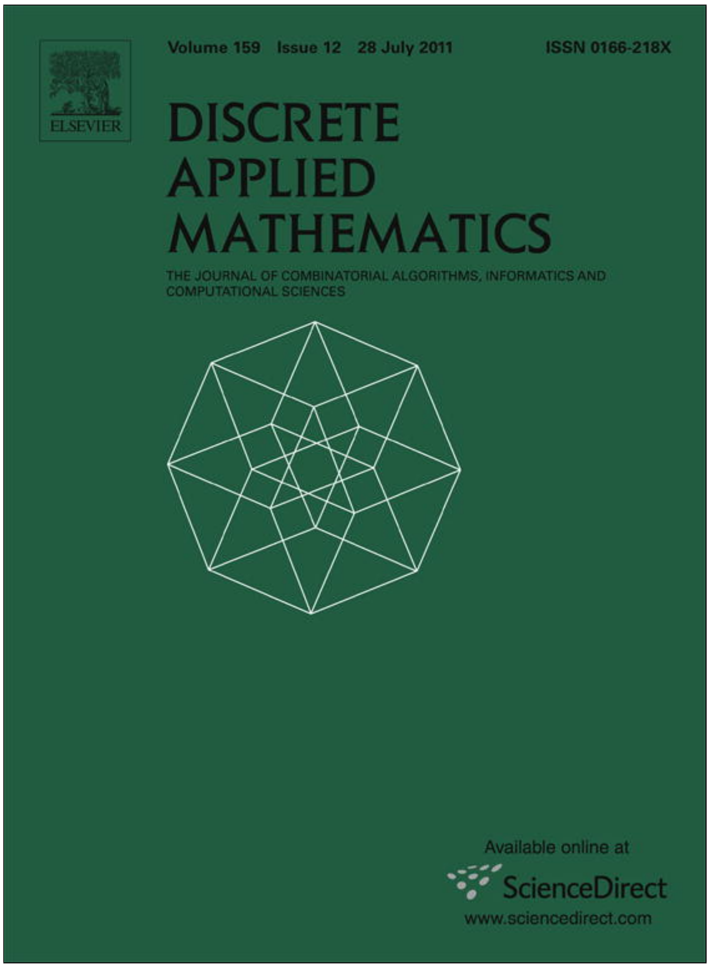
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The $L(2,1)$-labeling of unigraphs

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ABSTRACT

The $L(2,1)$-*labeling problem* consists of assigning colors from the integer set $0,\ldots,\lambda$ to the nodes of a graph G in such a way that nodes at a distance of at most two get different colors, while adjacent nodes get colors which are at least two apart. The aim of this problem is to minimize λ and it is in general NP-complete. In this paper the problem of $L(2,1)$-labeling unigraphs, i.e. graphs uniquely determined by their own degree sequence up to isomorphism, is addressed and a 3/2-approximate algorithm for $L(2,1)$-labeling unigraphs is designed. This algorithm runs in $O(n)$ time, improving the time of the algorithm based on the greedy technique, requiring $O(m)$ time, that may be near to $\Theta(n^2)$ for unigraphs.

1. Introduction

The $L(2,1)$-*labeling problem* [10] consists in assigning colors from the integer set $0,\ldots,\lambda$ to the nodes of a graph G in such a way that nodes at a distance of at most two get different colors, while adjacent nodes get colors which are at least two apart. The aim is to minimize λ.

This problem has its roots in mobile computing. The task is to assign radio frequencies to transmitters at different locations without causing interference. This situation can be modeled by a graph, whose nodes are the radio transmitters/receivers, and adjacencies indicate possible communications and, hence, interference. The aim is to minimize the frequency bandwidth, i.e. λ.

In general, both determining the minimum number of necessary colors [10] and deciding if this number is $<k$ for any fixed $k \geq 4$ [9] is NP-complete. Therefore, researchers have focused on some special classes of graphs. For some classes – such as paths, cycles, wheels, tilings and k-partite graphs – tight bounds for the number of colors necessary for an $L(2,1)$-labeling are well known in the literature and so a coloring can be computed efficiently. For many other classes of graphs – such as chordal graphs [14], interval graphs [8], split graphs [2], outerplanar and planar graphs [2,6], bipartite permutation graphs [1], and co-comparability graphs [5] – approximate bounds have been looked for. For a complete survey, see [4].

Unigraphs [11,12] are graphs uniquely determined by their own degree sequence up to isomorphism and are a superclass including *matrogenic graphs*, *matroidal graphs*, *split matrogenic graphs* and *threshold graphs*. The interested reader can find information related to these classes of graphs in [13].

In [7], all these subclasses are $L(2,1)$-labeled: threshold graphs can be optimally $L(2,1)$-labeled in time linear in Δ with $\lambda \leq 2\Delta$, while for matrogenic graphs the upper bound $\lambda \leq 3\Delta$ holds, where Δ is the maximum degree of the graph. In the same paper the problem of $L(2,1)$-labeling the whole superclass of unigraphs is left open.

In this paper, a 3/2-approximate algorithm for the $L(2,1)$-labeling of unigraphs is presented. This algorithm runs in $O(n)$ time, which is the best possible. Observe that a naive algorithm, based on the greedy technique, would obtain an $O(m)$ time complexity, that may be near to $\Theta(n^2)$ for unigraphs.

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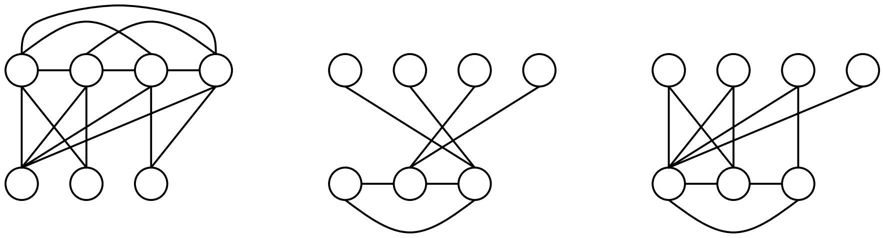

Fig. 1. (a) A split graph G; (b) its complement $\overline{G}$; (c) its inverse G^I.

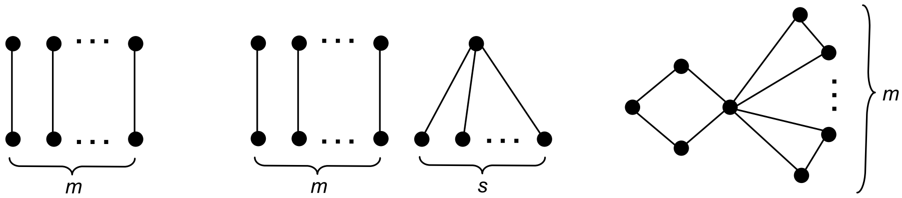

Fig. 2. (a) mK_2; (b) $U_2(m, s)$; (c) $U_3(m)$.

The technique used in the algorithm takes advantage of the degree sequence analysis. In particular, this algorithm exploits the concept of boxes, i.e. the equivalence classes of nodes in a graph under equality of degree.

This paper is organized as follows.

