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Quantum Relativity of Subsystems

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One of the most basic notions in physics is the partitioning of a system into subsystems and the study of correlations among its parts. In this Letter, we explore this notion in the context of quantum reference frame (QRF) covariance, in which this partitioning is subject to a symmetry constraint. We demonstrate that different reference frame perspectives induce different sets of subsystem observable algebras, which leads to a gauge-invariant, frame-dependent notion of subsystems and entanglement. We further demonstrate that subalgebras which commute *before* imposing the symmetry constraint can translate into noncommuting algebras in a given QRF perspective *after* symmetry imposition. Such a QRF perspective does not inherit the distinction between subsystems in terms of the corresponding tensor factorizability of the kinematical Hilbert space and observable algebra. Since the condition for this to occur is contingent on the choice of QRF, the notion of subsystem locality is frame dependent.

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Introduction.—Operationally, subsystems are distinguished by physically accessible measurements. Suppose that one can measure the set of observables described by the minimal algebra $\mathcal{A}$ containing a collection $\{\mathcal{A}_i\}_{i=1}^n$ of commuting subalgebras, $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$. This implies that observables in $\mathcal{A}_i$ and $\mathcal{A}_j$ are simultaneously measurable, as expected of observables associated with distinct subsystems. When these algebras admit Hilbert space representations $\mathcal{A} \simeq \mathcal{B}(\mathcal{H})$ and $\mathcal{A}_i \simeq \mathcal{B}(\mathcal{H}_i)$, the commuting subalgebra structure can induce a tensor product structure on the composite Hilbert space $\mathcal{H} \simeq \bigotimes_{i=1}^n \mathcal{H}_i$, where $\mathcal{H}_i$ is associated with the ith subsystem. Given that this tensor product structure is induced by the distinguished sets of observables $\mathcal{A}_i$, entanglement and the notion of subsystem itself is defined relative to these distinguished sets [1–5].

The physically accessible observables and states of a system are dictated by the symmetries of the situation under consideration [1–6]. For example, in a gauge theory physically accessible states and observables are invariant with respect to arbitrary gauge transformations [7]. In the canonical approach, going back to Dirac [8], this invariance requirement is implemented by introducing a kinematical Hilbert space $\mathcal{H}_{\rm kin} \simeq \bigotimes_{i=1}^n \mathcal{H}_i$ that may come equipped with a kinematical tensor product structure. Supposing that $\hat{C} \in \mathcal{L}(\mathcal{H}_{\rm kin})$ is a generator of a gauge symmetry, the physical, i.e., invariant states satisfy the constraint equation $\hat{C}|\psi_{\rm phys}\rangle = 0$. This is necessary, for example, for the counting of independent gauge-invariant degrees of freedom (such as photon polarizations). Solutions to this equation may lie outside of $\mathcal{H}_{\rm kin}$ as they may not be normalizable with respect to its inner product. To overcome this issue, one introduces a new physical inner product that is used to complete the solution space of the constraint equation to form the physical Hilbert space $\mathcal{H}_{\rm phys}$ (see, e.g., Refs. [9–14]). Physical observables are elements of the physical algebra $\mathcal{A}_{\rm phys} \simeq \mathcal{B}(\mathcal{H}_{\rm phys})$ known as Dirac observables, and commute with the constraint on physical states, $[\mathcal{A}_{\rm phys}, \hat{C}]|\psi_{\rm phys}\rangle = 0$. This ensures that $\mathcal{A}_{\rm phys}$ is invariant under gauge transformations generated by $\hat{C}$, which is necessary for the gauge invariance of physical expectation values. This constraint-based approach also applies to operational scenarios without bona fide gauge symmetry, where these constraints correspond to an agent using an internal quantum system as reference frame instead of an external classical one [15].

It is important to note that the physical Hilbert space $\mathcal{H}_{\rm phys}$ does not inherit the kinematical tensor product structure $\mathcal{H}_{\rm kin} \simeq \bigotimes_{i=1}^n \mathcal{H}_i$ and associated notion of subsystem. Instead, a notion of subsystem must be induced by commuting subalgebras of $\mathcal{A}_{\rm phys}$, and, in general, will be nonlocal with respect to the kinematical tensor product structure [16].

