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Pro-p groups with few normal subgroups

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A R T I C L E   I N F O

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A B S T R A C T

Motivated by the study of pro-p groups with finite coclass, we consider the class of pro-p groups with few normal subgroups. This is not a well defined class and we offer several different definitions and study the connections between them. Furthermore, we propose a definition of periodicity for pro-p groups, thus, providing a general framework for some periodic patterns that have already been observed in the existing literature. We then focus on examples and show that strikingly all the interesting examples not only have few normal subgroups, but in addition have periodicity in the lattice of normal subgroups.

1. Introduction

Let p be a prime and G be a pro-p group. There are several ways to require G to have few normal subgroups. In this paper we relate some of them to conditions on pairs of open normal subgroups N and M of G. Consider, for instance, the following conditions:

(a) there exists a constant c such that for every N and for every M not contained in N we have $|N : N \cap M| \leqslant p^c$.

We can weaken this condition in two different ways, obtaining:

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(b) there exists a constant c such that for every N and for every M with $|G:N| = |G:M|$ we have $|N : N \cap M| \leqslant p^c$,

and

(c) for every N there exists a constant c, depending on N, such that for every M not contained in N we have $|N : N \cap M| \leqslant p^c$.

As we will see soon these conditions are closely related to finite coclass and its generalizations. Let $\gamma_i(G)$ be the ith term of the lower central series of G. We recall that G has *finite coclass* if there exists a constant c such that $|G : \gamma_i(G)| \leqslant p^{i+c-1}$ for all i, the *coclass* of G being the minimal value of c for which the above is satisfied. If N and M are open normal subgroups of the same index p^i of a pro-p group G of finite coclass, then they both contain $\gamma_{i+1}(G)$, so $|N : N \cap M| \leqslant p^c$. Therefore, a pro-$p$ group G of finite coclass satisfies condition (b).

The study of pro-p groups of finite coclass was motivated by the attempt to classify finite p-groups according to their coclass. The first naive attempt to generalize the notion of finite coclass was to study pro-p groups G for which there exists w such that $|\gamma_i(G) : \gamma_{i+1}(G)| \leqslant p^w$, that is pro-$p$ groups with *finite width*. However, the class of pro-p groups with finite width seems to be too big to have a general theory. Indeed it contains many pro-p groups of various nature such as linear pro-p groups, the Nottingham group, some index subgroups of the Nottingham group like Fesenko subgroups, and some branch groups.

The next attempt to go beyond coclass was in [17] and [20], where the notion of finite obliquity was introduced. Finite obliquity is a rather technical definition, so we leave its presentation to Section 2. However, in Theorem 38 we show that finite obliquity is equivalent to condition (a).

In Theorem 36 we show that condition (c) is equivalent to many other conditions: the main one is that G is *just infinite*, that is G is infinite and its only non-trivial closed normal subgroups are of finite index. As condition (a) trivially implies condition (c), we have that a pro-p group of finite obliquity is just infinite.

Another very natural definition is that of a *sandwich* pro-p group: G is sandwich if there exists an integer r such that for every non-trivial open normal subgroup N of G we have $\gamma_i(G) \geqslant N \geqslant \gamma_{i+r}(G)$ for some i (depending on N). Clearly this notion is interesting only as far as G is non-nilpotent. Under this assumption it is known that being sandwich is equivalent to having finite obliquity, see [3,17]. Moreover, in Corollary 31 we show that in this case G has finite width. The sandwich pro-p groups already appeared in [9] under the name of constrained pro-p groups; a similar notion for graded Lie algebras is found on [8,10].

In [20, Section 7.4] it is shown that a pro-p group of finite coclass G has a finite normal subgroup H such that G/H is just infinite. It is also possible to see that for all big enough i there exists only one normal subgroup of index p^i in G/H (it is actually one of the subgroups of the lower central series). In particular, this implies that a pro-p group G of finite coclass has *Constant Normal Subgroup Growth* (*CNSG*), i.e. there is a bound for the number $a_n^{\triangleleft}(G)$ of normal subgroups of G of index n.

In Theorem 20 we prove that condition (b) is equivalent to CNSG. In Corollary 23 we show that a pro-p group G with CNSG has a finite normal subgroup H such that G/H is just infinite, thus generalizing the result of [20] mentioned above. In addition, we show that if G is non-nilpotent, then G has finite width.

The CNSG pro-p groups form a (proper) subclass of the class of pro-p groups having *Polynomial Normal Subgroup Growth* (*PNSG*), for which there is a constant c such that $a_n^{\triangleleft}(G) \leqslant n^c$ for all n. Very little is known about pro-p groups with PNSG. Indeed, Lubotzky and Segal posed the problem of finding a characterization of the finitely generated PNSG pro-p groups [21, Open Problems 2(a)]. The family of PNSG pro-p groups includes important examples such as pro-p groups of finite rank (i.e. p-adic analytic pro-p groups), and various classes of pro-p groups of infinite rank. Many of them, such as $\mathrm{SL}_d^1(\mathbb{Z}_p)$, $\mathrm{SL}_d^1(\mathbb{F}_p[\![t]\!])$ ($p \nmid d$), and the Nottingham group ($p > 2$), are actually CNSG pro-p groups.

As condition (a) clearly implies both conditions (b) and (c), finite obliquity implies CNSG and just infiniteness. However, CNSG does not imply just infiniteness and therefore it does not imply finite obliquity, in contrast with the conclusion suggested in [20, Exercises 12.1(3), (4), (5), (6)]. Suppose that G is an infinite pro-p group and H is a finite p-group. If G has finite width, then $G \times H$ also has finite width. If G has CNSG, then $G \times H$ also has CNSG. However, $G \times H$ contains a normal subgroup of infinite index, namely H.

To summarize if G is a pro-p group, then

(1) G satisfies condition (a) if and only if it has finite obliquity.
If G is non-nilpotent, then G has finite obliquity if and only if it is sandwich, and in this case G has finite width.

(2) G satisfies condition (b) if and only if it has CNSG.
If G is non-nilpotent, then G has CNSG if and only if there exists a constant c such that for every open normal subgroup N of G, there exists i such that $\gamma_i(G) \leqslant N$ and $|N : \gamma_i(G)| \leqslant p^c$ and in this case G has finite width (see Theorem 24).

(3) G satisfies condition (c) if and only if G is just infinite.

The second part of this paper introduces the definition of a periodic map for a pro-p group G and a notion of periodicity for the lattice of its open normal subgroups. This notion is coherent with the one found in [17, Chapter 4]. Indeed, the power map that is used there to show a periodic pattern in the sequence of the lower central factors of G satisfies the conditions listed in our Definition 42 below. We also prove (see Theorem 50) that, for just infinite non-abelian pro-p groups, the existence of a periodic map implies that the group has finite obliquity and therefore has CNSG. Moreover, in Theorem 49 we see that the existence of a periodic map on a just infinite pro-p group yields a periodicity in the normal subgroup growth, i.e. there exists d such that for all big enough n we have $a^{\triangleleft}_{p^n}(G) = a^{\triangleleft}_{p^{n+d}}(G)$. We remark that the idea of periodicity has already appeared in the literature in various contexts, see for example, [17] and [18].

As we have mentioned above, finite obliquity implies just infiniteness and CNSG implies just infiniteness up to a finite normal subgroup. Therefore, the rest of the paper is devoted to the study of known examples of just infinite pro-p groups with respect to our finiteness conditions. We recall that a group is *hereditarily just infinite* if all its open subgroups are just infinite. We also recall Wilson's dichotomy: if G is a just infinite pro-p group, then either G is a branch group or it contains a normal subgroup of finite index which is a direct sum of k copies of a hereditarily just infinite pro-p group H and G transitively permutes these copies by conjugation, see [12] and [26] for details and the definitions. We show in Corollary 60 that branch groups do not have CNSG, and thus do not have finite obliquity or carry a periodic map. Therefore, we only need to consider just infinite groups of the second type. For pro-p groups of this type we do not know what is the connection between G having our finiteness conditions and H having them. However, it seems reasonable to start with hereditarily just infinite pro-p groups.

Henceforth we focus on hereditarily just infinite pro-p groups. We show that open subgroups of the Nottingham group ($p > 2$), the Ershov groups, and most t-linear just infinite pro-p groups admit a periodic map, and therefore have finite obliquity (in some cases this was already known). (See Section 5 for the definition and results on t-linear just infinite pro-p groups, but in particular pro-p groups of finite rank admit a periodic map.) We should mention that t-linear just infinite pro-p groups admit a periodic map if and only if they have CNSG. Those with CNSG were classified in [19]. In addition, for some of these groups periodicity in the normal subgroup growth was proved before in [1]. It is worth noting here that an entirely different proof of the finite obliquity for the insoluble just infinite pro-p groups of finite rank was already given in [17, III(d)], while our proof deals with the soluble and the insoluble case at the same time.

The rest of the known hereditarily just infinite pro-p groups are: the groups of Cartan type which do not have finite width and thus do not have CNSG; the Fesenko subgroup for which we show that it does not have CNSG; some other index subgroups of the Nottingham group for which either we do not know whether they have finite width or, for those we do, we show they do not have CNSG.

Finally, let us discuss some problems. In all the examples a hereditarily just infinite pro-p group G with CNSG has finite obliquity. A striking fact is that in such case G admits a periodic map and has a periodicity in the normal subgroup growth. Another surprising fact is that all the open subgroups of G also have all of these properties. Therefore,

Problem 1. Suppose G is a (hereditarily) just infinite pro-p group which has CNSG, does G have finite obliquity?

Problem 2. Suppose G is a (hereditarily) just infinite pro-p group which has CNSG, does G have periodicity in the normal subgroup growth? Does it admit a periodic map?

Problem 3. Suppose G is a (hereditarily) just infinite pro-p group which satisfies one of our finiteness conditions, is it true that all open subgroups of G satisfy the same condition too?

Another very interesting observation is that for all the examples of hereditarily just infinite pro-p groups with CNSG the subgroup growth is relatively slow, that is $a_n(G) \leqslant n^{c \log n}$ for all n, where $a_n(G)$ is the number of subgroups of index n. Therefore,

Problem 4. Can one bound the subgroup growth of a pro-p group by the normal subgroup growth in a non-trivial way? In particular, is it true that if G has CNSG, then $a_n(G) \leqslant n^{c \log n}$ for all n?

One more possible line of investigation is suggested by Lemma 18 where it is proved that a pro-p-group G with CNSG has *finite normal rank* (*FNR*), that is $|H : \overline{[H,G]H^p}|$ is bounded for every open normal subgroup H of G. In turn FNR trivially implies PNSG. Observe that Lubotzky and Mann proved that PSG is equivalent to finite rank (see [5, Theorem 3.19]). It makes sense to ask

Problem 5. Can FNR be characterized in terms of normal subgroup growth (excluding the abelian case where FNR and finite rank coincide)?

