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Multivariate negative aging in an exchangeable model of heterogeneity

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Abstract

We consider a simple exchangeable model, which accounts for heterogeneity and dependence. Based on this model, we show how, and in which sense, situations of negative aging arise in a natural way from conditions of heterogeneity among items.

Keywords: Exchangeability; Mixtures; DFR distributions; Schur convexity

1. Introduction

It is well-known that situations of negative aging for one-dimensional lifetimes distributions can arise from mixtures. A classical result (see Barlow and Proschan, 1975) states that a mixture of decreasing failure rate (DFR) distributions is DFR (then, in particular, a mixture of exponential distributions is DFR). More generally, it has been noticed that properties of negative aging can even arise in the case of mixtures of non-necessarily DFR distributions (see, for instance, Vaupel and Yashin, 1985). Examples and results in this direction are also provided by Gurland and Sethuraman (1994, 1995) and Block and Joe (1997), where the notion of "ultimately DFR" (see Section 2 for a formal definition) is considered. These topics are strictly related with the study of optimality in burn-in procedures (see, for instance, Block and Savits 1997); they are also relevant in the field of survival analysis, where properties of mixtures have a central role (see, for instance, Aalen, 1998 and Hougaard, 1995).

In order to explain the purpose of this paper, let us shortly dwell on the relations between the concepts of mixture and heterogeneity.

Let a system be formed of n items and let (Z_i, T_i) be a pair of random variables corresponding to the ith item, for $i = 1, \dots, n$. We think of T_i as an observable variable of interest and Z_i as an unobservable

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(endogenous to the item) quantity which determines the distribution of T_i. Then, Z_i being random, the marginal distribution of T_i is a mixture (the mixture of conditional distributions given $\{Z_i = z\}$).

Often, in the literature, one assumes stochastic independence among the Z_i's, in which case $T_1, \ldots, T_n$ are independent as well. In this case, conditions of negative aging for $T_1, \ldots, T_n$ are expressed in terms of aging properties of the (mixture-type) one-dimensional distributions. In this paper, we are instead interested in the case when $Z_1, \ldots, Z_n$ are not independent and then $T_1, \ldots, T_n$ are dependent. In particular, we consider special multivariate exchangeable distributions for $T_1, \ldots, T_n$. These arise from the hierarchical model which will be described in Section 3. Our central aim is to show in which sense this model can give rise to (multivariate) negative aging.

In the case when $T_1, \ldots, T_n$ are dependent, negative aging cannot be described in terms of aging properties of the (predictive) one-dimensional distributions. Rather multivariate notions of aging are to be considered. We focus on multivariate notions for exchangeable lifetimes which can be thought of as of "Bayesian-type", and which are based on stochastic comparisons among residual lifetimes of items of different age and are related with the property of Schur convexity of the joint survival function of $T_1, \ldots, T_n$ (see Marshall and Olkin, 1979 for the definition of Schur-convex function).

The role of Schur convexity in the formalization of negative aging for exchangeable lifetimes will be briefly summarized in Section 2. In that section we shall also introduce a notion of negative aging which can be seen as an extension to the multivariate case of the ultimate DFR property. In Section 3 we describe a special case of multivariate exchangeable distributions for which we show aspects of negative aging. Our results will be given in Section 4. They can be seen as multivariate analogues of results about ultimate DFR, cited above. In fact, the multivariate property of aging introduced in Section 2 will be proved. As a special by-product we obtain sufficient conditions under which the joint survival function of the model described in Section 3 is Schur-convex; moreover we find sufficient conditions on mixtures which guarantee the ultimate DFR property for one-dimensional distributions. Section 5 is devoted to a discussion about different aspects of our results.

2. Some multivariate notions of negative aging

First, let us recall the very well-known notion of DFR distribution (see, for instance, Barlow and Proschan, 1975). We say that a lifetime T, with survival function $\bar{G}(t)$, has a DFR distribution if, for all $\tau > 0$,

$$P\{T > r + \tau | T > r\} = \frac{\bar{G}(r + \tau)}{\bar{G}(r)}$$

is a non-decreasing function of r. DFR describes a property of negative aging in that it is equivalent to the probability of survival being increasing with the time. Obviously, when $\bar{G}(\cdot)$ is absolutely continuous with a failure rate function $\lambda(\cdot)$, i.e. when

$$\bar{G}(t) = \exp\left\{-\int_0^t \lambda(\xi)\,\mathrm{d}\xi\right\}$$

the DFR property is equivalent to $\lambda(\cdot)$ being non-increasing.

More general notions of negative aging can be naturally defined by assuming that the function $P\{T > r + \tau | T > r\}$ is non-decreasing in an interval of the form $[0, a]$ or $[b, +\infty)$. We focus our attention to the latter case, which defines the property of *ultimately DFR* (with respect to b). Such notion is of interest since it can arise in cases of mixtures as shown in Gurland and Sethuraman (1994, 1995), and Block and Joe (1997). Trivially, the property of DFR is the particular case when $b = 0$.

