Des. Codes Cryptogr. (2010) 55:37–43
DOI 10.1007/s10623-009-9328-5

The group fixing a completely regular line–oval

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Received: 19 May 2009 / Revised: 14 September 2009 / Accepted: 14 September 2009 /
Published online: 27 September 2009

Abstract We prove that the action of the full collineation group of a symplectic translation plane of even order on the set of completely regular line–ovals is transitive. This provides us with a complete description of the group of collineations fixing a completely regular line–oval.

Keywords Line–oval · Translation plane · Symplectic spread

Mathematics Subject Classification (2000) 51A35 · 51A40 · 51E20; Secondary 51A50

1 Introduction

Symplectic translation planes of even order are interesting geometrical and combinatorial objects, because of their close relation with non-linear codes [1–4]. Recently, we gave a necessary and sufficient condition for a finite translation plane of even order to be symplectic in terms of the existence of *completely regular line–ovals* (see [5,6] and the next section). A line–conic in a desarguesian affine plane is the basic, and unique, example of a completely regular line–oval (see [7]). Line–conics are equivalent to one another with respect to the full collineation group of the plane. Therefore it is natural to ask whether such a result holds for completely regular line–ovals in *any* symplectic translation plane.

In this paper we investigate the action of the full collineation group of the symplectic translation plane on the set of all completely regular line–ovals. The main result is that such an action is transitive. Also, this result provides us with a complete description of the group of all collineations fixing a completely regular line–oval: it is isomorphic to the group fixing

Communicated by Guglielmo Lunardon.

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the symplectic spread, modulo K^*, the *kernel homology group*. As a consequence we prove that the number of completely regular line–ovals is $|T||K^*|$, where T is the translation group of the plane.

This paper relies on previous results (see [5,6]) and uses the *isomorphism theorem* for symplectic translation planes due to Kantor (see [2,8]). Therefore it can be viewed as a completion of the previous investigation of completely regular line–ovals.

We conclude with a remark on the notation.

If F is a finite field and $\alpha \in \mathrm{Aut}(F)$, then x^α is the image of $x \in F$ under α.

If $f: A \to B$ is a mapping, then $f(x)$ is the image of $x \in A$ under f. Also, if $S \subseteq A$ then $f(S) := \{f(x) \mid x \in S\}$ is the image of S.

If $f: A \to B$ and $g: B \to C$ are mappings, we write $g \circ f$ to denote the *composition* of f with g:

$$(g \circ f)(x) := g(f(x)), \text{ for all } x \in A.$$

2 Preliminary results

Let Π be a projective plane of *even* order q. A *line–hyperoval* is a set of $q+2$ lines, no three concurrent. This definition dualizes that of *hyperoval*, a set of $q+2$ points no three collinear. Pick a line of the line–hyperoval $\mathcal{H}$, say ℓ_∞, and consider the affine plane Π^{ℓ_∞}, obtained by deleting line ℓ_∞ and all its points. We get the set $\mathcal{O} = \mathcal{H} \setminus \{\ell_\infty\}$ consisting of $q+1$ lines, one for each parallel class and no three concurrent. The set $\mathcal{O}$ is currently called a *line–oval*. It is clear that, conversely, every line–oval in an affine plane determines a line–hyperoval in the corresponding projective plane.

Let $\mathfrak{A}$ be an affine plane of even order q and $\mathcal{O}$ a line–oval. In the following ℓ_∞ will denote the *line at infinity* of the plane, and $B(\mathcal{O})$ the set of points which are on the lines of the line–oval $\mathcal{O}$. With abuse of language, if P is a point at infinity and if a line r has direction P, then we say that r is on P or else that P belongs to r.

It is easy to prove that every point of $B(\mathcal{O})$ belongs to precisely two lines of $\mathcal{O}$. Therefore,

$$|B(\mathcal{O})| = q(q+1)/2 \tag{2.1}$$

and

$$|\ell \cap B(\mathcal{O})| = q/2\,, \text{ for every line } \ell \notin \mathcal{O}. \tag{2.2}$$

We let $\Gamma := \mathrm{Aut}(\mathfrak{A})$ denote the collineation group of $\mathfrak{A}$ and $\Gamma(\mathcal{O})$ the group of all collineations of $\mathfrak{A}$ fixing $\mathcal{O}$. In all this paper we assume that the order q of planes involved is greater than or equal to 8.