We observe that FNR is closed under extensions. In the easiest example of that, i.e. the direct product of a non-abelian CNSG pro-p group by itself, the normal subgroup growth turns out to be polynomial in the logarithm of the index.

In this paper, unless explicitly stated, ‘finitely generated’ means ‘topologically finitely generated.’

2. Finiteness conditions

Remark 6. If G is a finitely generated pro-p group, then every subgroup of finite index is open. Conversely if G is a pro-p group which is not finitely generated, then, for every $k \geqslant 0$, G has infinitely many open subgroups of index p^k (and, a fortiori, infinitely many not necessarily open subgroups of index p^k). Summarizing, for a pro-p group G, the number of open subgroups of a given index and the number of subgroups of the same index are either both infinite or both finite and equal.

Definition 7. A pro-p group G is *just infinite* if it is infinite and every non-trivial closed normal subgroup of G has finite index in G.

Since closed subgroups of finite index of a topological group are open and open subgroups of a compact topological group (in particular a pro-p group) have finite index, an infinite pro-p group G is just infinite if and only if every non-trivial closed normal subgroup of G is open in G.

It is well known that a just infinite pro-p group is finitely generated and that the sole just infinite pro-p group with non-trivial center is the procyclic group $\mathbb{Z}_p$ (in particular it is the sole nilpotent just infinite pro-p group).

As in [20, Definition 12.1.6] we have:

Definition 8. Given a pro-p group G, for every positive integer i we set:

$$\mu_i(G) := \gamma_i(G) \cap \bigcap_{\substack{N \trianglelefteq_C G \\ N \nleqslant \gamma_i(G)}} N.$$

It would be quite natural to consider a similar definition by taking the intersection over open normal subgroups. However this makes no difference:

Lemma 9. *Let S be a subset of a profinite group G. Then*:

$$\bigcap_{\substack{N \trianglelefteqslant_C G \\ N \nsubseteq S}} N = \bigcap_{\substack{N \trianglelefteqslant_O G \\ N \nsubseteq S}} N.$$

Proof. We write $C(S)$ for $\bigcap_{N \trianglelefteqslant_C G,\, N \nsubseteq S} N$ and $O(S)$ for $\bigcap_{N \trianglelefteqslant_O G,\, N \nsubseteq S} N$. As open subgroups are closed we clearly have $C(S) \leqslant O(S)$. Conversely, let $N \trianglelefteqslant_C G$ and $N \nsubseteq S$. If $M \trianglelefteqslant_O G$, then $NM \nsubseteq S$ and since $NM \trianglelefteqslant_O G$ we obtain that $NM \geqslant O(S)$. Because N, being closed, coincides with $\bigcap_{M \trianglelefteqslant_O G} NM$ we have $N \geqslant O(S)$. □

Again as in [20, Definition 12.1.6], we define the following definitions:

Definition 10. A pro-p group G has *finite obliquity* if $\sup_{i \geqslant 1} |\gamma_i(G) : \mu_i(G)|$ is finite. In this case $\log_p \sup_{i \geqslant 1} |\gamma_i(G) : \mu_i(G)|$ is called the *obliquity* of G.

Definition 11. A pro-p group G has *finite width* if $\sup_{i \geqslant 1} |\gamma_i(G) : \gamma_{i+1}(G)|$ is finite. In this case $\log_p \sup_{i \geqslant 1} |\gamma_i(G) : \gamma_{i+1}(G)|$ is called the *width* of G.

In [17] the definition of a pro-p group G of finite obliquity was given with the additional assumption that G has finite width. However, this is not a heavy requirement, in fact, in [20, Exercise 12.1(2)] the following proposition is posed as an exercise.

Proposition 12. *If G is a pro-p group of finite obliquity and $|G : \gamma_2(G)|$ is finite, then G has finite width.*

A slight improvement of this result comes from the trivial remark that an infinite pro-p group with finite derived subgroup has finite obliquity and does not have finite width. Excluding this case, in Corollary 30 we shall show that a pro-p group of finite obliquity has finite width.

Definition 13. Let G be a pro-p group and let r be a positive integer. The group G is *r-sandwich* if one of the following equivalent properties holds:

(i) for every non-trivial open normal subgroup N of G there exists a positive integer i such that $\gamma_i(G) \geqslant N \geqslant \gamma_{i+r}(G)$;
(ii) for every positive integer i and for every open normal subgroup N of G we have $\gamma_i(G) \geqslant N$ or $N \geqslant \gamma_{i+r-1}(G)$.

We shall say that G is *sandwich* if it is r-sandwich for some positive integer r.

Using Definition 8 and Lemma 9 we may rephrase the definition of r-sandwich pro-p group as follows:

Remark 14. A pro-p group G is r-sandwich if and only if $\mu_i(G) \geqslant \gamma_{i+r-1}(G)$ for every positive integer i.

We have the immediate

Remark 15. If K is a closed normal subgroup of an r-sandwich pro-p group G, then G/K is an r-sandwich pro-p group.

One could give a different definition of an r-sandwich pro-p group by replacing in Definition 13 either $\gamma_i(G)$ or $\gamma_{i+r-1}(G)$ (or both) with their closure. However, such (a priori different) definitions are actually equivalent. First of all, we note that an open normal subgroup, being closed, contains $\gamma_{i+r-1}(G)$ if and only if it contains its closure, so we only need to investigate what happens if we replace $\gamma_i(G)$ with its closure. This will be done in Lemma 17.

It is well known (see [24, 5.2.5]) that, for every group G, the subgroups $\gamma_i(G)$ for $i \geqslant 2$ either all have finite index in G or all have infinite index in G. More accurate information can be given when G is a pro-p group:

Lemma 16. *Let G be a pro-p group. Then exactly one of the following holds*:

(1) *for every $i \geqslant 2$ the subgroup $\overline{\gamma_i(G)}$ has infinite index in G, in this case $\mu_i(G) = 1$ for all $i \geqslant 2$;*
(2) *for every i the subgroup $\gamma_i(G)$ is open in G, in this case G is finitely generated.*

Proof. Assume that, for every $i \geqslant 2$, $\overline{\gamma_i(G)}$ has infinite index in G, in particular, $\gamma_i(G)$ does not contain any open normal subgroup of G. Therefore $\mu_i(G)$ is the intersection of all the open normal subgroups of G and thus it is the trivial subgroup.

Now assume that there exists an integer $j \geqslant 2$ such that $\overline{\gamma_j(G)}$ has finite index in G. Then $\overline{\gamma_2(G)}$ and $\Phi(G)$ (which contain $\overline{\gamma_j(G)}$) have finite index in G. By [5, Proposition 1.9(iii)] G is finitely generated and by [5, Proposition 1.19] we have that $\overline{\gamma_2(G)} = \gamma_2(G)$, hence $\gamma_2(G)$ has finite index in G. Therefore, for all i, $\gamma_i(G)$ has finite index in G so from [5, Theorem 1.17] it is open in G. □

We deduce the following lemma.

Lemma 17. *Let G be a pro-p group and let i and r be positive integers. Then the following conditions are equivalent*:

(i) *for every open normal subgroup N of G we have $\gamma_i(G) \geqslant N$ or $N \geqslant \gamma_{i+r-1}(G)$;*
(ii) *for every open normal subgroup N of G we have $\overline{\gamma_i(G)} \geqslant N$ or $N \geqslant \gamma_{i+r-1}(G)$.*

Moreover, if $i > 1$, $\gamma_{i+r-1}(G) \neq 1$ and one (and hence both) of the previous conditions holds, then G is finitely generated and $\gamma_j(G)$ is open in G for all j.

Proof. If $i = 1$ or $\gamma_{i+r-1}(G) = 1$ there is nothing to prove, so we may assume that $i > 1$ and $\gamma_{i+r-1}(G) \neq 1$. We suppose that condition (ii) (which is, a priori, weaker than condition (i)) holds and show that condition (i) holds. Since $\gamma_{i+r-1}(G) \neq 1$ there exists an open normal subgroup N which does not contain $\gamma_{i+r-1}(G)$ and is therefore contained in $\overline{\gamma_i(G)}$, which has then finite index in G. The result now follows from Lemma 16. □

Lemma 18. *Let G be a pro-p group. If there exists a constant c such that $|N : N \cap M| \leqslant p^c$ for all open normal subgroups M and N of the same index, then*

(1) *for every open normal subgroup H of G, we have $|H : \overline{[H, G]H^p}| \leqslant p^{2c+1}$;*
(2) *if K is a closed not open normal subgroup of G, then K is a finite group of order at most p^c.*

Proof. If the first claim were false, then there would exist open normal subgroups L and H of G such that $H \geqslant L \geqslant \overline{[H, G]H^p}$ and $|H : L| = p^{2c+2}$. In the elementary abelian p-group H/L, we choose two subgroups M/L and N/L of order p^{c+1} and trivial intersection. Both M and N are open (as L is open) and normal in G (since H/L is central in G/L). Clearly M and N have the same index in G. However, their intersection, that is L, has index p^{c+1} in N, contrary to the assumption.

We now assume that the second claim is false, hence, K has an open subgroup with index at least p^{c+1}. Such a subgroup is the intersection of an open subgroup H of G with K. Let M be an open normal subgroup of G contained in H. Since K is not open, there exists an open normal subgroup N

of G which contains K and such that $|G:M| = |G:N|$. Since $NM/M \cong N/N \cap M$, $KM/M \cong K/K \cap M$, and $KM \leqslant NM$ we have that $|N : N \cap M| \geqslant |K : K \cap M| \geqslant p^{c+1}$, a contradiction. □

Lemma 19. *Let M and N be open normal subgroups of a pro-p group G with the same index. If $|N : N \cap M| = p^d$, then there exists n such that $a_n^{\lhd}(G) > d$.*

Proof. For every integer $0 \leqslant i \leqslant d$ we may choose two normal subgroups M_i and N_i in G such that $N \cap M \leqslant M_i \leqslant M$, $N \cap M \leqslant N_i \leqslant N$ and $|M : M_i| = |N : N_i| = p^i$. The $d+1$ distinct normal subgroups $N_i M_{d-i}$ have the same index in G. □

Theorem 20. *Let G be a pro-p group. Then the following conditions are equivalent*:

(i) *G has CNSG*;
(ii) *there exists a constant k such that for all closed normal subgroups M and N either $|N : N \cap M| \leqslant p^k$ or $|M : N \cap M| \leqslant p^k$*;
(iii) *there exists a constant h such that for all open normal subgroups M and N either $|N : N \cap M| \leqslant p^h$ or $|M : N \cap M| \leqslant p^h$*;
(iv) *there exists a constant c such that for all open normal subgroups M and N of the same index $|N : N \cap M| \leqslant p^c$.*

Proof. By Lemma 19, condition (i) implies condition (iv). Trivially condition (ii) implies condition (iii) and this implies condition (iv).

