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# Fractional GP refinable functions

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*Dedicated to Laura Gori, for her 70th birthday*

Abstract: *A new class of refinable functions extending the GP class introduced in* [12] *is presented. It is characterized by a symbol with fractional exponent that gives rise to non-compactly supported refinable functions. Nevertheless, the decay and stability properties of these refinable functions allow them to generate a multiresolution analysis* (*MRA*) *of* $L^2(\mathbb{R})$. *For suitable values of their parameters these refinable functions reduce to the fractional B-splines introduced in* [16], *while, for integer* $\alpha$, *they interpolate the GP refinable functions. Furthermore, this class of refinable functions is proved to be closed with respect to convolution and fractional differentiation, allowing for its convenient the applicability to Sobolev spaces. The fractional refinable functions introduced here show an useful order of polynomial exactness.*

## 1 – Introduction

Cardinal B-splines are well known in the literature. Their appealing properties allow for their diverse applications in approximation theory. In particular, in the context of multiresolution analysis, they exhibit crucial properties such as refinability, polynomial reproduction, total positivity and useful differential and convolution formulae. A class of totally positive, centrally symmetric and compactly supported refinable functions is given in [12] as an extension of the B-spline class which we will refer to as the class of the GP refinable functions.

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Due to their localized support, in many applications these GP refinable functions give a better numerical results than the B-splines themselves [4], [8], [10] and [11], [13], [14].

Recently, in [16], the class of cardinal B-splines has been extended by introducing a non integer exponent in the discrete Fourier transform. Following this approach, we construct an extension of the GP refinable class and investigate its MRA's, approximation, differentiation and convolution properties. The paper is organized as follows. In Section 2, we review some preliminary facts and establish some notations to be used later. In Section 3, a brief description of the class of fractional B-splines is presented. In Section 4, a characterization and the main features of the class of GP refinable functions as well as a convolutional property are provided. In Section 5, the class of GP refinable functions is extended to a fractional index in the mask giving rise to symbols with fractional exponent. These new refinable functions generated in this way are no longer compactly supported nor positive. Nevertheless, we show that they generate a MRA of $L^2(\mathbb{R})$. Moreover they are characterized by a simple convolution relation between suitable minimally supported refinable functions and suitable fractional B-splines. Finally, in Section 6, the closure of this new class with respect to convolution and differentiation is proved and a surprisingly high order of polynomial exactness for the fractional refinable functions is proved.

## 2 – Preliminaries

In this section we introduce some notation and definitions. We call *refinable function* $\varphi$ a solution of a *refinement equation*

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(2x-k), \quad x \in \mathbb{R} \tag{1}$$

where the sequence

$$a = \{a_k\}_{k\in\mathbb{Z}} \tag{2}$$

is called the *mask* of $\varphi$ and here is supposed to satisfy

$$\sum_{k\in\mathbb{Z}} a_{2k} = \sum_{k\in\mathbb{Z}} a_{2k+1} = 1\,. \tag{3}$$

The Fourier transform of (1) yields

$$\hat{\varphi}(\omega) = \frac{1}{2} m(\omega/2)\hat{\varphi}(\omega/2)\,, \tag{4}$$

where $\hat{\varphi}$ is the Fourier transform of $\varphi$, namely

$$\hat{\varphi}(\omega) := \int \varphi(x) e^{-i\omega x}\, dx, \quad \omega \in \mathbb{R}, \tag{5}$$

and $m$ is the trigonometric series of the mask $a$, i.e.

$$m(\omega) := \sum_{k\in\mathbb{Z}} a_k e^{-i\omega k}, \quad \omega \in \mathbb{R}. \tag{6}$$

By the change of variable $z = e^{i\omega}$, (6) gives rise to the *symbol* associated to the mask $a$, i.e.

$$b(z) := \sum_{k\in\mathbb{Z}} a_k z^k . \tag{7}$$

In the following we shall need the *convolution product* between the functions $f$ and $g$, defined as

$$(f * g)(x) := \int_{\mathbb{R}} f(\tau) g(x-\tau)\, d\tau . \tag{8}$$

It is known that the convolution of two refinable functions is still refinable and its discrete Fourier transform can be expressed through the product of their discrete Fourier transforms [3]. It is also worthwhile to recall (see [7], [5], for instance) that a refinable function enjoying suitable conditions generates a multiresolution analysis (**MRA**) of $L^2(\mathbb{R})$, i.e. a nested sequence of closed subspaces $V_j,\ j \in \mathbb{Z}$ of $L^2(\mathbb{R})$, such that

- $\overline{(\cup_{j\in\mathbb{Z}} V_j)} = L^2(\mathbb{R})$;
- $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
- $\exists$ a Riesz basis of $V_0$, i.e. a basis $\{e_k\}_{k\in\mathbb{Z}}$ of $V_0$ satisfying the Riesz condition

$$A||c||_{l^2} \le \left\| \sum_{k\in\mathbb{Z}} c_k e_k \right\|_{L^2} \le B||c||_{l^2}, \quad \forall\, (c_k)_{k\in\mathbb{Z}} \in l_2(\mathbb{Z}), \tag{9}$$

for some positive constants $A$ and $B$ such that $0 < A \le B < \infty$.