Lemma 2.1 $\Gamma(\mathcal{O})$ *acts faithfully on* $\mathcal{O}$.

Proof Let $g \in \Gamma(\mathcal{O})$ such that $g(\ell) = \ell$ for all $\ell \in \mathcal{O}$. We claim that $g = 1$. First of all we note that g induces the identity on ℓ_∞, since on each point of ℓ_∞ there is exactly one line of $\mathcal{O}$ and $\Gamma(\mathcal{O})$ fixes ℓ_∞. Let $P \in B(\mathcal{O})$. There are two distinct lines of $\mathcal{O}$, say ℓ and m, such that $P = \ell \cap m$. Therefore $g(P) = g(\ell) \cap g(m) = \ell \cap m = P$. So g is the identity on $B(\mathcal{O})$. Let now r be a line not in $\mathcal{O}$. Then $|r \cap B(\mathcal{O})| = q/2$. As $q \geq 8$ there are on r at least two fixed points; hence $g(r) = r$. Therefore g also fixes every line not in $\mathcal{O}$; hence g is the identity. □

Corollary 2.2 *A collineation* $g \in \Gamma$ *induces the identity on the line at infinity and fixes* $\mathcal{O}$ *if and only if* $g = 1$.

Definition 2.3 A line–oval $\mathcal{O}$ (in the affine plane $\mathfrak{A}$) is called *completely regular* if for every point $P \in \ell_\infty$ and for every pair of distinct affine lines x, y on P, with $x, y \notin \mathcal{O}$, there is a third affine line z on P such that for every line u at least one of the points $u \cap x$, $u \cap y$ or $u \cap z$ belongs to $B(\mathcal{O})$. The triple of lines $\{x, y, z\}$ is called a *P-regular triple* (for $\mathcal{O}$).

Basic properties of completely regular line–ovals can be found in [9,5,6]. The main result of [6] states that only symplectic translation planes of even order admit completely regular line–ovals. For the theory of translation planes we refer to [10]. We only recall that any finite translation plane can be constructed from a $2n$-dimensional vector space V over a finite field F of order q, picking a *spread* of V, that is a family Σ of $q^n + 1$ subspaces of dimension n partitioning the set of nonzero vectors of V. The translation plane $\mathfrak{A}(\Sigma)$ has as *points* the vectors of V and as *lines* the cosets $S + v$, where $S \in \Sigma$ and $v \in V$. The translations of $\mathfrak{A}(\Sigma)$ are the maps $\tau_w : v \to v + w$, for all $v \in V$. They form a group isomorphic to the additive group of V.

The *kernel* of the spread Σ or of the plane $\mathfrak{A}(\Sigma)$ is the set of all semilinear endomorphisms k of V such that $k(S) \subseteq S$ for all $S \in \Sigma$. It is the largest field K such that V can be viewed as a vector space over K and the components of Σ are K-subspaces. The multiplicative group K^* consists of *homologies* with center 0 (see [10, Theorem 1. 12]). Every collineation of $\mathfrak{A}(\Sigma)$ fixing 0 is a semilinear automorphism of V as a K-vector space V (see [10, Theorem 1. 10]). Such collineations form a group, called the *translation complement* and denoted by Γ_0.

We are interested in the case where V is equipped with a nondegenerate, alternating, bilinear form β, which will be called a *symplectic form*. The pair (V, β) is a *symplectic space*. A subspace S of V is *totally isotropic* if $\beta(u, v) = 0$ for all $u, v \in S$. More on symplectic and orthogonal geometries can be found in [11]. A spread of V is *symplectic* if it consists of totally isotropic subspaces. The corresponding translation plane is called *symplectic* too.

Let $Q : V \to F$ be a *quadratic form*. We denote by

$$S(Q) := \{v \in V \mid Q(v) = 0\}$$

the set of *singular vectors* of Q (including also the zero vector). All the quadratic forms we consider will be non–degenerate. We say that the quadratic form Q is *hyperbolic* or *elliptic* if the maximal dimension of subspaces contained in $S(Q)$ is n or $n-1$ (where $\dim_F(V) = 2n$). The following result will be used later on.

