1}^{k} \left(a_j + \frac{b_j}{n}\right) p(n-j) \right] h^{*n}(x)$$
$$\text{for } x = 1, 2, \ldots, \tag{6}$$

with $g(0) = p(0)$.

He also proved that every discrete density $p \in P_0$ on the range $\{0, 1, \ldots, k\}$ with $k \leqslant \infty$ satisfies Eq. (5) with:

$$a_j = -\frac{p(j)}{p(0)}, \quad b_j = 2j\frac{p(j)}{p(0)} \quad \text{for } j = 1, \ldots, k$$

and $\omega = 0$.

Then, on a theoretical level, when $p \in P_0$ we can always evaluate Eq. (4) recursively by (6).

Not every discrete distribution, however, can fit into (5) with a finite number of parameters.

When $k = 1$ and $\omega = 0$, Eq. (6) reduces to the well-known Panjer's formula [13] and only Poisson, Negative Binomial and Binomial counting distributions satisfy Eq. (5). According to Panjer and Wang [15], various counting distributions satisfying (5) can also be found for $k \leqslant 2$ and $\omega \geqslant 1$ but many others can fit into (5) with an infinite k only. The authors mentioned for instance the Poisson–Inverse Gaussian distribution and the Generalized Poisson.

In this case and more generally for large k and ω, Eq. (6) may be difficult to use because of the high order of convolutions involved, that is for the same reason which had motivated recursive evaluation of Eq. (4).

Another difficult aspect of Eq. (6) is the underflow in which it can incur when practically applied. This problem has been discussed by several authors when $k = 1$ [16,14,27] but it has not been solved for any k and ω.

2.2. Methods based on De Pril Transform

According to Sundt [24], the De Pril Transform of a distribution $f \in P_s$ for any non-negative integer s is defined by the following recursion:

$$\varphi_f(x) = \frac{1}{f(s)}\left[xf(x+s) - \sum_{y=1}^{x-1}\varphi_f(y)f(x+s-y)\right] \quad \text{for } x = 1, 2, \ldots \tag{7}$$

with $\varphi_f(0) = s$.

Further, a distribution $f \in P_s$ is uniquely determined by its De Pril Transform. This was already proved by Sundt [23] in the special case of distributions in P_0.

In particular, by solving Eq. (7) with respect to $f(x)$ we obtain

$$f(x) = \frac{1}{x-s}\sum_{y=1}^{x-s}\varphi_f(y)f(x-y) \quad \text{for } x = s+1, s+2, \ldots \tag{8}$$

Conversely, we obtain Eq. (7) by solving (8) with respect to $\varphi_f(x)$.

As for the evaluation of the aggregate claims distribution, we can use De Pril Transforms for calculating the convolutions involved in Eq. (1).

According to Sundt [24], the convolution of a finite number of distributions (not necessarily identical) in $\overline{P}_0$ is a distribution in $\overline{P}_0$ and its De Pril Transform is the sum of the De Pril Transforms of these distributions. This is also proved by Dhaene and Sundt [8] in the special case of distributions in P_0.

Then, for calculating the convolution of the first n severity distributions:

- first of all, we can find the De Pril Transforms of the severity random variables by using Eq. (7);
- then, we can find the De Pril's Transform of their convolution by summing their De Pril Transforms;
- finally, we can derive the discrete density of the convolution recursively by (8).

Unfortunately, this procedure may be rather time-consuming for evaluating the aggregate claims distribution since we have to repeat it for all n.

According to Sundt [24], we can simplify this procedure by approximating the De Pril Transforms of the severity distributions with the De Pril Transforms of more general functions (not necessarily probability distributions themselves). Such approximations are also discussed by Dhaene and Sundt [8] and Sundt [25] in the special case of distributions in P_0.

When applying such approximations, however, it is not easy to deduce the error bounds for the resulting approximation to Eq. (1).

An alternative procedure using De Pril's Transforms can be applied for evaluating the aggregate distributions when the severity random variables are identically distributed.

According to Sundt [24], the De Pril Transform of a compound distribution g with $p \in \overline{P}_0$ and $h \in \overline{P}_1$ satisfies

$$\varphi_g(x) = \varphi_p(0)\varphi_h(x) + x\sum_{j=1}^{x}\frac{\varphi_p(j)}{j}h^{*j}(x) \quad \text{for } x = 0, 1, \ldots \tag{9}$$

Then, we can evaluate Eq. (4) in the following way:

- first of all, as we know the counting distribution and the severity distribution, we can find their De Pril Transform by using Eq. (7);
- then, we can find the De Pril Transform of g by using (9);
- finally, we can obtain g recursively by (8).

This procedure, however, does not simplify the evaluation of the aggregate claims distribution since Eq. (9) involves the same order of convolutions than the original equation (4).

A critical aspect of both the above-mentioned procedures concerns the application of Eq. (8). As a matter of fact, we have to preliminarily calculate $f(s)$, i.e. the first value of f not null, but in many practical applications this value may be so small as to cause underflow problems.

3. Methods using FFT

Buhlmann [6] discussed the application of FFT for evaluating a compound Poisson distribution. In this case and generally speaking when the Continuous Fourier Transform (CFT) is known, the aggregate claims distribution can be evaluated by sampling the CFT (in probability theory also called characteristic function) and by inverting it. Obviously, this implies a discretization error.

In this section, we discuss the application of FFT in the more general case in which the CFT is not known. In this case, no special assumption on the counting distribution nor the identical distribution of the severity random variables are needed but stability and overflow problems are possible.