In this Letter, we consider composite systems that are invariant under a gauge transformation admitting a tensor product representation across $\mathcal{H}_{\rm kin}$: that is, gauge transformations that act locally on the kinematical factors $\mathcal{H}_i$. We take one of these kinematical subsystems to serve as a reference frame from which the remaining subsystems are described. To do so, we make use of recent results from the theory of quantum reference frames (QRFs) to transform from the quantum theory on the physical Hilbert space $\mathcal{H}_{\rm phys}$, which encodes all QRF choices, to an isomorphic theory from the perspective of a subsystem serving as a reference frame [15,17–26] (see also Refs. [27–31] for a related formulation without constraints). We show that subsystems encoded by a perspective-dependent tensor product structure induce a partitioning of the physical Hilbert through the construction of sets of commuting subalgebras of so-called relational Dirac observables [9,10,15,17–23,32–39] associated with different reference frame perspectives. In general, different reference frames induce different partitions of the physical Hilbert space and invariant observable algebra, and thus the associated notion of subsystems is reference frame dependent. We identify the necessary and sufficient condition for when the physical Hilbert space inherits (some of) the kinematical subsystem partitioning in terms of the spectrum of the relevant constraint, and this condition is contingent on the choice of QRF. This allows us to develop a description of subsystems and entanglement in terms of physical Hilbert space structures that is manifestly gauge invariant and reference frame dependent.

From physical states to QRF perspectives.—Consider a kinematical Hilbert space that partitions into three factors, $\mathcal{H}_{\rm kin} = \mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B} \otimes \mathcal{H}_{\rm C}$, and a single constraint of the form $\hat{C} = \hat{C}_{\rm A} + \hat{C}_{\rm B} + \hat{C}_{\rm C}$, where each $\hat{C}_i$ is self-adjoint and acts only on $\mathcal{H}_i$. For simplicity, we assume that each $\hat{C}_i$ is nondegenerate, and treat degeneracies in the Supplemental Material [40]. Each $\hat{C}_i$ thus generates a unitary representation of either the translation group $\mathbb{R}$ or U(1) on $\mathcal{H}_i$ [44]. Consequently, $\hat{C}$ generates a one-parameter unitary representation of either $\mathbb{R}$ or U(1) on $\mathcal{H}_{\rm kin}$, depending on the combination of the $\hat{C}_i$ [23]. The constraint $\hat{C}$ may be a Hamiltonian constraint as in gravitational systems, generating temporal reparametrization invariance and dynamics [9,10,19–23,32–39,45–49], or it may be the generator of a spatial symmetry, such as spatial translation invariance [17,18]. For simplicity, we do not consider interactions between the subsystems A, B, and C in the constraint.

As noted above, physical states satisfy $\hat{C}|\psi_{\rm phys}\rangle = 0$, and together they constitute $\mathcal{H}_{\rm phys}$. This induces a redundancy with respect to $\mathcal{H}_{\rm kin}$, which can be removed by identifying the choice of redundant subsystem with the choice of QRF relative to which the other systems will be described. A physical state encodes each choice of QRF, therefore assuming the role of a *perspective-neutral* state, linking all the different perspectives [17–23]. Letting $i, j, k \in \{{\rm A}, {\rm B}, {\rm C}\}$, we denote the chosen reference system by k and the remaining kinematical factors by i and j. We then define

$$\sigma_{ij|k} \coloneqq {\rm spec}(\hat{C}_i + \hat{C}_j) \cap {\rm spec}(-\hat{C}_k), \tag{1}$$

allowing us to write an arbitrary physical state as

$$|\psi_{\rm phys}\rangle = \sum\!\!\!\!\!\!\!\!\int_{c_i + c_j \in \sigma_{ij|k}} \psi(c_i, c_j) |{-c_i - c_j}\rangle_k \otimes |c_i\rangle_i \otimes |c_j\rangle_j, \tag{2}$$

for some $\psi(c_i, c_j)$, where $|c_i\rangle_i$ is the eigenstate of $\hat{C}_i$ with eigenvalue c_i (likewise for j and k). Thus, if $\psi(c_i, c_j)$ has nontrivial support over various values of the eigenvalues c_i, c_j, then k is entangled with i, j relative to the *kinematical* tensor product structure. However, due to the redundancy, this entanglement is not gauge invariant [21].