Lemma 2.4 *Let Q and Q' be quadratic forms on V, regarded as a vector space over* GF(2). *Assume that Q and Q' have the same polar form β. Then*

1. *$Q - Q' = \beta_{v_0}$, for some $v_0 \in V$, and where $\beta_{v_0}(w) = \beta(v_0, w)$; and*
2. *Q and Q' have the same type (both are hyperbolic or elliptic) if and only if $Q(v_0) = 0$, where $v_0 \in V$ is as above.*

Proof Since Q and Q' have the same polar form, we get

$$Q(v + w) - Q(v) - Q(w) = Q'(v + w) - Q'(v) - Q'(w) = \beta(v, w).$$

Hence

$$(Q - Q')(v + w) = (Q - Q')(v) + (Q - Q')(w).$$

So $Q - Q'$ is a linear functional and there is $v_0 \in V$ such that $Q - Q' = \beta_{v_0}$. This prove the first claim. To prove the other claim, first of all note that Q and Q' have the same type if and only if $|S(Q)| = |S(Q')|$.

Assume that $Q' = Q + \beta_{v_0}$ with $Q(v_0) = 0$. Then a simple calculation gives $S(Q') = S(Q) + v_0$; hence $|S(Q)| = |S(Q')|$, and so Q and Q' have the same type. Conversely, if $Q(v_0) \neq 0$, then $S(Q') = \complement(S(Q)) + v_0$ (here $\complement(S(Q))$ is the complement of $S(Q)$) and so Q and Q' cannot have the same type, since $S(Q) \neq S(Q')$. □

We are now ready to state the theorem which characterizes symplectic translation planes of even order and provides a description of completely regular line–ovals too.

Theorem 2.5 *Let $\mathfrak{A}$ be an affine plane of even order $q \geq 8$ and $\mathcal{O}$ a completely regular line–oval. Then $q = 2^d$ and $\mathfrak{A}$ is a translation plane. Denote by V the set of points of $\mathfrak{A}$. Then V is a $2d-$dimensional vector space over* GF(2)*. Let Σ be the spread of V that defines the plane. Assume that $0 \in B(\mathcal{O})$. Then there exists a hyperbolic quadratic form Q on V (as a vector space over* GF(2)*) such that $B(\mathcal{O})$ is the set of singular vectors of Q. Moreover, if β is the symplectic form polarized by Q, then the spread Σ is symplectic with respect to β.*

Conversely, let (V, β) be a symplectic space of dimension $2d$ over GF(2)*, Σa symplectic spread and $\mathfrak{A}(\Sigma)$ the corresponding symplectic translation plane. Then there is a completely regular line–oval $\mathcal{O}$ such that $B(\mathcal{O})$ is the set of singular vectors of a hyperbolic quadratic form with polar form β.*

For the proof we refer to [6, Theorem 4.6] and [5, Theorems 6, 7, 9].

3 The action on the set of completely regular line–ovals

We fix a symplectic space (V, β) over the finite field $F = \mathrm{GF}(q)$, with $q = 2^d$, a symplectic spread Σ and the corresponding symplectic translation plane $\mathfrak{A}(\Sigma)$. Let K be the kernel of Σ. We want to prove that the full collineation group Γ of $\mathfrak{A}(\Sigma)$ acts transitively on the set $\mathcal{R}$ of all completely regular line–ovals.

First of all we show that Γ acts on $\mathcal{R}$.

Lemma 3.1 *Let $\mathcal{O}$ be a completely regular line–oval. Then for every $g \in \Gamma$ the line–oval $g(\mathcal{O})$ is completely regular.*

Proof First we observe that $B(g(\mathcal{O})) = g(B(\mathcal{O}))$. Next, we directly prove that if $\{x, y, z\}$ is a $P-$regular triple for $\mathcal{O}$, then $\{g(x), g(y), g(z)\}$ is a $g(P)-$regular triple for $g(\mathcal{O})$.