The space $V_j\ \ j \in \mathbb{Z}$ are generated from $\varphi$ in the following way

$$V_j := \overline{\operatorname{span}\{2^{j/2}\varphi(2^j \bullet - k),\ k \in \mathbb{Z}\}}, \tag{10}$$

or equivalently by the Riesz property as

$$V_j := \left\{ f : f = \sum_{l\in\mathbb{Z}} c_l 2^{j/2} \varphi(2^j \bullet - k), \quad (c_k)_{k\in\mathbb{Z}} \in l_2(\mathbb{Z}) \right\}. \tag{11}$$

## 3 – Fractional cardinal B-splines

In the paper [16], the extension of the class of the cardinal B-splines is provided by introducing the concept of fractional finite difference operator. Their starting point is the definition of the classical cardinal B-spline, i.e. the B-spline on integer knots. Let us consider the truncated power function

$$T^n(x) := (\max(0,x))^n, \quad x \in \mathbb{R} \tag{12}$$

and the finite difference operator

$$\Delta^n f(x) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x-k), , x \in \mathbb{R}\,. \tag{13}$$

Then, the classical B-spline is defined in the following way

$$B^n := \Delta^{n+1} T^n\,. \tag{14}$$

From this definition Fourier transform of $B^n$ comes

$$\hat{B}^n(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n+1}, \omega \in \mathbb{R}\,. \tag{15}$$

Now, to extend the definition (14) to a real index $\alpha$, consider the Gamma function

$$\begin{cases} \Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx & \text{for } \alpha > 0\,, \\ \Gamma(\alpha) = \alpha^{-1}\Gamma(\alpha+1) & \text{for } \alpha < 0\,, \end{cases} \tag{16}$$

and the generalized binomial coefficient,

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad k \in \mathbb{Z}\,, \tag{17}$$

that, for integer $\alpha$, reduces to the usual binomial coefficient. We note that, from (16) it follows that $\lim \Gamma(\alpha) = -\infty$ for $\alpha \to 0^-$ and $\lim \Gamma(\alpha) = +\infty$ for $\alpha \to 0^+$ and hence, by recursion, that $\lim \Gamma(\alpha) = (-1)^{|k|+1} \cdot \infty$ for $\alpha \to k^-$, and $\lim \Gamma(\alpha) = (-1)^{|k|} \cdot \infty$ for $\alpha \to k^+$. These facts imply

$$\binom{\alpha}{k} = 0, \quad \text{for } k < 0\,, \tag{18}$$

and

$$\operatorname{sign} \binom{\alpha}{k} = (-1)^k, \quad \text{for } k > \alpha - 1\,. \tag{19}$$

Hence, defining the *fractional finite difference operator*

$$\Delta^\alpha f(x) := \sum_{k\geq 0} \binom{\alpha}{k} (-1)^k f(x-k)\,, \tag{20}$$

leads to the *fractional cardinal B-spline*

$$B_+^\alpha := \frac{1}{\Gamma(\alpha+1)} \Delta^{\alpha+1} T^\alpha \tag{21}$$

of [16]. The binomial theorem

$$\begin{gathered} (1+z)^\beta = \sum_{k\geq 0} \binom{\beta}{k} z^k, \quad \beta \geq 0, \quad |z| \leq 1\,, \\ \beta \in \mathbb{R}, \quad z \in \mathbb{C}\,, \end{gathered} \tag{22}$$

allows one to obtain the following expression for the Fourier transform of $B_+^\alpha$,

$$\hat{B}_+^\alpha(\omega) = \left( \frac{1-e^{-i\omega}}{i\omega} \right)^{\alpha+1}, \quad \omega \in \mathbb{R}\,. \tag{23}$$

For non integer $\alpha$, the authors prove that two properties of the classical B-spline get lost, i.e. $B_+^\alpha$ is no longer positive and no longer compactly supported; nevertheless, they prove that $B_+^\alpha$, for $\alpha > -1$, is bounded and has a decay at infinity proportional to $|x|^{-\alpha-2}$, i.e.

$$B_+^\alpha(x) = O\left( \frac{1}{|x|^{\alpha+2}} \right), \quad \text{for } |x| \to \infty\,. \tag{24}$$

However, most of the B-splines properties are preserved and in some cases improved. They prove that $B_+^\alpha$ is refinable with mask

$$b_k^\alpha = \frac{1}{2^\alpha} \binom{\alpha+1}{k},$$

and corresponding symbol

$$b^\alpha(z) = \frac{1}{2^\alpha}(1+z)^{\alpha+1}\,. \tag{25}$$

Moreover, the functions $\{B_+^\alpha(\cdot - k)\}_{k\in\mathbb{Z}}$ form a Riesz basis of $V_0$, giving rise, hence, to a MRA of $L^2(\mathbb{R})$.

