ARTICLE IN PRESS

Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Antibandwidth of complete k-ary trees

Tiziana Calamoneri [a], Annalisa Massini [a], L'ubomír Török [b,*], Imrich Vrt'o [c]

[a] Computer Science Department, University of Rome "La Sapienza", Via Salaria, 113, 00198 Rome, Italy

[b] Institute of Mathematics and Computer Science, Slovak Academy of Sciences, Severná 5, 974 01, Banská Bystrica, Slovak Republic

[c] Institute of Mathematics, Slovak Academy of Sciences, Dúbravská 9, 841 04 Bratislava, Slovak Republic

ARTICLE INFO

Article history:
Received 29 September 2006
Accepted 22 October 2008
Available online xxxx

Keywords:
Antibandwidth
Complete k-ary tree

ABSTRACT

The antibandwidth problem is to label vertices of a n-vertex graph injectively by $1, 2, 3, \ldots n$, so that the minimum difference between labels of adjacent vertices is maximised. The problem is motivated by the obnoxious facility location problem, radiocolouring, work and game scheduling and is dual to the well known bandwidth problem. We prove exact results for the antibandwidth of complete k-ary trees, k even, and estimate the parameter for odd k up to the second order term. This extends previous results for complete binary trees.

1. Introduction

The antibandwidth problem consists of labelling vertices of an n-vertex graph $G = (V, E)$ injectively by $1, 2, 3, \ldots, n$, so that the minimum difference between labels of adjacent vertices is maximised. The corresponding maxmin value is denoted by $\mathrm{ab}(G)$. This problem is the dual one of the classical bandwidth problem [3]. It is naturally motivated by obnoxious facility location problems [1], radiocolouring [5] and work and game scheduling tasks [7]. It also belongs to the broad family of graph labelling problems [4]. In the literature it is known under different names: separation number [7], dual bandwidth [8] and antibandwidth [11].

The antibandwidth problem is NP-hard [7]. So far it is known to be polynomially solvable for 3 classes of graphs: the complements of interval, arborescent comparability and threshold graphs [2,6]. Known results include simple relations of the antibandwidth invariant to the minimum, maximum degree, chromatic index and powers of hamiltonian paths in the complement graph [7–9]. Exact results and tight bounds are known for paths, cycles, special trees, meshes, hypercubes [8,9,11,12]. The class of n-vertex forests with $\mathrm{ab(F)} = \lfloor n/2 \rfloor$ is characterized in [9], which for complete binary trees gives a value of $(n-1)/2$. The same result for complete binary trees was also independently proved in [12].

In our paper, we prove that the antibandwidth of the n-vertex complete k-ary tree, for $k \geq 4$ even, is $(n-k+1)/2$. For odd k, we show tight bounds up to the second order term. In particular, the antibandwidth of the n-vertex complete ternary tree of height h is $n/2 - \Theta(h)$. For $h = 2$ and odd k the antibandwidth equals $(k^2+1)/2$.

2. Basic notions

Let $T(k, n)$ be the n-vertex, complete k-ary tree. Note that $n = 1 + k + k^2 + \cdots + k^h = (k^{h+1} - 1)/(k - 1)$, where h is the height of the tree. Divide vertices of the tree into $h + 1$ levels according to their distances from the root, which is on level 1. Let $d(v)$ be the degree of a vertex v. Of course $d(v)$ can be either 1 (if v is a leaf), or k (if v is the root) or $k + 1$ (if v is an internal vertex).

* Corresponding author.
E-mail addresses: calamo@di.uniroma1.it (T. Calamoneri), massini@di.uniroma1.it (A. Massini), torok@savbb.sk (L'. Török), vrto@savba.sk (I. Vrt'o).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.10.019

Please cite this article in press as: T. Calamoneri, et al., Antibandwidth of complete k-ary trees, Discrete Mathematics (2008), doi:10.1016/j.disc.2008.10.019

For a nonempty graph $G = (V, E)$, let f be a one-to-one labelling $f : V \to \{1, 2, 3, \ldots |V|\}$. Define the *antibandwidth of G according to f* as

$$\mathrm{ab}(G, f) = \min_{uv \in E} |f(u) - f(v)|.$$

The *antibandwidth of G* is defined as

$$\mathrm{ab}(G) = \max_{f} \mathrm{ab}(G, f).$$

It is useful to imagine the antibandwidth problem as a linear layout problem. The vertices are mapped into integer points $\{1, \ldots, |V|\}$ on a line such that the minimal distance of adjacent vertices is maximised.