Let us show that condition (iv) yields condition (ii) with $k = c$. So suppose condition (iv) holds and that M and N are closed normal subgroups of G. If one of them is not open, say N, then by Lemma 18 we have that $|N : N \cap M| \leqslant |N| \leqslant p^c$. If N and M are both open, then we may assume that $|G : N| \geqslant |G : M|$. Let $\tilde{M}$ be an open normal subgroup of G such that $|G : \tilde{M}| = |G : N|$ and $M \geqslant \tilde{M}$. Therefore $|N : N \cap M| \leqslant |N : N \cap \tilde{M}| \leqslant p^c$.

Suppose again that condition (iv) holds, and we will prove condition (i). The Frattini subgroup $\overline{[G,G]G^p}$ has finite index in G by Lemma 18, so G is finitely generated. Moreover, for every open normal subgroup H of G, we have that $\overline{[H,G]H^p}$ has finite index in G and it is therefore open in G. We fix a positive integer n and consider an open normal subgroup M of G of index p^n. We set $M_0 := M$ and $M_i := \overline{[M_{i-1}, G]M_{i-1}^p}$ for every $i > 0$. By the above remark each M_i is an open normal subgroup of G. We extend the definition of M_i inductively to negative i's as follows: M_i is the subgroup of G such that $M_i/M_{i+1} = \Omega_1(Z(G/M_{i+1}))$. Clearly, the subgroups M_i are open (as they contain M) and normal in G. Let us consider another open normal subgroup N of index p^n. We know that $|M : M \cap N| \leqslant p^c$ so if we choose a sequence $M = K_0 \geqslant K_1 \geqslant \cdots \geqslant K_{r-1} \geqslant K_r = M \cap N$ of open normal subgroups of G with $|K_i : K_{i+1}| = p$ for every i, then $r \leqslant c$. An easy induction shows that $K_i \geqslant M_i$ for every $i \geqslant 0$. In particular, $N \geqslant K_r \geqslant M_r \geqslant M_c$. In a similar (dual) way we may prove that $N \leqslant M_{-c}$. As N is arbitrary, we may associate to every open normal subgroup of G of index p^n a different subgroup of M_{-c}/M_c. By Lemma 18 the order of each M_i/M_{i+1} is bounded in terms of c, hence, the order of M_{-c}/M_c can also be bounded in terms of c. Since the number of subgroups of a group of bounded order is bounded, $a_n^{\lhd}(G)$ is also bounded, therefore, G has CNSG. □

Remark 21. In [7, Section 5] Ershov refers to G satisfying condition (iii) of the previous theorem as *rigid*. In our terminology, his Lemma 5.1 states that a pro-p group with an open subgroup with CNSG has itself CNSG. A dual property also holds: if H is a finite normal subgroup of a pro-p group G and G/H has CNSG, then G has CNSG too.

Corollary 22. *If G is a pro-p group with CNSG and $\gamma_2(G)$ is open in G, then G has finite width.*

Proof. By Lemma 16 every $\gamma_i(G)$ is open in G. If the exponent of the section $\gamma_i(G)/\gamma_{i+1}(G)$ is p^{e_i}, then we may find a sequence of open normal subgroups $\gamma_i(G) = L_0 \geqslant L_1 \geqslant \cdots \geqslant L_{e_i} = \gamma_{i+1}(G)$ such that $L_i^p \leqslant L_{i+1}$ for every i. Since G satisfies condition (iv) of Theorem 20, then from Lemma 18 the order of each L_i/L_{i+1} is bounded. Then to show that G has finite width we need only to prove that e_i can be bounded, but p^{e_i} is bounded by the exponent of $G/\gamma_2(G)$ due to [14, Satz 2.13 b]. $\square$

We now characterize CNSG pro-p groups which do not have finite width:

Corollary 23. *Let G be an infinite pro-p group with CNSG. Then there exists a unique maximal normal finite subgroup H of G. Moreover, G/H is just infinite either of finite width or it is isomorphic to $\mathbb{Z}_p$. In the former case, G has finite width, H is the hypercenter of G, and G/H has trivial center, in the latter case, G is nilpotent and does not have finite width.*

Proof. By Lemma 18 there exists a unique maximal normal finite subgroup H of G. Furthermore, closed normal subgroups of G strictly containing H are necessarily open, so G/H is just infinite. Since a non-trivial finite normal subgroup of a pro-p group has non-trivial intersection with the center of the pro-p group, we have that H is contained in $Z_i(G)$ for some i. If G/H has non-trivial center, then it is isomorphic to $\mathbb{Z}_p$, hence, G is nilpotent and it does not have finite width. If G/H has trivial center, then obviously H is the hypercenter of G and $\gamma_2(G/H)$, being closed and non-trivial, is open in G/H. Since a quotient of a group with CNSG has CNSG, by Corollary 22, we have that G/H has finite width and subsequently G has finite width too. $\square$

Assuming that $\gamma_2(G)$ is open in G we have a further characterization of a group G with CNSG:

Theorem 24. *Let G be a pro-p group such that $\gamma_2(G)$ is open. Then G has CNSG if and only if there exists a constant c such that, for every open normal subgroup N, there exists an integer i such that $\gamma_i(G) \leqslant N$ and $|N : \gamma_i(G)| \leqslant p^c$.*

Proof. Suppose that G has CNSG. By Corollary 22, G has finite width, say w. Let N be an open normal subgroup of G and let i be the minimum integer such that $|G : \gamma_i(G)| \geqslant |G : N|$. Since $\gamma_i(G)$ is open in G (see Lemma 16) we have, thanks to condition (iii) of Theorem 20, that $|\gamma_i(G) : N \cap \gamma_i(G)| \leqslant p^h$ with h not depending on N. Therefore, $\gamma_{i+h}(G) \leqslant N \cap \gamma_i(G)$. As a consequence $\gamma_{i+h}(G) \leqslant N$ and $|N : \gamma_{i+h}(G)| \leqslant |\gamma_{i-1}(G) : \gamma_{i+h}(G)| \leqslant p^{w(h+1)}$.

Conversely, if N and M are two open normal subgroups of G, then there exist integers i and j such that $|N : \gamma_i(G)| \leqslant p^c$ and $|M : \gamma_j(G)| \leqslant p^c$. Suppose that $i \geqslant j$. Then $|N : N \cap M| \leqslant |N : \gamma_i(G)| \leqslant p^c$ so from Theorem 20 the group G has CNSG. $\square$

Corollary 25. *If G is a pro-p group with CNSG and $\gamma_2(G)$ is open in G, then G has finite central width, that is there exists a constant c such that $|N : [G, N]| \leqslant p^c$ for every open normal subgroup N of G.*

Proof. From the previous theorem there exists an integer i such that N contains $\gamma_i(G)$ and $|N : \gamma_i(G)|$ is bounded. As a consequence $[G, N]$ contains γ_{i+1} so $|N : [G, N]|$ can be bounded by $|N : \gamma_i(G)||\gamma_i(G) : \gamma_{i+1}(G)|$. By Corollary 22, $|\gamma_i(G) : \gamma_{i+1}(G)|$ is bounded, hence the claim follows. $\square$

For a pro-p group G we set $P_1(G) := G$ and $P_i(G) := \overline{[P_{i-1}(G), G]P_{i-1}(G)^p}$ for all $i > 1$. Note that if G has CNSG, then G is finitely generated and therefore $P_i(G)$ are open in G for all i and actually it is enough to define $P_i(G) := [P_{i-1}(G), G]P_{i-1}(G)^p$ for all $i > 1$ (see [5, Proposition 1.16]). By Lemma 18, $|P_i(G) : P_{i+1}(G)|$ is bounded. Therefore, using a similar argument as in the proof of Theorem 24, we obtain

Theorem 26. *Let G be a pro-p group. Then G has CNSG if and only if there exists a constant c such that for any open normal subgroup N there exists an integer i such that $P_i(G) \leqslant N$ with $|N : P_i(G)| \leqslant p^c$.*

In [3] it is proved that a sandwich pro-p group G of weak finite width (namely $\gamma_i(G)/\gamma_{i+1}(G)$ are finite for all i) is just infinite (or finite) and that a pro-p group G of finite width is sandwich if and only if it has finite obliquity. We show that the hypothesis on the width can be removed with the exclusion of an obvious exception. We start with the following trivial remark.

Remark 27. Let G be a pro-p group. If G is nilpotent of class c, then G is r-sandwich for all $r \geqslant c$.

We are now ready to prove the following:

Theorem 28. *Let G be an r-sandwich pro-p group. If G is not nilpotent of class at most r, then*

(1) *G is finitely generated*;
(2) *$\gamma_i(G)$ is an open subgroup of G for every i*;
(3) *G has finite width*;
(4) *G has CNSG.*

Proof. We apply Lemma 17 with $i = 2$ and obtain the first two statements.

Since every quotient $\gamma_i(G)/\gamma_{i+1}(G)$ is a finite abelian group of exponent dividing the exponent of $G/\gamma_2(G)$ (see [14, Hilffsatz III.2.14]), to prove the third statement it is enough to find an upper bound for the number of generators of every quotient $\gamma_i(G)/\gamma_{i+1}(G)$ for all i large enough.

Let $\{g_1, g_2, \ldots, g_d\}$ be a generating set for G. We claim that, for all $i \geqslant 2r$, the number of generators of $\gamma_i(G)/\gamma_{i+1}(G)$ is bounded above by $1 + d + d^2 + \cdots + d^r$. If $\gamma_{i-r}(G) = \gamma_{i-r+1}(G)$, then $\gamma_i(G) = 1$ and we are done, otherwise choose an element x in $\gamma_{i-r}(G) - \gamma_{i-r+1}(G)$ and consider the subset S formed by x and the commutators $(x, g_{i_1}, g_{i_2}, \ldots, g_{i_s})$ with $1 \leqslant s \leqslant r$ and $1 \leqslant i_1, i_2, \ldots, i_s \leqslant d$. The subgroup H generated by S and $\gamma_{i+1}(G)$ is normal and open, as it contains $\gamma_{i+1}(G)$. As $H \nleqslant \gamma_{i-r+1}(G)$ we have that $\gamma_i(G)$ is a subgroup of H. Since $i \geqslant 2r$, the elements of S commute modulo $\gamma_{i+1}(G)$ so $H/\gamma_{i+1}(G)$ is an abelian group generated by $1 + d + d^2 + \cdots + d^r$ elements. Our claim then follows from the fact that $\gamma_i(G)/\gamma_{i+1}(G)$ is a subgroup of $H/\gamma_{i+1}(G)$.