They prove also that $B_+^\alpha$ fulfills other interesting properties, as the following *convolution* relation

$$B_+^{\alpha_1} * B_+^{\alpha_2} = B_+^{\alpha_1+\alpha_2+1}, \quad \alpha_1, \alpha_2 > -1\,, \tag{26}$$

the following *fractional differentiation* relation

$$D^\gamma B_+^\alpha(x) = \Delta^\gamma B_+^{\alpha-\gamma}(x), \quad \alpha > -1, \gamma \in \mathbb{R}\,, \tag{27}$$

where the operator $D^\gamma$ is defined in the Fourier domain

$$(\hat{D}^\gamma f)(\omega) := (i\omega)^\gamma \hat{f}(\omega)\,. \tag{28}$$

Finally, $B_+^\alpha$ has *order of polynomial exactness* $d = \lceil\alpha\rceil + 1$, i.e. $\exists$ a sequence $e_k^l$ such that

$$x^l = \sum_{k\in\mathbb{Z}} e_k^l B_+^\alpha(x-k), \quad l = 0, \dots, \lceil\alpha\rceil\,. \tag{29}$$

REMARK 1. Note that, for integer values of $\alpha$, (20) reduces to (13), i.e. the fractional B-splines interpolate the cardinal B-splines.

REMARK 2. The decay in (24) implies that $B_+^\alpha \in L_1(\mathbb{R})$.

REMARK 3. It is worthwhile to remark that (27) allows for the application of such fractional B-splines to partial differential problems in Sobolev spaces with fractional indices.

REMARK 4. As for (29), we recall that the cardinal B-spline $B^n$ reproduces polynomials up to the degree $n$.

## 4 – On a class of MRAs on $L^2(\mathbb{R})$

Our starting point is the class of refinable functions $\varphi_{n,h}$ presented in [12] and defined through an *explicit* expression of the masks. We show two different subclasses of it. The first comes out to be

$$a_{h,k} = \frac{1}{2^h}\left[\binom{n+1}{k} + 4(2^{h-n}-1)\binom{n-1}{k-1}\right], \qquad k = 0, 1, \dots, n+1\,, \tag{30}$$

where $h \geq n$ is a real parameter and $n \geq 2$, while the second is

$$a_{l,m,k} = \frac{1}{2^l}\left[\binom{n+1}{k} + (2^m-4)\binom{n-1}{k-1} + (2^{-n+4+l} - 2^{m+2})\binom{n-3}{k-2}\right], \tag{31}$$

$$k=0,1,\dots,n+1,\ \text{with}\ l\geq n-2+\log_2(1+2^{m-1}),\quad m\geq n\ \text{real parameters and}\ n\geq 2\,. \tag{32}$$

We emphasize that (30) and (31) can be considered both as generalizations of the masks of the cardinal B-splines to whom $\varphi_{n,h}$ reduce for $h=n$, preserving all their appealing properties. In fact, the refinable functions $\varphi$ are *compactly supported* on $[0,n+1]$, *centrally symmetric* and *totally positive*. Moreover, they belong to $C^{n-2}(\mathbb{R})$ and $C^{n-4}(\mathbb{R})$ respectively and they have *order of exactness* $d=n-1\geq 0$ and $d=n-3\geq 0$ respectively. Moreover, they give rise to a **MRA** of $L^2(\mathbb{R})$. We shall call $\Phi_2$ and $\Phi_4$, the spaces generated by the refinable functions $\varphi_{n,h}\in C^{n-2}(\mathbb{R})$ corresponding to the masks in (30) and by the refinable functions $\varphi_{n,H}\in C^{n-4}(\mathbb{R})$, $H=(l,m)$ corresponding to the masks in (31), respectively. Moreover, from now on, the indices $n$, $h$ in $\varphi_{n,h}$ are assumed to be $n\in\mathbb{N}$ and $h\in\mathbb{R}$. In [12], it is proved that the GP class is closed with respect to the convolution and in [13] a relation for the convolution of two functions in $C^{n-2}(\mathbb{R})$, depending on the same parameter $h$, is given. Instead, for any two refinable functions in $C^{n-2}(\mathbb{R})$, the following general result holds.