We can then describe physics from k's perspective via either of two paths [21–23]: a "relational Schrödinger picture" (known in the context of Hamiltonian constraints as the Page-Wootters formalism [46,47]) and a "relational Heisenberg picture" [17–20]. In both cases, observables on i, j are described relative to outcomes of an observable on k, namely an element of a positive operator-valued measure (POVM). The elements of this POVM can be constructed via projectors onto *orientation* states of the reference frame:

$$|g\rangle_k \coloneqq \sum\!\!\!\!\!\!\!\int_{c_k} e^{i[\theta(c_k) - c_k g]} |c_k\rangle_k, \tag{3}$$

where $\theta(c_k)$ are arbitrary phases and g is a coordinate on G_k, the group generated by $\hat{C}_k$. These orientation states transform covariantly under G_k, $|g'\rangle = e^{-i(g'-g)\hat{C}_k}|g\rangle$ [21–23,48–51]. The QRF perspective corresponding to k is obtained by conditioning physical states on k being in the orientation g, thus fixing the gauge, leading to a reduced physical Hilbert space $\mathcal{H}_{ij|k}$. In the relational Schrödinger picture, this proceeds via the reduction map $\mathcal{R}_k^{(S)}(g)\colon \mathcal{H}_{\rm phys} \to \mathcal{H}_{ij|k}$ given by $\mathcal{R}_k^{(S)}(g) \coloneqq \langle g|_k \otimes \mathbf{1}_{ij}$ (with its domain restricted to $\mathcal{H}_{\rm phys}$). This leads to the orientation-dependent relational Schrödinger state $|\psi_{ij|k}(g)\rangle \coloneqq \mathcal{R}_k^{(S)}(g)|\psi_{\rm phys}\rangle \in \mathcal{H}_{ij|k}$ and the decomposition $|\psi_{\rm phys}\rangle = \mu \int_{G_k} dg\, |g\rangle_k \otimes |\psi_{ij|k}(g)\rangle$ exhibiting the kinematical entanglement between k and i, j, where μ is a normalization factor.

On the other hand, in the relational Heisenberg picture one first transforms $|\psi_{\rm phys}\rangle$ to shift the nonredundant information into the i, j partition with a *frame disentangler* ("trivialization") that is a shift conditional on frame k, $\mathcal{T}_{k,\varepsilon} \coloneqq \mu \int_{G_k} dg\, e^{i\varepsilon_k g} |g\rangle\langle g|_k \otimes e^{i(\hat{C}_i + \hat{C}_j)g}$. This factors out the QRF, removing the kinematical entanglement between k and i, j (see Supplemental Material [40]):

$$\mathcal{T}_{k,\varepsilon}|\psi_{\rm phys}\rangle = |\varepsilon_k\rangle_k \otimes |\psi_{ij|k}\rangle, \tag{4}$$

where $\otimes$ denotes the kinematical tensor product between k and i, j, $|\psi_{ij|k}\rangle = e^{i(\hat{C}_i + \hat{C}_j)g}|\psi_{ij|k}(g)\rangle \in \mathcal{H}_{ij|k}$ is the corresponding "relational Heisenberg state," and $|\varepsilon_k\rangle_k = \mu \int_{G_k} dg\, e^{i\varepsilon_k g}|g\rangle_k$. Here, $-\varepsilon_k$ must be a fixed, but arbitrary element of $\sigma_{ij|k}$ in which case Eq. (4) satisfies the transformed constraint $\mathcal{T}_{k,\varepsilon}\hat{C}\mathcal{T}_{k,\varepsilon}^{-1} \sim (\hat{C}_k - \varepsilon_k \mathbf{1})$, which fixes the now-redundant QRF k and preserves gauge invariance. One then conditions on the reference frame being in a given orientation of k, leading to the relational Heisenberg picture reduction map $\mathcal{R}_k^{(H)}: \mathcal{H}_{\rm phys} \to \mathcal{H}_{ij|k}$ given by

$$\mathcal{R}_k^{(H)} \coloneqq \mathcal{R}_k^{(S)}(g)\mathcal{N}(g,\varepsilon_k)\mathcal{T}_{k,\varepsilon}, \tag{5}$$

where $\mathcal{N}(g,\varepsilon_k)$ is a normalization factor such that $\mathcal{R}_k^{(H)}|\psi_{\rm phys}\rangle = |\psi_{ij|k}\rangle$, which is (weakly) independent of g and ε, and we therefore do not include these labels in $\mathcal{R}_k^{(H)}$. This reduction is unitarily equivalent to acting with $\mathcal{R}_k^{(S)}(g)$ on *physical states* [21–23]. We will use the relational Heisenberg picture in what follows, denoting the reduction map by $\mathcal{R}_k \equiv \mathcal{R}_k^{(H)}$ for simplicity.