Let ℓ be any line such that $\ell \cap \ell_\infty = A \neq P$. If none of the points $\ell \cap g(x)$, $\ell \cap g(y)$, $\ell \cap g(z)$ were in $B(g(\mathcal{O}))$, then none of the points $g^{-1}(l) \cap x$, $g^{-1}(l) \cap y$, $g^{-1}(l) \cap z$ would be in $B(\mathcal{O})$; but this is a contradiction, since $\mathcal{O}$ is completely regular and $\{x, y, z\}$ is P-regular for $\mathcal{O}$. □

Lemma 3.2 *Let β and $\overline{\beta}$ be symplectic forms on V (as an F-vector space) with respect to which Σ is symplectic. Then there is $k \in K^*$ such that $\overline{\beta}(u, v) = \beta(k(u), k(v))$, for all $u, v \in V$.*

Proof (See also [8]) The symplectic form β defines a polarity θ of the projective space $\mathrm{PG}(V)$. Then θ restricts to a polarity θ_K of $\mathrm{PG}(V_K)$, and every polarity θ'' of $\mathrm{PG}(V)$ that restricts to θ_K has the form $a\theta'$, where $a \in K^*$ and $\theta' = f \circ \beta_K$, for some nonzero $F-$linear functional $f : K \to F$. Therefore there are at most $|K^*/F^*|$ such polarities of $\mathrm{PG}(V)$, and θ is one of them. On the other hand, there are precisely $|K^*/F^*|$ polarities of $\mathrm{PG}(V)$ that restricts to θ_K, and they are defined by the symplectic forms $\gamma_{m,f}$, where $m \in K^*$:

$$\gamma_{m,f}(u, v) := f(m(\beta_K(u, v)), \text{ for all } u, v \in V.$$

Now to finish the proof it suffices to note that the set of symplectic forms

$$\{\overline{\beta} \mid \overline{\beta}(u, v) = \beta(k(u), k(v)), k \in K^*\}$$

has size $|K^*|$, and that $\{f \circ m \mid m \in K^*\}$ is the set of all nonzero F-linear functionals from K to F. □

Remark 3.3 The proof of the above lemma is essentially due to Kantor [8] and can be used to prove that if Σ is a symplectic spread of the F-vector space V, then Σ is again a symplectic spread of the K-vector space V. Note also that the proof that Σ is again a symplectic spread of the K-vector space V has been independently obtained by Lunardon [12].

Theorem 3.4 *The action of* Γ *on the set* $\mathcal{R}$ *of all completely regular line–ovals of* $\mathfrak{A}(\Sigma)$ *is transitive. More precisely, let* $\mathcal{O}$ *and* $\overline{\mathcal{O}}$ *be completely regular line ovals of* $\mathfrak{A}(\Sigma)$. *Then there are a translation* $\tau \in T$ *and a kernel homology* $k \in K^*$ *such that* $k \circ \tau(\mathcal{O}) = \overline{\mathcal{O}}$.

Proof First of all note that, up to translations, we can assume that the zero vector belongs to $B(\mathcal{O})$ and $B(\overline{\mathcal{O}})$. By Theorem 2.5 there are hyperbolic quadratic forms Q and $\overline{Q}$ on V (regarded as a GF(2)−space) such that $B(\mathcal{O})$ and $B(\overline{\mathcal{O}})$ are the sets of singular vectors (including 0) of Q and $\overline{Q}$, respectively. Let β and $\overline{\beta}$ be the respective symplectic forms. Then Σ is symplectic with respect to both symplectic forms. By Lemma 3.2, there is $k \in K^*$ such that

$$\overline{\beta}(u, v) = \beta(k(u), k(v)), \text{ for all } u, v \in V.$$

The quadratic form $Q'\colon V \to \mathrm{GF}(2)$ such that $Q'(v) = Q(k(v))$ polarizes to $\overline{\beta}$. Therefore by Lemma 2.4 there is $v_0 \in V$ such that $\overline{Q} - Q' = \overline{\beta}_{v_0}$, that is,

$$\overline{Q}(v) = Q(k(v)) + \beta(k(v_0), k(v)). \tag{3.1}$$

Then $B(\overline{\mathcal{O}}) = S(Q') + v_0$.