THEOREM 1. *Let be $\varphi_{n_1,h_1}$, $\varphi_{n_2,h_2}\in\Phi_2$, with $h_1\geq n_1$ and $h_2\geq n_2$. Then*

$$\varphi_{n_1,h_1}*\varphi_{n_2,h_2}=\varphi_{n_1+n_2+1,H}\in C^{n_1+n_2-4}\,, \tag{33}$$

*being $H=(l,m)$, with $l=h_1+h_2+1$ and $m=\log_2(2^{h_1-n_1+2}+2^{h_2-n_2+2}-4)$.*

PROOF. We call $b_{n_1,h_1}$, $b_{n_2,h_2}$ and $b_{n_1+n_2+1,H}$ the symbols of $\varphi_{n_1,h_1}$, $\varphi_{n_2,h_2}$, and $\varphi_{n_1+n_2+1,H}$, respectively. Hence, it is sufficient to prove that

$$b_{n_1,h_1}(z)\,b_{n_2,h_2}(z)=b_{n_1+n_2+1,H}(z)\,. \tag{34}$$

As matter of fact, one has

$$\begin{aligned} &b_{n_1,h_1}(z)\,b_{n_2,h_2}(z)=\\ &=\frac{(1+z)^{n_1-1}}{2^{h_1}}(z^2+(2^{h_1-n_1+2}-2)z+1)\times\\ &\times\frac{(1+z)^{n_2-1}}{2^{h_2}}(z^2+(2^{h_2-n_2+2}-2)z+1)=\\ &=(1+z)^{(n_1+n_2+1)-4+1}\,q_H(z)=:b_{N,H}(z)\,, \end{aligned} \tag{35}$$

with $N=n_1+n_2+1$, $q_H(z)=\frac{1}{2^l}(z^4+2^m z^3+(2^{-N+4+l}-2-2^{m+1})z^2+2^m z+1)$, $l=h_1+h_2+1$ and $m=\log_2(2^{h_1-n_1+2}+2^{h_2-n_2+2}-4)$. ∎

In the following, we will need an explicit expression of the Fourier transform of $\varphi_{n,h}$.

Let us begin with a first result on the zeros of $\hat{\varphi}_{n,h}$.

THEOREM 2. *Let be $\hat{\varphi}_{n,h}$ the Fourier transform of $\varphi_{n,h}$, with $n \geq 2$ and $h \geq n$, $n \in \mathbb{N}, h \in \mathbb{R}$.*

*Then*

$$\hat{\varphi}_{n,h}(\omega) = 0 \qquad \text{if and only if } \omega = 2k\pi, \quad k \in \mathbb{Z}\backslash\{0\}\,, \tag{36}$$

*i.e. $\hat{\varphi}_{n,h}$ has the same zeros as the function sinc$(\omega/2)$.*

PROOF. Recalling that $\varphi_{n,h}$ has order of exactness $d = n-1$ if $h > n$ and $d = n+1$ if $h = n$, with $n \geq 2$, the Strang and Fix condition

$$\hat{\varphi}(0)=1, \quad \frac{d^l\hat{\varphi}}{d\omega^l}(2k\pi) = 0, \quad k \in \mathbb{Z}\backslash\{0\},\ l = 0, \dots, d-1\,, \tag{37}$$

implies that the points $\omega = 2k\pi, k \in \mathbb{Z}\backslash\{0\}$, are zeros of $\hat{\varphi}_{n,h}$. Viceversa, if $\theta \neq 2k\pi$ were a zero of $\hat{\varphi}_{n,h}$, then, for (4) all the values $\frac{\theta}{2^l}$, $l \in \mathbb{Z}$, should be zeros of $\hat{\varphi}_{n,h}$. The continuity of $\hat{\varphi}_{n,h}$ would imply $\hat{\varphi}_{n,h}(0) = 0$ in contradiction with (37). □

Now we can prove semi-explicit formulae for $\hat{\varphi}_{n,h}$ and for $\varphi_{n,h} \in \Phi_2$.

THEOREM 3. *Let be $\hat{\varphi}_{n,h}$ the Fourier transform of $\varphi_{n,h} \in \Phi_2$, with $n \geq 2$ and $h \geq n$.*

*Then $\hat{\varphi}_{n,h}$ can be written in the following way*

$$\hat{\varphi}_{n,h}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{n-1} f_{n,h}(\omega)\,, \tag{38}$$

*where*

$$\begin{cases} f_{n,h}(\omega) = \left(\dfrac{1-e^{-i\omega}}{i\omega}\right)^2, & \text{for } h = n\,, \\ f_{n,h}(2k\pi) \neq 0,\ \forall k \in \mathbb{Z}, & \text{for } h > n\,, \end{cases} \tag{39}$$

*moreover, in the last case, $f_{n,h}$ verifies the following relation*

$$f_{n,h}(\omega) = \mu(\omega/2) f_{n,h}(\omega/2)\,,$$

*with*

$$\mu(\omega/2) = 2^{n-h-2}[e^{-i\omega} + (2^{h-n+2}-2)e^{-i\omega/2} + 1]\,. \tag{40}$$

Proof. (38) and (39) are deduced from the Strang and Fix condition (37) with the same observation on the exactness order of $\varphi_{n,h}$ as in the previous theorem, while (40) comes out substituting (38) in (4) and using the explicit expression for the function $m(\omega)$ [12]. ∎

From (38), by antitransforming one deduces the following "zero at $\pi$"-condition.