In the next section all the information required for the rest of the paper is summarized. A recognition algorithm for unigraphs and the corresponding characterization theorem on which it is based are outlined in Section 3. The core of the paper comes in the following three sections. Section 4 provides optimal $L(2, 1)$-labeling without repetitions (i.e. $L'(2, 1)$-labeling) for those graphs listed in the characterization of unigraphs, while an $L(2, 1)$-labeling for the same graphs is presented in Section 5. Finally, in Section 6 a linear time (in n and in Δ) $3/2$ approximate algorithm for $L(2, 1)$-labeling of unigraphs is presented. Concluding remarks and open problems complete the paper.

2. Preliminaries

In this section all the definitions and known results that will be used in the rest of the paper are summarized.

We consider only finite, simple, loopless graphs $G = (V, E)$, where V and E are the node and edge sets of G with cardinality n and m, respectively. Where no confusion arises, $G = (V, E)$ is called simply G.

Let $DS(G) = \delta_1, \delta_2, \ldots, \delta_n$ be the degree sequence of a graph G sorted by non-increasing values: $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq 0$. We call *boxes* the equivalence classes of nodes in G under equality of degree. In terms of boxes the degree sequence can be compressed as $d_1^{m_1}, d_2^{m_2}, \ldots, d_r^{m_r}, d_1 > d_2 > \cdots > d_r \geq 0$, where d_i is the degree of the m_i nodes contained in box $B_i(G)$, $1 \leq m_i \leq n$; hence $\sum_{i=1}^{r} m_i = n$ and $\sum_{i=1}^{r} d_i m_i = 2m$.

We call a box *universal (isolated)* if it contains only universal (isolated) nodes, where a node $x \in V$ is called *universal* (*isolated*) if it is adjacent to all other nodes of V (no other node in V); if x is a universal (isolated) node, then its degree is $d(x) = n - 1$ ($d(x) = 0$).

A graph I induced by subset $V_I \subseteq V$ is called *complete* or *clique* if any two distinct nodes in V_I are adjacent in G, *stable* or *null* if no two nodes in V_I are adjacent in G.

A graph G is said to be *split* if there is a partition $V = V_K \cup V_S$ of its nodes such that the induced subgraphs K and S are complete and stable, respectively (see Fig. 1(a)).

If $G = (V, E)$ is a graph, its *complement* is $\overline{G} = (V, V \times V - E)$ (see Fig. 1(b)). If $G = (V_K \cup V_S, E)$ is a split graph, its *inverse* G^I is obtained from G by deleting the set of edges $\{\{a_1, a_2\} : a_1, a_2 \in V_K\}$ and adding the set of edges $\{\{b_1, b_2\} : b_1, b_2 \in V_S\}$ (see Fig. 1(c)).

Given a graph G, if its node set V can be partitioned into three disjoint sets V_K, V_S and V_C such that K is a clique, S is a stable set and every node in V_C is adjacent to every node in V_K and to no node in V_S, then the subgraph induced by V_C is called *crown*.

In the following the definitions of some special graphs are recalled [15]:

mK_2: it is the union of m node-disjoint edges $m \geq 1$, also called perfect matching (see Fig. 2(a)).

$U_2(m, s)$: it is the disjoint union of a perfect matching mK_2 and a star $K_{1,s}$, for $m \geq 1$, $s \geq 2$ (see Fig. 2(b)).

$U_3(m)$: for $m \geq 1$, this graph is constructed as follows: fix a node in each component of the graph obtained as disjoint union of the chordless cycle C_4 and m triangles K_3, and merge all these nodes in one (see Fig. 2(c)).

$S_2 = (p_1, q_1; \ldots; p_t, q_t)$: to obtain this graph, add all the edges connecting the centers of l non-isomorphic arbitrary stars K_{1,p_i}, $i = 1, \ldots, t$, each one occurring q_i times, where $p_i, q_i, t \geq 1$, $q_1 + \cdots + q_t = l \geq 2$ (see Fig. 3(a)). Without loss of generality, in the following we assume $p_1 \leq \cdots \leq p_t$.

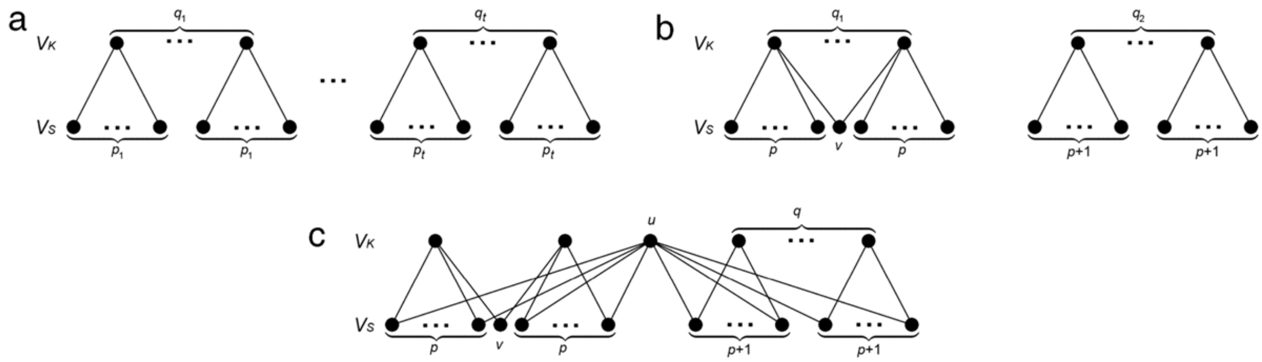

Fig. 3. (a) $S_2(p_1, q_1; \ldots; p_t, q_t)$; (b) $S_3(p, q_1; q_2)$; (c) $S_4(p, q)$.

$S_3(p, q_1; q_2)$: take a graph $S_2(p, q_1; p+1, q_2)$ where $p \geq 1, q_1 \geq 2$ and $q_2 \geq 1$; add a new node v to the stable part of the graph and add the set of q_1 edges $\{\{v, w\} : w \in V_K$ and $\deg_{V_S}(w) = p\}$: the obtained graph is S_3 (see Fig. 3(b)).

$S_4(p, q)$: it is constructed taking a graph $S_3(p, 2; q)$, $q \geq 1$, adding a new node u to the clique part and connecting it with each node of the stable except v (see Fig. 3(c)).

It is easy to see that S_2, S_3 and S_4 are split graphs, where the clique part is constituted by the centers of the stars for S_2 and S_3, and by the centers of the stars and u for S_4.

3. Characterization and recognition of unigraphs

In this section we recall a characterization of unigraphs in terms of superposition of a red and a black graph.

Theorem 3.1 ([3]). *A graph G is a unigraph if and only if its node set can be partitioned into three disjoint sets V_K, V_S and V_C such that:*

(i) *$V_K \cup V_S$ induces a split unigraph F in which K is the clique and S is the stable set;*
(ii) *V_C induces a crown H and either H or $\overline{H}$ is one of the following graphs:*

$$C_5, \qquad mK_2, \quad m \geq 2, \qquad U_2(m, s), \qquad U_3(m);$$

(iii) *the edges of G can be colored red and black so that:*
  a. *the red partial graph is the union of the crown H and of node-disjoint pieces P_i, $i = 1, \ldots, z$. Each piece P_i (or $\overline{P_i}$, or P_i^I or $\overline{P_i^I}$) is one of the following graphs:*
  K_1, $S_2(p_1, q_1; \ldots; p_t, q_t)$, $S_3(p, q_1; q_2)$, $S_4(p, q)$,
  considered without the edges in the clique;
  b. *the linear ordering $P_1, \ldots, P_z$ is such that each node in V_K belonging to P_i is not linked to any node in V_S belonging to P_j, $j = 1, \ldots, i-1$, but is linked by a black edge to every node in V_S belonging to P_j, $j = i+1, \ldots, z$. Furthermore, any edge connecting either two nodes in V_K or a node in V_K and a node in V_C is black.*

In view of the previous lemma, although not explicitly mentioned, when we speak about a unigraph G we mean that its node set is partitioned into the three sets V_C, V_K and V_S, inducing the crown, the clique and the stable part, respectively.