FFT is considered a valid alternative to the original method of Heckman and Meyers [10] for evaluating the aggregate claims distribution by inversion of its characteristic function. As such, it has been included in the recent version 2.2 of the program CrimCalc [12] used in actuarial practice.

3.1. Formalization

We can start by defining the Discrete Fourier Transform (DFT) of the aggregate claims distribution. From Eq. (1), we obtain

$$\phi_g(y) = \sum_{x=0}^{mx} g(x)\exp\left\{\mathrm{i}y\frac{2\pi}{mx+1}x\right\} \tag{10}$$

$$= \sum_{n=0}^{mn} p(n)\left(\sum_{x=0}^{mx}(*_{i=1}^{n}f_i)(x)\exp\left\{\mathrm{i}y\frac{2\pi}{mx+1}x\right\}\right) \tag{11}$$

$$= \sum_{n=0}^{mn} p(n)\phi_{(*_{i=1}^{n}f_i)}(y) \tag{12}$$

for $y = 0, 1, \ldots, mx$

with $\phi_{(*_{i=1}^{0}f_i)}(y) = 0\ \forall y$. For the independence of the severity random variables, we also obtain for $n = 1, \ldots, mn$:

$$\phi_{(*_{i=1}^{n}f_i)}(y) = \prod_{i=1}^{n}\phi_{f_i}(y) = \phi_{(*_{i=1}^{n-1}f_i)}(y)\phi_{f_n}(y), \tag{13}$$

where

$$\phi_{f_i}(y) = \sum_{x=0}^{mx} f_i(x)\exp\left\{\mathrm{i}y\frac{2\pi}{mx+1}x\right\}.$$

It is easily seen that g is the Inverse Fourier Transform (IDFT) of Eq. (10) and, for each n, $(*_{i=1}^{n}f_i)$ is the IDFT of Eq. (13).

Then, for evaluating Eq. (1), we can proceed in two different ways:

(a) We first compute Eq. (12) and then we perform its inverse by

$$g(x) = \frac{1}{mx+1}\sum_{y=0}^{mx}\phi_g(y)\exp\left\{-\mathrm{i}x\frac{2\pi}{mx+1}y\right\} \quad \text{for } x = 0, 1, \ldots, mx. \tag{14}$$

(b) We directly apply Eq. (1) after computing the convolutions for each n ($n = 1, \ldots, mn$) by Eq. (13) and by

$$(*_{i=1}^{n}f_i)(x) = \frac{1}{mx+1}\sum_{y=0}^{mx}\phi_{(*_{i=1}^{n}f_i)}(y)\exp\left\{-\mathrm{i}x\frac{2\pi}{mx+1}y\right\} \quad \text{for } x = 0, 1, \ldots, mx. \tag{15}$$

With respect to the former, the latter procedure requires the computation of a greater number of transforms but each transform has a lower dimension. As a matter of fact, for each n, the values of x greater than $mx^{(n)}$ have a null probability, thus we do not have to consider them when calculating Eqs. (13) and (15).

This makes the two procedures virtually equivalent in the applications.

In both cases, we can apply FFT for speeding up the computations. This method consists in computing DFTs and IDFTs stepwise thus reducing the number of complex multiplications.

For a detailed description of FFT, we refer to the extensive literature on the subject [3,4,19,20,26,28].

We remind however that for each transform with $(mx^{(n)} + 1)$ data points, the FFT requires $2^{\gamma_n}\gamma_n/2$ complex multiplications, where

$$\gamma_n = \mathrm{INT}\left(\log_2 mx^{(n)}\right) + 1. \tag{16}$$

This condition allows for the fact that FFT is usually applied for transforms with a length equal to a power of 2. As a matter of fact, there exist algorithms with a different base but the time saving resulting from the lower number of multiplications does not balance the waste due to other complications [19, pp. 509–510].

3.2. Number of real multiplications

For calculating the number of multiplications involved in the evaluation of the aggregate claims distribution by FFT, we have to consider that each forward transform yields a complex vector whereas each inverse transform (expressing a probability) yields a real vector. Besides, each complex multiplication implies four real ones. Therefore:

- each FFT with 2^{γ_n} data points (below denoted by 2^{γ_n}-FFT) requires $2^{\gamma_n+1}\gamma_n$ real multiplications;
- the point by point product of two forward transforms of dimension 2^{γ_n} (below denoted by 2^{γ_n}-CC) requires 2^{γ_n+2} real multiplications;
- the point by point product of a probability and a forward transform of dimension 2^{γ_n} (below denoted by 2^{γ_n}-RC) requires 2^{γ_n+1} real multiplications;
- the point by point product of a probability and an inverse transform of dimension $(mx^{(n)} + 1)$ (below denoted by $(mx^{(n)} + 1)$-RR) requires $(mx^{(n)} + 1)$ real multiplications.

3.2.1. An introductory example

In order to illustrate the problem as clearly as possible, we start with an example.

Let us assume $mx_i = 60\ \forall i$.

For calculating the convolution of order 2, we should consider a number of data points equal to the product between the maximum numerical realization of the severity distributions (say 60) and the order of convolution (say 2), hence 120. However, as the number of data points must be a power of 2, the same value rises to $2^7 = 128$ (with $\gamma_2 = \mathrm{INT}(\log_2 120) + 1 = 7$).

Besides, notice that the convolution requires:

- three transforms (one for the random variable "sum" resulting from the previous order convolution, another one for the convolving random variable and a last one for calculating the inverse that is the result of convolution);
- the product between two forward transforms.