Our purpose in this section is to introduce a multivariate definition of ultimately DFR. To this aim, it is convenient to start by noticing that the DFR property can be equivalently formulated as follows: let $T_1,\dots,T_n$ be i.i.d. lifetimes with a common survival function $\bar{G}(t)$. Then $\bar{G}(\cdot)$ is DFR if and only if, for any choice of ages $r_1,\dots,r_n$ and any $\tau > 0$, the following implication holds:

$$r_i < r_j \Rightarrow P\{T_i > r_i + \tau | D_n\} \leqslant P\{T_j > r_j + \tau | D_n\}, \tag{2.1}$$

where

$$D_n \equiv \{T_1 > r_1,\dots,T_n > r_n\} \tag{2.2}$$

and $i \neq j$ is any pair from $\{1,2,\dots,n\}$. Indeed, in the i.i.d. case

$$P\{T_i > r_i + \tau | D_n\} = P\{T_i > r_i + \tau | T_i > r_i\}$$

and therefore (2.1) is equivalent to $P\{T_i > r + \tau | T_i > r\}$ being non-decreasing in r.

The formulation (2.1) of the one-dimensional DFR property actually introduces redundant conditions; however this is convenient in the present frame. Indeed, the validity of the implication (2.1) provides a notion of multivariate DFR also for the case when the lifetimes $T_1,\dots,T_n$ are assumed to be exchangeable, rather than being i.i.d. It can be easily shown that the validity of (2.1) is equivalent to Schur convexity of the joint survival function of the lifetimes

$$\bar{F}(t_1,\dots,t_n) \equiv P\{T_1 > t_1,\dots,T_n > t_n\}$$

(for more detailed discussions on this point of view see Bassan and Spizzichino, 1999).

Now we want to formulate a multivariate analogue of the property of ultimately DFR for exchangeable lifetimes. We shall follow the same line as above, i.e. we start by rephrasing the univariate ultimately DFR as a property of n i.i.d. lifetimes. Let $T_1,\dots,T_n$ be independent and with a common survival function $\bar{G}(\cdot)$. Then $\bar{G}(\cdot)$ is ultimately DFR if and only if, for some $\bar{r} \geqslant 0$, for any choice of ages $r_1,\dots,r_n \geqslant \bar{r}$ and for any $\tau > 0$, the implication (2.1) holds.

This fact suggests the following definition for vectors of exchangeable lifetimes $T_1,\dots,T_n$:

Definition 1. The vector of exchangeable lifetimes $(T_1,\dots,T_n)$ has the multivariate ultimately negative aging (m.u.n.a.) property with respect to $\bar{r} \geqslant 0$ if (2.1) holds for any $\tau > 0$ and any $r_1,\dots,r_n \geqslant \bar{r}$.

In words, $(T_1,\dots,T_n)$ has the m.u.n.a. property if, between two items which are "old enough", probabilities of surviving for an extra period τ are larger for the "elder" one than for the "younger".

In Section 4, we shall find sufficient conditions under which exchangeable models, to be described in Section 3, fulfill the m.u.n.a. property. As corollaries, we obtain sufficient conditions for the corresponding joint survival function being Schur-convex or for the one-dimensional distribution being ultimately DFR.

3. A case of a-priori exchangeability arising from conditions of heterogeneity

Let us consider a lot of n items and let us denote by $T_1,\dots,T_n$ the corresponding lifetimes. We think of the case when a real random variable Z_i $(i = 1,2,\dots,n)$ is attached to each of the lifetimes $T_1,\dots,T_n$. $Z_1,\dots,Z_n$ take their values in the set $\mathscr{Z} \subset \mathbb{R}$ and Z_i has the meaning of weakness (or frailty) of the ith component.

More precisely, we make the following assumptions:

(A1) The distribution of $\mathbf{Z} \equiv (Z_1,\dots,Z_n)$ has a (priori) exchangeable density function $\pi^{(n)}(\mathbf{z})$.

(A2) $T_1,\dots,T_n$ are conditionally independent given $Z_1,\dots,Z_n$.

(A3) A family of one-dimensional survival functions $\{\bar{G}(t|z): z \in \mathscr{Z}\}$ is given such that the conditional survival function of T_i, given $\mathbf{Z} = \mathbf{z}$, is $\bar{G}(t|z_i)$.

(A4) The distributions $\bar{G}(t|z)$ are such that, for a suitable $\bar{t} \geqslant 0$, $\bar{G}(t|z)/\bar{G}(t|z')$ is non-decreasing in $t \geqslant \bar{t}$, for $z < z'$.