It is worthy to be noticed that there is a basic difference between a matching inside the split part of a unigraph and a matching constituting the crown: the nodes of the first one induce an $S_2(1, q)$ (i.e. the red edges of the matching plus the black edges connecting as a clique the nodes that are in V_K); the second one corresponds to an mK_2. An analogous difference holds between the graph induced by the nodes of an antimatching inside the split part of a unigraph ($\overline{S_2(1, q)}$) and the crown inducing an antimatching ($\overline{mK_2}$). This difference will be very important when we will $L(2, 1)$-label the pieces of the unigraph, as we underline in Sections 4 and 5.

In Fig. 4 a unigraph is depicted, and its red and black partial graphs are highlighted. The pieces P_i defined by the previous theorem are included in dotted rectangles. Observe that in this figure, and all over the paper, we depict all nodes belonging to V_K above all nodes belonging to V_s, that always lie on the bottom part of the drawing; moreover, we avoid to draw all the edges of the clique, but we include the nodes of V_K in a rectangle to underline that they induce a clique.

From the characterization stated in Theorem 3.1, and recalling that a unigraph is a graph uniquely determined by its own degree sequence up to isomorphism, it is possible to derive a linear time recognition algorithm for unigraphs that identifies the structure of the graph analyzing only its degree sequence [3]. In particular, this algorithm exploits the concept of boxes. If there is not an isolated or universal box (K_1 in item (iii).a of Theorem 3.1), a group of boxes can induce either a crown as specified in item (ii), or one of the graphs S_2, S_3, S_4 (or their complement, their inverse, or the inverse of their complement) in item (iii).a. This algorithm for recognizing unigraphs works pruning the degree sequence $d_1^{m_1}, \ldots, d_r^{m_r}$ of a given graph G. At each step, the algorithm finds one of the node-disjoint pieces P_i of G, checking the first p and the last q boxes, according to part (iii).a. of Theorem 3.1. The algorithm proceeds on the pruned graph $G - P_i$, that represents a unigraph if G is a unigraph (part (iii).b of Theorem 3.1). This step is iterated until either G is recognized to be a unigraph or some contradiction is highlighted because either part (iii).a or part (iii).b are recognized to be not true.

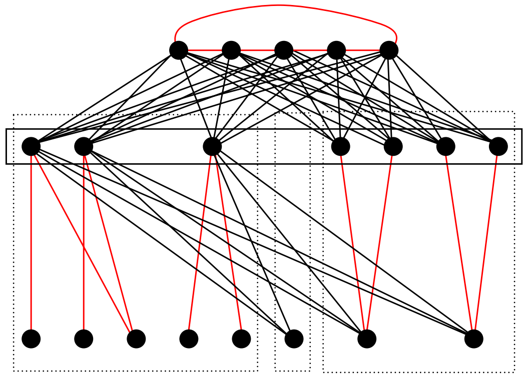

Fig. 4. A unigraph where its crown C_5 and its pieces $S_3(1, 2; 1)$, K_1 and $S_2(2, 2)^I$ are highlighted by dotted rectangles. Edges are colored according to Theorem 3.1 (edges completely contained into the dotted rectangles and the edges of the crown C_5 are red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

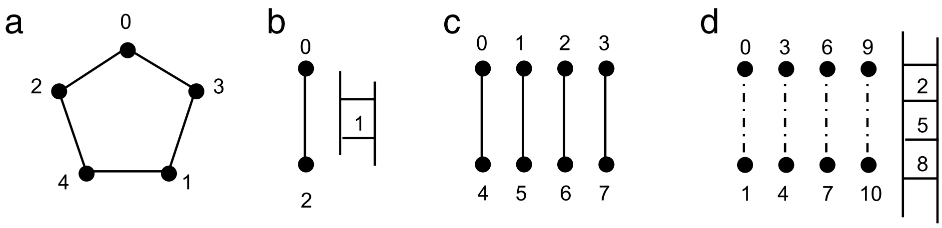

Fig. 5. Optimal $L'(2, 1)$-labeling: (a) of a C_5; (b) of a K_2; (c) of a $4K_2$; (d) of a $\overline{4K_2}$.

4. $L'(2, 1)$-labeling of the crown and of the pieces

An $L'(2, 1)$-*labeling* (also called $L(2, 1)$-labeling without repetitions) [8] is a one-to-one $L(2, 1)$-labeling into the set $0, \ldots, \lambda'$, with the aim of minimizing λ'.

In order to design the $L(2, 1)$-labeling algorithm for unigraphs, in this section we will show how to optimally $L'(2, 1)$-label the graphs cited in Theorem 3.1 and, for each of them, we provide the number of used colors, taking into account the black connections of Theorem 3.1. In the following it will be clear why we need to $L'(2, 1)$-label some pieces of a unigraph in order to get an $L(2, 1)$-labeling of it.

We underline that, from now on, in the figures, when we depict complement and inverse graphs, we omit to draw all the edges, except the absent ones, represented by dotted lines. Moreover, the unused colors are highlighted in a queue.

4.1. Crown

In order to $L(2, 1)$-label the V_C nodes of the crown of a unigraph G, we have to consider whether there are other nodes in the unigraph or it is constituted by the only crown; in other words, we have to distinguish if the crown is the only piece in the graph or not. If at least another (split) piece exists, all the nodes in V_C are at mutual distance two, since the crown is completely connected to the nodes of V_K. When $V_K = \emptyset$ this condition is not required. It follows that in the first case we have to $L'(2, 1)$-label the crown, while in the second case we have to $L(2, 1)$-label it.

In the following we will show how to optimally $L'(2, 1)$-label the crown.

Lemma 4.1 ([3]). *Let G be a unigraph with $V_K \neq \emptyset$. If its crown H is:*

- *the cycle $C_5 = \overline{C_5}$ then it can be optimally $L'(2, 1)$-labeled with 5 consecutive colors;*
- *a matching mK_2, then it can be optimally $L'(2, 1)$-labeled with $2m$ consecutive colors if $m > 1$ and with $3 = 2m + 1$ colors if $m = 1$; in this latter case one color remains unused.*
- *a hyperoctahedron $\overline{mK_2}$, then it can be optimally $L'(2, 1)$-labeled with $3m - 1$ consecutive colors and $m - 1$ colors remain unused.*

In Fig. 5(a)–(d) $L'(2, 1)$-labeling of C_5, K_2, mK_2 and $\overline{mK_2}$ when $m = 4$ are reported. It is to notice that the $L(2, 1)$- and $L'(2, 1)$-labelings coincide for C_5 and $\overline{4K_2}$.