Then, the valued number of multiplications is approximately

$$3 \cdot 2 \cdot (2^{\gamma_2} \cdot \gamma_2) + 4 \cdot 2^{\gamma_2} = 6 \cdot (128 \cdot 7) + 4 \cdot 128 = 5888.$$

For calculating the convolution of order 3, we have to develop the calculation between the convolution of order 2 and the third severity random variable. In this case, the maximum numerical realization is $60 \cdot 3 = 180$ and the maximum transforms dimension becomes $2^8 = 256$ (with $\gamma_3 = \mathrm{INT}(\log_2 180) + 1 = 8$).

Then, the valued number of multiplications is about

$$3 \cdot 2 \cdot (2^{\gamma_3} \cdot \gamma_3) + 4 \cdot 2^{\gamma_3} = 6 \cdot (256 \cdot 8) + 4 \cdot 256 = 13,312$$

and so on with a more than proportional increasing.

Generally speaking, the number of multiplications for the convolution of order n is approximately

$$3 \cdot 2 \cdot (2^{\gamma_n} \cdot \gamma_n) + 4 \cdot 2^{\gamma_n}.$$

It follows that the number of multiplications for all the convolutions up to the one of maximum order mn is about

$$\sum_{n=2}^{mn} [3 \cdot 2 \cdot (2^{\gamma_n} \cdot \gamma_n) + 4 \cdot 2^{\gamma_n}].$$

Thus, for a total of $mn = 3$ convolutions, 5888 + 13,312 = 19,200 real multiplications.

3.2.2. The complete count

In the previous example, we have computed only the number of real multiplications needed for developing the convolutions according to the procedure (b) illustrated in Section 3.1. Besides, we have not accounted for some possible simplifications of the calculation.

Below, we instead show the number of real multiplications needed for calculating the entire aggregate claims distribution (in the general circumstances) according both to procedure (a) and to procedure (b). In the latter case, we also consider the reductions due to the transforms reassessment.

The number of real multiplications needed for developing the procedure (a) by FFT is

$$\mathrm{MTCD}_{\mathrm{FFT_a}} = 2^{\gamma_{mn}}[2\gamma_{mn}(mn+1) + 6mn - 4].$$

As a matter of fact, we have to perform:

- for $n = 1$, one $2^{\gamma_{mn}}$-FFT for calculating ϕ_{f_1} and one $2^{\gamma_{mn}}$-RC for calculating $p(1)\phi_{(*_{i=1}^{1} f_i)}$ (being $\phi_{(*_{i=1}^{1} f_i)} = \phi_{f_1}$);
- for $n = 2, \ldots, mn$, one $2^{\gamma_{mn}}$-FFT for calculating ϕ_{f_n}, one $2^{\gamma_{mn}}$-CC for calculating $\phi_{(*_{i=1}^{n} f_i)}$ (being $\phi_{(*_{i=1}^{n} f_i)} = \phi_{(*_{i=1}^{n-1} f_i)}\phi_{f_n}$) and one $2^{\gamma_{mn}}$-RC for calculating $p(n)\phi_{(*_{i=1}^{n} f_i)}$;
- finally, one $2^{\gamma_{mn}}$-FFT for calculating (14).

When the severity random variables are identically distributed, this number drops to

$$\mathrm{MTCD}_{\mathrm{FFT_a}} = 2^{\gamma_{mn}}(4\gamma_{mn} + 6mn - 4). \tag{17}$$

This is because in this case, for $n = 2, \ldots, mn$, $\phi_{f_n} = \phi_{f_1}$ and hence we have not to compute a new FFT at each convolution.

Similarly, the number of real multiplications needed for carrying out the procedure (b) by FFT is in the general case

$$\mathrm{MTCD}_{\mathrm{FFT_b}} = (mx^{(1)} + 1) + \sum_{n=2}^{mn} \left[2^{\gamma_n}(4\gamma_n + 4) + (mx^{(n)} + 1)\right] + \sum_{j=\gamma_2}^{\gamma_{mn}} 2^{j+1} j.$$

Actually, we have to develop:

- for $n = 1$, one $(mx^{(1)} + 1)$-RR for calculating $p(1)(*_{i=1}^{1} f_i)$ (being $(*_{i=1}^{1} f_i) = f_1$);
- for $n = 2, \ldots, mn$, one 2^{γ_n}-FFT for calculating ϕ_{f_n}, one 2^{γ_n}-CC for calculating $\phi_{(*_{i=1}^{n} f_i)}$ (being $\phi_{(*_{i=1}^{n} f_i)} = \phi_{(*_{i=1}^{n-1} f_i)}\phi_{f_n}$ but with $\phi_{(*_{i=1}^{n-1} f_i)}$ to be recalculated in the new dimension 2^{γ_n} as explained below), one 2^{γ_n}-FFT for calculating Eq. (15) and one $(mx^{(n)} + 1)$-RR for calculating $p(n)(*_{i=1}^{n} f_i)$;
- for $j = \gamma_2, \ldots, \gamma_{mn}$, one 2^j-FFT for calculating $\phi_{(*_{i=1}^{n-1} f_i)}$ (according to the expression in brackets in Eq. (11)). This follows by considering that, n being equal, $\phi_{(*_{i=1}^{n-1} f_i)}$ changes when the dimension changes, thus we have to compute it again at each reassessment. When using FFT, the reassessment does not occur at each convolution but only when n gives rise to a new power of 2, that is $(\gamma_{mn} - \gamma_2 + 1)$ times.