The reduced physical Hilbert space $\mathcal{H}_{ij|k}$ and observables on it encode the physics of i, j as described from the internal perspective of QRF k. When $G_k = \mathrm{U}(1)$, $\mathcal{H}_{ij|k}$ need not be a subspace of the kinematical factors $\mathcal{H}_i \otimes \mathcal{H}_j$ [23]. Furthermore, thanks to the redundancy in describing $\mathcal{H}_{\rm phys}$, the reduction is invertible on physical states (but not on $\mathcal{H}_{\rm kin}$), so that $\mathcal{H}_{ij|k}$ is isometric to $\mathcal{H}_{\rm phys}$ [17–23]. Hence, the algebraic properties of observables are preserved. This permits us to change QRF: the change from k to i takes the compositional form of a "quantum coordinate transformation," $\Lambda_{k\to i} \coloneqq \mathcal{R}_i \circ \mathcal{R}_k^{-1}$, transforming both states and observables via the structure on $\mathcal{H}_{\rm phys}$ which is *a priori* neutral with respect to QRF perspectives [17–23]; see Fig. 1. The same physical situation, encoded in the perspective-neutral state $|\psi_{\rm phys}\rangle$, is thus described from different internal QRF perspectives. We shall now exploit this gauge-invariant, perspective-neutral framework to explain dependence of, first, subsystem locality and correlations and, second, tensor factorizability on the choice of QRF.

Frame-dependent subsystems and correlations.—The QRF dependence of correlations has been observed in Ref. [27], with the conclusion that superposition in one frame manifests as entanglement in another frame. Later, the formalism for changing QRFs introduced in Ref. [27] was shown to be equivalent to the frame-change map in Fig. 1 [17], and the frame dependence of correlations was studied in a variety of contexts [15,17,21,24,29]. Here, we use the perspective-neutral architecture to describe the QRF relativity of subsystems and correlations.

The Heisenberg picture reduction illustrates why a fixed perspective-neutral state $|\psi_{\rm phys}\rangle$ generally leads to different properties, such as correlations, in A's and B's perspective (see Fig. 1): when going to A's perspective, the now-redundant A becomes kinematically disentangled from the nonredundant B, C, while B becomes disentangled from A,

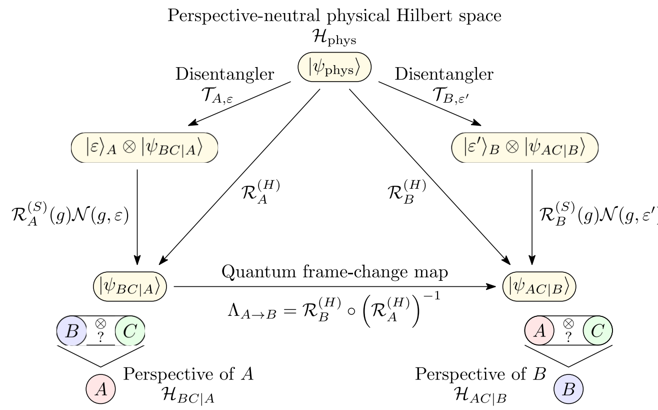

FIG. 1. The change from perspective A to perspective B takes the compositional form of a "quantum coordinate transformation," $\Lambda_{A\to B} \coloneqq \mathcal{R}_B^{(H)} \circ (\mathcal{R}_A^{(H)})^{-1}$ [17–23]. This induces a transformation on the algebra observables from the perspective of A to the perspective of B, namely, $\Lambda_{A\to B}\mathcal{A}_{BC|A}\Lambda_{A\to B}^{-1} \subseteq \mathcal{A}_{AC|B}$, where $\mathcal{A}_{ij|k} \simeq \mathcal{B}(\mathcal{H}_{ij|k})$.

C when proceeding to B's perspective. Which kinematical tensor factor in $|\psi_{\rm phys}\rangle$ is chosen as redundant and which as independent changes. In other words, the nonredundant (physical) information in $|\psi_{\rm phys}\rangle$ is shifted among different kinematical tensor factors when going to different QRF perspectives. Indeed, the wave function $\psi(c_i, c_j)$ will look different for different choices of i, j. As we shall see later, $\mathcal{H}_{ij|k}$ may not even be factorizable across i and j.