Assumption (A1) can be motivated in cases where items are apparently similar, that is the variables $Z_1, \ldots, Z_n$ are not observable and there is a situation of symmetry among them. Assumptions (A2) and (A3) mean that, if we could observe the values of $Z_1, \ldots, Z_n$, the distribution of each T_i would only depend on the actual value of Z_i and $T_1, \ldots, T_n$ would be independent: dependence among $T_1, \ldots, T_n$ only arises from dependence among $Z_1, \ldots, Z_n$. As far as the assumption (A4) is concerned we note that, in the case when $\bar{G}(t|z)$ admits a density function $g(t|z)$ and a failure rate function

$$\lambda(t|z) \equiv \frac{g(t|z)}{\bar{G}(t|z)},$$

it means that $\lambda(t|z)$ is a non-decreasing function of z whenever $t \geqslant \bar{t}$. Indeed, assumption (A4) is equivalent to the function

$$\exp\left\{ -\int_0^t (\lambda(\xi|z) - \lambda(\xi|z'))\, \mathrm{d}\xi \right\}$$

being non-decreasing in $t \geqslant \bar{t}$, for $z < z'$. This explains the meaning of $Z_1, \ldots, Z_n$ as weakness of the components.

We note that assumptions (A1)–(A3) imply that also $T_1, \ldots, T_n$ are exchangeable (see Gerardi et al., 2000, where other aspects of the distribution of $T_1, \ldots, T_n$ are discussed). In this paper our interest will be focused on conditional probabilities of the type

$$P\{T_i > r_i + \tau | D_n\}, \tag{3.1}$$

where D_n is the event defined by (2.2). As a simple consequence of (A2) and (A3), the conditional probability in (3.1) can be written in the form

$$P\{T_i > r_i + \tau | D_n\} = \mathbb{E}\left[\frac{\bar{G}(r_i + \tau | Z_i)}{\bar{G}(r_i | Z_i)} | D_n \right]. \tag{3.2}$$

We also note that, by Bayes' formula, the conditional density of Z_i, Z_j $(i \neq j)$, given D_n, is

$$\pi^{(2)}(z_i, z_j | D_n) \propto \pi^{(2)} \left(z_i, z_j \middle| \bigcap_{l \neq i,j} \{T_l > r_l\} \right) \bar{G}(r_i|z_i) \bar{G}(r_j|z_j), \tag{3.3}$$

where

$$\pi^{(2)} \left(\cdot, \cdot \middle| \bigcap_{l \neq i,j} \{T_l > r_l\} \right)$$

is the conditional density of Z_i, Z_j, given $\bigcap_{l \neq i,j} \{T_l > r_l\}$. Formulae (3.2) and (3.3) will be used in Section 4. We shall also use Lemma 3.1 stated below. Prior to that, it is convenient to recall some notation and terminology about notions of stochastic orderings.

For one-dimensional random variables X and Y, X is less than Y in the *usual stochastic order* (written $X \leqslant_{\mathrm{st}} Y$) if and only if $P\{X > z\} \leqslant P\{Y > z\}$, for all $z \in \mathbb{R}$. As it is very well-known, this condition is equivalent to the inequality $\mathbb{E}[\varphi(X)] \leqslant \mathbb{E}[\varphi(Y)]$ for any non-decreasing function $\varphi: \mathbb{R} \to \mathbb{R}$ such that the expected values exist.

X is less than Y in the *hazard rate order* (written $X \leqslant_{\rm hr} Y$) if and only if the ratio $\bar{F}_X(z)/\bar{F}_Y(z)$ is a non-increasing function (then note that the above assumption (A4) with $\bar{t}=0$ means that $T_1,\dots,T_n$ are ordered with respect to the variable z in the sense of the hazard rate order).

X is less than Y in the *likelihood ratio order* (written $X \leqslant_{\rm lr} Y$) if and only if the ratio $f_X(z)/f_Y(z)$ is a non-increasing function, f_X and f_Y being densities with respect to a same dominating measure. It is also very well-known that

$$X \leqslant_{\rm lr} Y \Rightarrow X \leqslant_{\rm hr} Y \Rightarrow X \leqslant_{\rm st} Y$$

(see, for instance, Shaked and Shanthikumar, 1994).

For n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X}$ is less than $\mathbf{Y}$ in the usual stochastic order (written $\mathbf{X} \leqslant_{\rm st} \mathbf{Y}$) if and only if $\mathbb{E}[\varphi(\mathbf{X})] \leqslant \mathbb{E}[\varphi(\mathbf{Y})]$ for any non-decreasing function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that the expected values exist.

Let $\mathbf{X}$ be a n-dimensional random vector and let U be a one-dimensional random variable. We say that $\mathbf{X}$ is *stochastically increasing* in U if and only if

$$u' < u'' \Rightarrow [\mathbf{X}|U=u'] \leqslant_{\rm st} [\mathbf{X}|U=u''],$$

where $[\mathbf{X}|U=u]$ denotes a random vector that has as its distribution, the conditional distribution of $\mathbf{X}$ given the event $\{U=u\}$.

Our next result we assume that (A1)–(A4) hold.