Lemma 4.2. *Let G be a unigraph with $V_K \neq \emptyset$. If its crown H is:*

- *$U_2(m, s)$ then it can be optimally $L'(2, 1)$-labeled with $2m + s + 1$ consecutive colors;*
- *$\overline{U_2(m, s)}$ then it can be optimally $L'(2, 1)$-labeled with $3m + 2s - 1$ colors and $m + s - 2$ colors remain unused;*

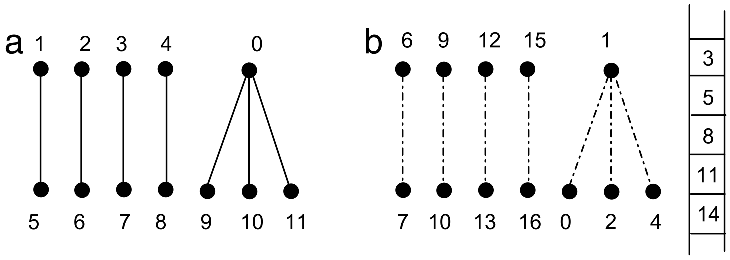

Fig. 6. Optimal $L'(2,1)$-labeling of: (a) $U_2(4,3)$; (b) $\overline{U_2(4,3)}$.

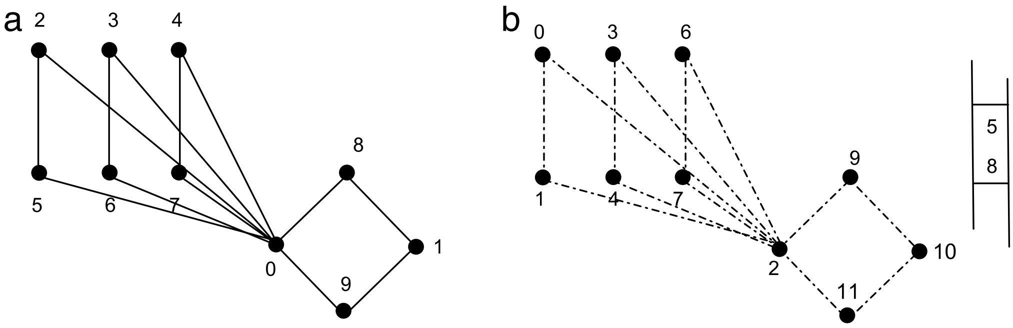

Fig. 7. Optimal $L'(2,1)$-labelings of: (a) $U_3(3)$; (b) $\overline{U_3(3)}$.

- *$U_3(m)$ then it can be optimally $L'(2,1)$-labeled with $2m+4$ consecutive colors;*
- *$\overline{U_3(m)}$ then it can be optimally $L'(2,1)$-labeled with $3m+3$ colors and $m-1$ colors remain unused.*

Proof. As $U_2(m,s)$ is the disjoint union of an mK_2 and a star $K_{1,s}$, in view of Lemma 4.1, if $m \geq 2$, we need $2m$ colors for mK_2, let them be $1,\ldots,2m$, while the star can be easily optimally $L'(2,1)$-labeled with s+1 colors, assigning 0 to the center of the star and colors $2m+1,\ldots,2m+s$ to the leaves (see Fig. 6(a)). If $m=1$ we need the same number of colors simply arranging them in a different way (e.g. using 1 and 3 for the K_2, 0 for the center of the star, and the other ones for its leaves).

$\overline{U_2(m,s)}$ is given by a hyperoctahedron $\overline{mK_2}$ completely connected with the complement of a star $K_{1,s}$. $3m-1$ colors are needed to optimally $L'(2,1)$-label the hyperoctahedron (see Lemma 4.1), and the unused colors cannot be used inside the same U_2 in view of the complete connection with $\overline{mK_2}$. In order to label the complement of the star we need $2s-1$ colors more, whose $s-2$ are unused. Finally, we have to add a further color between the colors of the hyperoctahedron and of $\overline{K_{1,s}}$ because they are completely connected (see Fig. 6(b)), so also one more color remains unused. By summing all the contributions, the thesis follows.

In order to label $U_3(m)$, let 0 be the color of the maximum degree node. It is not difficult to give different colors to all the other nodes in order to get an optimal $L'(2,1)$-labeling with a number of colors equal to the number of nodes (see Fig. 7(a)).

Observe that $\overline{U_3(m)}$ is constituted by a hyperoctahedron $\overline{mK_2}$, completely connected to three nodes, two of which are connected, and the third one is adjacent to a degree 1 node. Consequently, $3m-1$ colors are necessary to label the hyperoctahedron (see Lemma 4.1); one of the colors unused by the hyperoctahedron can be used for the degree 1 node. 4 more consecutive colors are necessary for the remaining three nodes, since the first of them cannot be used (see Fig. 7(b)). The unused colors are hence $m-1$. □

We highlight that the $L(2,1)$- and $L'(2,1)$-labelings coincide on $\overline{U_2}$ and $\overline{U_3}$ graphs as they have diameter 2.

4.2. Split pieces

Each split piece P_i (S_2, S_3 and S_4 of Theorem 3.1) must be colored using colors at mutual distance at least two in the clique part.

For what concerns the stable part, we have to distinguish two cases, according to the fact that P_i is the first piece in the linear ordering of item iii.(b) of Theorem 3.1 (i.e. $i=1$) or not (i.e. $i>1$). Indeed, black edges defined in item iii.(b) impose to use different colors for the nodes in the stable part of each P_i, $i>1$, hence for this piece we have to provide an $L'(2,1)$-labeling. Only colors in the stable part of P_1 can be eventually repeated.

In this subsection we show how to $L'(2,1)$-label split pieces.

Lemma 4.3. *Let G be a unigraph. If one of its pieces P_i, $i>1$, is*

- *$S_2(p_1,q_1;\ldots;p_t,q_t)$ then it can be optimally $L'(2,1)$-labeled with $\sum_{i=1}^{t}(p_i+1)q_i$ consecutive colors;*
- *$\overline{S_2(p_1,q_1;\ldots;p_t,q_t)}^I$ then it can be optimally $L'(2,1)$-labeled with $\sum_{i=1}^{t}(p_i+1)q_i$ colors; if $q_1>2$ and $p_1=1$ then it can be optimally $L'(2,1)$-labeled with $\sum_{i=1}^{t}(p_i+1)q_i+\lfloor q_1/2\rfloor$ colors and $\lfloor q_1/2\rfloor$ of them remain unused;*

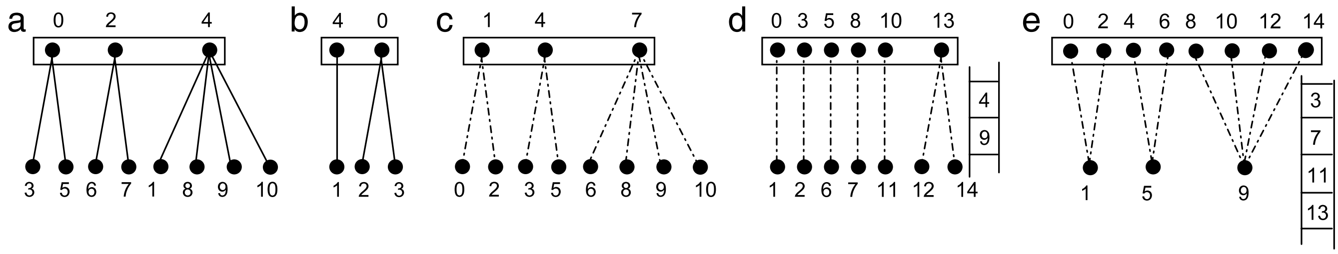

Fig. 8. Optimal $L'(2,1)$-labelings of: (a) $S_2(2,2;4,1)$; (b) $S_2(1,1;2,1)$; (c) $\overline{S_2^I(2,2;4,1)}$; (d) $\overline{S_2^I(1,5;2,1)}$; (e) $\overline{S_2(2,2;4,1)}$.