We say that a set of vertices U in a graph $G = (V, E)$ is a *vertex r-bisector* if removing U the remaining vertices can be partitioned into disjoint sets V_1, V_2, s.t. $|V_1|, |V_2| \le r$ and every path between V_1 and V_2 contains a vertex from U.

Similarly, we say that a set of edges F in a graph $G = (V, E)$ is an *edge* $\lceil n/2 \rceil$*-bisector* if removing F the vertices are partitioned into disjoint sets V_1, V_2, s.t. $|V_1|, |V_2| \le \lceil n/2 \rceil$ and every edge between V_1 and V_2 belongs to F.

3. Even *k* case

In this section we will provide the exact value of the antibandwidth of a complete k-ary tree, where k is even.

Theorem 1. *For even $k \ge 4$,*

$$\mathrm{ab}(T(k, n)) = \frac{n + 1 - k}{2}.$$

Proof. Lower bound. We prove the lower bound by providing a labelling.

Split the set $\{1, 2, \ldots, n\}$, where $n = 1 + k + k^2 + \cdots + k^h$ into segments of consecutive integers. Listing the segments consecutively as their elements increase, we take

$$L_1, L_{h-1}, L_{h-3}, \ldots, L_4, L_2, L_3, L_5, \ldots, L_{h-2}, L_h, M,$$
$$R_h, R_{h-2}, \ldots, R_5, R_3, R_2, R_4, \ldots, R_{h-3}, R_{h-1} R_1$$

for h odd, and

$$L_1, L_{h-1}, L_{h-3}, \ldots, L_5, L_3, L_2, L_4, \ldots, L_{h-2}, L_h, M,$$
$$R_h, R_{h-2}, \ldots, R_4, R_2, R_3, R_5, \ldots, R_{h-3}, R_{h-1}, R_1$$

for even h, where M contains one element and both L_i and R_i contain $k^i/2$ elements for $i = 1, 2, \ldots, h$.

For example, with $k = 4$ and $h = 4$, we get segments $L_1, L_3, L_2, L_4, M, R_4, R_2, R_3, R_1$ that are of cardinalities $2, 32, 8, 128, 1, 8, 32, 2$ respectively. Explicitly, the segments are

$$\{1, 2\}, \{3, \ldots, 34\}, \{35, \ldots, 42\}, \{43, \ldots, 170\}, \{171\}$$
$$\{172, \ldots, 299\}, \{300, \ldots, 307\}, \{308, \ldots, 339\}, \{340, 341\}.$$

Label the root with the element M. Label the nodes of the second level with the elements of L_1 followed by the elements of R_1 consecutively. Label the nodes of the third level with the elements of R_2 followed by the elements of L_2, and so on. In general, if i is odd to label the nodes at level $i + 1$ use the elements of L_i followed by the elemnts of R_i, and if i is even use the elements of R_i followed by the elements of L_i.

In our example, the root is labeled with 171, the nodes at the second level are labeled consecutively with $1, 2, 340, 341$, at the third with $300, \ldots, 307, 35, \ldots, 42$, at the fourth with $3, \ldots, 34, 308, \ldots, 339$ and at the fifth with $172, \ldots, 299, 43, \ldots, 170$.