Lemma 3.1. *If $\bar{t} \leqslant r_1 < r_2$ then, for all $z \in \mathscr{Z}$,*

$$P\{Z_1 > z|D_n\} \geqslant P\{Z_2 > z|D_n\}.$$

Proof. We must prove that, for any non-decreasing function $\varphi : \mathscr{Z} \to \mathbb{R}$, such that the expected values exist,

$$\mathbb{E}[\varphi(Z_1)|D_n] \geqslant \mathbb{E}[\varphi(Z_2)|D_n],$$

that is

$$\int_{\mathscr{Z}} \int_{\mathscr{Z}} [\varphi(z_1) - \varphi(z_2)]\pi^{(2)}(z_1, z_2|D_n)\,\mathrm{d}z_1\,\mathrm{d}z_2 \geqslant 0. \tag{3.4}$$

Taking into account Eq. (3.3) and the symmetry property of

$$\pi^{(2)}\left(z_1, z_2 \,\middle|\, \bigcap_{l \neq 1,2} \{T_l > r_l\}\right),$$

inequality (3.4) can be written as

$$\int\int_{\{z_1 \geqslant z_2\}} [\varphi(z_1) - \varphi(z_2)]\pi^{(2)}\left(z_1, z_2 \,\middle|\, \bigcap_{l \neq 1,2} \{T_l > r_l\}\right) \bar{G}(r_1|z_1)\bar{G}(r_2|z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$$\geqslant \int\int_{\{z_1 \geqslant z_2\}} [\varphi(z_1) - \varphi(z_2)]\pi^{(2)}\left(z_1, z_2 \,\middle|\, \bigcap_{l \neq 1,2} \{T_l > r_l\}\right) \bar{G}(r_2|z_1)\bar{G}(r_1|z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

which can be seen to hold as an immediate consequence of the assumption (A4). □

4. Sufficient conditions for the m.u.n.a. property in the exchangeable model

In this Section we find additional conditions, for the exchangeable model defined by assumptions (A1)–(A4), which are sufficient to ensure the m.u.n.a. property defined in Section 2. Let us consider, for the moment, the following condition:

(C1) $\bar{G}(t|z)$ is ultimately DFR with respect to $\bar{r} \geqslant 0$, for all $z \in \mathscr{Z}$.

Proposition 4.1. *Under assumptions* (A1)–(A4) *and condition* (C1), $(T_1,\dots,T_n)$ *has the m.u.n.a. property with respect to* $\hat{t} \equiv \max\{\bar{t},\bar{r}\}$.

Proof. We observe that there is no loss of generality in fixing $i=1$, $j=2$. In view of the identity (3.2), we must prove

$$\mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)}\bigg| D_n\right] - \mathbb{E}\left[\frac{\bar{G}(r_1+\tau|Z_1)}{\bar{G}(r_1|Z_1)}\bigg| D_n\right] \geqslant 0$$

for any $r_3,\dots,r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$, and $\tau > 0$. By Lemma 3.1 and (A4), we have, for any r, $r_3,\dots,r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$, and $\tau > 0$,

$$\mathbb{E}\left[\frac{\bar{G}(r+\tau|Z_2)}{\bar{G}(r|Z_2)}\bigg| D_n\right] \geqslant \mathbb{E}\left[\frac{\bar{G}(r+\tau|Z_1)}{\bar{G}(r|Z_1)}\bigg| D_n\right].$$

Therefore,

$$\mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)}\bigg| D_n\right] - \mathbb{E}\left[\frac{\bar{G}(r_1+\tau|Z_1)}{\bar{G}(r_1|Z_1)}\bigg| D_n\right]$$
$$\geqslant \mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)} - \frac{\bar{G}(r_1+\tau|Z_2)}{\bar{G}(r_1|Z_2)}\bigg| D_n\right].$$

Let $\pi^{(1)}(z_2|D_n)$ denote the conditional density of Z_2 given D_n. Then, for any $r_3,\dots,r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$, and $\tau > 0$,

$$\mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)} - \frac{\bar{G}(r_1+\tau|Z_2)}{\bar{G}(r_1|Z_2)}\bigg| D_n\right]$$
$$= \int_{\mathscr{Z}} \left[\frac{\bar{G}(r_2+\tau|z_2)}{\bar{G}(r_2|z_2)} - \frac{\bar{G}(r_1+\tau|z_2)}{\bar{G}(r_1|z_2)}\right] \pi^{(1)}(z_2|D_n)\,\mathrm{d}z$$
$$\geqslant \inf_{z_2 \in \mathscr{Z}} \left[\frac{\bar{G}(r_2+\tau|z_2)}{\bar{G}(r_2|z_2)} - \frac{\bar{G}(r_1+\tau|z_2)}{\bar{G}(r_1|z_2)}\right]$$

and the latter quantity is non-negative by condition (C1). □

By Proposition 4.1, we immediately obtain

Corollary 4.2. *For the model defined by* (A4) *with* $\bar{t}=0$ *and* (A1)–(A3), *let* $\bar{G}(t|z)$ *be DFR, for all* $z \in \mathscr{Z}$. *Then* $\bar{F}(t_1,\dots,t_n)$ *is Schur-convex.*

Proposition 4.1 can be seen as the analogue, in our multivariate frame, of the fact that mixture of ultimately DFR distributions is ultimately DFR. As recalled in Section 1, mixture distributions can exhibit the ultimately DFR property even when single members of the mixture are not necessarily such. Here, we want to show an analogous result valid in our multivariate setting. Namely, for models defined by assumptions (A1)–(A4), we want to show that the m.u.n.a. property can be obtained even relaxing condition (C1). Indeed, let us consider the alternative set of conditions

(C1′) For some $\bar{z} \in \mathscr{Z}$ and $\bar{r} \geqslant 0$, $z \leqslant \bar{z}$ implies $\bar{G}(t|z)$ ultimately DFR with respect to $\bar{r}$.