Theorem 4. *Let be $h \geq n$ and $n \geq 2$. Then for any GP refinable function $\varphi_{n,h} \in \Phi_2$ one has*

$$\varphi_{n,h} = (B^{n-2} * g_{n,h})\,,$$

*where $B^{n-2}$ is the B-spline of degree $n-2$ and $\hat{g}_{n,h}$ does not vanish on the points $\omega = 2k\pi$, $k \in \mathbb{Z}\backslash\{0\}$.*

Proof. The result follows directly from (38) and (39), recalling (14) and observing that, since $\hat{g}_{n,h}(\omega) \equiv f_{n,h}(\omega)$, for the Strang and Fix condition, $g_{n,h}$ does not have any order of exactness. ∎

Remark 5. Theorem 3 claims that any GP refinable function of $\Phi_2$, with exactness order equal to $n-1$ is the convolution of the B-spline having the same exactness order and a distribution. Hence, the part of the GP refinable function that carries out the order of exactness is known explicitly, and that might allow one to obtain explicit approximation formulae.

As for the numerical applications, when $h$ is increasing, these refinable functions show better performances if compared with the B-splines, due to the better localization of their supports (see, for instance [4], [10], [11], [13], and [14]).

## 5 – Fractional refinable functions

Starting from (31), we consider an index $\alpha \in \mathbb{R}^+$ and exploiting (17), (18) and (19), we define the *mask* $a_h^\alpha$ whose components are

$$a_{h,k}^\alpha = \frac{1}{2^h}\left[\binom{\alpha+1}{k} + 4(2^{h-\alpha}-1)\binom{\alpha-1}{k-1}\right], \tag{41}$$

$$\text{with } k = 0,1,\dots\,, \tag{42}$$

and $h \geq \alpha$. Note that, for $h = \alpha$, one obtains

$$a_{\alpha,k}^\alpha = \frac{1}{2^\alpha}\binom{\alpha+1}{k}, \quad k \in \mathbb{Z}\,, \tag{43}$$

that is the mask of the fractional B-spline $B_+^\alpha$. Due to (22), the *symbol* associated with the sequence $a_h^\alpha$ is given, for $|z| \le 1$, by the formula

$$b_h^\alpha(z) = \frac{1}{2^h}[(1+z)^{\alpha+1} + 4(2^{h-\alpha}-1)z(1+z)^{\alpha-1}]\,, \tag{44}$$

that can be extended analytically to the whole complex plane $\mathbb{C}$; the corresponding *refinement equation* turns out to be

$$\varphi_h^\alpha(x) = \sum_{k\ge 0} a_{h,k}^\alpha \varphi_h^\alpha(2x-k), \quad x \in \mathbb{R}\,. \tag{45}$$

As first point, it comes interesting to characterize the class of functions $\varphi_h^\alpha$ through a convolution formula between a suitable minimally supported GP refinable function and a suitable fractional B-spline.

THEOREM 5. *For any $\alpha \ge 2$ and for any $h \ge \alpha$, the following convolution relation holds*

$$\varphi_h^\alpha = (\varphi_{2,\hat h} * B_+^{\alpha-3})\,, \tag{46}$$

*where $\hat h = h - \alpha + 2$, $\varphi_{2,\hat h}$ is a GP refinable function of $\Phi_2$ with support $[0,3]$ and $B_+^{\alpha-3}$ is the fractional B-spline of fractional degree $\alpha - 3$.*

PROOF. The claim follows from Theorem 1, observing that, for $\hat h = h-\alpha+2$, one has

$$b_h^\alpha(z) = \frac{1}{2} b_{2,\hat h}(z) b^{\alpha-3}(z)\,, \tag{47}$$

being $b_{2,\hat h}(z) = \frac{1}{2^{\hat h}}(1+z)(z^2 + (2^{\hat h}-2)z + 1)$ the symbol associated to $\varphi_{2,\hat h}$ and $b^{\alpha-3}(z) = \frac{1}{2^{\alpha-3}}(1+z)^{\alpha-2}$ the symbol associated to $B_+^{\alpha-3}$. ∎

As first consequence, (46) guarantees the *refinability* of the fractional functions $\varphi_h^\alpha$, since $\varphi_h^\alpha$ is the convolution of two refinable functions. Moreover, it suggests to derive some properties of the fractional refinable functions from the analogous properties of the fractional B-splines. In this regard, through (46) we can analyze the decay of $\varphi_h^\alpha$ to the infinity.