- $\overline{S_2(p_1,q_1;\ldots;p_t,q_t)}$ *then it can be optimally* $L'(2,1)$*-labeled with* $2\sum_{i=1}^t p_iq_i - 1$ *colors and* $\sum_{i=1}^t q_i(p_i-1) - 1$ *of them remain unused; if* $p_1 = 1$ *then both the number of used and unused colors must be incremented by* $\lfloor q_1/2 \rfloor$*;*
- $\overline{S_2(p_1,q_1;\ldots;p_t,q_t)^I}$ *then it can be optimally* $L'(2,1)$*-labeled with* $2\sum_{i=1}^t p_iq_i - 1$ *colors and* $\sum_{i=1}^t q_i(p_i-1) - 1$ *of them remain unused; if* $t = 1$ *and* $q_1 = 1$ *then it can be optimally* $L'(2,1)$*-labeled with* $2p_1 + 1$ *colors and* p_1 *of them remain unused.*

Proof. For the $\sum_{i=1}^t q_i$ centers of the stars of S_2, that are connected in a clique, $2\sum_{i=1}^t q_i - 1$ colors are necessary, and $\sum_{i=1}^t q_i - 1$ of them are unused. Let U be the set of these unused colors. Colors from U are assigned to the leaves of each star taking into account to avoid those colors at distance one from the color assigned to the center (see Fig. 8(a)). In order to complete the labeling, further $\sum_{i=1}^t (p_i-1)q_i + 1$ consecutive colors will be necessary. The number of used colors is hence $\sum_{i=1}^t p_iq_i + \sum_{i=1}^t q_i$, that is exactly the number of nodes of S_2. Observe that, if $\sum_{i=1}^t q_i = 2$, in order not to discard any color, the nodes in the clique must be labeled with a different rule (see Fig. 8(b)). Indeed, if the clique was labeled with 0 and 2, color 1 would be discarded.

For what concerns $\overline{S_2^I}$, again a number of colors equal to the number of nodes is necessary and sufficient, but the labeling must be performed in the following way: label the first of the p_i leaves of each star with the first available color c; label the center of the star with color $c+1$, and the remaining $p_i - 1$ leaves with colors $c+2,\ldots,c+p_i$ (see Fig. 8(c)). This method works if $p_1 \geq 2$. But, if it holds that $q_1 > 2$ and $p_1 = 1$, then the first q_1 stars constitute a matching and more colors are necessary. Namely, for each color g assigned to a node of the matching in the clique, both $g-1$ and $g+1$ cannot be assigned to any node in the clique and to any node in the stable set, except its mate; hence one between $g-1$ and $g+1$ must remain unused (see Fig. 8(d)).

It is easy to see that for labeling $\overline{S_2(p_1,q_1;\ldots;p_t,q_t)}$ and $S_2(p_1,q_1;\ldots;p_t,q_t)^I$, $2\sum_{i=1}^t p_iq_i - 1$ colors are always necessary and sufficient. Indeed, they are necessary for $L'(2,1)$-labeling the clique containing all the leaves of the stars, and each center of a star may be colored with one of the colors unused during the labeling of the leaves opportunely chosen (see Fig. 8(e)). It follows that $\sum_{i=1}^t q_i(p_i-1) - 1$ colors remain unused. Observe that if $p_1 = 1$ in $\overline{S_2}$, arguments similar to those explained for S_2 can be used, and the thesis follows. Finally, if $t = 1$ and $q_1 = 1$ in S_2^I, it is easy to see that 2 colors more are needed since S_2^I is a clique with $p+1$ nodes. □

Lemma 4.4. *Let G be a unigraph. If one of its pieces* P_i, $i > 1$, *is*

- $S_3(p,q_1;q_2)$ *then it can be optimally* $L'(2,1)$*-labeled with* $pq_1 + (p+1)q_2 + q_1 + q_2 + 1$ *consecutive colors;*
- $\overline{S_3(p,q_1;q_2)^I}$ *then it can be optimally* $L'(2,1)$*-labeled with* $pq_1 + (p+1)q_2 + q_1 + q_2 + 1$ *consecutive colors; if* $q_1 > 2$ *and* $p_1 = 1$ *then it can be optimally* $L'(2,1)$*-labeled with* $pq_1 + (p+1)q_2 + q_1 + q_2 + 1 + \lfloor q_1/2 \rfloor$ *colors and* $\lfloor q_1/2 \rfloor$ *of them are unused;*
- $\overline{S_3(p,q_1;q_2)}$ *then it can be optimally* $L'(2,1)$*-labeled with* $2p(q_1+q_2) + 2q_2 + 1$ *colors and* $pq_1 + pq_2 - q_1 - 1$ *of them remain unused; if* $p_1 = 1$ *then it can be optimally* $L'(2,1)$*-labeled with* $2p(q_1+q_2) + 2q_2 + 1 + \lfloor q_1/2 \rfloor$ *colors and* $pq_1 + pq_2 - q_1 - 1 + \lfloor q_1/2 \rfloor$ *of them remain unused;*
- $S_3(p,q_1;q_2)^I$ *then it can be optimally* $L'(2,1)$*-labeled with* $2p(q_1+q_2) + 2q_2 + 1$ *colors and* $pq_1 + pq_2 - q_1 - 1$ *of them remain unused.*

Proof. Remind that $S_3(p,q_1;q_2)$ is obtained adding a node v to the stable set of a graph $S_2(p,q_1;p+1,q_2)$, $p \geq 1$, $q_1 \geq 2$, $q_2 \geq 1$ and v is connected to the first q_1 centers. Consequently the methods for labeling S_3, $\overline{S_3}$, S_3^I and $\overline{S_3^I}$ are the same presented for S_2, $\overline{S_2}$, S_2^I and $\overline{S_2^I}$ only taking care of node v. In order not to overburden the exposition we omit further details and we present only Fig. 9(a) in which the labeling of $S_3(2,1;1)$ is depicted. □

Lemma 4.5. *Let G be a unigraph. If one of its pieces* P_i *(cf. Theorem 3.1) is*

- $S_4(p,q)$ *then it can be optimally* $L'(2,1)$*-labeled with* $2p + qp + 2q + 4$*;*
- $\overline{S_4(p,q)}$ *then it can be optimally* $L'(2,1)$*-labeled with* $2pq + 4p + 2q + 1$ *colors and* $pq + 2p - 3$ *of them remain unused; if* $p = 1$ *then it can be labeled with* $2pq + 4p + 2q + 2$ *colors and* $pq + 2p - 2$ *of them remain unused;*

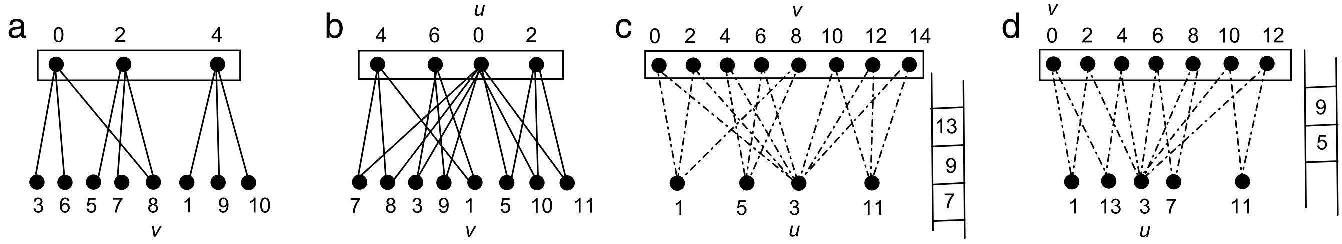

Fig. 9. Optimal $L'(2,1)$-labelings of: (a) $S_3(2,2;1)$; (b) $S_4(2,1)$; (c) $\overline{S_4(2,1)}$; (d) $\overline{S_4(1,2)}$.