However, when there *is* a physical tensor product structure this generically leads to different correlations in different frames. This can be understood by examining the observables that probe the respective tensor factorizations. Suppose the reduced physical Hilbert space $\mathcal{H}_{BC|A} \simeq \mathcal{H}_{B|A} \otimes \mathcal{H}_{C|A}$ admits a tensor factorization across the subsystems B, C from A's perspective, induced from the original tensor product structure of $\mathcal{H}_{\rm kin}$, and similarly that $\mathcal{H}_{AC|B} \simeq \mathcal{H}_{A|B} \otimes \mathcal{H}_{C|B}$ so that we can consider entanglement across these subsystems. The subsystem physical Hilbert spaces $\mathcal{H}_{i|k}$ may be different from their kinematical counterparts $\mathcal{H}_i$ of which they may [21,22] or may not be [23] subspaces. We consider the algebra generated by local subsystem observables on $\mathcal{H}_{ij|k}$, namely, $\mathcal{A}_{ij|k} \coloneqq \mathcal{A}_{i|k} \otimes \mathcal{A}_{j|k}$, where $\mathcal{A}_{i|k} \coloneqq \mathcal{B}(\mathcal{H}_{i|k})$ (hence a type I factor), so that $[\mathcal{A}_{i|k} \otimes \mathbf{1}_{j|k}, \mathbf{1}_{i|k} \otimes \mathcal{A}_{j|k}] = 0$. Since $\mathcal{A}_{ij|k}$ is dense in $\mathcal{B}(\mathcal{H}_{ij|k})$ with respect to the strong operator topology [52,53], we can treat $\mathcal{A}_{ij|k}$ for all practical purposes as the observable algebra of the tensor product space $\mathcal{H}_{ij|k}$ (in finite dimensions the algebras are isomorphic).

Using the fact that $\mathcal{R}_k$ is an invertible isometry, these observable algebras can be embedded into the algebra of relational Dirac observables $\mathcal{A}_{\rm phys} \coloneqq \mathcal{B}(\mathcal{H}_{\rm phys})$ as $\mathcal{A}_{\rm phys}^{i|k} \coloneqq \mathcal{R}_k^{-1}(\mathcal{A}_{i|k} \otimes \mathbf{1}_j)\mathcal{R}_k \subset \mathcal{A}_{\rm phys}$. This yields $[\mathcal{A}_{\rm phys}^{i|k}, \mathcal{A}_{\rm phys}^{j|k}] = 0$. Since the properties of $\mathcal{R}_k$ thus imply that $\mathcal{A}_{\rm phys}^{i|k}$ and $\mathcal{A}_{\rm phys}^{j|k}$ are commuting type I factors, and that moreover $\mathcal{R}_k^{-1}\mathcal{A}_{ij|k}\mathcal{R}_k = \mathcal{A}^{i|k}_{\text{phys}} \cdot \mathcal{A}^{j|k}_{\text{phys}}$ is dense in $\mathcal{A}_{\text{phys}}$, it follows [52] that these factors induce a *physical* tensor product on $\mathcal{H}_{\text{phys}} \simeq \mathcal{H}^{i|k}_{\text{phys}} \otimes \mathcal{H}^{j|k}_{\text{phys}}$. The following theorem shows this to be QRF dependent in general (see Supplemental Material [40], also for the degenerate case).

Theorem 1.—The algebra $\mathcal{A}^{C|A}_{\text{phys}}$ of relational observables of C relative to A is distinct from the algebra $\mathcal{A}^{C|B}_{\text{phys}}$ of relational observables of C relative to B, $\mathcal{A}^{C|A}_{\text{phys}} \neq \mathcal{A}^{C|B}_{\text{phys}}$.

However, note that $\mathcal{A}^{C|A}_{\text{phys}}$ is isomorphic to $\mathcal{A}^{C|B}_{\text{phys}}$ when $\mathcal{H}^{C|A}_{\text{phys}} \simeq \mathcal{H}^{C|B}_{\text{phys}}$. The two tensor factorizations $\mathcal{H}_{\text{phys}} \simeq \mathcal{H}^{B|A}_{\text{phys}} \otimes \mathcal{H}^{C|A}_{\text{phys}}$ and $\mathcal{H}_{\text{phys}} \simeq \mathcal{H}^{A|B}_{\text{phys}} \otimes \mathcal{H}^{C|B}_{\text{phys}}$ therefore constitute *different* physical tensor factorizations. It is therefore clear that a given physical state $|\psi_{\text{phys}}\rangle \in \mathcal{H}_{\text{phys}}$ exhibits different correlations in the two different factorizations.