(C2) The joint distribution of $(Z_1, \dots, Z_n)$ is such that $(Z_1, \dots, Z_{j-1}, Z_{j+1}, \dots, Z_n)$ is stochastically increasing in Z_j with respect to the $\leqslant_{\mathrm{st}}$ ordering.

(C3) $\bar{G}(\bar{t}|z) \geqslant \bar{G}(\bar{t}|z')$, if $z < z'$.

Trivially (C1′), is a weaker condition than (C1), and (C2) is a condition of positive dependence for the vector $(Z_1, \dots, Z_n)$. Next, we show that under (C1′), (C2) and (C3) the m.u.n.a. property holds, provided that the probability $P(Z_i \leqslant \bar{z})$ is large enough. Let us denote, for an arbitrary subset $W \subset \mathscr{Z}$,

$$A_{r,v,\tau}(W) \equiv \inf_{z \in W} \left[\frac{\bar{G}(r+v+\tau|z)}{\bar{G}(r+v|z)} - \frac{\bar{G}(r+\tau|z)}{\bar{G}(r|z)} \right]$$

and

$$\mathscr{Z}' \equiv \{z \in \mathscr{Z} | z \leqslant \bar{z}\}, \qquad (\mathscr{Z}')^{\mathrm{c}} \equiv \mathscr{Z} \setminus \mathscr{Z}'.$$

Moreover, let us define

$$\bar{p}(\hat{t}, \bar{z}) \equiv \sup_{(r,v,\tau) \in R_{\hat{t},\bar{z}}} \frac{A_{r,v,\tau}((\mathscr{Z}')^{\mathrm{c}})}{A_{r,v,\tau}((\mathscr{Z}')^{\mathrm{c}}) - A_{r,v,\tau}(\mathscr{Z}')},$$

where

$$R_{\hat{t},\bar{z}} \equiv \{(r,v,\tau) | r \geqslant \hat{t},\ v > 0,\ \tau > 0;\ A_{r,v,\tau}((\mathscr{Z}')^{\mathrm{c}}) < 0\}, \quad \hat{t} \equiv \max\{\bar{t}, \bar{r}\}.$$

In the following results we mantain assumptions (A1)–(A4)

Proposition 4.3. *Let the conditions* (C1′), (C2), (C3) *hold. If moreover*

$$\bar{p}(\hat{t}, \bar{z}) < 1 \tag{4.1}$$

and the marginal distribution of Z_i is such that

$$P(Z_i \leqslant \bar{z}) \geqslant \bar{p}(\hat{t}, \bar{z}) \tag{4.2}$$

then $(T_1, \dots, T_n)$ has the m.u.n.a. property (with respect to $\hat{t}$).

Before proving Proposition 4.3, we need two preliminary lemmas.

Lemma 4.4. *Let us assume* (C1′). *Then*

$$\inf_{r \geqslant \bar{r}, v>0, \tau>0} A_{r,v,\tau}(\mathscr{Z}') \geqslant 0 \tag{4.3}$$

and

$$\bar{p}(\hat{t},\bar{z}) \leqslant 1. \tag{4.4}$$

Proof. By (C1′) it is, for all $r \geqslant \bar{r}$, $v>0$, $\tau>0$ and for $z \leqslant \bar{z}$,

$$\frac{\bar{G}(r+v+\tau|z)}{\bar{G}(r+v|z)} - \frac{\bar{G}(r+\tau|z)}{\bar{G}(r|z)} \geqslant 0.$$

Taking the infimum over $\mathscr{Z}'$, we have $A_{r,v,\tau}(\mathscr{Z}') \geqslant 0$, and therefore inequality (4.3) follows. By (4.3), we trivially have, for any $(r,v,\tau)\in R_{\hat{t},\bar{z}}$,

$$\frac{A_{r,v,\tau}((\mathscr{Z}')^{\mathrm{c}})}{A_{r,v,\tau}((\mathscr{Z}')^{\mathrm{c}}) - A_{r,v,\tau}(\mathscr{Z}')} \leqslant 1$$

which gives (4.4). □

Under conditions of positive dependence among the $Z_1,\dots,Z_n$, the following lemma explains the effect of conditioning with respect to the event D_n defined by (2.2).