We now consider an open normal subgroup N of G. Then there exists an integer i such that $\gamma_i(G) \geqslant N \geqslant \gamma_{i+r}(G)$, hence, $|N : \gamma_{i+r}(G)| \leqslant |\gamma_i(G) : \gamma_{i+r}(G)| \leqslant p^{rw}$, where w is the width of G. By Theorem 24, G has CNSG. □

In the proof of [17, Lemma II.3] it is (implicitly) shown that a pro-p group of finite obliquity is r-sandwich for some integer r. More explicitly what is proved in the cited lemma is the following.

Proposition 29. *If G is a pro-p group of finite obliquity r, then G is $(r + 1)$-sandwich.*

Proof. For every positive i we have $|\gamma_i(G) : \mu_i(G)| \leqslant p^r$, therefore, $\mu_i(G) \geqslant \gamma_{i+r}(G)$. □

Corollary 30. *Let G be a pro-p group of finite obliquity. If $\gamma_2(G)$ is not finite, then G has finite width.*

Proof. As $\mu_2(G)$ has finite index in $\gamma_2(G)$, we have that $\mu_2(G) \neq 1$, hence every $\gamma_i(G)$ is open in G. By Lemma 16, since G is infinite, it is not nilpotent. By Proposition 29, G is sandwich, therefore, Theorem 28 implies the result. □

Combining the already cited results from [3] with Proposition 29, Corollary 30, and Theorem 28 we obtain

Corollary 31. *Let G be a non-nilpotent pro-p group. Then the following two conditions are equivalent*:

(i) *G has finite obliquity*;
(ii) *G is sandwich.*

Moreover, if one of the these conditions holds, then G has CNSG and it is just infinite of finite width.

Remark 32. In the previous corollary the assumption that G is non-nilpotent can be replaced by the hypothesis that $\gamma_2(G)$ is open and G is infinite.

Definition 33. Given a subset S of a pro-p group G we denote by $V(S)$ the set of the open normal subgroups of G which are not contained in S.

Remark 34. Using Lemma 9 the definition of $\mu_i(G)$ can be rewritten as

$$\mu_i(G) = \gamma_i(G) \cap \bigcap_{N \in V(\gamma_i(G))} N.$$

Definition 35. Let H be an open subgroup of a pro-p group G. We set

$$\omega(H) := \sup_{N \in V(H)} |H : N \cap H|.$$

For completeness we set $\omega(G) := 1$.

We give a number of different characterizations of just infinite pro-p groups:

Theorem 36. *Let G be a pro-p group. The following properties are equivalent*:

(i) *G is just infinite or finite*;
(ii) *$V(H)$ is finite for every open subgroup H of G*;
(iii) *$\omega(H)$ is finite for every open subgroup H of G*;
(iv) *every infinite family $\mathcal{B}$ of open normal subgroups of G is a basis for the neighborhoods of* 1;
(v) *there exists a family $\mathcal{F}$ of open subgroups of G with trivial intersection such that $V(H)$ is finite for every H in $\mathcal{F}$*;
(vi) *there exists a family $\mathcal{F}$ of open subgroups of G with trivial intersection such that $\omega(H)$ is finite for every H in $\mathcal{F}$.*

Proof. Clearly (ii) implies (iii), (iii) implies (vi), (ii) implies (v) and (v) implies (vi).

We now assume that (i) holds and prove that (ii) holds. This is clear if G is nilpotent, hence, procyclic or finite. We may then assume that G is not nilpotent, therefore by Lemma 16, the subgroups $\gamma_i(G)$ form a basis for the neighborhoods of 1. Thus, the closure of a normal subgroup N of G is the intersection $\bar{N} = \bigcap_{i \geqslant 1} N\gamma_i(G)$. So, if $\bar{N}$ contains $\gamma_i(G)$, then $N\gamma_{i+1}(G) \geqslant \gamma_i(G)$. The converse also holds, $N\gamma_{i+1}(G) \geqslant \gamma_i(G)$ implies, with an easy induction on j, that $N\gamma_{i+j}(G) \geqslant \gamma_i(G)$ for every positive j.

We now define, for every positive integer i, the subset

$$C_i := \left\{ x \in G \mid \overline{\langle x \rangle^G} \geqslant \gamma_i(G) \right\}.$$

Since G is just infinite we have $\bigcup_{i \geqslant 1} C_i = G - \{1\}$.

By the above discussion $x \in C_i$ if and only if $\langle x \rangle^G \gamma_{i+1}(G) \geqslant \gamma_i(G)$. If x and y are congruent modulo $\gamma_{i+1}(G)$, then $\langle x \rangle^G \gamma_{i+1}(G) = \langle y \rangle^G \gamma_{i+1}(G)$, so if $x \in C_i$, then C_i contains the whole neighborhood $x\gamma_{i+1}(G)$. Therefore, C_i is open. We note that the subsets C_i form a chain and their union contains the closed (hence compact) subset $G - H$. Thus, there exists i such that $C_i \geqslant G - H$. This implies that every element of $V(H)$ contains the open subgroup $\gamma_i(G)$, since $G/\gamma_i(G)$ is finite this implies (ii).

We now assume that (ii) holds and prove that (iv) holds. Let $\mathcal{B}$ be an infinite family of open normal subgroups of G. Given an open subgroup H of G, the family $V(H)$ is finite, hence, there exists some element N of $\mathcal{B}$ which does not belong to $V(H)$, that is $N \leqslant H$. Therefore, $\mathcal{B}$ is a basis for the neighborhoods of 1, as claimed.

Conversely if we assume that (iv) holds, then for every open subgroup H of G, the family $V(H)$ which is not a basis for the neighborhoods of 1, is necessarily finite, that is (ii) holds.

Finally, let us assume that (vi) holds and prove that (i) holds. Let K be a non-trivial closed normal subgroup of G and let H be an element of $\mathcal{F}$ which does not contain K. Then K, being the intersection of the open normal subgroups of G containing K, is the intersection of some elements of $V(H)$. Thus, there exists an open normal subgroup N containing K such that $|G:N|$ is maximum possible. This implies that N is contained in every open normal subgroup of G containing K, hence, $N = K$. Therefore, K is open, i.e., (i) holds. □

We recall that a pro-p group is said to be *hereditarily just infinite* if all its open subgroups are just infinite. Using the previous theorem it is possible to give the following characterization.

Corollary 37. *Let G be an infinite pro-p group. Then G is hereditarily just infinite if and only if for every open subgroup H of G there are finitely many open subgroups K of G such that $K \nleqslant H$ and $H \leqslant N_G(K)$.*

Proof. If $H \leqslant_O G$, we set $U(H) := \{K \leqslant_O G \mid K \nleqslant H, H \leqslant N_G(K)\}$. If $H \leqslant_O L \leqslant_O G$, we set $V_L(H) := \{K \trianglelefteqslant_O L \mid K \nleqslant H\}$. We claim that for every open subgroup H of G we have $U(H) = \bigcup_{H \leqslant L \leqslant_O G} V_L(H)$. In fact if K is an element of $U(H)$, then K belongs to $V_{HK}(H)$. On the other hand the elements of $V_L(H)$ are normalized by H and thus belong to $U(H)$. By Theorem 36, G is hereditarily just infinite if and only if $V_L(H)$ is finite for every $H \leqslant_O L \leqslant_O G$. Since for a given open subgroup H of G there are only a finite number of open subgroups of G containing H this means that G is hereditarily just infinite if and only if $U(H)$ is finite for every $H \leqslant_O G$, as required. □

Theorem 38. *Let G be a pro-p group. Then the following are equivalent*:

(i) *G has finite obliquity and finite width*;
(ii) *$\sup_{N \trianglelefteqslant_O G} \omega(N)$ is finite and $G \ncong \mathbb{Z}_p$.*

Proof. We first assume that G has finite obliquity and finite width, say o and w respectively. Clearly $G \ncong \mathbb{Z}_p$. Let N be an open normal subgroup of G. By Proposition 29 there exists i such that $\gamma_i(G) \geqslant N \geqslant \gamma_{i+o+1}(G)$. If $K \in V(N)$, then $K \in V(\gamma_{i+o+1}(G))$ so $K \geqslant \mu_{i+o+1}(G)$ and, as a consequence, $|\gamma_{i+o+1}(G) : K \cap \gamma_{i+o+1}(G)| \leqslant p^o$. Therefore,

$$|N : K \cap N| \leqslant \left|\gamma_i(G) : \gamma_{i+o+1}(G)\right|\left|\gamma_{i+o+1}(G) : K \cap \gamma_{i+o+1}(G)\right| \leqslant p^{w(o+1)+o}.$$

Conversely, assume that (ii) holds. By Theorem 36, G is just infinite, as $G \ncong \mathbb{Z}_p$ this implies that $\gamma_i(G)$ are open for all i and G is not nilpotent. Hence, there exists an integer r such that $\omega(\gamma_i(G)) \leqslant p^{r-1}$ for every i. Let N be an open normal subgroup of G. If $N \nleqslant \gamma_i(G)$, then $|\gamma_i(G) : N \cap \gamma_i(G)| \leqslant p^{r-1}$. Therefore, $N \geqslant \gamma_{i+r-1}(G)$, that is G is r-sandwich. By Corollary 31, G has finite obliquity. □

It is not true, in general, that a pro-p group with an open subgroup of finite obliquity has itself finite obliquity. As a simple example consider the direct product of a pro-p group of finite obliquity and finite width with a finite p-group. This group has finite width and it is not just infinite and hence, by Corollary 31, it does not have finite obliquity. However we have the following result.

Corollary 39. *Let G be a just infinite insoluble pro-p group. If G has an open subgroup H of finite obliquity, then G has finite obliquity.*

Proof. Since G is insoluble, the subgroup H is non-nilpotent. By Corollary 31, H has finite width. According to Theorem 38, to prove our claim we need an upper bound for $\omega(N)$ as N ranges over the set of the open normal subgroups of G. By Theorem 36, there exists a finite number of open normal subgroups of G which are not contained in H and for each of them $\omega(N)$ is finite. Hence, we may focus our attention on the open normal subgroups of G which are contained in H. So let N be such an open normal subgroup and let M be an open normal subgroup of G which is not contained in N. If M is also contained in H, then $|N : N \cap M|$ can be bounded, by Theorem 38, independently of M

and N. If M is not contained in H, then $|N : N \cap M| \leqslant |G : M|$. Since we already noticed that there exist only a finite number of open normal subgroups of G which are not contained in H, we find an upper bound for $|N : N \cap M|$. $\square$

Remark 40. We shall prove later (see Remark 55) that soluble just infinite pro-p groups have finite obliquity, so the insolubility hypothesis in the previous corollary can be dropped.