THEOREM 6. *For any $\alpha > 2$ and for any $h \ge \alpha$, one has*

$$\varphi_h^\alpha \in L^2(\mathbb{R})\,. \tag{48}$$

Proof. The Fourier transform applied to (47) yields

$$\hat{\varphi}_h^\alpha(\omega) = \hat{\varphi}_{2,\hat{h}}(\omega)\hat{B}^{\alpha-3}(\omega)\,. \tag{49}$$

We observe that $\hat{\varphi}_{2,\hat{h}}(\omega)$ and $\hat{B}^{\alpha-3}(\omega)$ are bounded on $\mathbb{R}$. Also, since $\varphi_{2,\hat{h}}$ is compactly supported and $B^{\alpha-3}$ belongs to $L^2(\mathbb{R})$ for $\alpha > 2$ [16], then for the Parseval's identity one has that $\hat{\varphi}_{2,\hat{h}}$ and $\hat{B}^{\alpha-3}$ are bounded functions of $L^2(\mathbb{R})$; that implies the square integrability of $\hat{\varphi}_h^\alpha$ and hence of $\varphi_h^\alpha$ on $\mathbb{R}$. □

The convolution formula (46) allows one also to prove that the functions $\varphi_h^\alpha(x-k)$ form a stable basis for the space $V_0$.

Theorem 7. *For any $\alpha > 2$ and for any $h \geq \alpha$, the refinable functions $\{\varphi_h^\alpha(\cdot - k)\}_{k\in\mathbb{Z}}$ fulfill the Riesz condition*:

$$A||c||_{l^2} \leq \left\|\sum_{k\in\mathbb{Z}} c(k)\varphi_h^\alpha(x-k)\right\|_{L^2} \leq B||c||_{l^2}, \quad \forall c \in l^2(\mathbb{Z})\,, \tag{50}$$

*with* $0 < A \leq B < \infty$.

Proof. It is known (see for instance [5]) that (50) is equivalent to bound the function

$$a(\omega) := \sum_{l\in\mathbb{Z}} |\hat{\varphi}_h^\alpha(\omega + 2l\pi)|^2$$

from above and from below with the constants $A$ and $B$ respectively. To find a lower bound one observes that since $a(\omega)$ is symmetric and $2\pi$-periodic, we can restrict its study to $\omega \in [0, 2\pi]$. So, for (46), we have

$$\begin{aligned} a(\omega) &= \sum_{l\in\mathbb{Z}} |\hat{\varphi}_{\hat{h},2}(\omega + 2l\pi)|^2 |\hat{B}_+^{\alpha-3}(\omega+2l\pi)|^2 \geq |\hat{\varphi}_{\hat{h},2}(\omega)|^2 |\hat{B}_+^{\alpha-3}(\omega)|^2 \geq \\ &\geq m^2 \left|\mathrm{sinc}\frac{\omega}{2}\right|^{2\alpha-4} \geq m^2 \left(\frac{2}{\pi}\right)^{2\alpha-4} =: A \end{aligned} \tag{51}$$

where $m := \min|\hat{\varphi}_{\hat{h},2}(\omega)|^2$, $\omega \in [0, 2\pi]$ is non-zero since $\hat{\varphi}$ has zeros only in $\omega_k = 2k\pi$, $k \in \mathbb{Z}\backslash\{0\}$.

As for the upper bound, one has

$$\begin{aligned} &\sum_{l\in\mathbb{Z}} |\hat{\varphi}_h^\alpha(\omega+2l\pi)|^2 = \sum_{l\in\mathbb{Z}} |\hat{\varphi}_{\hat{h},2}(\omega+2l\pi)|^2 \hat{B}_+^{\alpha-3}(\omega+2l\pi)|^2 \leq \\ &\leq \sum_{l\in\mathbb{Z}} |\hat{\varphi}_{\hat{h},2}(\omega+2l\pi)|^2 \sum_{l\in\mathbb{Z}} |\hat{B}_+^{\alpha-3}(\omega+2l\pi)|^2 \leq B_1\,B_2 =: B\,, \end{aligned} \tag{52}$$

where $B_1$ and $B_2$ are the right Riesz constants associated to $\varphi_{\hat{h},2}$ and $B_+^{\alpha-3}$ respectively. □

This theorem guaranties that the set $\{\varphi_h^\alpha(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for the space $V_0$ that it generates. Moreover, as consequence of the refinability and of Theorems 4-5, it is possible to claim that $\varphi_h^\alpha$ gives rise to a MRA of $L^2(\mathbb{R})$. More precisely, one has the following theorem.