To show that this labelling works in general we have to examine the differences between labels of vertices in neighbouring levels. The most straightforward is to determine the minimal difference between the label of the root M and labels from the sets L_1, R_1. This is, clearly, equal to $\frac{n-k+1}{2}$. Moreover, we need to prove the following four cases for h odd:

1. for even i, $\min\{R_i\} - \min\{L_{i-1}\} \ge \frac{n-k+1}{2}$
2. for even i, $\max\{R_{i-1}\} - \max\{L_i\} \ge \frac{n-k+1}{2}$
3. for odd i, $\min\{R_i\} - \min\{L_{i-1}\} \ge \frac{n-k+1}{2}$
4. for odd i, $\max\{R_{i-1}\} - \max\{L_{i-1}\} \ge \frac{n-k+1}{2}$,

where $\min(\max)\{R_i\}, \min(\max)\{L_i\}$ stand for the minimal (maximal) label from the set R_i, L_i respectively. Note that the same four cases have to be examined for h even. These proofs are rather technical, so we show only one case in detail, the rest of them follow in the same way.

Please cite this article in press as: T. Calamoneri, et al., Antibandwidth of complete k-ary trees, Discrete Mathematics (2008), doi:10.1016/j.disc.2008.10.019

Assume i even, h odd. We examine the case (i). According to labeling algorithm we have

$$\begin{aligned}\min\{R_i\} &= |L_1| + |L_{h-1}| + |L_{h-3}| + \cdots + |L_4| + |L_2| + |L_3| + |L_5| + \cdots \\ &\quad + |L_{h-2}| + |L_h| + |M| + |R_h| + |R_{h-2}| + \cdots + |R_5| + |R_3| + |R_2| \\ &\quad + |R_4| + \cdots + |R_{i-2}| + 1\end{aligned}$$

$$\min\{L_i\} = |L_1| + |L_{h-1}| + |L_{h-3}| + \cdots + |L_4| + |L_2| + |L_3| + |L_5| + \cdots + |L_{i-3}| + 1.$$

Then the difference

$$\begin{aligned}\min\{R_i\} - \min\{L_i\} &= |L_{i-1}| + |L_{i+1}| + \cdots + |L_{h-2}| + |L_h| + |M| \\ &\quad + |R_h| + |R_{h-2}| + \cdots + |R_5| + |R_3| + |R_2| + |R_4| + \cdots + |R_{i-2}|.\end{aligned}$$

After some algebraic manipulations and using the fact that $|L_i| = |R_i|$ we have

$$\min\{R_i\} - \min\{L_i\} = |M| + |R_2| + |R_3| + |R_4| + \cdots + |R_{i-2}| + 2(|R_{i-1}| + |R_{i+1}| + \cdots + |R_h|).$$

Now, using $|R_i| = \frac{k^i}{2}$ we get

$$\min\{R_i\} - \min\{L_i\} = 1 + \frac{k^2 + k^3 + \cdots + k^{i-1}}{2} + (k^{i-1} + k^{i+1} + \cdots + k^h).$$

This term has to be greater or equal to $\frac{n-k+1}{2}$:

$$1 + \frac{k^2 + k^3 + \cdots + k^{i-1}}{2} + (k^{i-1} + k^{i+1} + \cdots + k^h) \geq \frac{n-k+1}{2}.$$

Note that $n = 1 + k + k^2 + \cdots + k^h$, and we finally get

$$k^{i-1} + k^{i+1} + \cdots + k^h \geq k^i + k^{i+2} + \cdots + k^{h-1}.$$

Since all terms are positive and there is one term more on the left side we conclude that this inequality is true for all $i = 1, 2, \ldots, h$.

Upper Bound. We proceed by contradiction, so let us assume that

$$\text{ab}(T(k, n)) \geq \frac{n+1-k}{2} + 1.$$

Let $f : V_T \to \{1, 2, \ldots, n\}$ be a bijective labelling of the vertices of $T(k, n)$. Then, two cases can arise:

1. There exists a vertex v with neighbours u and w, such that $f(u) < f(v) < f(w)$. Hence $d(v) \geq k$. Then, we can define two integer values l and $r = d(v) - l$, both ≥ 1 such that $u_1, u_2, \ldots, u_l$ and $w_1, w_2, \ldots, w_r$ are neighbours of v and $f(u_1) < f(u_2) < \cdots < f(u_l) < f(v) < f(w_1) < f(w_2) < \cdots < f(w_r)$

   It follows that $f(w_1) - f(u_l) \leq n - 1 - (l-1) - (r-1) \leq n + 1 - k$ since $l + r \geq k$.