- $S_4(p,q)^I$ *then it can be optimally* $L'(2,1)$*-labeled with* $2pq+4p+2q+2$ *colors and* $pq+2p-2$ *of them remain unused;*
- $\overline{S_4(p,q)}^I$ *then it can be optimally* $L'(2,1)$*-labeled with* $3q+2p+qp+4$ *colors and* q *of them remain unused.*

Proof. The clique part of $S_4(p,q)$ requires $2(q+3)-1$ colors, whose $q+3$ are used. Node u in the clique part is colored with 0. All the nodes in the stable part (except v) are connected to u and hence at mutual distance two and require distinct colors. The colors discarded while labeling the clique can be opportunely used in the stable part, and in particular node v is labeled with 1. The centers of the two stars with p leaves cannot receive color 2. The remaining nodes are labeled with consecutive new colors (see Fig. 9(b)). Totally, the number of necessary colors is the same as the number of nodes of S_4.

The clique part of $\overline{S_4}$ can be labeled with all the even colors from 0 to $e = 2(2p+(p+1)q)$. The odd colors, opportunely used, are sufficient to label the stable part, and $pq+2p-3$ colors remain unused (see Fig. 9(c)). In the special case when $p = 1$, one color must be unused, hence one color more is necessary, as shown in Fig. 9(d).

Analogous considerations hold for S_4^I. In this case, color e is assigned to node u and node v must have a new odd color $e+1$. The number of unused colors is one more than in the previous case, i.e. $pq+2p-2$.

Finally, an optimal labeling of $\overline{S_4^I}$ is obtained using $qp+2p+3q+5$ colors. Indeed $2(q+3)-1$ colors are required by the clique. Let 0, 2 and 4 the colors assigned to v and to the centers of the stars with p leaves each one. Color 3 is suitable for labeling u. Moreover color 1 can be assigned to one of the leaves of the star with center labeled with 2. No other odd colors from 5 to $2(q+3)-3$ can be utilized in the stable part so, q colors must remain unused. Since nodes in the stable part must have different colors (in view of the fact that each pair is at distance two), we have to add $2p+qp+q-1$ consecutive different colors for completing the coloring of $\overline{S_4^I}$. □

5. $L(2,1)$-labeling of the crown and of the pieces

5.1. Crown

We recall that the $L(2,1)$- and $L'(2,1)$-labelings of C_5 and $\overline{mK_2}$ coincide (see Fig. 5(a) and (d). Furthermore, in view of their structure, $\overline{U_2}$ and $\overline{U_3}$ are graphs with diameter 2, hence even their $L(2,1)$- and $L'(2,1)$-labelings coincide and they must be labeled with all different colors, independently of the rest of the unigraph. Finally, in U_3 only one node can be labeled re-using a color (the node labeled by 1 in Fig. 7(a)) hence the number of colors necessary to $L'(2,1)$- and to $L(2,1)$-label U_3 is the same, but in the latter case one color remains unused. So, it remains to prove the following lemma.

Lemma 5.1. *Let G be a unigraph constituted only by its crown. If G is:*

- mK_2*, then it can be optimally* $L(2,1)$*-labeled with 3 colors and one color remains unused;*
- $U_2(m,s)$ *then it can be optimally* $L(2,1)$*-labeled with* $s+2$ *colors and one color remains unused.*

Proof. The $L(2,1)$-labeling of a matching with 3 colors is trivial: 0 and 2 can be used for adjacent nodes, and color 1 remains unused (see Fig. 10(a)).

As U_2 is constituted by the disjoint union of a matching and a star, we can optimally label the star with $s+2$ colors, whose one color is unused; for the matching, it is possible to re-use a couple of not adjacent already used colors (see Fig. 10(b)). It is to notice that the pair (1, 3) for labeling the matching is also feasible, but the choice of not using a color will be useful later and will be clear during the presentation of the algorithm. □

5.2. Split pieces

Observe that $\overline{S_2}$ is a diameter 2 graph, if $\sum_{i=1}^{t} q_i > 2$, hence there is no difference between the $L(2,1)$- and $L'(2,1)$-labelings. Furthermore, if $\sum_{i=1}^{t} q_i = 2$, then the centers of the two stars are at distance three, but there is no way to assign them the same color using the minimum number of colors.

For what concerns S_2^I, the number of used colors is the same as in the case without repetitions, as the maximum number of necessary colors is given by the clique part, but some colors can be replicated in the stable part, hence $\sum_{i=1}^{t} p_i q_i - 3$ colors remain unused.

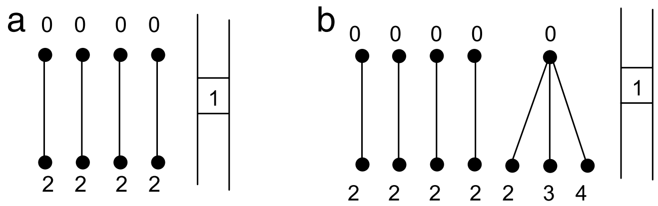

Fig. 10. Optimal $L(2,1)$-labelings of: (a) $4K_2$; (b) $U_2(4,3)$.

$\overline{S_2^I}$ is a diameter 2 graph, when $\sum_{i=1}^{t} q_i > 2$ and hence its $L(2,1)$-labeling coincides with its $L'(2,1)$-labeling. If $\sum_{i=1}^{t} q_i = 2$, $\overline{S_2^I}$ coincides with S_2.

Similar considerations hold for $\overline{S_3}$, S_3^I, $\overline{S_3^I}$, $\overline{S_4}$, S_4^I and $\overline{S_4^I}$. Finally, in S_4 the only node that is at distance three from some leaves is u, and hence it is the only node that can receive a repeated color. It follows that the $L(2,1)$-labeling of S_4 is identical with respect to the $L'(2,1)$-labeling, except for node u. Hence, in order to study the $L(2,1)$-labeling of the split graphs in item iii(a) of Theorem 3.1, it is enough to prove the following result.