THEOREM 8. *For any $\alpha > 2$ and for any $h \geq \alpha$, $\varphi_h^\alpha$ generates a MRA of $L^2(\mathbb{R})$.*

PROOF. The claim follows since $\varphi_h^\alpha$ belongs to $L^2(\mathbb{R})$, is refinable and $\{\varphi_h^\alpha(\cdot - k)\}_{k\in\mathbb{Z}}$ form a Riesz basis of $V_0$. ∎

## 6 – Some further properties

As for the convolution of the two fractional refinable functions the following result holds.

THEOREM 9. *For any $\alpha_1, \alpha_2 > 2$ and for any $h_1 \geq \alpha_1$ and $h_2 \geq \alpha_2$, the following convolution formula holds.*

$$(\varphi_{h_1}^{\alpha_1} * \varphi_{h_2}^{\alpha_2})(x) = (\varphi_{5,H} * B^{\alpha_1+\alpha_2-5})(x)\,, \tag{53}$$

*with $\varphi_{5,H} \in \Phi_4$ and $H$ as in Theorem 1.*

PROOF. From (46) and Theorem 1, the thesis follows by observing that

$$(\varphi_{h_1}^{\alpha_1} * \varphi_{h_2}^{\alpha_2})(x) = (\varphi_{2,\hat{h}_1} * \varphi_{2,\hat{h}_2}) * (B_+^{\alpha_1-3} * B_+^{\alpha_2-3})(x) = (\varphi_{5,H} * B_+^{\alpha_1+\alpha_2-5})(x)\,. \quad ∎$$

As for the order of polynomial exactness, we prove for the fractional refinable functions the following surprising property.

THEOREM 10. *Let be $\alpha > 2$ and $h \geq \alpha$.*

*Then the fractional refinable function $\varphi_h^\alpha$ has order of polynomial exactness $d = \lceil \alpha \rceil - 1$, i.e. there exist sequences $p_k^l$ such that*

$$x^l = \sum_{k\in\mathbb{Z}} p_k^l \varphi^\alpha(x-k), \quad l = 0, \ldots, \lceil \alpha \rceil - 2\,. \tag{54}$$

PROOF. It is known (see for example [3]) that (54) is equivalent to prove the Strang and Fix condition (37) for $\hat{\varphi}^\alpha$. Moreover, due to (46), one has

$$\hat{\varphi}^\alpha(\omega) = \hat{\varphi}_{2,\hat{h}}(\omega)\hat{B}_+^{\alpha-3}(\omega)\,, \tag{55}$$

where

$$\hat{B}_+^{\alpha-3}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\alpha-2},$$

and for (38) $\hat{\varphi}_{2,\hat{h}}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right) f_{n,h}(\omega)$ with $f_{n,h}(2k\pi) \neq 0$, $\forall k \in \mathbb{Z}$. Then, the thesis comes by following steps similar to the derivation of relation (4.1) in [16]. □

For the fractional differentiation, a formula similar to that of the fractional B-splines holds.

THEOREM 11. *For any real $\alpha > 2$ and for any real $h \geq \alpha$, the following differential formula holds.*

$$D^\gamma\varphi^\alpha(x) = \Delta^\gamma\varphi^{\alpha-\gamma}(x), \quad \alpha > 2, \quad \gamma \in \mathbb{R}^+\,. \tag{56}$$

PROOF. Let $\hat{\Delta}$ the operator defined as follows

$$\hat{\Delta}^\gamma(\omega) = \sum_{k\geq 0}(-1)^k\gamma k e^{-i\omega k} = (1-e^{-i\omega})^\gamma. \tag{57}$$

We observe that the operator $\Delta^\gamma$ is a convolution operator, then, in the Fourier domain, it becomes

$$(i\omega)^\gamma\hat{\varphi}^\alpha(\omega) = \hat{\Delta}^\gamma\hat{\varphi}^{\alpha-\gamma}(\omega), \quad \alpha > 2, \quad \gamma \in \mathbb{R}^+\,. \tag{58}$$ □

So the result follows from (48).

REMARK 6. As for (54), we recall that the GP refinable functions $\varphi_{n,h}$ reproduce polynomials up to the degree $n-2$.