We conclude this section with some remarks on pro-p groups of finite coclass. If G is such a pro-p group, then it is finitely generated and there exists an integer l such that, for all $i \geqslant l$, the sections $\gamma_i(G)/\gamma_{i+1}(G)$ have order p and, thanks to the results in [20, Section 7.4], all non-trivial closed normal subgroups of G contained in $\gamma_l(G)$ are terms of the lower central series of G (in particular they are open). This does not necessarily mean that a pro-p group of finite coclass has only one normal subgroup of index p^k, for large enough k. As a simple example take the direct product of a finite coclass infinite pro-p group with a finite p-group. On the contrary, the unique infinite pro-p group of maximal class M is an example of a pro-p group of finite coclass with only one normal subgroup of index p^k for $k \geqslant 2$. We have indeed

Proposition 41. *Let G be an infinite pro-p group of finite coclass. The following properties are equivalent*:

(i) *G is just infinite*;
(ii) *G has only one normal subgroup of index p^k for all large enough k*;
(iii) *G has finite obliquity.*

Proof. Condition (iii) implies condition (i) by Corollary 31.

Assume now that condition (ii) holds. If N is an open normal subgroup of G, then N contains every open normal subgroup of G of large enough index. Since G is finitely generated this implies that the number of open normal subgroups of G which are not contained in N is finite. Theorem 36 implies condition (i).

Finally we assume that G is just infinite. Choose l such that $\gamma_i(G)/\gamma_{i+1}(G)$ have order p for all $i \geqslant l$. By Theorem 36, $\gamma_l(G)$ contains all open normal subgroups of G of large enough index. As $\gamma_l(G)$ contains exactly one normal subgroup of G of index p^k for all k large enough, condition (ii) holds. As a consequence $\omega(N) = 1$ for every open normal subgroup N of G of large enough index, so there is only a finite number of open normal subgroups N of G for which $\omega(N) > 1$. For such subgroups, $\omega(N)$ is finite by Theorem 36, hence, there exists an upper bound for $\omega(N)$ as N ranges over the set of the open normal subgroups of G. By Theorem 38, G has finite obliquity. Summarizing, we have proved that condition (i) implies conditions (ii) and (iii). $\square$

3. Periodic maps in just infinite pro-*p* groups

Definition 42. Let G be a pro-p group. A *periodic map* of G or simply a *period* of G is a map $\tau : M \to G$ defined on an open normal subgroup M of G satisfying the following requirements:

(1) $\tau(M)$ is an open subgroup of G;
(2) for every open normal subgroup H of G contained in $\tau(M)$, $\tau^{-1}(H)$ is an open normal subgroup of G such that $|G : \tau^{-1}(H)| < |G : H|$.

Lemma 43. *Let $\tau : M \to G$ be a period of a pro-p group G. Let H and K be two open normal subgroups contained in $\tau(M)$. Then $H \geqslant K$ if and only if $\tau^{-1}(H) \geqslant \tau^{-1}(K)$ (in particular $H = K$ if and only if $\tau^{-1}(H) = \tau^{-1}(K)$). If the inclusion $H \geqslant K$ holds, then $|H : K| \leqslant |\tau^{-1}(H) : \tau^{-1}(K)|$.*

Proof. The first claim is just elementary set theory.

Assume that $H \geqslant K$: let

$$H = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_{r-1} \geqslant H_r = K$$

be a chain of open normal subgroups of G such that $|H_i : H_{i+1}| = p$ for every i (so $|H : K| = p^r$). For every i, the subgroup H_i strictly contains H_{i+1}, hence, $\tau^{-1}(H_i)$ strictly contains $\tau^{-1}(H_{i+1})$, that is $|\tau^{-1}(H_i) : \tau^{-1}(H_{i+1})| \geqslant p$. The result follows. □

As a particular case of the previous Lemma we have the following:

Corollary 44. *Let $\tau : M \to G$ be a period of a pro-p group G. Let N and H be two open normal subgroups contained in $\tau(M)$ and such that $N \ngeqslant H$. Then*

$$\tau^{-1}(N) \ngeqslant \tau^{-1}(H) \quad \textit{and} \quad |N : H \cap N| \leqslant \left|\tau^{-1}(N) : \tau^{-1}(H) \cap \tau^{-1}(N)\right|.$$

The proof of the following remark is immediate.

Remark 45. Let $\tau : M \to G$ be a period of a pro-p group. If N is an open normal subgroup of G contained in $\tau(M)$, then the restriction of τ to $\tau^{-1}(N)$ is a period of G.

Definition 46. If $\tau : M \to G$ is a period of a pro-p group, the *degree* of τ, written as $\deg \tau$, is

$$\deg \tau := \log_p \min_{\substack{N \trianglelefteqslant_O G \\ N \leqslant \tau(M)}} \frac{|G : N|}{|G : \tau^{-1}(N)|}.$$

Definition 47. A period $\tau : M \to G$ is said to be *uniform* if $|G : N| = p^{\deg \tau} |G : \tau^{-1}(N)|$ for all open normal subgroups N of G contained in $\tau(M)$.

Proposition 48. *Let $\tau : M \to G$ be a period of degree d of a pro-p group. Then there exists an open normal subgroup $\hat{M}$ of G contained in M such that the restriction of τ to $\hat{M}$ is a uniform period of degree d.*

Proof. Let H be an open normal subgroup of G contained in $\tau(M)$ and satisfying the relation $|G : H| = p^d |G : \tau^{-1}(H)|$. We set $\hat{M} := \tau^{-1}(H)$. Let K be an open normal subgroup of G contained in H. By Lemma 43 we have that

$$\frac{|G : K|}{|G : \tau^{-1}(K)|} \leqslant \frac{|G : H|}{|G : \tau^{-1}(H)|} = p^d.$$

The result follows from the definition of degree. □

Theorem 49. *Let G be a just infinite pro-p group and let $\tau : M \to G$ be a period of degree d. Then, for all n large enough, $\tau(K)$ is an open normal subgroup of G for every open normal subgroup K of G of index p^n and τ induces a bijection between the set of the open normal subgroups of G of index p^n and the set of the open normal subgroups of G of index p^{n+d}. In particular $a^{\triangleleft}_{p^n}(G) = a^{\triangleleft}_{p^{n+d}}(G)$ for large enough n.*

Proof. By Proposition 48 we may assume that τ is uniform. By Theorem 36, the set $V(\tau(M))$ is finite, hence, $\tau(M)$ contains all the open normal subgroups of index p^{n+d} for large enough n. The inverse images via τ of these groups are, according to Lemma 43, $a^{\triangleleft}_{p^{n+d}}(G)$ distinct open normal subgroups of index p^n. Therefore, $a^{\triangleleft}_{p^n}(G) \geqslant a^{\triangleleft}_{p^{n+d}}(G)$. As a consequence $a^{\triangleleft}_{p^n}(G) = a^{\triangleleft}_{p^{n+d}}(G)$ for all large enough n and the inverse images via τ of the open normal subgroups of G of index p^{n+d} are all the open normal subgroups of G of index p^n. In particular, $\tau(K)$ is an open normal subgroup of G if K is an open normal subgroup of G of large enough index in G. □

We now prove

Theorem 50. *Let G be a just infinite pro-p group with a periodic map. Then either $G \cong \mathbb{Z}_p$ or G has finite obliquity.*

Proof. Let $\tau : M \to G$ be a period. By Theorem 36, the set $V(\tau(M))$ is finite. Hence, we may set $\omega := \max_{K\in V(\tau(M))}\omega(K)$ and $i := \max_{K\in V(\tau(M))}|G : K|$. By Theorem 38, we only need to show that $\sup_{N\trianglelefteqslant_O G}\omega(N)$ is finite. More precisely, we claim that $\omega(N) \leqslant \max(\omega, i)$ for every $N \trianglelefteqslant_O G$. The proof is by induction on $|G : N|$, the inductive basis being trivial. If $N \nleqslant \tau(M)$, then $\omega(N) \leqslant \omega$. If $N \leqslant \tau(M)$, let H be an element of $V(N)$. If $H \nleqslant \tau(M)$, then $|N : H \cap N| \leqslant |G : H| \leqslant i$. If $H \leqslant \tau(M)$, then Corollary 44 yields $|N : H \cap N| \leqslant |\tau^{-1}(N) : \tau^{-1}(H) \cap \tau^{-1}(N)| \leqslant \omega(\tau^{-1}(N))$. The result follows from the inductive hypothesis. □

4. Open subgroups of the Nottingham group

Let q be a power of a prime number p. The Nottingham group over $\mathbb{F}_q$, denoted by N_q, is defined to be the group of normalized automorphisms of the ring of formal power series $\mathbb{F}_q[\![t]\!]$:

$$N_q := \{\phi \in \operatorname{Aut}(\mathbb{F}_q[\![t]\!]) \mid t\phi = t(1+tf),\ \ f \in \mathbb{F}_q[\![t]\!]\}.$$

In this section we fix q and denote the Nottingham group N_q by J. We define the following subgroups of J:

$$J_k := \{\phi \in \operatorname{Aut}(\mathbb{F}_q[\![t]\!]) \mid t\phi = t(1+t^k f),\ \ f \in \mathbb{F}_q[\![t]\!]\}.$$

It is well known that two elements ϕ and ψ of J are congruent modulo J_k if and only if $t\phi$ and $t\psi$ are congruent modulo t^{k+1}.

For every positive integer m let τ_m be the map from J to itself defined as follows: if $\psi \in J$ is the element such that $t\psi = t + s(t)$, then $\tau_m(\psi)$ is the element that sends t to $t + t^{p^m}s(t)$. Note that $\tau_m(J_k) = J_{k+p^m}$ for every k. We will use these maps to define periodic maps for open subgroups of J.

We begin with an auxiliary result which is probably already known.

Lemma 51. *Let $\phi \in J$, $\psi \in J_k$ and suppose that $t\phi = a(t)$ and $t\psi = t + s(t)$. Then*

$$t(\phi^{-1} \circ \psi \circ \phi) \equiv t + \frac{s(a(t))}{a'(t)} \bmod t^{2k+2},$$

where a' is the derivative of a.