In the special case of identically distributed severity random variables, the number of real multiplications becomes

$$\mathrm{MTCD}_{\mathrm{FFT_b}} = (mx^{(1)} + 1) + \sum_{n=2}^{mn} \left[2^{\gamma_n}(2\gamma_n + 4) + (mx^{(n)} + 1)\right] + \sum_{j=\gamma_2}^{\gamma_{mn}} 2^{j+2} j. \tag{18}$$

As a matter of fact in this case, for $n = 2, \ldots, mn$ and dimension being equal, $\phi_{f_n} = \phi_{f_1}$, thus we have to compute again this transform at each reassessment only.

In this case, the number of real multiplications needed for computing solely the convolutions is

$$\mathrm{MCONV}_{\mathrm{FFT_b}} = \sum_{n=2}^{mn} 2^{\gamma_n}(2\gamma_n + 4) + \sum_{j=\gamma_2}^{\gamma_{mn}} 2^{j+2} j. \tag{19}$$

The number of multiplications is useful to compare the efficiency of the two different evaluation procedures using FFT but it is purely an indication of the computational effort needed for their execution.

As a matter of fact, this number does not show the complexity due to various other additional operations required by FFT.

Notice for instance that, for each transform, we have to define two tables (one for the real part and the other for the imaginary part) each one of dimension up to $2^{\gamma_{mn}}$ (or only one table with a double dimension). The definition and management of these tables take up memory and slow down the procedures execution.

Besides, for each distribution, the numerical realizations are expressed as a function of the position of the corresponding probability inside the tables. Then, in order to avoid to lose this connection (or in order to re-establish it) after transforms calculation, we have to rearrange the input data (or the output data) in a bit-reverse order. This requires a further computational effort.

For the above-mentioned reasons, the computational effort rises more than proportionally as the number of convolutions goes up.

Further, the various complex operations usually cause approximation problems in the applications.

4. The new method proposed

In this section, we propose a new method for evaluating the aggregate claims distribution.

It requires no special assumption on the counting distribution nor the identical distribution of the severity random variables. Besides, it develops the direct calculation of Eq. (1) without applying any transform and it does not incur in underflow and overflow problems.

Obviously, this is an approximated method but unlike the other approximated methods (among which the methods based on De Pril Transform):

- the approximation concerns both the counting distribution and the convolution of the severity distributions;
- the approximation does not consist in truncating the original distribution up to a given number of terms nor in replacing it with another distribution or a more general function (but simply in considering only the significant numerical realizations and in neglecting the others);
- the resulting approximation of the aggregate claims distribution has a prefixed maximum error.

The method is actually composed of two logically distinct phases even if they are performed by the same software:

1. during the former, we identify the values of the counting distribution significant for the following evaluation of the aggregate claims distribution;
2. during the latter, we perform all the convolutions up to the one of maximum order by identifying after each convolution the significant distinct numerical realizations and by neglecting the others in the calculation of the next order convolution.

In the first phase, we consider significant those values of n with probability not lower than a suitably determined value. This value is the maximum number not greater than 10^{-21} such that the sum of the probabilities not lower than it is equal to one with a given maximum error and simultaneously the first three moments of the aggregate claims distribution obtained by neglecting the others differ in relative terms from the corresponding exact values with the same maximum error (lower than 10^{-10} in our applications).

In the second phase, we consider significant for each convolution those values of x with probability not smaller than a suitably determined value. This value is the maximum number not greater than 10^{-21} such that the distribution obtained by neglecting the realizations with probabilities smaller than it is exact with a desired maximum error (lower than 10^{-6} in our applications). We say that a distribution is exact with a given error if the sum of its probabilities is equal to one with that error and simultaneously the first three moments differ in relative terms from the corresponding exact values with the same error.

In the end, we perform the direct calculation of Eq. (1) for only the significant values of n and x determined in the previous phases.

The aggregate claims distribution obtained in such a way is exact in the above-mentioned sense with a maximum error equal to the greater between the one chosen in the approximation to the counting distribution and the one chosen in the approximation to the convolutions (say, an error lower than 10^{-6} in our applications). They are also exact at least up to the decimal point defined by this error (the sixth in our applications) the probabilities of all the numerical realizations.

4.1. Calculation phases

Let $E_{(*_{i=1}^{n}f_i)}$, $E^2_{(*_{i=1}^{n}f_i)}$ and $E^3_{(*_{i=1}^{n}f_i)}$ be the first three exact moments of each convolution. They are automatically calculated by the software during the reading of the severity random variables input data.

Further, let E_g, E_g^2 and E_g^3 be the first three exact moments of the aggregate claims distribution. For Eq. (1), we have

$$E_g = \sum_{n=0}^{mn} p(n) E_{(*_{i=1}^{n}f_i)},$$

$$E_g^2 = \sum_{n=0}^{mn} p(n) E^2_{(*_{i=1}^{n}f_i)},$$

$$E_g^3 = \sum_{n=0}^{mn} p(n) E^3_{(*_{i=1}^{n}f_i)}.$$

The first phase of the proposed method concerns the identification of the significant values of the counting distribution.

Let Ω denote the set of such values. This set is given by those values n of the counting distribution with probability not smaller than the maximum number not greater than 10^{-21} such that the following conditions are simultaneously fulfilled:

$$1 - \sum_{n\in\Omega} p(n) < 10^{-10}, \tag{20}$$

$$1 - \frac{\sum_{n\in\Omega} p(n) E_{(*_{i=1}^{n}f_i)}}{E_g} < 10^{-10}, \tag{21}$$

$$1 - \frac{\sum_{n\in\Omega} p(n) E^2_{(*_{i=1}^{n}f_i)}}{E_g^2} < 10^{-10}, \tag{22}$$

$$1 - \frac{\sum_{n\in\Omega} p(n) E^3_{(*_{i=1}^{n}f_i)}}{E_g^3} < 10^{-10}. \tag{23}$$

Notice that we have chosen the value 10^{-10} according to the results obtained in a number of practical applications from 1996. We can however furtherly reduce this value if necessary.