   Hence $\min\{f(v) - f(u_l), f(w_1) - f(v)\} \leq \frac{n+1-k}{2}$, a contradiction.
2. For every v with neighbours $u_1, u_2, \ldots, u_{d(v)}$ either $f(u_i) < f(v)$, for all $i = 1, 2, \ldots, d(v)$ or $f(u_i) > f(v)$, for all $i = 1, 2, \ldots, d(v)$. Let I be the interval $[(n+1-k)/2, (n+1+k)/2]$ and let us focus on the vertices with degree strictly greater than 1.

   (a) Assume there exists v, with $d(v) > 1$, s.t. $f(v) \in I$. W.l.o.g. assume that $f(v) \leq (n+1)/2$.

   If for all neighbours $u_1, u_2, \ldots, u_{d(v)}$ of v it holds $f(u_1) < f(u_2) < \cdots < f(u_j) < f(v)$, then

   $$f(v) - f(u_j) \leq \frac{n+1}{2} - 1 - (j-1) \leq \frac{n+1-k}{2},$$

   a contradiction.

   If, on the contrary, for all neighbours $u_1, u_2, \ldots, u_{d(v)}$ of v, it holds $f(v) < f(u_1) < f(u_2) < \cdots < f(u_j)$ then

   $$f(u_1) - f(v) \leq n - \frac{n+1}{2} + \frac{k}{2} - (j-1) \leq \frac{n+1-k}{2},$$

   again a contradiction.

   (b) Assume that for all v with $d(v) > 1$, it holds $f(v) \notin I$. Consider the root r. As $f(r) \notin I$, w.l.o.g. assume that $f(r) \leq \frac{n+1-k}{2} - 1$. Then for all vertices w on level 2 we have $f(w) \geq \frac{n+1-k}{2} + 1$. Similarly, for vertices w on level 3 we have $f(w) \leq \frac{n+1-k}{2} - 1$, etc., until we reach the vertices on level h. Depending on the parity of h we have two cases.

   First, assume that for all vertices p on level h we have $f(p) \geq \frac{n+1+k}{2} + 1$.

   As $k^h \geq \frac{n-1+k}{2}$, at least one leaf w satisfies (note that leafs are on level $h+1$):

   $$f(w) \geq \frac{n-1+k}{2}.$$

   Clearly, for the parent p of w: $f(w) < f(p)$. Hence

   $$f(p) - f(w) \leq n - \frac{n-1+k}{2} = \frac{n+1-k}{2},$$

   a contradiction.

Second, assume that for all vertices p on level h we have $f(p) \leq \frac{n+1-k}{2} - 1$. Again, as in previous case, we have the following reasoning. As $k^h \geq \frac{n-1+k}{2}$, at least one leaf (in the level $h+1$) w satisfies:

$$f(w) \geq \frac{n-1+k}{2}.$$

Clearly, in this case for the parent p of w: $f(p) < f(w)$. Hence

$$f(w) - f(p) \leq \frac{n-1+k}{2} - \frac{n+1-k}{2} = k-1$$

again, a contradiction. □

4. Odd *k* case

In this section we provide upper and lower bounds for the antibandwidth that differ in a lower order term, in the case k odd. Unfortunately, in this case, the symmetric construction exploited in the even case cannot be applied, so we will use a completely different technique.

Theorem 2. *For odd $k \geq 3$ and $h \geq 3$*

$$\mathrm{ab}(T(k,n)) \leq \frac{n}{2} - \max\left\{\frac{k}{2}, \frac{h}{8} - o(h)\right\}.$$

Proof. The upper bound of the form $(n-k)/2$ can be obtained in a similar way as for the k even case. For the second upper bound assume that h is odd. The even h case can be proven similarly. Let S be a smallest set of vertices after whose removal the vertices of the resulting forest can be divided into independent sets X and Y, s.t. $|X|, |Y| \leq n/2$. We claim that

$$\mathrm{ab}(T(k,n)) \leq \frac{n-|S|}{2}.$$

To prove this, consider an optimal layout. Removing the last $n-2\mathrm{ab}(T(k,n))$ vertices we get 2 independent sets: the first one is the set on positions $1, 2, 3, \ldots, \mathrm{ab}(T(k,n))$ and the second one is the set on the positions $\mathrm{ab}(T(k,n))+1, \ldots, 2\mathrm{ab}(T(k,n))$. Note that there are possible edges between the two sets only, otherwise we get an edge of length smaller than $\mathrm{ab}(T(k,n))$.