Lemma 4.5. *Let us assume* (C2) *and* (C3). *Then, for* $j=1,\dots,n$ *and* $r_1 \geqslant \bar{t},\dots,r_n \geqslant \bar{t}$,

$$Z_j \geqslant_{lr} [Z_j|D_n],$$

where $[Z_j|D_n]$ *denotes a random variable that has as its distribution the conditional distribution of* Z_j *given* D_n.

Proof. Due to Bayes' formula,

$$\pi^{(1)}(z_j|D_n) \propto \pi^{(1)}(z_j)P\{D_n|Z_j=z_j\},$$

where $\pi^{(1)}(z_j|D_n)$ denotes the conditional density of Z_j given D_n. Therefore, we only need to show that, for any $r_1 \geqslant \bar{t},\dots,r_n \geqslant \bar{t}$, $P\{D_n|Z_j=z_j\}$ is a non-increasing function of z_j. For this, we note that

$$\begin{aligned} &P\{D_n|Z_j=z_j\}\\ &\quad= \int_{\mathscr{Z}}\cdots\int_{\mathscr{Z}} P\{D_n|\mathbf{Z}=\mathbf{z}\}\pi^{(n-1)}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n|z_j)\,\mathrm{d}z_1\dots\mathrm{d}z_{j-1}\,\mathrm{d}z_{j+1}\dots\mathrm{d}z_n\\ &\quad= \bar{G}(r_j|z_j)\int_{\mathscr{Z}}\cdots\int_{\mathscr{Z}} \prod_{i\neq j}\bar{G}(r_i|z_i)\pi^{(n-1)}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n|z_j)\,\mathrm{d}z_1\dots\mathrm{d}z_{j-1}\,\mathrm{d}z_{j+1}\dots\mathrm{d}z_n \end{aligned}$$

which, recalling (A4), is immediately seen to be a non-increasing function of z_j, in view of (C2) and (C3). □

We now prove Proposition 4.3.

Proof. As in the proof of Proposition 4.1, we can limit ourselves to show that, for any $r_3 \geqslant \hat{t},\dots,r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$, $\tau>0$,

$$\mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)} - \frac{\bar{G}(r_1+\tau|Z_2)}{\bar{G}(r_1|Z_2)}\middle| D_n\right] \geqslant 0. \tag{4.5}$$

Let us denote by $\pi^{(1)}(z_2|D_n)$ the conditional density of Z_2 given D_n. Then, for any $r_3 \geqslant \hat{t}, \ldots, r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$ and $\tau > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)} - \frac{\bar{G}(r_1+\tau|Z_2)}{\bar{G}(r_1|Z_2)}\middle| D_n\right] \\
&= \int_{\mathscr{Z}'}\left[\frac{\bar{G}(r_2+\tau|z_2)}{\bar{G}(r_2|z_2)} - \frac{\bar{G}(r_1+\tau|z_2)}{\bar{G}(r_1|z_2)}\right]\pi^{(1)}(z_2|D_n)\,\mathrm{d}z_2 \\
&\quad + \int_{(\mathscr{Z}')^{\mathrm{c}}}\left[\frac{\bar{G}(r_2+\tau|z_2)}{\bar{G}(r_2|z_2)} - \frac{\bar{G}(r_1+\tau|z_2)}{\bar{G}(r_1|z_2)}\right]\pi^{(1)}(z_2|D_n)\,\mathrm{d}z_2 \\
&\geqslant A_{r_1,r_2-r_1,\tau}(\mathscr{Z}')P\{Z_2\in\mathscr{Z}'|D_n\} + A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}})P\{Z_2\in(\mathscr{Z}')^{\mathrm{c}}|D_n\}.
\end{aligned} \tag{4.6}$$

By Lemma 4.4

$$A_{r_1,r_2-r_1,\tau}(\mathscr{Z}') \geqslant 0$$

thus, (4.6) is non-negative for those $r_2 > r_1 \geqslant \hat{t}$, $\tau > 0$ such that

$$A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}}) \geqslant 0.$$

Let us now consider those $r_2 > r_1 \geqslant \hat{t}$, $\tau > 0$ such that $A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}}) < 0$. Since

$$\begin{aligned}
&\mathbb{E}\left[\frac{\bar{G}(r_2+\tau|Z_2)}{\bar{G}(r_2|Z_2)} - \frac{\bar{G}(r_1+\tau|Z_2)}{\bar{G}(r_1|Z_2)}\middle| D_n\right] \\
&\geqslant [A_{r_1,r_2-r_1,\tau}(\mathscr{Z}') - A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}})]P\{Z_2\in\mathscr{Z}'|D_n\} + A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}})
\end{aligned}$$

by Lemma 4.4 the proof would be completed by showing that

$$P\{Z_2\in\mathscr{Z}'|D_n\} \geqslant \frac{A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}})}{A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}}) - A_{r_1,r_2-r_1,\tau}(\mathscr{Z}')}, \tag{4.7}$$

for all $r_3, \ldots, r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$ and $\tau > 0$, for which $A_{r_1,r_2-r_1,\tau}((\mathscr{Z}')^{\mathrm{c}}) < 0$. Since, by Lemma 4.5,

$$P\{Z_2\in\mathscr{Z}'|D_n\} \geqslant P\{Z_2\in\mathscr{Z}'\}$$

for all $r_3, \ldots, r_n \geqslant \hat{t}$, $r_2 > r_1 \geqslant \hat{t}$, inequality (4.7) follows by condition (4.2). □

As in Corollary 4.2 we now consider the special case with $\hat{t} = 0$. By Proposition 4.3 it follows.