Let $v \in B(\overline{\mathcal{O}})$. Then $v + v_0 \in S(Q')$, that is,

$$Q'(v + v_0) = 0 = Q(k(v)) + \beta(k(v_0), k(v));$$

hence $k(v) \in S(Q) + k(v_0)$. It follows

$$k(B(\overline{\mathcal{O}})) = B(\mathcal{O}) + k(v_0)$$

or, equivalently,

$$B(\overline{\mathcal{O}}) = k^{-1}(B(\mathcal{O})) + v_0 = B(\tau_{v_0} \circ k^{-1}(\mathcal{O})),$$

where τ_{v_0} is the translation defined by vector v_0. From this,

$$\overline{\mathcal{O}} = \tau_{v_0} \circ k^{-1}(\mathcal{O})$$

follows. For, if ℓ is a line of $\overline{\mathcal{O}}$, then $\ell \subset B(\overline{\mathcal{O}})$; since $k \circ \tau_{v_0}$ is a collineation fixing the line at infinity, $k \circ \tau_{v_0}(\ell)$ is a line contained in $B(\mathcal{O})$, and so $k \circ \tau_{v_0}(\ell) \in \mathcal{O}$. □

The above theorem allows us to give a description of $\Gamma(\mathcal{O})$. We need a result due to Kantor. If (V, β) is a symplectic space over F, we denote by $\Gamma\mathrm{Sp}(V)$ the group of all semilinear automorphisms $g \in \Gamma\mathrm{L}(V)$ such that $\beta(g(u), g(v)) = a\beta(u, v)^\alpha$ for some $a \in F^*, \alpha \in \mathrm{Aut}(F)$, and all $u, v \in V$.

Theorem 3.5 [8, Theorem 2] *Let Σ_1 and Σ_2 be symplectic spreads of the finite symplectic space (V, β) over F. Let K_2 be the kernel of Σ_2. Assume that either $|F|$ is even or $[K_2 : F]$ is odd. If $g \in \Gamma\mathrm{L}(V)$ sends Σ_1 to Σ_2, then $g = h \circ s$ with $h \in K_2^*$ and $s \in \Gamma\mathrm{Sp}(V)$ sending Σ_1 to Σ_2.*

In particular by letting $\Sigma_1 = \Sigma_2 = \Sigma$ in the above theorem, we get

Corollary 3.6 *Let Σ be a symplectic spread of the F-symplectic space (V, β) and let $\mathfrak{A}(\Sigma)$ be the corresponding symplectic translation plane. Assume that the kernel K of Σ contains F and that either $|F|$ is even or $[K : F]$ is odd. Then the translation complement of $\Gamma = \mathrm{Aut}(\mathfrak{A}(\Sigma))$ can be factored as the product of its homologies with centre 0 and its intersection with $\Gamma\mathrm{Sp}(V)$.*

In view of this corollary we denote by $\Gamma\mathrm{Sp}(V)(\Sigma)$ the group of all $g \in \Gamma\mathrm{Sp}(V)$ fixing Σ. Then $\Gamma\mathrm{Sp}(V)(\Sigma) = \Gamma_0 \cap \Gamma\mathrm{Sp}(V)$. If $K = F$ the corollary says that Γ_0 coincides with $\Gamma\mathrm{Sp}(V_K)(\Sigma)$.

Now we can describe the group $\Gamma(\mathcal{O})$. Let T be the translation group of the plane, K the kernel of Σ, and assume that $F \subseteq K$. Denote by V_K the $K-$vector space V and let

$$\Lambda(\Sigma) := \{g \in \Gamma\mathrm{Sp}(V_K)(\Sigma) \mid \sigma \circ g \in \Gamma(\mathcal{O}),\ \text{some } \sigma \in T\}.$$