The second phase concerns the calculation of the severity random variable convolutions.

For the purpose, we use the approximated procedure proposed by Bruno et al. [5] but we extend the approximation control to the third moment of the resulting distribution.

We proceed recursively for $n = 1, 2, \ldots, mn$. In particular, for each n, we approximate the convolution by the distribution $(*_{i=1}^{n} f_i)^{(a)}$ obtained by considering only the significant distinct values of $(*_{i=1}^{n-1} f_i)^{(a)} * f_n$, with $(*_{i=1}^{0} f_i)^{(a)} * f_1 = f_1$.

Let $\Lambda(n)$ be the set of such values for each n. This set is given by those values x of $(*_{i=1}^{n-1} f_i)^{(a)} * f_n$ with probability not smaller than the maximum number not greater than 10^{-21} such that the following conditions are simultaneously fulfilled:

$$1 - \sum_{x \in \Lambda(n)} (*_{i=1}^{n} f_i)^{(a)}(x) < 10^{-6}, \tag{24}$$

$$1 - \frac{E_{(*_{i=1}^{n} f_i)^{(a)}}}{E_{(*_{i=1}^{n} f_i)}} < 10^{-6}, \tag{25}$$

$$1 - \frac{E^2_{(*_{i=1}^{n} f_i)^{(a)}}}{E^2_{(*_{i=1}^{n} f_i)}} < 10^{-6}, \tag{26}$$

$$1 - \frac{E^3_{(*_{i=1}^{n} f_i)^{(a)}}}{E^3_{(*_{i=1}^{n} f_i)}} < 10^{-6}, \tag{27}$$

where $E_{(*_{i=1}^{n} f_i)^{(a)}}$, $E^2_{(*_{i=1}^{n} f_i)^{(a)}}$, and $E^3_{(*_{i=1}^{n} f_i)^{(a)}}$ are the first three moments of the approximated convolution.

This procedure has various advantages in the calculation of convolutions.

Notice that, thanks to the conditions (24)–(27), this procedure gives exact results at least up to the sixth decimal point for the individual probabilities of each convolution.

Besides, it is easier and more efficient than the FFT for two main reasons:

- it enables a substantial reduction of the numerical realizations to consider (hence of the multiplications to perform) at each convolution;
- it does not present the complications of the FFT due to complex operations, bit-reversal, and so on.

At the end of the two previous phases, we perform the direct calculation of the aggregate claims distribution for only the identified values of n and x.

Virtually, we approximate Eq. (1) by

$$g^{(a)}(x) = \sum_{n \in \Omega} p(n) (*_{i=1}^{n} f_i)^{(a)}(x) \quad \text{for } x \in \bigcup_{n \in \Omega} \Lambda(n) \tag{28}$$

with $g^{(a)}(0) = p(0)$.

This drastically speeds up the computation, excludes underflow possibilities and avoids overflow problems.

We also show that the maximum approximation is lower than 10^{-6}, that is of an order equal to the greater between the one chosen in the approximation to the counting distribution and the one chosen in the approximation to the convolutions.

4.2. Error bounds

We can easily prove that:

$$1 - \sum_{x \in \bigcup_{n \in \Omega} \Lambda(n)} g^{(a)}(x) < 10^{-6}.$$

As a matter of fact, according to Eqs. (28), (24) and (20), we have

$$1 - \sum_{x \in \bigcup_{n \in \Omega} \Lambda(n)} g^{(a)}(x) = 1 - \sum_{n \in \Omega} p(n) \sum_{x \in \bigcup_{n \in \Omega} \Lambda(n)} (*_{i=1}^{n} f_i)^{(a)}(x) < 1 - (1 - 10^{-10})(1 - 10^{-6}) \cong 10^{-6}.$$

We can also show that for $k = 1, 2, 3$:

$$1 - \frac{E^k_{g^{(a)}}}{E^k_g} < 10^{-6}.$$

As a matter of fact, according to Eq. (28) and, respectively, to Eqs. (25), (21), Eqs. (26), (22) and Eqs. (27), (23), we have

$$\begin{aligned}1 - \frac{E^k_{g^{(a)}}}{E^k_g} &= 1 - \frac{\sum_{n\in\Omega} p(n) E^k_{(*^n_{i=1} f_i)^{(a)}}}{E^k_g} \\ &< 1 - \frac{\sum_{n\in\Omega} p(n)(1-10^{-6}) E^k_{(*^n_{i=1} f_i)}}{E^k_g} \\ &< 1 - \frac{(1-10^{-10})E^k_g(1-10^{-6})}{E^k_g} \cong 10^{-6}.\end{aligned}$$

5. Applications

In this section, we show some applications of the proposed method.

For simplicity, we assume that the severity random variables are identically distributed with common probability density h [17, pp. 178, 229]: Table 1.

Therefore, we have $mx_i = 60\ \forall i$ and hence $mx^{(n)} = n \cdot 60$.

On the other hand, we take as mn the superior extreme of the reduced set Ω identified during the first phase of the proposed method.

We first show some results using a Poisson counting distribution. The method gives the same results of the other methods quoted in the paper (at least up to the sixth decimal point) and it takes the lowest processing time. Besides, it does not incur in underflow and overflow problems.

We also show some results using a Generalized Pareto mixture of Poisson distributions as counting distributions.

All the calculations have been carried out using a P.C. Pentium IV 2000 MHz and algorithms written in C++. For the FFT computation, we have used the algorithm illustrated in Press et al. [19, pp. 507–508].