Corollary 4.6. *Let us set* $\hat{t} = 0$. *Under conditions* (C1′), (C2), (4.1) and (4.2), *the joint survival function* $\bar{F}(t_1, \ldots, t_n)$ *is Schur-convex.*

Condition (C1) implies both the m.u.n.a. property for $\bar{F}(t_1, \ldots, t_n)$ and the ultimately DFR property for its one-dimensional marginal $\bar{F}_1$. Similarly, the next result shows that those conditions, alternative to (C1), which are sufficient to guarantee the m.u.n.a. property for $\bar{F}(t_1, \ldots, t_n)$ also imply the ultimately DFR property for its one-dimensional marginal $\bar{F}_1$.

Corollary 4.7. *Under conditions* (C1′), (C3), (4.1) *and* (4.2) *the marginal* (*one-dimensional*) *distribution of* $T_1, \ldots, T_n$ *is ultimately DFR* (*with respect to* $\hat{t}$).

Proof. Let us first consider the case in which $Z_1,\dots,Z_n$ are i.i.d. Then condition (C2) is trivially verified and, by Proposition 4.3, we can conclude that $(T_1,\dots,T_n)$ has the m.u.n.a. property (with respect to $\hat{t}$). Now we remark that $Z_1,\dots,Z_n$ are i.i.d. if and only if $T_1,\dots,T_n$ are i.i.d and that, for this case, the m.u.n.a. property of $T_1,\dots,T_n$ is equivalent to the one-dimensional distribution being ultimately DFR. Let us now consider the case when $Z_1,\dots,Z_n$ are exchangeable but not-necessarily independent. It is easy to see that the one-dimensional distribution of $T_1,\dots,T_n$ is given by

$$\bar{G}(t) = \int_{\mathscr{Z}} \bar{G}(t|z)\pi^{(1)}(z)\,\mathrm{d}z$$

irrespectively of $Z_1,\dots,Z_n$ being independent or not. □

5. Discussion

Here we present a few comments regarding the results of Section 4.

(1) As an example of application of Proposition 4.1 we consider a case with $\bar{G}(t|z)$ defined arbitrarily for $t<1$ and, for $t \geqslant 1$,

$$\bar{G}(t|z) = \exp\{-\phi(z)t^{\psi(z)}\}, \tag{5.1}$$

where ϕ, $\psi : [0,+\infty) \to [0,+\infty)$ are non-decreasing functions with $\psi(z) \leqslant 1$, for all z. It is immediately checked that conditions (A4) and (C1) are satisfied with $\bar{t}=1$ and $\bar{r}=1$, respectively. Then, $(T_1,\dots,T_n)$ has the m.u.n.a. property with respect to $\hat{t}=1$.

In particular, Proposition 4.1 can be applied to *multiplicative frailty models* specified by conditional hazard rates of the form

$$\lambda(t|z) = z\mu(t)$$

with $\mu(t)$ being non-increasing on some interval $[\bar{r},\infty)$. These models have been considered by Hougaard (1995), where $Z_1,\dots,Z_n$ are assumed i.i.d.. Here we suppose $Z_1,\dots,Z_n$ being, more in general, exchangeable.

(2) Different situations in which (one-dimensional) mixtures of IFR (Increasing Failure Rate) distributions turn out to be ultimately DFR have been presented in the recent literature on survival data analysis (see, for instance, Gurland and Sethuraman, 1995; Block and Joe, 1997). Those cases can actually provide further examples where conditional survival functions $\bar{G}(\cdot|z)$, for our model, are ultimately DFR, which is a fundamental condition in Propositions 4.1 and 4.3. In fact, let us assume, for instance, that the distribution of any observable lifetime T_i $(i=1,\dots,n)$ depends on a pair Z_i, Γ_i with $\mathbf{Z}$, Γ stochastically independent. Let us moreover assume

(a) $T_1,\dots,T_n$ are conditionally independent given $\mathbf{Z}$ and Γ with

$$P(T_i > t|\mathbf{z},\gamma) = \bar{H}(t|z_i,\gamma_i),$$

where $\{\bar{H}(\cdot|z,\gamma)\colon z\in\mathscr{Z}, \gamma\in L\}$ is a given family of conditional survival functions

(b) $\Gamma_1,\dots,\Gamma_n$ are independent with marginal prior $p_0(\cdot)$.