Proof. Let $t\phi^{-1} = b(t)$. From Taylor's approximation we obtain the following congruence:

$$t(\phi^{-1} \circ \psi) = [b(t)]\psi = b(t + s(t)) \equiv b(t) + b'(t)s(t) \bmod t^{2k+2}.$$

Note that

$$t = b(a(t)).$$

In particular,

$$1 = b'(a(t))a'(t).$$

Hence,

$$t(\phi^{-1} \circ \psi \circ \phi) \equiv b(a(t)) + b'(a(t))s(a(t)) \equiv t + \frac{s(a(t))}{a'(t)} \bmod t^{2k+2}. \qquad \square$$

Corollary 52. *For $k \geqslant p^m$, the map τ_m induces a J-isomorphism from J_k/J_{k+p^m} onto J_{k+p^m}/J_{k+2p^m}.*

Proof. It is straightforward to show that τ_m induces an isomorphism from J_k/J_{k+p^m} onto J_{k+p^m}/J_{k+2p^m}. In order to prove that this isomorphism commutes with conjugation by elements of J, we consider $\phi \in J$ and $\psi \in J_k$ like in Lemma 51. From Lemma 51 and the definition of τ_m we have that

$$t\bigl(\tau_m\bigl(\phi^{-1} \circ \psi \circ \phi\bigr)\bigr) \equiv t + t^{p^m} \frac{s(a(t))}{a'(t)} \bmod t^{2k+p^m+2}$$

and

$$t\bigl(\phi^{-1} \circ \tau_m(\psi) \circ \phi\bigr) \equiv t + a(t)^{p^m} \frac{s(a(t))}{a'(t)} \bmod t^{2k+2p^m+2}.$$

Since $a(t)^{p^m} s(a(t)) \equiv t^{p^m} s(a(t)) \bmod t^{k+2p^m+1}$ we obtain our claim. □

Theorem 53. *Let p be an odd prime, q a power of p, and $J = N(\mathbb{F}_q)$. Then every open subgroup H of J is a just infinite pro-p group with a periodic map.*

Proof. In the proof of [4, Proposition 3] it is shown that H is just infinite, that $H \geqslant J_n$ for some integer n, and that for every non-trivial closed normal subgroup N of J_n there exists an integer k such that $J_k \geqslant N \geqslant J_{k+2n+2}$. Let m be an integer such that $p^m \geqslant 2n+2$. Then for every non-trivial closed normal subgroup N of J_n there exists an integer k such that $J_k \geqslant N \geqslant J_{k+p^m}$.

We now claim that the restriction τ of τ_m to J_{p^m} is a periodic map of H. First of all, J_{p^m} is an open normal subgroup of H and $\tau(J_{p^m}) = J_{2p^m}$ is an open (normal) subgroup of H.

Let N be an open normal subgroup of H contained in $\tau(J_{p^m}) = J_{2p^m}$. We know that there exists k such that $J_k \geqslant N \geqslant J_{k+p^m}$. Since $N \leqslant J_{2p^m}$ we have that $k \geqslant 2p^m$. By Corollary 52, τ induces a J-isomorphism between J_{k-p^m}/J_k and J_k/J_{k+p^m}. In particular, τ is also an H-isomorphism, so as a consequence $\tau^{-1}(N)/J_k$ is a normal subgroup of H/J_k contained in J_{k-p^m}/J_k and

$$\bigl|J_{k-p^m}/J_k : \tau^{-1}(N)/J_k\bigr| = |J_k/J_{k+p^m} : N/J_{k+p^m}|.$$

Therefore, $\tau^{-1}(N)$ is an open normal subgroup of H and

$$\bigl|H : \tau^{-1}(N)\bigr| = |H : J_{k-p^m}|\bigl|J_{k-p^m} : \tau^{-1}(N)\bigr| = |H : J_{k-p^m}||J_k : N| = \frac{|H : N|}{|J_{k-p^m} : J_k|}.$$

Our claim follows because $|J_{k-p^m} : J_k| = p^m > 1$. □

5. *t*-linear just infinite pro-*p* groups

A pro-p group Γ is said to be *t-linear* (see [15]) if it is a closed subgroup of $\mathrm{GL}_n(R)$ for some commutative profinite ring R.

It is shown in [15] that a t-linear just infinite pro-p group is linear over $\mathbb{Z}_p$ or $\mathbb{F}_p[\![t]\!]$. Moreover, t-linear just infinite pro-p groups are naturally divided into three families:

(1) Soluble just infinite pro-p groups (e.g. $\mathbb{Z}_p$).
(2) Non-soluble p-adic analytic just infinite pro-p groups (e.g. $\mathrm{SL}_n^1(\mathbb{Z}_p)$).
(3) $\mathbb{F}_p[\![t]\!]$-linear just infinite pro-p groups (e.g. $\mathrm{SL}_n^1(\mathbb{F}_p[\![t]\!])$).

The groups in the first and second family are p-adic analytic just infinite pro-p groups, and are also called *p-adically simple* groups. In [17] it is shown that the groups in the second family are open subgroups of $\mathrm{Aut}(\mathcal{L}) \wr W$, where $\mathcal{L}$ is a simple Lie algebra over $\mathbb{Q}_p$ and W is the iterated wreath product of finitely many copies of the cyclic group of order p. By [23, Corollary 0.5] every group in the third family is commensurable with an open subgroup of the group $G(F)$ of the F-rational points of a simple simply connected algebraic group G defined over a local field F (e.g. $\mathrm{SL}_n(\mathbb{F}_p((t)))$).

These three families are actually disjoint. Indeed a p-adic analytic just infinite pro-p group is not $\mathbb{F}_p[\![t]\!]$-analytic (see [16] and [25, Proposition 5.6] for the soluble and insoluble case respectively). We shall say that the groups in the first and second families have characteristic 0 and the groups in the third family have characteristic p.

Let us first consider p-adic analytic groups, i.e. groups included in either one of the first two families of t-linear just infinite pro-p groups.

Proposition 54. *A non-trivial pro-p group G of finite rank has a periodic map.*

Proof. As in the proof of [5, Theorem 4.8] there exists an open normal subgroup M of G such that every open normal subgroup of G contained in M is powerful. We may assume that $M \neq 1$. Indeed, if G is infinite, an open subgroup is necessarily non-trivial, while if G is finite we may, for instance, choose $M = Z(G)$. We claim that the map $\tau : M \to G$ defined by $\tau(g) = g^p$, is a period.

By [5, Lemma 3.4], $\tau(M)$ is an open subgroup of G. As $\tau(M)$ is contained in the Frattini subgroup of M we have that $M \neq \tau(M)$. Let H be an open normal subgroup of G contained in $\tau(M)$, and let K be the subgroup generated by $\tau^{-1}(H)$. Since τ commutes with conjugations by elements of G we have that K is normal in G, and it is open since $\tau^{-1}(H)$ contains H. We claim that $K = \tau^{-1}(H)$, that is $K^p \leqslant H$. Our choice of M implies that K is a powerful pro-p group. The quotient K/H is then a powerful finite p-group. As K/H is generated by $\tau^{-1}(H)/H$, that is by elements of period p, [5, Lemma 2.5] implies that $K^p \leqslant H$, as claimed.

By definition of τ, K/H contains the elements of M/H of period p. Therefore, K/H is non-trivial as M/H is a non-trivial finite p-group (we recall that M strictly contains $\tau(M)$ and hence H). We may conclude that $|G : K| < |G : H|$. $\square$

Remark 55. Since it is well known that just infinite soluble pro-p groups have finite rank (e.g. [20, Chapter 12]), these groups admit periodic maps and have, therefore, finite obliquity.

Now let Γ be a t-linear insoluble just infinite pro-p group, hence, Γ belongs either to the second or to the third family. Then by [23, Corollary 0.5], there exist an open normal subgroup Γ_0 of Γ, local fields $F_1, \dots, F_k$ of the same characteristic as Γ, and absolutely simple simply connected and connected algebraic groups $\tilde{G}_i$ defined over F_i such that Γ_0 is an open subgroup of $\prod_{i=1}^k \tilde{G}_i(F_i)$ (in the notation of [23], $\tilde{G}(F) = \prod_{i=1}^k \tilde{G}_i(F_i)$, where $\tilde{G} = \prod_{i=1}^k \tilde{G}_i$ and $F = \bigoplus_{i=1}^k F_i$).

By [23, Corollary 0.3], the action of Γ on Γ_0 can be extended to an action on $\tilde{G}(F)$. Moreover, Γ permutes the direct factors $\tilde{G}_i(F_i)$. Let $H_i = \Gamma_0 \cap \tilde{G}_i(F_i)$. The normal closure N of H_1 in Γ is the direct product of some of the subgroups H_i. Since Γ is just infinite, N is open in Γ, so $N = \prod_{i=1}^k H_i$. Therefore, the direct factors H_i, and a fortiori the groups $\tilde{G}_i(F_i)$, are permuted transitively by Γ. Thus, the fields F_i are isomorphic and the groups $\tilde{G}_i$ correspond to the same Dynkin diagram. We will then say that Γ is of *type X*, where X is the type of any of the groups $\tilde{G}_i$.

We recall a result from [13] (see also [23, Proposition 1.11]).

Proposition 56. *Let G be a simple algebraic group defined over an algebraically closed field K of characteristic $p \geqslant 0$. Let $\tilde{G}$ be the universal cover of G and $\mathcal{Z}(\tilde{G})$ its scheme-theoretic center. Then either the action of G on its Lie algebra is irreducible or one of the following holds*

(1) *the characteristic of K divides $|\mathcal{Z}(\tilde{G})|$;*
(2) *$p = 2$ and G is of type F_4;*
(3) *$p = 3$ and G is of type G_2.*

We will say that a t-linear just infinite pro-p group Γ is of *irreducible type* if $\tilde{G}_i$ acts irreducible on its Lie algebra.

Theorem 57. *Let Γ be a t-linear just infinite pro-p group. Then Γ has CNSG if and only if Γ is of irreducible type. Moreover, in this case Γ has a periodic map.*

Proof. The first statement follows from [19, Theorem 2.1].

Suppose, now, that Γ is of irreducible type. The case when the characteristic of Γ is equal to 0 is done in Proposition 54. Therefore, we fix our attention on the case when Γ has characteristic p. We assume that $\tilde{G} = \tilde{G}_1$ and $F = F_1$ as the proof of the general case follows the same ideas.

Let R be the ring of integers of F with a maximal ideal $\mathfrak{m}$ and let π be a generator of $\mathfrak{m}$ and $q = |R/\mathfrak{m}|$. Let $\mathcal{O}_{\tilde{G},1}$ denote the completion of the affine ring of $\tilde{G}$ with respect to the ideal defining identity section. Then $\mathcal{O}_{\tilde{G},1} = F[\![x_1,\dots,x_n]\!]$ for any system $x_1,\dots,x_n$ of local parameters of $\tilde{G}$ in 1. Note that the morphism $d: G \times_F G \to G$ that sends (g,h) to gh^{-1} induces an F-algebra homomorphism

$$d^*: \mathcal{O}_{\tilde{G},1} \to \mathcal{O}_{\tilde{G},1} \hat{\otimes}_F \mathcal{O}_{\tilde{G},1}.$$

For any element $\gamma \in \Gamma$, the conjugation by γ extends to a unique isomorphism of algebraic groups $\tilde{G} \to \tilde{G}$ over a unique isomorphism local fields $F \to F$ [23, Corollary 0.3]. This isomorphism induces an isomorphism

$$\alpha_\gamma: \mathcal{O}_{\tilde{G},1} \to \mathcal{O}_{\tilde{G},1}.$$

We are interested in local parameters $x_1,\dots,x_n$ such that d^* sends $\mathcal{F} = R[\![x_1,\dots,x_n]\!]$ to $\mathcal{F} \hat{\otimes}_F \mathcal{F}$ and α_γ sends $\mathcal{F}$ to $\mathcal{F}$ for any $\gamma \in \Gamma$. The existence of such local parameters $x_1,\dots,x_n$ is shown in the proof of [23, Proposition 5.3].