Table 1
Severity distribution

Realizations	Probabilities
14	0.0103301
15	0.0307990
16	0.0293511
17	0.0103301
18	0.0730414
19	0.0111568
20	0.0264554
24	0.1002133
26	0.0815418
28	0.0252146
30	0.0212857
31	0.0254214
55	0.0991756
60	0.4556837
Otherwise	0

5.1. Poisson counting distribution

Here, we use a Poisson counting distribution with parameter $\lambda > 0$:

$$p(n) = e^{-\lambda}\frac{\lambda^n}{n!}.$$

In Table 2, for different values of λ, we show some data concerning the proposed method which are useful for a comparison with the other methods. In particular:

- in the first column, we show the different values of λ;
- in the second column, we show the extremes of the reduced set Ω identified during the first phase of the method. Remember that the superior extreme is the total number of convolutions mn to develop during the second phase;
- in the third column, we show the total number of distinct numerical realizations identified during the second phase after each convolution up to the one of order $mn - 1$.

The values in the last column of Table 2 multiplied the number of distinct numerical realizations of the severity distribution (say 14, in the case under study) provides an indication of the number of real multiplications needed for calculating all the convolutions up to the one of maximum order mn.

In Table 3, we compare such number of multiplications with the one needed for calculating the same number of convolutions by FFT. In particular:

- in the first column, we show the total number of convolutions to perform derived by Table 2;
- in the second column, we show the number of real multiplications for the proposed method given by:

$$\mathrm{MCONV_{PM}} = \text{Num. distinct realizations} \cdot 14.$$

- in the third column, we show the number of real multiplications for FFT given by Eq. (19), with γ_n given by (16);
- in the fourth column, we show the ratio between the values of the third column and those of the second one.

Table 2
Extremes of the reduced set Ω[a] and cumulative number of distinct realizations using a Poisson counting distribution with parameter λ

λ	Extremes of Ω	Num. distinct realizations
104.814259	20–180	519,182
204.814259	77–306	1,213,573
304.814259	145–426	2,031,981
504.814259	294–658	3,959,604
1004.814259	699–1217	10,051,481
5004.814259	4301–5475	95,554,905

[a] Ω is the set of the significant values of the counting distribution considered in the calculation.

Table 3
Number of real multiplications for calculating the convolutions

Number of convolutions	Proposed method	FFT	FFT/Proposed method
180	7,268,548	46,565,120	6.4
306	16,990,022	134,055,680	7.9
426	28,447,734	267,749,120	9.4
658	55,434,456	669,878,016	12.1
1217	140,720,734	2,325,317,376	16.5
5475	1,337,768,670	54,466,021,120	40.7

According to the data in Table 3, the application of FFT for calculating the convolutions requires a greater load in terms of number of real multiplications with respect to the proposed method (results being equal at least up to the sixth decimal point). Really, the load is even greater since for FFT we have not considered the additional operations of bit-reversal and for the proposed method we have not allowed for the further simplification obtained by eliminating the non-significant numerical realizations.

Besides, remember that the number of multiplications is only one of so many elements affecting the procedures processing time.

In Table 4, for different values of λ, we show the processing time in seconds for evaluating the entire aggregate claims distributions using the proposed method and all the other methods quoted in the paper. In particular:

- in the first column, we show the different values of λ;
- in the second column, we show the processing time of the proposed method;
- in the third column, we show the processing time taken by applying Eq. (6) with $k = 1$, $\omega = 0$, $a_1 = 0$ and $b_1 = \lambda$ and with mn derived from Table 2;
- in the fourth column, we show the processing time taken by applying Eq. (8) with $f = g$, $s = 0$ and $\varphi_g(x) = x\lambda h(x)$;
- in the fifth column, we show the processing time taken by applying FFT according to the procedure (b) illustrated in Section 3.1.

According to Table 4, the other methods require a higher processing time with respect to the proposed method (conditions and results being equal). Besides, the time ratio rises (exponentially in the case of FFT) as λ rises and for large λ the methods based on Panjer's formula and De Pril Transform incur in underflow problems.

It may be useful to notice that if we would evaluate the aggregate claims distribution using FFT according to the procedure (a) illustrated in Section 3.1, we could perhaps obtain a time saving. However, supposed that the time is proportional to the number of real multiplications, the possible reduction would not modify the result of the comparison with the proposed method.

Table 4
Processing time (in seconds)[a] for evaluating the aggregate claims distribution using a Poisson counting distribution with parameter λ

λ	Proposed method	Panjer	De Pril	FFT_b
104.814259	<1	2	2	2
204.814259	1	5	4	11
304.814259	2	9	10	24
504.814259	3	21	23	67
1004.814259	4	Underflow	Underflow	261
5004.814259	41	Underflow	Underflow	3850

[a] All the calculations have been carried out using a P.C. Pentium IV 2000 MHz and algorithms written in C++. For the FFT computation, we have used the algorithm illustrated in Press et al. [19, pp. 507–508].

Table 5
Number of real multiplications for evaluating the aggregate claims distribution using a Poisson counting distribution with parameter λ

λ	FFT_a	FFT_b	FFT_b/FFT_a
104.814259	18,546,688	47,542,700	2.6
204.814259	61,997,056	136,874,246	2.2
304.814259	85,590,016	273,206,606	3.2
504.814259	262,668,288	682,887,334	2.6
1004.814259	965,476,352	2,369,787,773	2.4
5004.814259	17,260,609,536	55,365,459,595	3.2

In Table 5, we compare the number of real multiplications required for calculating the entire aggregate claims distribution by procedure (a) and procedure (b). In particular:

- in the first column, we show the different values of λ;
- in the second column, we show the number of real multiplications given by Eq. (17) with mn derived by Table 2 and γ_{mn} given by Eq. (16);
- in the third column, we show the number of real multiplications given by Eq. (18) with γ_n given by (16);
- in the fourth column, we show the ratio between the data in the third column and those in the second one.