Lemma 3.7 $\Lambda(\Sigma)$ *is a subgroup of* $\Gamma\mathrm{Sp}(V_K)(\Sigma)$. *Moreover,* $K^* \cap \Lambda(\Sigma) = \{1\}$ *and*

$$\Gamma\mathrm{Sp}(V_K)(\Sigma) = K^*\Lambda(\Sigma).$$

Proof Let $g, h \in \Lambda(\Sigma)$. By definition there are $\sigma, \tau \in T$ such that $\sigma \circ g, \tau \circ h \in \Gamma(\mathcal{O})$. Therefore

$$\sigma \circ g \circ (\tau \circ h)^{-1} = \sigma \circ g \circ h^{-1} \circ \tau \in \Gamma(\mathcal{O}).$$

As T is a normal subgroup, there is $\tau' \in T$ such that

$$\sigma \circ g \circ h^{-1} \circ \tau = \sigma \circ \tau' \circ g \circ h^{-1} \in \Gamma(\mathcal{O}).$$

Thus $g \circ h^{-1}\Lambda(\Sigma)$.

Let $k \in K^* \cap \Lambda(\Sigma)$. Then $\sigma \circ k \in \Gamma(\mathcal{O})$, and so $k \in \Gamma(\sigma(\mathcal{O}))$. By Corollary 2.2 we get $k = 1$.

To prove the last assertion, let $g \in \Gamma\mathrm{Sp}(V_K)(\Sigma)$. Note that if $g \in \Gamma(\mathcal{O})$, then clearly $g \in \Lambda(\Sigma)$. Let then $g \notin \Gamma(\mathcal{O})$. Since $g(\mathcal{O})$ is a completely regular line–oval, by Theorem 3.4 there are $\tau \in T$ and $k \in K^*$ such that $k \circ \tau(g(\mathcal{O})) = O$; whence $\tau \circ (k \circ g)(\mathcal{O}) = \mathcal{O}$, and so $k \circ g \in \Lambda(\Sigma)$. □

Theorem 3.8

$$\Gamma(\mathcal{O}) \cong \Lambda(\Sigma) \cong \Gamma\mathrm{Sp}(V_K)(\Sigma)/K^* = \Gamma_0/K^*.$$

Proof We only need to check that the map $\psi : \Lambda(\Sigma) \to \Gamma(\mathcal{O})$ such that $\psi(g) = \sigma \circ g$ is a group homomorphism. Let $g, h \in \Lambda(\Sigma)$ and let $\tau, \tau', \tau'' \in T$ such that

$$\psi(g) = \tau \circ g,\ \psi(h) = \tau' \circ h,\ \psi(g \circ h) = \tau'' \circ g \circ h.$$

Now $\tau \circ g \circ \tau' \circ h \in \Gamma(\mathcal{O})$ and $\tau \circ g \circ \tau' \circ h = \tau \circ \sigma \circ g \circ h$, some $\sigma \in T$. Since ψ is bijective then $\tau \circ \sigma \circ g \circ h = \tau'' \circ g \circ h$ and so $\tau \circ g \circ \tau' \circ h = \psi(g) \circ \psi(h) = \tau'' \circ g \circ h$. □

A standard result in the theory of permutation groups allows us to calculate the number of completely regular line–ovals. Let us denote by $\mathcal{O}^\Gamma$ the orbit of $\mathcal{O}$ under the action of Γ. Then the stabilizer of $\mathcal{O}$ is the group $\Gamma(\mathcal{O})$; hence

$$|\mathcal{O}^\Gamma| = |\Gamma|/|\Gamma(\mathcal{O})| = |T||\Gamma_0|/|\Gamma(\mathcal{O})| = |T||K^*|. \tag{3.2}$$

Corollary 3.9 *The number of completely regular line–ovals of* $\mathfrak{A}(\Sigma)$ *is* $|T||K^*|$*, where* T *is the translation group and* K *the kernel of* Σ.

In case $\mathfrak{A}(\Sigma)$ is desarguesian of order q, then every completely regular line–oval is a line–conic (see [7]). Thus with respect to the line at infinity there are $q^2(q-1)$ line–conics. In the projective completion $\mathrm{PG}(2, q)$ the collineation group is transitive on the set of lines, which are q^2+q+1 in number. Therefore the total number of line–conics of $\mathrm{PG}(2, q)$ is

$$(q^2+q+1)q^2(q-1)$$

which is clearly also the number of irreducible conics of $\mathrm{PG}(2, q)$.

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