As $\mathrm{ab}(T(k,n)) \leq n/2$ we have

$$|S| \leq n - 2\mathrm{ab}(T(k,n)),$$

which proves the claim.

In what follows we prove that $|S| \geq h/4 - o(h)$. We need some new notations. Let L_i, for $i = 1, 2, 3, \ldots, h+1$ denote the set of vertices of the i-th level of the tree, while L_1 contains the root. Set $x_i = |L_i \cap X|$, $y_i = |L_i \cap Y|$, $s_i = |L_i \cap S|$. Observe that, for $i \geq 2$, as X, Y and S are defined, and in view of the structure of a complete k-ary tree, we have that

$$k(x_{i-1} + y_{i-1} + s_{i-1}) = x_i + y_i + s_i. \tag{1}$$

Furthermore, the properties of X, Y and S imply that the children of vertices in $L_{i-1} \cap X$ must be in $L_i \cap (S \cup Y)$, hence $y_i + s_i \geq kx_{i-1}$. By (1), this is equivalent to $ky_{i-1} + ks_{i-1} \geq x_i$. Repeating this argument for $L_{i-1} \cap Y$ we derive the following:

$$x_i - ks_{i-1} \leq ky_{i-1} \leq x_i + s_i \tag{2}$$

$$y_i - ks_{i-1} \leq kx_{i-1} \leq y_i + s_i. \tag{3}$$

Now we show that S is a vertex $(n/2 + 7|S|/2)$-bisector. It is easy to see that the sets

$$V_1 = (\cup_{\text{even } i}(L_i \cap X)) \cup (\cup_{\text{odd } i}(L_i \cap Y)),\ V_2 = (\cup_{\text{odd } i}(L_i \cap X)) \cup (\cup_{\text{even } i}(L_i \cap Y))$$

are distinct and any path between them contains a vertex from S. Hence S is a vertex r-bisector. Let us estimate r.

$$|V_1| = \sum_{\text{even } i} x_i + \sum_{\text{odd } i} y_i \leq \sum_{\text{even } i} x_i + \frac{1}{k}\sum_{\text{even } i}(x_i + s_i) \leq \frac{k+1}{k}\sum_{\text{even } i} x_i + \frac{1}{k}|S|. \tag{4}$$

To estimate the last sum we need estimations for every x_i, for even i. From the left hand side of inequality (3) we have

$$\sum_{i=2}^{h+1}(y_i - ks_{i-1}) \leq k\sum_{i=2}^{h+1} x_{i-1}$$

$$|Y| - y_1 - k|S| \leq k(|X| - x_{h+1})$$

$$n - |X| - |S| - y_1 - k|S| \leq k(|X| - x_{h+1})$$

$$kx_{h+1} \leq (k+1)|X| - n + (k+1)|S| + 1 \leq \frac{(k+1)n}{2} - n + (k+2)|S|$$

$$x_{h+1} \leq \frac{k-1}{2k}n + \frac{k+2}{k}|S|. \tag{5}$$

Combining right hand sides of inequalities (2) and (3) we have:

$$x_{i-2} \leq \frac{1}{k}(y_{i-1} + s_{i-1}) \leq \frac{1}{k}\left(\frac{1}{k}(x_i + s_i) + s_{i-1}\right) = \frac{1}{k^2}(x_i + s_i + ks_{i-1}).$$

Iterating this inequality backwards, starting with $i = h + 1$ we get for even $i \geq 2$

$$x_i \leq \frac{1}{k^{h-i+1}}\left(x_{h+1} + \sum_{j=i+1}^{h+1} k^{h+1-j} s_j\right).$$