Lemma 5.2. *Let G be a unigraph. If its first split piece P_1 (cf. Theorem 3.1) is*

- $S_2(p_1, q_1; \ldots; p_t, q_t)$ *then it can be optimally $L(2,1)$-labeled with* $\left(2\sum_{i=1}^{t} q_i - 1\right) + \max\left\{0, p_t + x - \left(\sum_{i=1}^{t} q_i - 1\right)\right\}$ *colors, where $x = 1$ if $q_t \le 2$ and $x = 2$ otherwise;*
- $S_3(p, q_1; q_2)$ *then it can be optimally $L(2,1)$-labeled with* $\left(2\sum_{i=1}^{t} q_i - 1\right) + y + \max\{0, p_t + x - \left(\sum_{i=1}^{t} q_i - 1\right)\}$ *colors, where $x = 1$ and $y = 1$ if $q_t \le 2$ and $x = 2$ and $y = 0$ otherwise.*

Proof. S_2 is composed by stars whose $\sum_{i=1}^{t} q_i$ centers are connected in a clique. So, at least $2\sum_{i=1}^{t} q_i - 1$ colors are necessary. The first color must be assigned to one among the q_t centers of the maximum size stars. Each time two distance 2 colors are assigned in the clique, the color in between remains unused. All such colors can be opportunely assigned to some nodes in the stable part, possibly many times, paying attention that no leaf of a center of a star labeled c takes label $c-1$ or $c+1$. Observe that the p_i leaves of each star must receive all different colors, as they are at mutual distance two. Consider now the q_t stars of maximum size p_t. If the unused colors are not enough to label its leaves, some colors must be added. Their number is $p_t - \left(\sum_{i=1}^{t} q_i - 2\right)$ if $q_t \le 2$ (indeed at most one unused color must be discarded, see Fig. 11(a)) and is one color more if $q_t \ge 3$. Finally, if p_t is sufficiently small, the unused colors are enough to label all the leaves of the maximum size stars and then no other colors must be added.

S_3 is obtained from an S_2 by adding a node to the stable part. It is easy to see that the number of colors necessary to label S_2 are enough for S_3, as the added node v can receive either an already used color or one among the colors unused during the labeling of the clique part (see Fig. 11(b)). Only if $q_t \le 2$ then one color more is necessary for v, and it must be labeled 1, as shown in Fig. 11(c). This is the meaning of y in the formula of the number of colors. □

6. An algorithm for $L(2,1)$-labeling unigraphs

The labelings presented in the previous two sections will be used for the linear time algorithm for labeling the whole unigraph detailed in this section.

In Section 3, we have claimed that it is possible to identify the structure of a connected unigraph analyzing only its degree sequence, so the following $L(2,1)$-labeling algorithm will deal with the representation of a graph $G = (V, E)$ in terms of boxes with degree sequence $d_1^{m_1}, d_2^{m_2}, \ldots, d_r^{m_r}$, $d_1 > d_2 > \cdots > d_r$.

Let us call k_i the largest color used for labeling the clique part of P_i separately, considering that each split piece P_i must be colored using colors at mutual distance at least two in the clique part.

The algorithm labels each piece in two phases. In the first phase, only $k_i + 1$ colors are considered, and in the second phase the labeling is completed. In particular, the algorithm first puts in a queue S the pieces P_i, with clique part K_i and stable part S_i described in Theorem 3.1, that it recognizes according to the algorithm in [3], and the crown H, if it exists. Then, the algorithm partially labels each piece P_i dequeued from the queue according to its own structure. In order to explain the partial labeling of piece P_i, let $c_{i-1} - 1$ be the last color used for the partial labeling of pieces $P_1, \ldots, P_{i-1}$. We label with colors from c_{i-1} to $c_i - 1 = c_{i-1} + k_i + 1$ all nodes in the clique and possibly some nodes in the stable set according to the rules of the previous section. In general, some nodes in the stable set remain unlabeled.

Not used colors from c_{i-1} to $c_i - 1$ will be inserted into a queue Q together with the information that they have been enqueued by P_i.

If some nodes in S_i remain uncolored, P_i is again queued in S together with the information of the number of its uncolored nodes u_i. The labeling of the partially labeled pieces will be completed by the last part of the algorithm. Only the crown and the first piece are immediately completely labeled.

The crown, if it is not the unique piece of G, is completely $L'(2,1)$-labeled while the first piece, independently from which piece it is, is completely $L(2,1)$-labeled since the nodes in its stable part are not extremes of any black edge and so repetitions of colors are possible. Notice that, if the unigraph is constituted by the only crown, it is the first (and unique) piece, and hence it is correctly $L(2,1)$-labeled.

Observe that a disconnected unigraph consists in a connected one and an isolated box. Hence, if the unigraph is not connected, we can assign the same color to all nodes of the isolated box and run the algorithm for $L(2,1)$-labeling the non-trivial connected component. For this reason, as input of the algorithm only connected unigraphs are considered.

Finally, we say that color k is *thrown out* if we decide not to use it; after k has been thrown out it is not available anymore. Procedure `Recognize-Pieces`(G, S, num) takes in input unigraph G, recognizes its num pieces P_i and put them in S.

The $L(2,1)$-labeling algorithm is the following:

ALGORITHM $L(2,1)$-`Label-Unigraphs`
INPUT:a connected unigraph G by means of its degree sequence $d_1^{m_1}, \ldots, d_r^{m_r}$
OUTPUT: an $L(2,1)$-labeling for G.
`Initialize-QueueColors` $Q = \emptyset$;
`Recognize-Pieces(G,S,num)`;
PHASE 1.
REPEAT
  `DequeuePiece` P_i from S;
  Step 1 // P_1 is completely $L(2,1)$-labeled;
  IF $i = 1$
   THEN completely $L(2,1)$-label P_1
      (details in Sections 5.1 and 5.2);
  ELSE
  Step 2
  IF P_i split component
    THEN Partially $L'(2,1)$-label P_i appropriately with new colors from c_{i-1} to $c_{i-1} + k_i + 1$
      (details in Section 4.2);
    FOR EACH unused color d between c_{i-1} and $c_{i-1} + k_i + 1$
      `EnqueueColor` (d, P_i) in Q;
      $c_i \leftarrow c_{i-1} + k_i + 2$;
    IF P_i is partially $L'(2,1)$-labeled and u_i among its nodes are not labeled
      THEN `EnqueuePiece`(P_i, u_i) in S;
  Step 3
  IF P_i crown
    $L'(2,1)$-label P_i appropriately with new colors starting from c_{i-1}
      (details in Section 4.1);
    FOR EACH unused color u in the $L'(2,1)$-labeling of the crown
    `EnqueueColor`(d, P_i) in Q;
UNTIL(i = num);
PHASE 2.
REPEAT
  `DequeuePiece`(P_i, u_i) from S;
   WHILE $(u_i > 0$ AND $Q \neq \emptyset)$ DO
    `DequeueColor`(d, P_j) from Q;
     IF $(j \leq i)$
     THEN throw d out;
    ELSE use d to $L'(2,1)$-label one uncolored node in P_i;
      decrease u_i by 1;
  IF $Q = \emptyset$
  THEN $L'(2,1)$-label the u_i uncolored nodes of P_i with m_i consecutive new colors from
     c_{i-1} to $c_{i-1} + m_i - 1$;
UNTIL $(S = \emptyset)$.