Let $\tilde{\mathcal{G}} := \mathbf{Spf}\,\mathcal{F}$ be a smooth formal model of $\tilde{G}$ (see [23, Definition 5.4]). The formal model $\tilde{\mathcal{G}}$ determines a collection of principal congruence subgroups $\tilde{\mathcal{G}}(\mathfrak{m}^k)$ of $\tilde{G}(F)$. Note that they are invariant by conjugation of the elements from Γ. Since $\tilde{\mathcal{G}}(\mathfrak{m}^i)$ form a base for the neighborhoods of 1 in $\tilde{G}(F)$, there exists k such that $\tilde{\mathcal{G}}(\mathfrak{m}^k) \leqslant \Gamma_0$.

The Lie algebra $L =: \operatorname{Lie}\tilde{\mathcal{G}}$ of a smooth formal model $\tilde{\mathcal{G}}$ is defined in [23, p. 484]. It is an R-algebra. Denote by L_F the algebra $L \otimes F$. Then L_F is a Lie F-algebra isomorphic to the Lie algebra of $\tilde{G}(F)$.

Since Γ is of irreducible type, L_F is an irreducible $\tilde{G}(F)$-module. Denote by B the closed associative $\mathbb{F}_p$-subalgebra generated by the elements of Γ_0 in $\operatorname{End}_R(L)$. Since $\tilde{G}$ is absolutely simple over F and L_F is an irreducible $\tilde{G}(F)$-module, there exists j such that $\mathfrak{m}^j \operatorname{End}_R(L) \leqslant B$ (see, for example, [23, Theorem 2.3]). Hence if $a \in L - pL$, then the Γ_0-module generated by a contains $\mathfrak{m}^j L$.

Suppose now that $s \geqslant 2j$ and N is an open normal subgroup of Γ such that $N \leqslant \tilde{\mathcal{G}}(\mathfrak{m}^s)$ and $N \nleqslant \tilde{\mathcal{G}}(\mathfrak{m}^{s+1})$. We claim that $\tilde{\mathcal{G}}(\mathfrak{m}^{s+j}) \leqslant N$. First note that $\tilde{\mathcal{G}}(\mathfrak{m}^s)/\tilde{\mathcal{G}}(\mathfrak{m}^{2s})$ is an abelian group and it is isomorphic to $L/\mathfrak{m}^s L$ as a Γ_0-module. Since $\mathfrak{m}^j \operatorname{End}_R(L) \leqslant B$, it follows that if $n \in N$ and $n \in \tilde{\mathcal{G}}(\mathfrak{m}^s) - \tilde{\mathcal{G}}(\mathfrak{m}^{s+1})$, then

$$\tilde{\mathcal{G}}\bigl(\mathfrak{m}^{s+j}\bigr) \leqslant N\tilde{\mathcal{G}}\bigl(\mathfrak{m}^{2s}\bigr).$$

In particular, there exists $n_1 \in N$ such that $n_1 \in \tilde{\mathcal{G}}(\mathfrak{m}^{s+j}) - \tilde{\mathcal{G}}(\mathfrak{m}^{s+j+1})$. Replacing s by $s+j$ and repeating the argument, we obtain that

$$\tilde{\mathcal{G}}\bigl(\mathfrak{m}^{2s}\bigr) \leqslant \tilde{\mathcal{G}}\bigl(\mathfrak{m}^{s+2j}\bigr) \leqslant N\tilde{\mathcal{G}}\bigl(\mathfrak{m}^{2s+2j}\bigr).$$

So $\tilde{\mathcal{G}}(\mathfrak{m}^{s+j}) \leqslant N\tilde{\mathcal{G}}(\mathfrak{m}^{2s+2j})$. Continuing by induction we obtain that $N\tilde{\mathcal{G}}(\mathfrak{m}^r)$ contains $\tilde{\mathcal{G}}(\mathfrak{m}^{s+j})$ for any r and so $\tilde{\mathcal{G}}(\mathfrak{m}^{s+j}) \leqslant N$.

Let $f = \sum_{\gamma \in \Gamma_0} a_\gamma(\gamma - 1) \in \mathbb{Z}[\Gamma_0]$ be an element whose image a in $\operatorname{End}_R(L)$ is central and different from zero. The existence of such an element follows, for example, from the existence of a multilinear central polynomial for B [22, Corollary 13.6.3]. The element a is in the maximal ideal of B, as f is in the augmentation ideal of $\mathbb{Z}[\Gamma]$: therefore a is not invertible. Hence $a \in \mathfrak{m}\,\mathrm{Id}_L$, where Id_L is the identity map of L. Suppose that $a = \pi^l \bmod \mathfrak{m}^{l+1}\,\mathrm{End}_R(L)$ (note that $l \geqslant 1$). Define $\tau: \Gamma \to \Gamma$ by

$$\tau(g) = \prod [g, \gamma]^{a_\gamma}.$$

Put $r = \max\{k, 2l + 2j\}$. We claim that $\tau: \tilde{\mathcal{G}}(\mathfrak{m}^r) \to \Gamma$ is a periodic map.

First we claim that if $g \in \tilde{\mathcal{G}}(\mathfrak{m}^w)$ and $r \leqslant w \leqslant s \leqslant 2w - l$, then

$$\tau\big(g\tilde{\mathcal{G}}(\mathfrak{m}^s)\big) = \tau(g)\tilde{\mathcal{G}}\big(\mathfrak{m}^{l+s}\big).$$

The inclusion $\tau(g\tilde{\mathcal{G}}(\mathfrak{m}^s)) \leqslant \tau(g)\tilde{\mathcal{G}}(\mathfrak{m}^{l+s})$ is clear, for τ acts on $\tilde{\mathcal{G}}(\mathfrak{m}^w)/\tilde{\mathcal{G}}(\mathfrak{m}^{2w})$ as a acts on $L/\mathfrak{m}^w L$ and $\tilde{\mathcal{G}}(\mathfrak{m}^{2w})$ is contained in $\tilde{\mathcal{G}}(\mathfrak{m}^{l+s})$. To prove the reverse inclusion we show that $\tau(g\tilde{\mathcal{G}}(\mathfrak{m}^s))\tilde{\mathcal{G}}(\mathfrak{m}^e) \geqslant \tau(g)\tilde{\mathcal{G}}(\mathfrak{m}^{l+s})$ for every e. The base of induction $e = l + s$ is clear. Assume that it holds for e and let us prove it for $e + 1$. Let $h \in \tilde{\mathcal{G}}(\mathfrak{m}^{l+s})$. We want to show that $\tau(g)h \in \tau(g\tilde{\mathcal{G}}(\mathfrak{m}^s))\tilde{\mathcal{G}}(\mathfrak{m}^{e+1})$. By inductive hypothesis, there are $a \in \tilde{\mathcal{G}}(\mathfrak{m}^e)$ and $f \in \tilde{\mathcal{G}}(\mathfrak{m}^s)$ such that $\tau(g)ha = \tau(gf)$. Let $b \in \tilde{\mathcal{G}}(\mathfrak{m}^{e-l})$ be such that $\tau(b) = a^{-1} \bmod \tilde{\mathcal{G}}(\mathfrak{m}^{e+1})$. Since τ is defined as a product of commutators, it is easy to see that

$$\tau(gfb) = \tau(gf)\cdot\tau(b) = \tau(g)h \bmod \tilde{\mathcal{G}}\big(\mathfrak{m}^{e+1}\big),$$

as claimed.

Put $M := \tilde{\mathcal{G}}(\mathfrak{m}^r)$. Then $\tau(M) = \tilde{\mathcal{G}}(\mathfrak{m}^{r+l})$ is an open normal subgroup of Γ. If N is an open normal subgroup of Γ contained in $\tau(M)$, then there exists $s \geqslant r + l$ such that $N \leqslant \tilde{\mathcal{G}}(\mathfrak{m}^s)$ and $N \nleqslant \tilde{\mathcal{G}}(\mathfrak{m}^{s+1})$. Then $\tilde{\mathcal{G}}(\mathfrak{m}^{s+j}) \leqslant N$, and so

$$N = \bigcup g_i\tilde{\mathcal{G}}\big(\mathfrak{m}^{s+j}\big).$$

Then

$$\tau^{-1}(N) = \bigcup h_i\tilde{\mathcal{G}}\big(\mathfrak{m}^{s+j-l}\big),$$

where $\tau(h_i) \equiv g_i \bmod \tilde{\mathcal{G}}(\mathfrak{m}^{s+j})$. Since $2(s-l) \geqslant s + j$, $\tau^{-1}(N)$ is a normal subgroup of Γ (recall that τ acts on $\tilde{\mathcal{G}}(\mathfrak{m}^{s-l})/\tilde{\mathcal{G}}(\mathfrak{m}^{2(s-l)})$ as a acts on $L/\mathfrak{m}^{s-l}L$). Moreover,

$$|\Gamma : N| = |L/\mathfrak{m}L|^l\big|\Gamma : \tau^{-1}(N)\big|. \qquad \square$$

6. The Ershov groups

Assume that $p > 2$, and let r and s be positive integers such that $0 < r < p^s/2$ and $p \nmid r$. The following groups were introduced in [7]. Let $\mathcal{Q} := \mathcal{Q}^1(s, r)$ be the subgroup of the Nottingham group $N(p)$ which consists of the elements ϕ such that

$$t\phi = \sqrt[r]{\frac{at^r + b}{ct^r + d}}$$

for some a and d in $1 + t^q\mathbb{F}_p[\![t^q]\!]$ and b and c in $t^q\mathbb{F}_p[\![t^q]\!]$ where $q = p^s$. The group $\mathcal{Q}$ is a deformation of the group $\mathrm{SL}_2^1(\mathbb{F}_p[\![t]\!])$. In fact, there is a natural bijection α between $\mathrm{SL}_2^1(\mathbb{F}_p[\![t]\!])$ and $\mathcal{Q}$:

$$\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \phi : t \mapsto \sqrt[r]{\frac{a^q t^r + b^q}{c^q t^r + d^q}}.$$

The map α is slightly different from the map defined in [7, Section 4], because in our paper the elements of the Nottingham group acts on the right and in Ershov's paper they act on the left.

Put $G = \mathrm{SL}_2^1(\mathbb{F}_p[[t]])$ and $H = \mathcal{Q}$. We have seen in Section 5 that G has a period $\tau : M \to G$. We use this map to construct a periodic map for H. It is proved in [7, Proposition 4.3], that the map α is an approximation map with respect to the filtrations $\{\gamma_n(G)\}$ and $\{\gamma_n(H)\}$. The group G is sandwich: any normal subgroup of G lies between $\gamma_n(G)$ and $\gamma_{n+2}(G)$. We may assume that $M = \gamma_n(G)$ for some $n > 2$. Then α establishes an isomorphism between the lattice of the open normal subgroups of G contained in M and the lattice of the open normal subgroups of H contained in $\alpha(M)$. Thus $\alpha \circ \tau \circ \alpha^{-1}$ is a periodic map for H.