Using this estimation we compute

$$\begin{aligned}\sum_{\text{even } i\geq 2}^{h+1} x_i &\leq \sum_{\text{even } i\geq 2}^{h+1} \frac{x_{h+1}}{k^{h-i+1}} + \sum_{\text{even } i\geq 2}^{h+1} \sum_{j=i+1}^{h+1} \frac{s_j}{k^{i-j}} \\ &< x_{h+1} \sum_{\text{even } t\geq 0}^{h-1} \frac{1}{k^t} + \sum_{j=3}^{h+1}\left(\frac{1}{k} + \frac{1}{k^3} + \cdots + \frac{1}{k^{h-2}}\right) s_j \\ &< x_{h+1} \sum_{\text{even } t\geq 0}^{\infty} \frac{1}{k^t} + \sum_{j=3}^{h+1} \frac{k}{k^2-1} s_j \\ &< \frac{k^2}{k^2-1} x_{h+1} + \frac{k}{k^2-1}|S|.\end{aligned}$$

Thus

$$\sum_{\text{even } i\geq 2}^{h+1} x_i < \frac{k^2}{k^2-1} x_{h+1} + \frac{k}{k^2-1}|S|. \tag{6}$$

Substituting (6) into (4) and using (5) we obtain

$$\begin{aligned}|V_1| &\leq \frac{k}{k-1} x_{h+1} + \frac{2}{k-1}|S| \leq \frac{k}{k-1}\left(\frac{k-1}{2k} n + \frac{k+2}{k}|S|\right) + \frac{2}{k-1}|S| \leq \frac{n}{2} + \frac{k+4}{k-1}|S| \\ &\leq \frac{n}{2} + \frac{7}{2}|S|.\end{aligned}$$

Repeating the same calculations for $|V_2|$ we get the same bound, hence concluding that S is a vertex $(n/2+7|S|/2)$-bisector. Assume $|V_1| \leq |V_2|$. Let $|V_2| = n/2 + p$. By deleting a suitable set of at most $\log_k p + 1$ vertices we can separate p vertices from V_2 and add them to V_1. To see this, observe that p can be expressed in the form

$$p = \sum_{i=1}^{z} \alpha_i \frac{k^i - 1}{k-1},$$

where $0 \leq \alpha_i \leq k$ are integers, and z is the smallest number s.t. $(k^{z+1} - 1)/(k-1) > p$, i.e., $z \leq \log_k p + 1$. And note that by removing a suitable vertex from V_2 we get k complete subtrees of size $(k^j - 1)/(k-1)$, where $j \leq z$.

Thus we get a vertex $n/2$-bisector. Its size is

$$|S| + \log_k p + 1 \leq |S| + \log_k \frac{7}{2}|S| + 1.$$

Further, removing all edges incident to the vertices of the vertex $n/2$-bisector and distributing the isolated vertices among the current sets V_1 and V_2 in such a way that neither of them contains more than $n/2$ vertices we get an edge $n/2$-bisector of the size at most

$$(k+1)\left(|S| + \log_k \frac{7}{2}|S| + 1\right).$$

It is known [10] that the size of the smallest edge $\lceil n/2 \rceil$-bisector of the complete k-ary n-vertex tree of height h is at least

$$\frac{k-1}{2}(h - \log_k h - 1).$$

Thus we have

$$(k+1)\left(|S| + \log_k \frac{7}{2}|S| + 1\right) \geq \frac{k-1}{2}(h - \log_k h - 1).$$

Hence

$$|S| \geq \frac{k-1}{2(k+1)}(h - \log_k h - 1) - \log_k \frac{7}{2}|S| - 1.$$

As $|S| \leq h$, this yields

$$|S| \geq \frac{k-1}{2(k+1)}h - o(h) \geq \frac{h}{4} - o(h). \quad \square$$

In the following paragraphs, for the sake of completeness, we shortly repeat the algorithm by Miller and Pritikin [9]. This algorithm provides reasonably good layout for forests and we use its slight modification in the lower bound construction in the next theorem.

For a bipartite graph B with a specified bipartition M, N with $|M| \leq |N|$, we refer to the *minority* $MIN(B) = |M|$ and *majority* $MAJ(B) = |N|$ of B and refer to M and N as being the minority and majority sides, respectively.