In Table 6, we show the aggregate claims distribution and the cumulative distribution obtained by applying the proposed method for $\lambda = 504.814259$. For space reasons, we show the values corresponding only to some numerical realizations (expressed as a function of the mean value and the standard deviation of the distribution).

The results in Table 6 (compared with those of the other methods) allow to verify that the probabilities of the individual numerical realizations are exact at least up to the sixth decimal point. As proved in Section 4.2, the same result holds for the first three moments of the distribution. Besides, the sum of the probabilities is equal to one with an error lower than 10^{-6}.

In the end, we remind that the proposed method can be applied even for values of λ for which the other methods cannot be applied.

In Table 7, we show the cumulative distribution obtained for $\lambda = 91{,}000$ (for a total of $mn = 92{,}832$ convolutions). The processing time taken for the entire calculation is 1950 s.

Table 6
Aggregate claims distribution and cumulative distribution using a Poisson counting distribution with parameter $\lambda = 504.814259$

Aggregate claims amount[a]	Probabilities	Cumulative distribution values
$\mu - 5.00\sigma \cong 16{,}347$	0.000000	0.000000
$\mu - 4.00\sigma \cong 17{,}395$	0.000000	0.000017
$\mu - 3.00\sigma \cong 18{,}443$	0.000004	0.001051
$\mu - 2.50\sigma \cong 18{,}967$	0.000015	0.005405
$\mu - 2.00\sigma \cong 19{,}491$	0.000051	0.021317
$\mu - 1.90\sigma \cong 19{,}595$	0.000062	0.027153
$\mu - 1.80\sigma \cong 19{,}700$	0.000075	0.034326
$\mu - 1.75\sigma \cong 19{,}752$	0.000082	0.038411
$\mu - 1.70\sigma \cong 19{,}805$	0.000090	0.042970
$\mu - 1.65\sigma \cong 19{,}857$	0.000098	0.047852
$\mu - 1.60\sigma \cong 19{,}910$	0.000107	0.053271
$\mu - 1.50\sigma \cong 20{,}014$	0.000125	0.065289
$\mu - 1.00\sigma \cong 20{,}538$	0.000235	0.158591
$\mu - 0.75\sigma \cong 20{,}800$	0.000292	0.227754
$\mu - 0.50\sigma \cong 21{,}062$	0.000340	0.310857
$\mu - 0.25\sigma \cong 21{,}324$	0.000371	0.404499
$\mu - 0.00\sigma \cong 21{,}586$	0.000381	0.503536
$\mu + 0.25\sigma \cong 21{,}848$	0.000367	0.601928
$\mu + 0.50\sigma \cong 22{,}110$	0.000332	0.693829
$\mu + 0.75\sigma \cong 22{,}372$	0.000283	0.774593
$\mu + 1.00\sigma \cong 22{,}634$	0.000227	0.841425
$\mu + 1.50\sigma \cong 23{,}158$	0.000122	0.931859
$\mu + 2.00\sigma \cong 23{,}681$	0.000052	0.975826
$\mu + 2.50\sigma \cong 24{,}205$	0.000018	0.992972
$\mu + 3.00\sigma \cong 24{,}729$	0.000005	0.998325
$\mu + 4.00\sigma \cong 25{,}777$	0.000000	0.999947
$\mu + 5.00\sigma \cong 26{,}824$	0.000000	0.999999
$\mu + 8.00\sigma \cong 29{,}968$	0.000000	1.000000

[a] $\mu = 21586.462129$ and $\sigma = 1047.695834$. They are, respectively, the mean value and the standard deviation of the aggregate claims distribution.

Table 7
Aggregate claims cumulative distribution using a Poisson counting distribution with parameter $\lambda = 91{,}000$

Aggregate claims amount[a]	Cumulative distribution values
$\mu - 5.00\sigma \cong 3{,}820{,}935$	0.000000
$\mu - 4.00\sigma \cong 3{,}835{,}002$	0.000030
$\mu - 3.00\sigma \cong 3{,}849{,}069$	0.001327
$\mu - 2.50\sigma \cong 3{,}856{,}102$	0.006149
$\mu - 2.00\sigma \cong 3{,}863{,}135$	0.022643
$\mu - 1.90\sigma \cong 3{,}864{,}542$	0.028604
$\mu - 1.80\sigma \cong 3{,}865{,}949$	0.035817
$\mu - 1.75\sigma \cong 3{,}866{,}652$	0.039943
$\mu - 1.70\sigma \cong 3{,}867{,}355$	0.044447
$\mu - 1.65\sigma \cong 3{,}868{,}059$	0.049359
$\mu - 1.60\sigma \cong 3{,}868{,}762$	0.054687
$\mu - 1.50\sigma \cong 3{,}870{,}169$	0.066705
$\mu - 1.00\sigma \cong 3{,}877{,}202$	0.158658
$\mu - 0.75\sigma \cong 3{,}880{,}718$	0.226704
$\mu - 0.50\sigma \cong 3{,}884{,}235$	0.308708
$\mu - 0.25\sigma \cong 3{,}887{,}752$	0.401538
$\mu - 0.00\sigma \cong 3{,}891{,}268$	0.500249
$\mu + 0.25\sigma \cong 3{,}894{,}785$	0.598941
$\mu + 0.50\sigma \cong 3{,}898{,}302$	0.691642
$\mu + 0.75\sigma \cong 3{,}901{,}818$	0.773450
$\mu + 1.00\sigma \cong 3{,}905{,}335$	0.841343
$\mu + 1.50\sigma \cong 3{,}912{,}368$	0.933083
$\mu + 2.00\sigma \cong 3{,}919{,}402$	0.977145
$\mu + 2.50\sigma \cong 3{,}926{,}435$	0.993730
$\mu + 3.00\sigma \cong 3{,}933{,}468$	0.998627
$\mu + 4.00\sigma \cong 3{,}947{,}535$	0.999967
$\mu + 5.00\sigma \cong 3{,}961{,}602$	1.000000
$\mu + 8.00\sigma \cong 4{,}003{,}802$	1.000000