Then, when unconditioning with respect to $\Gamma_1,\dots,\Gamma_n$, the conditional survival function of $(T_1,\dots,T_n)$, given $\mathbf{Z}$, is

$$\bar{F}(t_1,\dots,t_n|\mathbf{z}) = \int_{L^n} \prod_{i=1}^{n} \bar{H}(t_i|z_i,\gamma_i)p_0(\gamma_i)\,\mathrm{d}\gamma_1\dots\mathrm{d}\gamma_n = \prod_{i=1}^{n} \bar{G}(t_i|z_i),$$

where

$$\bar{G}(t|z) \equiv \int_L \bar{H}(t|z,\gamma)\, p_0(\gamma)\, \mathrm{d}\gamma.$$

The afore-mentioned conditions on $p_0(\gamma)$ and $\{\bar{H}(t|z,\gamma)\colon z\in\mathscr{Z}, \gamma\in L\}$, which guarantee that $\bar{G}(t|z)$ is ultimately DFR, for any z, can then be used to build examples of ultimately DFR $\bar{G}(t|z)$.

(3) As pointed out in Corollary 4.7, we obtained sufficient conditions for ultimately DFR properties of mixtures of one-dimensional distributions. Our conditions are different from those obtained in Gurland and Sethuraman (1995), and Block and Joe (1997).

(4) As is well-known, mixtures of DFR (one-dimensional) distributions are DFR. According to the line followed in this paper, we rephrase such property in the following terms: let $T_1,\dots,T_n$ be conditionally independent, given $Z_1,\dots,Z_n$, with

$$P(T_i > t|\mathbf{Z}=\mathbf{z}) = \bar{G}(t|z_i).$$

If $Z_1,\dots,Z_n$ are i.i.d., then $T_1,\dots,T_n$ are also i.i.d. and, if $\bar{G}(t|z)$ is DFR, for all z, then $T_1,\dots,T_n$ are i.i.d. with a DFR distribution. For the joint survival function of $(T_1,\dots,T_n)$ the implication

$$0 \leqslant r_i < r_j \Rightarrow P\{T_i > r_i+\tau|D_n\} \leqslant P\{T_j > r_j+\tau|D_n\} \tag{5.2}$$

holds for any choice of $r_1,\dots,r_n \geqslant 0$, $\tau > 0$.

Our Corollary 4.2 states that the implication (5.2) still holds under the more general condition that $Z_1,\dots,Z_n$ are exchangeable, provided the assumption (A4) (with $\bar{t}=0$) is verified. The above arguments can be easily extended to the case of the ultimately DFR property.

(5) In the case when

$$P\{Z_1=\cdots=Z_n\}=1, \tag{5.3}$$

i.e. when $T_1,\dots,T_n$ are conditionally i.i.d., Corollary 4.2 reduces to a special case of a well-known result: the joint survival function of conditionally i.i.d. DFR lifetimes is Schur-convex.

Let us now consider

$$\bar{G}(t|z) = \exp\{-zt\}.$$

Under the condition (5.3), $\bar{F}(t_1,\dots,t_n)$ is even Schur constant, that is the equality holds in the implication (5.2). We notice that, on the contrary, an inequality in the strict sense holds if condition (5.3) is replaced by $Z_1,\dots,Z_n$ i.i.d..

(6) As far as condition (C2) is concerned, we note that, due to monotonicity of the distribution of each T_i in the variable Z_i, it also implies that the lifetimes $T_1,\dots,T_n$ are positively dependent, as one can heuristically expect; for specific results in this direction see Jogdeo (1978), and Shaked and Spizzichino (1998). In this way, Corollary 4.6 provides examples where vectors of strongly positively dependent lifetimes have Schur-convex survival functions.

(7) The extension of our results to non-exchangeable dependence among the lifetimes is presently under investigation by the authors. Here we limit ourselves to observe what follows: In many cases, properties of aging for a vector of lifetimes are interesting in view of the fact that can be converted into corresponding probabilistic assessments on the behavior of the vector of order statistics (see, for instance, Spizzichino, 2001). In this paper we considered notions of aging for exchangeable lifetimes. However, it is well-known that for an arbitrary vector of random variables $\mathbf{X} \equiv (X_1,\dots,X_n)$ we can build a vector of exchangeable lifetimes $\mathbf{T} \equiv (T_1,\dots,T_n)$ in such a way that the two corresponding vectors of order statistics have the same law; this

is true by letting, for instance,

$$T_i \equiv X_{\pi_i} \quad i = 1, \ldots, n,$$

where π is a random permutation of $\{1, \ldots, n\}$. If $f_{\mathbf{X}}$ is the joint density of $\mathbf{X}$, then the joint density $f_{\mathbf{T}}$ of $\mathbf{T}$ is the exchangeable density obtained by symmetrization

$$f_{\mathbf{T}}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\pi} f_{\mathbf{X}}(t_{\pi_1}, \ldots, t_{\pi_n}),$$

where the sum is extended to all permutations of $\{1, \ldots, n\}$. Therefore, a property of aging for $f_{\mathbf{T}}$ can, in a sense, be seen as a property of aging for $f_{\mathbf{X}}$.

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