Given any bipartition X, Y of a forest with $|X| \leq |Y|$, there always exists a vertex $y \in Y$ of degree 0 or 1 since the average degree of the majority side vertices is at most $(|X| + |Y| - 1)/|Y|$, which is less than two.

Let a forest F_1 have minority side M_1 and majority side N_1. For each $i \in [1, MAJ(F_1)]$, recursively define y_i, x_i, M_i, N_i as follows. Let $y_i \in N_i$ be a vertex of degree 0 or 1 in F_i. If y_i has degree 1 in F_i choose x_i as its sole neighbour. If M_i is empty, choose $x_i = y_i$. In any other case, choose x_i to be any element of M_i. Let $F_{i+1} = F_i - x_i - y_i$, $M_{i+1} = M_i - x_i$, $N_{i+1} = N_i - y_i$. The resulting layout is obtained by the following labeling. Assign $f(x_i) = i$ for each $i \in [1, MIN(F_1)]$ and $f(y_i) = MIN(F_1) + i$ for each $i \in [1, MAJ(F_1)]$. This leads to a construction with

$$\text{ab}(F) \geq MIN(F).$$

Theorem 3. *For odd $k \geq 3$ and $h \geq 3$*

$$\text{ab}(T(k, n)) \geq \frac{n}{2} - O(k^2 h).$$

Proof. Sketch. We proceed with the following construction.

1. Number the levels of the tree by $1, 2, \ldots, h+1$. First, delete the root vertex and its adjacent edges. For every level $i : i \geq 2$ number the vertices from left to right by integers $1, 2, \ldots, k^{i-1}$. Then delete the vertex with label $\lfloor \frac{k^{i-1}}{2} \rfloor$ together with its adjacent edges. Define the set D to consist of deleted vertices. The remaining parts of the tree define the forest F.
2. Divide the vertices of F into two parts X and Y s.t. $|Y| - 1 \leq |X| \leq |Y|$.
3. For every $v : v \in Y$ such that v was adjacent to some $d \in D$ define the priority to be equal 2. For all neighbours of every such v define the priority value to be equal 1. The rest of vertices of F obtain priority value 3. The higher priority is denoted by the lower number, i.e. 1 is higher priority than 2 for example.
4. Use the modified Miller/Pritikin algorithm to get the layout of F with $ab(F) \geq \lfloor \frac{n-h}{2} \rfloor$. The modification of used algorithm simply follows the priorities of vertices defined in the previous step. If it is not possible to label a vertex w with priority 1 directly, i.e. the vertex w does not have any neighbour from Y of degree 1 or there is no vertex from Y with degree 0, label one of the leaves from Y of degree 1 and its parent from X and remove them from the forest. This operation creates $k - 1$ isolated vertices from Y which can be used for labeling the vertices with priority 1.
5. Place the vertices from set D in the middle of the layout, between the sets X and Y.

The algorithm places the vertices from the sets X, Y, D in the order X, D, Y. For the final lower bound the distance from the neighbours of D to D is important. Let P_i be the set of vertices of priority i. Since every deleted vertex except the last one has $(k + 1)$ neighbours, approximately half of them belongs to the set Y, i.e. $|P_2| = (k + 1)h/2$. Every vertex from P_2 has k neighbours from X, i.e. $|P_1| = (k + 1)kh/2$. To label the vertices of P_1 we need $|P_1|$ vertices from Y of degree 0. These can be easily produced from leaves (see step 4 of the algorithm). With a simple analysis we get that the labeling of P_1 needs

$$\frac{hk(k+1)}{2} \cdot \frac{(k+1)}{k} = \frac{h(k+1)^2}{2}$$

leaves. Labeling of P_1 vertices will make all of P_2 vertices from Y isolated and therefore they can be used to label the second half of P_2 vertices from X. In resulting layout there will be $h(k+1)^2/2 P_1$ vertices, then $h(k+1)/2 P_2$ vertices. Since P_2 are the neighbours of D, then

$$ab(T(k, n)) \geq n/2 - h(k+1)^2/2 - h(k+1)/2 = n/2 - O(hk^2). \quad \square$$

Combining our methods we are able to prove that:

Theorem 4. *For odd $k \geq 3$ and $h = 2$*

$$\text{ab}(T(k, n)) = \frac{k^2 + 1}{2}.$$

Please cite this article in press as: T. Calamoneri, et al., Antibandwidth of complete k-ary trees, Discrete Mathematics (2008), doi:10.1016/j.disc.2008.10.019

Proof. For the upper bound we use the proof of the upper bound from Theorem 1. The only difference is in parity of k. The value from the claim of this theorem is

$$\frac{k^2+1}{2} = \frac{k^2+k+1-k}{2} = \frac{n-k}{2}.$$

The proof of the upper bound in Theorem 1 is, in fact, not based on the parity of k. Therefore for $h = 2$

$$\mathrm{ab}(T(k, n)) \le \frac{n-k+1}{2}$$

which, for odd k, and $h = 2$, is the same as

$$\mathrm{ab}(T(k, n)) \le \frac{n-k}{2} = \frac{k^2+1}{2}.$$

In the lower bound construction the root obtains the label $\frac{n+1}{2}$. The second level is labelled from left to right with labels

$$1, 2, \ldots, \frac{k+1}{2}, n - \frac{k-1}{2} + 1, \ldots, n$$

and the third level obtains labels (from left to right):

$$\frac{k+1}{2} + k\frac{k-1}{2} + 1, \ldots, \frac{n-1}{2}, \frac{n+1}{2} + 1, \ldots, n - \frac{k-1}{2}, \frac{k+1}{2} + 1, \ldots, \frac{k+1}{2} + \frac{k-1}{2}k.$$

Checking the minimal differencees between neighbouring vertices we get the claim. □

Acknowledgements

The authors wish to thank to referees who significantly contributed to the readability of this paper. The labeling in Theorem 1 is due to referee's suggestion.

Third author's research was supported partially by VEGA grant 1/0722/08 and APVV grant No. 51-009605.

Fourth author was supported by a fellowship from the Computer Science Department, University of Rome "La Sapienza", VEGA grant No. 2/0111/09 and APVV grant 0433/06.

References

[1] P. Cappanera, A survey on obnoxious facility location problems, Technical Report TR-99-11, Dipartimento di Informatica, Uni. di Pisa, 1999.
[2] S. Donnely, G. Isaak, Hamiltonian powers in treshold and arborescent comparability graphs, Discrete Mathematics 202 (1999) 33–44.
[3] J. Díaz, J. Petit, M. Serna, A survey of graph layout problems, ACM Computing Surveys 34 (2002) 313–356.
[4] J.A. Gallian, Graph labeling, Electronic Journal on Combinatorics DS6 (2005).
[5] W.K. Hale, Frequency assignment: theory and applications, Proceedings of IEEE 60 (1980) 1497–1514.
[6] G. Isaak, Powers of Hamiltonian paths in interval graphs, Journal of Graph Theory 28 (1998) 31–38.
[7] J.Y-T. Leung, O. Vornberger, J.D. Witthoff, On some variants of the bandwidth minimization problem, SIAM Journal of Computing 13 (1984) 650–667.
[8] Yixun Lin, JinJiang Yuan, The dual bandwidth problem for graphs, Journal of Zhengzhou University Natural Science Edition 35 (2003) 1–5.
[9] Z. Miller, D. Pritikin, On the separation number of a graph, Networks 19 (1989) 651–666.
[10] L. Palios, Upper and lower bounds for the optimal tree partitions, Technical Report GCG 47, The Geometry Center, University of Minnesota, Minneapolis, 1994.
[11] A. Raspaud, H. Schröder, O. Sýkora, L. Török, I. Vrt'o, Antibandwidth and cyclic antibandwidth of meshes and hypercubes, Discrete Mathematics (in press).
[12] Weili Yao, Ju Zhou, Xiaoxu Lu, Dual bandwidth of some special trees, Journal of Zhengzhou University Natural Science Edition 35 (2003) 16–19.