[a] $\mu = 3891268.954300$ and $\sigma = 14066.631372$. They are, respectively, the mean value and the standard deviation of the aggregate claims distribution.

5.2. Generalized Pareto mixture of Poisson counting distributions

We show some results obtained using as counting distribution a mixture distribution of the form:

$$\text{Poisson}(\Theta) \underset{\Theta}{\wedge} \text{ Generalized Pareto}(\alpha, \beta, \gamma),$$

i.e. a mixture of Poisson distributions with parameter Θ, where Θ is a random variable with generical realization λ ($\lambda > 0$) and probability density:

$$\varphi(\lambda) = \frac{\alpha^{\beta}\Gamma[\beta+\gamma]}{\Gamma[\beta]\Gamma[\gamma]} \frac{\lambda^{\gamma-1}}{(\alpha+\lambda)^{\beta+\gamma}} \quad \text{for } \lambda > 0,$$

which is the density function of a Generalized Pareto distribution with parameters $\alpha > 0$, $\beta > 0$, $\gamma > 0$, where Γ is the Gamma function.

This is a continuous mixture [11]. Thus, as well known, its probability mixed function is obtained by integration over the mixing parameter Θ. In other words,

$$p(n) = \int_0^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^{\gamma+n-1}}{n!} \frac{\alpha^{\beta}\Gamma[\beta+\gamma]}{\Gamma[\beta]\Gamma[\gamma](\alpha+\lambda)^{\beta+\gamma}} \, \mathrm{d}\lambda \quad \text{for } n = 0, 1, 2, \ldots$$

Table 8
Aggregate claims cumulative distribution using a Poisson-Generalized Pareto mixture counting distribution with parameters $\alpha = 3.4959$, $\beta = 30.1234$, $\gamma = 1.99321$

Aggregate claims amount[a]	Cumulative distribution values
$\mu - 0.42\sigma \cong 0$	0.798708
$\mu - 0.25\sigma \cong 4$	0.798708
$\mu - 0.00\sigma \cong 10$	0.798708
$\mu + 0.25\sigma \cong 16$	0.810615
$\mu + 0.50\sigma \cong 22$	0.831052
$\mu + 0.75\sigma \cong 28$	0.866018
$\mu + 1.00\sigma \cong 34$	0.874409
$\mu + 1.50\sigma \cong 46$	0.877189
$\mu + 2.00\sigma \cong 58$	0.896084
$\mu + 2.50\sigma \cong 70$	0.973562
$\mu + 3.00\sigma \cong 82$	0.980304
$\mu + 4.00\sigma \cong 106$	0.987965
$\mu + 5.00\sigma \cong 130$	0.997091
$\mu + 8.00\sigma \cong 202$	0.999813
$\mu + 10.00\sigma \cong 250$	0.999972
$\mu + 20.00\sigma \cong 491$	1.000000
$\mu + 50.00\sigma \cong 1212$	1.000000

[a] $\mu = 10.2310$ and $\sigma = 24.0489$. They are, respectively, the mean value and the standard deviation of the aggregate claims distribution.

By solving, we obtain,

$$p(n) = \frac{\alpha^{\beta}\Gamma[\beta+\gamma]\Gamma[n-\beta]\,{}_1F_1[\beta+\gamma, 1+\beta-n, \alpha]}{\Gamma[\beta]\Gamma[\gamma]\Gamma[1+n]} + \frac{\alpha^{n}\Gamma[\gamma+n]\Gamma[\beta-n]\,{}_1F_1[\gamma+n, 1-\beta+n, \alpha]}{\Gamma[\beta]\Gamma[\gamma]\Gamma[1+n]} \quad \text{for } n = 0, 1, 2, \ldots$$

where ${}_1F_1$ is the Kummer function.

In Table 8, we show the cumulative aggregate claims distribution obtained for $\alpha = 3.4959$, $\beta = 30.1234$, $\gamma = 1.99321$.

6. Conclusions

In the present paper, we propose a method of practical utility for calculating the aggregate claims distribution in a discrete framework.

It is an approximated method but the resulting approximation of the aggregate claims distribution is lower than a prefixed error (10^{-6} in our applications). In particular, the probability distribution and also the first three moments are exact with the above-mentioned prefixed maximum error.

This method does not require special assumptions on the counting distribution nor the identical distribution of the severity random variables and it does not incur in underflow and overflow computational problems.

Besides, it proves to be more flexible, easier and cheaper than the (exact and approximated) methods using recursion and FFT. In particular, the advantages of the proposed method rise as the number of convolutions rises.

In addition to the specific application proposed in this paper, the method can be applied in many other (life and non-life) actuarial fields where the sum of discrete random variables (with integer or referable to integer – positive, negative and also null – numerical realizations) and the calculation of compound distributions are involved. Besides, it can be extended in multivariate situations.

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