Put differently, transforming the algebra $\mathcal{A}_{C|A}$ from A's to B's perspective does *not* yield C's algebra relative to B: $\Lambda_{A\to B}(\mathbf{1}_B \otimes \mathcal{A}_{C|A})\Lambda^{-1}_{A\to B} \neq \mathbf{1}_A \otimes \mathcal{A}_{C|B}$. The tensor factorization between B, C relative to A thus does not map under QRF transformations into the tensor factorization between A, C relative to B. Instead, since $\Lambda_{A\to B}$ is an invertible isometry too, it maps into a different tensor factorization relative to B, namely one between combinations of A and C degrees of freedom. Consequently, the notion of subsystem locality is QRF dependent, as are the correlations inherited from a given physical state. We illustrate this observation in an example in the Supplemental Material [40].

When are kinematical subsystems physical?—As noted above, it is in general not the case that the reduced physical Hilbert space and the observable algebra can be factorized into the same subsystems that constitute tensor factors of the kinematical Hilbert space. In other words, imposing a given symmetry can remove the distinction between what might have been expected to be physical subsystems.

To see this, note that one can also obtain the reduced physical Hilbert space from k's perspective *directly* from the kinematical i, j tensor factors via the (possibly improper) projector $\Pi_{\sigma_{ij|k}}: \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_{ij|k}$ given by [21–23]

$$\Pi_{\sigma_{ij|k}} := \sum\!\!\!\!\!\!\!\int_{c_i,c_j|c_i+c_j\in\sigma_{ij|k}} |c_i\rangle\langle c_i|_i \otimes |c_j\rangle\langle c_j|_j. \quad (6)$$

Observe that $\sigma_{ij|k}$ is symmetric in i and j but not in i and k. Furthermore, defining $\sigma_i := \text{spec}(\hat{C}_i)$, note that $\sigma_{ij|k} := (\hat{C}_i + \hat{C}_j)$ whenever $\sigma_k = \mathbb{R}$. The projector $\Pi_{\sigma_{ij|k}}$ is improper if $\sigma_{ij|k}$ is discrete, while at least one of $\hat{C}_i, \hat{C}_j$ has continuous spectrum [23]. The reduced physical Hilbert space $\mathcal{H}_{ij|k}$ factorizes into i and j subsystems $\mathcal{H}_{i|k}$ and $\mathcal{H}_{j|k}$ if and only if $\Pi_{\sigma_{ij|k}}$ does as well. This is only the case if (see Supplemental Material [40] for proof)

$$\sigma_{ij|k} = M(\sigma_{i|jk}, \sigma_{j|ik}), \quad (7)$$

where $\sigma_{i|jk} := \sigma_i \cap \text{spec}(-\hat{C}_j - \hat{C}_k)$ is the subset of σ_i compatible with the constraint equation, and where $M(\cdot,\cdot)$ denotes Minkowski addition, defined by $M(X,Y) := \{x+y | x \in X, y \in Y\}$. When this is satisfied, the projector defined in Eq. (6) becomes

$$\Pi_{\sigma_{ij|k}} = \left(\sum\!\!\!\!\!\!\!\int_{c_i\in\sigma_{i|jk}} |c_i\rangle\langle c_i|_i\right) \otimes \left(\sum\!\!\!\!\!\!\!\int_{c_j\in\sigma_{j|ik}} |c_j\rangle\langle c_j|_j\right), \quad (8)$$

and thus $\Pi_{\sigma_{ij|k}}(\mathcal{H}_i \otimes \mathcal{H}_j) = \mathcal{H}_{i|k} \otimes \mathcal{H}_{j|k}$. Here, $\mathcal{H}_{i|k} \subseteq \mathcal{H}_i$, unless $\hat{C}_k$ has discrete and $\hat{C}_i$ continuous spectrum [21–23] (likewise for $\mathcal{H}_{j|k}$). Note that when $\sigma_{i|jk} = \sigma_i$, then $\mathcal{H}_{i|k} = \mathcal{H}_i$. This holds for both i and j if $\sigma_k = \mathbb{R}$. We give an example of nonfactorizability of the physical Hilbert space in the Supplemental Material [40].