## 7 – Some numerical features

We report, in figure 1, on the left side, the plots of $\varphi_h^\alpha$ for $\alpha = 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5, 7$ and $h = \alpha$, which provide some examples of fractional B-splines. Note that, for integer values of $\alpha$ they reduce to the classical cardinal B-splines. On the other side, setting $h = \alpha + 5$, some examples of fractional GP refinable functions which do not reduce to the classical B-splines, are shown. We observe that both figures exhibit a decay that not only confirms what already claimed in Theorem 6, but also makes the fractional refinable functions to look like compactly supported functions. Moreover, we observe that the support of $\varphi_h^\alpha$ for non integer $\alpha$ and $\lceil\alpha\rceil = n$, looks strictly contained in the support $[0, n+1]$ of $\varphi_{n,h}$, even though the order of exactness of $\varphi_h^\alpha$ is exactly the same as that of $\varphi_{n,h}$, i.e. $n - 1$.

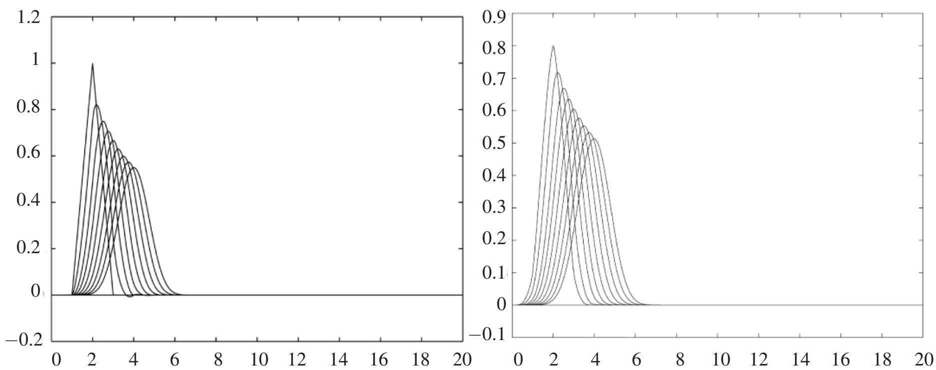


Fig. 1: Plots of the fractional B-splines for $\alpha = 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5, 7$ (left) and of $\varphi_h^\alpha$, $h = \alpha + 5$ (right).

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## REFERENCES

[1] A. S. Cavaretta – W. Dahmen – C. A. Micchelli: *Stationary Subdivision*, Mem. Amer. Math. Soc., **93** (1991), 1-186.

[2] C. K. Chui: *An Introduction to Wavelets*, Academic Press, Boston, 1992.

[3] A. Cohen: *Numerical Analysis of Wavelets Methods*, Elsevier, 2003.

[4] M. Cotronei – M. L. Lo Cascio: *Performance of a class of biorthogonal filters*, in "Proceedings SIMAI 2000", Ischia, July 200, 2000.

[5] I. Daubechies: *Ten Lectures on Wavelets*, CBMF Conference Series in Applied Mathematics, **61**, SIAM, Philadelphia, 1992.

[6] C. Gasquet – P. Witomski: *Fourier analysis and applications. Filtering, numerical computation, wavelets*, Texts in Applied Mathematics, **30**, Springer, 1998.

[7] T. N. T. Goodman – C. A. Micchelli: *On refinement equations determined by Pòlya frequency sequence*, SIAM J. Math. Anal., **23** 3 (1992), 766-784.

[8] L. Gori – L. Pezza: *On some applications of the wavelet Galerkin method for boundary value problems*, Mat. Mod., **15** n. 5 (2003), 61-70.

[9] L. Gori – L. Pezza – F. Pitolli: *A class of totally positive blending B-bases, in Curves and Surface Design*, S. Malo, 1999, P-J Laurent, P. Sablonnière, L. L. Schumaker (eds.), 119-126.

[10] L. Gori – L. Pezza – F. Pitolli: *Recent results on Wavelet Bases on the interval generated by GP refinable functions*, Appl. Numer. Math. 51, **4** (2004), 549-563.

[11] L. Pezza: *A new class of biorthogonal wavelets on the interval*, Wavelets and Splines, St. Petersburg Univ. Press, St. Petersburg, 2005, 40-57.

[12] L. Gori – F. Pitolli: *A class of totally positive refinable functions*, Rend. Mat. Serie VII, **20** (2000), 305-322.

[13] L. Gori – F. Pitolli – L. Pezza: *On the Galerkin method based on a particular class of refinable functions*, Num. Alg., **28** (2001), 187-198.

[14] L. Gori – F. Pitolli – E. Santi: *Positive refinable operators*, Num. Alg., **28** (2001), 199-213.

[15] S. Mallat: *A wavelet tour of signal processing*, 2. ed., Academic Press, 1999.

[16] M. Unser – T. Blu: *Fractional splines and wavelets*, SIAM, **42** (1) (2000), 43-67.

[17] M. Unser – T. Blu: *Wavelet theory demistified*, IEEE Trans. on Signal Processing, **51** (2) (2003), 470-483.

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