Theorem 6.1. *Algorithm* $L(2,1)$-`Label-Unigraphs` *correctly* $L(2,1)$-*labels a unigraph* G *in* $O(n)$ *time.*

Proof. The correctness of procedure `Recognize-Pieces` follows from [3]. We will prove that the labeling found by the algorithm is feasible. Indeed, nodes in V_K are labeled with colors at mutual distance at least two. Moreover, each node in V_S cannot be colored with a color at distance ≤ 1 to the colors of all its adjacent nodes (in V_K) in view of the following three facts:

1. Each piece P_i is feasibly labeled according to Sections 4 and 5;
2. The only $L(2,1)$-labeled piece is the first one, since its nodes in the stable part are not extreme of any black edge;

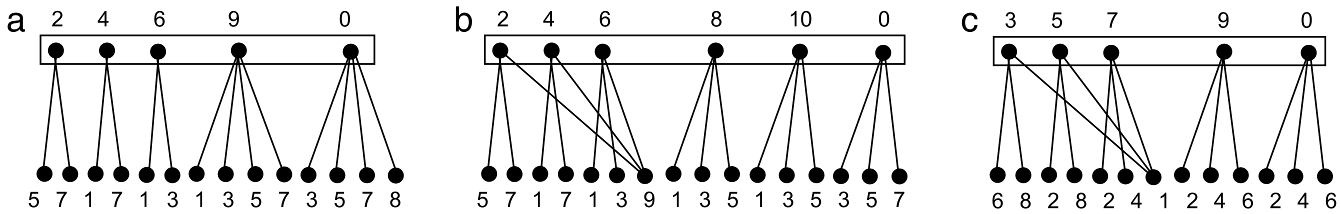

Fig. 11. Optimal $L(2, 1)$-labelings of: (a) $S_2(2, 3; 4, 2)$; (b) $S_3(2, 3; 3)$; (c) $S_3(2, 3; 2)$.

3. Each dequeued color d (enqueued by P_j) is used only for labeling nodes in the stable part of piece P_i with $i < j$, so that black edges cannot join the node labeled w with nodes labeled either $w + 1$ or $w - 1$.

In order to compute the time complexity, we have to add the contribution of the following four actions: the recognition procedure—requiring $O(n)$ time [3], the labeling of P_1, the partial labeling of each piece and the completion of the labeling. In order to label each piece P_i with n_i nodes we need $O(n_i)$ time. Each piece P_i is enqueued in S at most twice, once when it is recognized and possibly a second time if it is only partially labeled. It follows that the algorithm, without the recognition part, requires no more than $\sum_{i=1}^{t} O(n_i) = O(n)$ time; consequently, the whole algorithm needs $O(n)$ time. □

Theorem 6.2. *Algorithm $L(2, 1)$-Label-Unigraphs has a performance ratio of* $3/2$.

Proof. The nodes of a unigraph are partitioned into three classes, V_K, V_S and V_C.

Nodes of the clique induced by V_K must be labeled with colors at mutual distance at least two. Hence, $2|V_K| - 1$ colors are necessary in any labeling for these nodes, but only $|V_K|$ of them are used to label V_K. Due to the unigraph structure, the $V_K - 1$ remaining colors could be used for some nodes in V_S but not for the nodes in the crown, as each of them is connected to every node in V_K. For this reason, the nodes in V_C must be at distance of at least two from the colors used for V_K. Hence the color successive to the maximum used for the clique cannot be used for the crown, so one more color must be added.

Moreover, nodes in the crown induced by V_C must all be different from each other (except for the special case when the unigraph coincides with its crown). Let $|V_C| + \alpha$, where $0 \le \alpha \le |V_C|/2 - 1$, be the optimum number of colors necessary for labeling these nodes. Among the $|V_C| + \alpha$ colors, only $|V_C|$ are really used, while α colors could be used for other nodes in V_S.

For nodes in V_S, we have to distinguish whether they belong to P_1 or not, as only in the first case some colors can be repeated (cf. Section 5). Let us call β, $\beta \le |P_1 \cap S|$ the optimum number of colors necessary to label nodes of $P_1 \cap V_S$ and S' the set of nodes in S not belonging to P_1, i.e. $S' = S - \{P_1 \cap S\}$.

In the worst case, algorithm $L(2, 1)$-Label-Unigraphs is not able to use colors that remain unused after the coloring of V_K and V_C. So, the number of used colors is upper bounded by $2|V_K| - 1 + |V_C| + \alpha + 1 + \beta + |S'|$.

Let us now consider the optimum solution. We have to distinguish two cases according to the fact that the number of colors not used in $V_K \cup V_C$ is sufficient for labeling V_S or not:

- If $\beta + |S'| \le |V_K| + \alpha$, the number of colors used by the optimum solution is lower bounded simply by $2|V_K| - 1 + |V_C| + \alpha + 1$.
- If, on the contrary, $\beta + |S'| > |V_K| + \alpha$, we have to add $|S'| + \beta - |V_K| - \alpha$ colors in order to obtain a lower bound for the optimum solution of $2|V_K| - 1 + |V_C| + \alpha + 1 + (|S'| + \beta - |V_K| - \alpha) = |V_K| + |V_C| + |S'| + \beta$.

Now we compute the approximation ratio in the two cases, using as measure the ratio between the number of colors used by our algorithm and the number of colors used by the optimum solution, i.e. $\frac{\lambda+1}{\lambda^*+1}$. By exploiting that $\alpha \le |V_C|/2 - 1$ and hence $|V_C| \ge 2\alpha + 2 > 2\alpha$, that $\alpha \ge 0$ and the relationships between $\beta + |S'|$ and $|V_K| + \alpha$ we have:

- If $\beta + |S'| \le |V_K| + \alpha$ then
$$\frac{\lambda + 1}{\lambda^* + 1} \le \frac{2|V_K| + |V_C| + |S'| + \alpha + \beta}{2|V_K| + |V_C| + \alpha} \le 1 + \frac{|S'| + \beta}{2|V_K| + |V_C| + \alpha} < \frac{3}{2}.$$
- If $\beta + |S'| > |V_K| + \alpha$ then
$$\frac{\lambda + 1}{\lambda^* + 1} \le \frac{2|V_K| + |V_C| + |S'| + \alpha + \beta}{|V_K| + |V_C| + |S'| + \beta} \le 1 + \frac{|V_K| + \alpha}{|V_K| + |V_C| + |S'| + \beta} < \frac{3}{2}. \quad \square$$

Observe that when the unigraph is constituted only by its crown our algorithm provides the optimum labeling, according to the theorems of Section 5.1. Furthermore, it is not difficult to see that, if the input unigraph is either a threshold or a matrogenic graph then our algorithm behaves exactly in the same way as the known algorithms specifically designed for these classes of graphs and hence, in the case of threshold graphs, it provides the optimum labeling.

7. Concluding remarks and open problems

In this paper we have answered the open problem left in [7] to present an $L(2,1)$-labeling for unigraphs. In Theorem 6.2 we prove that its approximation ratio is 3/2, nevertheless a large number of examples show that our algorithm discards very few colors, thus achieving a number of used colors which is very close to the optimal value, so we suspect that its performance is even better.

We would like to conclude this paper with two considerations concerning the number of colors used by our algorithm in comparison to the minimum value λ.

First, observe that the number of colors used by any optimal $L(2,1)$-labeling must respect the following facts:

1. the nodes in V_K must be labeled with colors at a mutual distance of at most two;
2. the nodes in V_C must all be different from each other and at distance of at least two from the colors used for V_K (except for the special case when the unigraph coincides with its crown);
3. the nodes in $V_S \cap P_i$, $i > 1$ must all use different colors.

The foregoing facts imply that our algorithm may use a larger number of colors than is strictly necessary in the worst case; this number may be equal to the number of discarded colors.

Second, it is important to note that the use of optimal labelings for all pieces allows us to get a minimal (not a minimum) labeling of the whole unigraph so we cannot guarantee it is optimal. The reason for this is that a different arrangement of colors of the nodes in the clique may lead to less new colors being used in the second REPEAT cycle. From this consideration we conjecture that the $L(2,1)$-labeling problem may be NP-hard for unigraphs, and we leave the proof of this as an open problem.

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