To understand this more explicitly, let $\hat{A}_i \otimes \hat{A}_j$ be a kinematical basis element of $\mathcal{B}(\mathcal{H}_i) \otimes \mathcal{B}(\mathcal{H}_j)$. Condition (7) must be satisfied in order for the physical representation of this operator from k's perspective to factorize across i and j. Otherwise, the degrees of freedom of i and j become combined indivisibly into $\Pi_{\sigma_{ij|k}}(\hat{A}_i \otimes \hat{A}_j)$. This includes the case when $\hat{A}_i \otimes \hat{A}_j$ is diagonal in the eigenbases of $\hat{C}_i$ and $\hat{C}_j$, and thus commutes with $\Pi_{\sigma_{ij|k}}$ (see Supplemental Material [40]). Specifically, kinematical i subsystem observables of the form $\hat{A}_i \otimes \mathbf{1}_j$ will *not* translate into a product form on $\mathcal{H}_{ij|k}$. In fact, the following theorem holds (see Supplemental Material [40]).

Theorem 2.—There exist $\hat{A}_i \otimes \mathbf{1}_j$ and $\mathbf{1}_i \otimes \hat{A}_j$ in $\mathcal{B}(\mathcal{H}_i) \otimes \mathcal{B}(\mathcal{H}_j)$ whose images under $\Pi_{\sigma_{ij|k}}$ in $\mathcal{B}(\mathcal{H}_{ij|k})$ do *not* commute unless condition (7) is met.

By linearity, these conclusions extend to an arbitrary element of $\mathcal{B}(\mathcal{H}_i) \otimes \mathcal{B}(\mathcal{H}_j)$. Consequently, when condition (7) is not satisfied, the algebra of observables loses its distinction between parties i and j from the perspective of k's reference frame.

Frame-dependent factorizability.—When Eq. (7) holds in one frame but not in another, the preservation of the kinematical factorization on $\mathcal{H}_{ij|k}$ likewise depends on the frame, as we now illustrate. For concreteness, consider any constraint such that $\sigma_{\text{A}} = \mathbb{R}_+$, $\sigma_{\text{B}} = \mathbb{R}_+$, and $\sigma_{\text{C}} = \mathbb{R}$. Considering first C's perspective, we have that $\sigma_{\text{AB|C}} = \mathbb{R}_+ = M(\mathbb{R}_+, \mathbb{R}_+)$, i.e., condition (7) is satisfied, with $\sigma_{\text{A|BC}} = \sigma_{\text{A}}$ (likewise $\sigma_{\text{B|AC}}$), and $\Pi_{\sigma_{\text{AB|C}}}(\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}) = \mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$. From B's perspective, on the other hand, one can prove by contradiction that condition (7) is not satisfied (see Supplemental Material [40]), and therefore the reduced physical Hilbert space does not factor into A and C parts. This latter fact does not, however, imply that there exists no tensor factorization of $\mathcal{H}_{AC|B}$. Indeed, one can use the tensor factorization of $\mathcal{H}_{AB|C}$ to construct one on $\mathcal{H}_{AC|B}$ via the frame-change map $\Lambda_{C\to B}$, as in our discussion of frame-dependent correlations above. In this case the algebra of local observables from C's perspective, namely, $\mathcal{A}_{A|C} \otimes \mathcal{A}_{B|C}$, maps to a tensor factorization between *combinations* of A and C degrees of freedom, and therefore does not correspond to a partitioning into subsystems A and C.

As a particular example, consider a reparametrization-invariant system consisting of two free (nonrelativistic) unit-mass particles, A and B, and an ideal clock C (i.e., one whose Hamiltonian is equivalent to a momentum operator [19,24,48,54–56]). This corresponds to the constraint