Of course, the same argument shows that open subgroups of Ershov groups have periodic maps.

7. Groups which do not have CNSG

Let G be a pro-p group. We say that G is of *finite p-coclass* if there exists a constant c such that $|G : P_k(G)| \leqslant p^{k+c}$ for all natural numbers k. By [5, Exercise 7, p. 269, Theorem 10.1] we have

Proposition 58. *Let G be a pro-p group of finite p-coclass. Then G is virtually abelian.*

Theorem 59. *Let G be a pro-p group and let $K := \prod_{i=1}^m H_i$ be an open normal subgroup of G isomorphic to the direct product of m copies of the closed group H_1. Suppose that G permutes the set $\{H_i\}_{i=1}^m$. Then either G is virtually abelian or there exists n such that $a_n^{\lhd}(G) > m$.*

Proof. If G is not finitely generated, then it does not have CNSG and we are done, so we may assume that G is finitely generated. If H_1 is virtually abelian, then the same is true for K and G. We may then assume that H_1 is not virtually abelian: in particular H_1 is infinite. If G does not act transitively on the set $\{H_i\}_{i=1}^m$, then H_1^G is an infinite closed normal subgroup of G with infinite index in K and hence in G. By Corollary 23, G does not have CNSG in this case and we are done. We are left to consider the case in which G acts transitively on the set $\{H_i\}_{i=1}^m$. Therefore, if we set $N := N_G(H_1)$, we have $|G : N| = m$. Note that $N \geqslant K$. Moreover, since N normalizes H_1 it permutes the set $\{H_i\}_{i=2}^m$. Hence, $\hat{K} := \prod_{i=2}^m H_i$ is a closed normal subgroup of N. Since $K = H_1 \oplus \hat{K}$ we have that $K/\hat{K} \cong H_1$. Therefore, we can identify H_1 with an open subgroup of $N/\hat{K}$. Note that $N/\hat{K}$, being the quotient of an open subgroup of a finitely generated pro-p group, is finitely generated. By [5, Proposition 1.16(iii)] there exists an integer i such that $H_1 \geqslant P_i(N/\hat{K})$. As we are assuming that H_1 is not virtually abelian, $N/\hat{K}$ is not virtually abelian too. Hence, Proposition 58 implies that $|P_k(N/\hat{K}) : P_{k+1}(N/\hat{K})| > p$ for some $k \geqslant i$. Therefore, as $P_k(N/\hat{K})/P_{k+1}(N/\hat{K})$ is elementary abelian, we may choose 2 distinct subgroups L and J such that $P_k(N/\hat{K}) \geqslant L \geqslant P_{k+1}(N/\hat{K})$, $P_k(N/\hat{K}) \geqslant J \geqslant P_{k+1}(N/\hat{K})$ and L and J have the same (finite) index in $P_k(N/\hat{K})$ and hence in H_1. Clearly L and J are normal in $N/\hat{K}$. When we consider once again H_1 as a subgroup of G, we have that L and J are normal in N. We set $|H_1 : L| := p^t$. As L is normal in N, it has at most $|G : N| = m$ conjugates. On the other hand, since G acts transitively on $\{H_i\}_{i=1}^m$, every H_i contains at least one conjugate of L. Therefore L has exactly m conjugates and its normal closure L^G has index p^{tm} in K: in particular L^G, having finite index in the finitely generated pro-p group G, is open in G. Similarly J^G is an open normal subgroup of G of the same index in G as L^G. It is easy to see that $(L \cap J)^G = L^G \cap J^G$ and that since $|L : L \cap J| \geqslant p$, we have that $|L^G : L^G \cap J^G| \geqslant p^m$. By Lemma 19 there exists n such that $a_n^{\lhd}(G) > m$: this concludes the proof. □

Corollary 60. *Let G be a branch pro-p group. Then G does not have CNSG.*

Proof. It follows directly from the structure of branch groups (see [12]) and the previous theorem. □

Corollary 61. *Let G be a just infinite pro-p group. Then either G does not have CNSG or G contains an open normal subgroup N which is isomorphic to the direct product of a finite number of copies of a hereditarily just infinite pro-p group.*

Proof. It follows from Wilson's dichotomy concerning the structure of just infinite pro-p groups (see [12, Theorem 3]) and the previous theorem. □

The Fesenko group $T := T(r)$ is a subgroup of the Nottingham group $N(p)$ and it is defined as follows: let $q = p^r$ and define

$$T := \left\{ \phi \in N(p) \;\middle|\; t\phi = t + \sum_{k \geqslant 1} a_{qk+1} t^{qk+1},\ a_{qk+1} \in \mathbb{F}_p \right\}.$$

Theorem 62. *The Fesenko group T does not have CNSG.*

Proof. Let

$$R_n := \left\{ \phi \in N(p) \;\middle|\; t\phi = t + \sum_{k \geqslant nq+2} a_{qk+1} t^{qk+1}, a_{qk+1} \in \mathbb{F}_p \right\}.$$

Then from [11] we know that $\gamma_n(T) \nleqslant R_n$ and $\gamma_{n+1}(T) \leqslant R_n$. On the other hand, using again [11], we obtain

$$\left| R_n : \gamma_{n+1}(T) \right| \geqslant p^{\frac{qn(p-1)}{p}}.$$

Hence, using Theorem 24, we obtain that T does not have CNSG. □

We recall that for $q = p^r$ the *index subgroup* of type $B_{r,r}$ is the subgroup of the Nottingham group $N(p)$ defined as follows:

$$B_{r,r} := \left\{ \phi \in N(p) \;\middle|\; t\phi = t + \sum_{k \geqslant 1} a_{qk} t^{qk} + a_{qk+1} t^{qk+1},\ a_i \in \mathbb{F}_p \right\}.$$

Theorem 63. *The group $B := B_{r,r}$ does not have CNSG.*

Proof. We set $B_n := B \cap J_n$. By [2, Lemma 8.1] we have that B_{nq-1} contains $\gamma_n(B)$ but does not contain $\gamma_{n-1}(B)$. On the other hand, from the same lemma, we have that

$$\left| B_{nq-1} : \gamma_n(B) \right| \geqslant p^{n(q-1)-q}.$$

Hence, using Theorem 24, we obtain that B does not have CNSG. □

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References

[1] Y. Barnea, R. Guralnick, Subgroup growth in some pro-p groups, Proc. Amer. Math. Soc. 130 (3) (2002) 653–659.
[2] Y. Barnea, B. Klopsch, Index-subgroups of the Nottingham group, Adv. Math. 180 (1) (2003) 187–221.
[3] A.R. Camina, R.D. Camina, Pro-p groups of finite width, Comm. Algebra 29 (4) (2001) 1583–1593.
[4] R. Camina, The Nottingham group, in: M. du Sautoy, et al. (Eds.), [6], pp. 205–221.
[5] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal, Analytic pro-p Groups, 2nd ed., Cambridge Stud. Adv. Math., vol. 61, Cambridge University Press, Cambridge, 1999.

[6] M. du Sautoy, D. Segal, A. Shalev (Eds.), New Horizons in pro-p Groups, Progr. Math., vol. 184, Birkhäuser Boston Inc., Boston, MA, 2000.
[7] M. Ershov, New just-infinite pro-p groups of finite width and subgroups of the Nottingham group, J. Algebra 275 (1) (2004) 419–449.
[8] N. Gavioli, V. Monti, Ideally constrained Lie algebras, J. Algebra 253 (1) (2002) 31–49.
[9] N. Gavioli, V. Monti, C.M. Scoppola, Soluble pro-p-groups and soluble Lie algebras, in: Advances in Group Theory 2002, Aracne, Rome, 2003, pp. 261–267.
[10] N. Gavioli, V. Monti, C.M. Scoppola, Just infinite periodic Lie algebras, in: Finite Groups 2003, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 73–85.
[11] C. Griffin, The Fesenko groups have finite width, Q. J. Math. 56 (3) (2005) 337–344.
[12] R.I. Grigorchuk, Just infinite branch groups, in: M. du Sautoy, et al. (Eds.), [6], pp. 121–179.
[13] G.M.D. Hogeweij, Almost-classical Lie algebras. I, II, Nederl. Akad. Wetensch. Indag. Math. 44 (4) (1982) 441–452, 453–460.
[14] B. Huppert, Endliche Gruppen. I, Grundlehren Math. Wiss., vol. 134, Springer-Verlag, Berlin, 1967.
[15] A. Jaikin-Zapirain, On linear just infinite pro-p groups, J. Algebra 255 (2) (2002) 392–404.
[16] A. Jaikin-Zapirain, B. Klopsch, Analytic groups over general pro-p domains, J. Lond. Math. Soc. (2) 76 (2) (2007) 365–383.
[17] G. Klaas, C.R. Leedham-Green, W. Plesken, Linear pro-p-Groups of Finite Width, Lecture Notes in Math., vol. 1674, Springer-Verlag, Berlin, 1997.
[18] B. Klopsch, Zeta functions related to the pro-p group $\mathrm{SL}_1(\Delta_p)$, Math. Proc. Cambridge Philos. Soc. 135 (1) (2003) 45–57.
[19] M. Larsen, A. Lubotzky, Normal subgroup growth of linear groups: The (G_2, F_4, E_8)-theorem, in: Algebraic Groups and Arithmetic, Tata Inst. Fund. Res., Mumbai, 2004, pp. 441–468.
[20] C.R. Leedham-Green, S. McKay, The Structure of Groups of Prime Power Order, London Math. Soc. Monogr. Ser. (N.S.), vol. 27, Oxford University Press, Oxford, 2002.
[21] A. Lubotzky, D. Segal, Subgroup Growth, Progr. Math., vol. 212, Birkhäuser Verlag, Basel, 2003.
[22] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, revised ed., Grad. Stud. Math., vol. 30, American Mathematical Society, Providence, RI, 2001, with the cooperation of L.W. Small.
[23] R. Pink, Compact subgroups of linear algebraic groups, J. Algebra 206 (2) (1998) 438–504.
[24] D.J.S. Robinson, A Course in the Theory of Groups, 2nd ed., Grad. Texts in Math., vol. 80, Springer-Verlag, New York, 1996.
[25] A. Shalev, Lie methods in the theory of pro-p groups, in: M. du Sautoy, et al. (Eds.), [6], pp. 1–54.
[26] J.S. Wilson, On just infinite abstract and profinite groups, in: M. du Sautoy, et al. (Eds.), [6], pp. 181–203.