$$\hat{C} = \frac{\hat{p}_A^2}{2} + \frac{\hat{p}_B^2}{2} + \hat{p}_C. \tag{9}$$

Thus $\sigma_\mathrm{A} = \mathbb{R}_+$, $\sigma_\mathrm{B} = \mathbb{R}_+$, as above, and therefore the distinction between kinematical subsystems survives on $\mathcal{H}_{AB|C}$, but not on $\mathcal{H}^{d_B}_{AC|B}$, where $d_B = \pm 1$ labels the degeneracy of $\hat{p}_B^2$ (see Supplemental Material [40] for a discussion of degeneracies). In the Supplemental Material [40], we illustrate this by examining how the kinematical canonical pairs $(\hat{x}_i, \hat{p}_i)$ on $\mathcal{H}_i$ and $(\hat{x}_j, \hat{p}_j)$ on $\mathcal{H}_j$ appear from B's and C's perspectives. We show that their commutation relations are preserved on $\mathcal{H}_{AB|C}$, but not on $\mathcal{H}^{d_B}_{AC|B}$, where they yield mutually noncommuting canonical pairs, in line with Theorem 2, explaining the absence of a tensor factorization across A and C relative to B. The same conclusion holds for the corresponding relational observables on $\mathcal{H}_\mathrm{phys}$. This further highlights the distinction between local observables on $\mathcal{H}_\mathrm{kin}$ and the observables in a given physical reference frame.

Another example of the above class of constraints is obtained by replacing systems A and B in Eq. (9) with the nondegenerate Hamiltonian $\hat{H} = \hat{p}^2/2m + a_1 e^{a_2 \hat{q}}$ with a_1, $a_2 > 0$. This observation can also be easily extended to the case when A, B are harmonic oscillators.

Discussion and conclusions.—We have established a gauge-invariant and quantum frame-dependent notion of subsystems, locality, and correlations using relational observables. We have exploited the perspective-neutral approach to QRF covariance [15,17–26], showing algebraically how different QRF choices necessarily induce distinct tensor factorizations of the physical Hilbert space when the latter admits such structures. Further, we have identified the necessary and sufficient condition for QRF perspectives to inherit the kinematical partitioning of subsystems. Specifically, we have illustrated that the kinematical definition of subsystems may survive in some QRF perspectives, but dissolve in others.

The ensuing QRF dependence of subsystems and entanglement is a particular realization of the proposal for an observer-dependent notion of generalized entanglement put forward in Refs. [2,3,5] in terms of relational observables and QRFs. It is also related to the generic feature of the dependence of entanglement on classical coordinate choices [57]. However, here, as in previous work on QRFs [15,17,21,24,27,29], specific choices of coordinates are associated to the internal perspectives of different systems to form quantum reference systems and these may be in superpositions of "orientations." This provides a physical interpretation to the coordinate choices, and in turn the *quantum* relativity of subsystems and entanglement.

Note that the notion of QRF here is physically distinct from one sometimes used in the context of quantum information theory [6,58–62] (see, e.g., the discussion in Ref. [15]), where entanglement must be operationally defined relative to observables that are independent of the choice of an external frame not shared by two parties, for example, by appending an ancilla system. This approach also resonates with the proposal in Refs. [2,3,5], but does not involve the adoption of an internal perspective relative to a subsystem through the reduction maps employed here. In particular, the aim in that context is to define an external-frame-independent notion of entanglement, in contrast with our investigation of an internal-frame-dependent entanglement.

In the relativistic case, the entanglement between spin and momentum degrees of freedom for relativistic particles [63,64], as well as the momentum mode decomposition in quantum field theory [65], leading to the Unruh effect [66], Hawking radiation [67], and particle creation due to the expansion of the Universe [68], are also dependent on the choice of spacetime frame. In contrast to QRFs, these coordinate frames are not associated to dynamically evolving quantum systems, but are idealized noninteracting classical entities external to the physics being considered.

The frame dependence of factorizability demonstrated in this Letter implies that frameworks for general physical theories which take system composition as a primitive concept [69–74] are not currently able to describe fully general physical scenarios with multiple frames.

Finally, it will be fruitful to connect our observations with the currently widely explored notions of local subsystems and entanglement in gauge theories and gravity [75–86]. For example, defining local subsystems in gravity nonperturbatively in terms of commuting subalgebras of relational observables can complement the perturbative investigation of subsystems in terms of dressed observables in Refs. [76–78]. Conversely, the possible nonfactorizability of the physical Hilbert space observed here calls for a revision of the notion of subsystems. This question seems to be related to the construction of entangling products using edge modes and extended Hilbert spaces in gauge theories [80,86–90].

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P. A. H. proved theorems 1 and 2. M. P. E. L. established the condition relating kinematical and physical factorization. All authors contributed actively through discussions and revisions of technical aspects, as well as in the preparation of the manuscript.

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