Adv. Geom. **6** (2006), 323–332
DOI 10.1515/ADVGEOM.2006.019

Advances in Geometry

Two-transitive ovals

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(Communicated by W. M. Kantor)

Abstract. An oval $\mathcal{O}$ of a projective plane is called two-transitive if there is a collineation group G fixing $\mathcal{O}$ and acting 2-transitively on its points. If the plane has odd order, then the plane is desarguesian and the oval is a conic. In the present paper we prove that if a plane has order a power of two and admits a two-transitive oval, then either the plane is desarguesian and the oval is a conic, or the plane is dual to a Lüneburg plane.

Key words. Suzuki group, projective plane, oval, spread, symplectic translation plane, Lüneburg plane.

2000 Mathematics Subject Classification. Primary 20B25, 51E21; Secondary 51A50, 51A35, 05B25, 05E20

1 Introduction

Let $q = 2^{2d+1}$ with $d \geqslant 1$. The Suzuki simple group $\mathrm{Sz}(q)$ (see [25] and [26]) can be represented faithfully as a 2-transitive permutation group on q^2+1 letters. Known representations of $\mathrm{Sz}(q)$ as an automorphism group of geometric structures comprise

(1) the Tits ovoid in the projective geometry $\mathrm{PG}(3, q)$ (see [19, Chapter IV]);

(2) the Lüneburg plane of order q^2 (see [19, Chapter IV]); and

(3) the Suzuki–Tits inversive plane of order q (see [6, Chapter 6]).

These structures are equivalent, in the sense that, up to isomorphisms, each of them determines the others.

In this paper we are interested in the following question: Can a non-Lüneburg projective plane of order q^2, or its dual, admit a collineation group G isomorphic to $\mathrm{Sz}(q)$?

The possible actions of such a group are described in [19, Theorem 28.11], which are here recalled for the convenience of the reader.

Theorem 1.1. *Let Π be a projective plane of order q^2, and let G be a collineation group isomorphic to* $\mathrm{Sz}(q)$. *Then one of the following holds.*

(1) *G fixes a non-incident point line pair* (P, ℓ) *and G acts* 2*-transitively on the set of lines through P as well as on the set of points of* ℓ. *Moreover, G has two further point orbits of length* $(q^2+1)(q-1)$ *and* $(q^2+1)q(q-1)$ *and two further line orbits of length* $(q^2+1)(q-1)$ *and* $(q^2+1)q(q-1)$.

(2) *G fixes an oval* $\mathcal{O}$ *and its nucleus N and acts* 2*-transitively on the set of lines through N as well as on the set of points of* $\mathcal{O}$. *It also acts transitively on the set of secants lines to* $\mathcal{O}$ *and splits the set of exterior lines to* $\mathcal{O}$ *into two orbits of length* $\frac{1}{4}(q-r+1)q^2(q-1)$ *and* $\frac{1}{4}(q+r+1)q^2(q-1)$, *where* $r^2 = 2q$. *Furthermore, G has two point orbits of length* $(q^2+1)(q-1)$ *and* $(q^2+1)q(q-1)$.

(3) *The dual to* (2).

All three cases really occur. Cases (1) and (3) hold in Lüneburg planes ([9], [15], [19, Chapter IV], [22]). Therefore Case (2) holds in the dual Lüneburg planes. Our main result states that Case (2) occurs *only* in the dual Lüneburg planes:

Theorem 1.2. *Let* Π *be a projective plane of even order* q^2, *where* $q = 2^{2d+1}$ *with* $d \geqslant 1$. *Assume that* Π *admits a collineation group G isomorphic to* $\mathrm{Sz}(q)$ *and that* Π *has an oval* $\mathcal{O}$ *on which G acts* 2*-transitively. Then* Π *is the dual Lüneburg plane of order* q^2.

This theorem also solves the open problem about *two-transitive* ovals, in the case where the plane has order a power of 2. We briefly recall this problem.

Let Π be a projective plane of order n. An oval in Π is a set of $n+1$ points, no three of which are collinear. For the theory of ovals the reader is referred to [7] and to the survey paper [17]. An oval is called *two-transitive* if there is a collineation group G of Π fixing the oval and acting 2-transitively on its points.

Two-transitive ovals were firstly considered by Cofman [5] in order to give a local version of the Ostrom–Wagner theorem [24]. Cofman [5] proved: *if* Π *has odd order and if every involution of G is a central collineation, then the plane is desarguesian and* $\mathcal{O}$ *is a conic.*

Later Kantor [8] weakened the condition on the involutions by requiring only that G contained some nonidentical central collineation. Finally, in 1986, Biliotti and Korchmaros [2] gave a strong generalization of the foregoing results by requiring only the primitivity of G. Therefore, for planes of odd order, the problem of two-transitive ovals admits only the classical solution.

In case n is even the situation is more complicated. In view of [1], [4], [17] and Theorem 1.2 there are the following possibilities.

Theorem 1.3. *Let* Π *be a projective plane of even order n and let* $\mathcal{O}$ *be an oval. If G is a collineation group of* Π *fixing* $\mathcal{O}$ *and acting* 2*-transitively on its points, then G contains non-trivial elations. If* Δ *is the set of all non-trivial elations of G and if H is the subgroup generated by* Δ, *then exactly one of the following cases holds.*

(A) $|\Delta| = n+1$ *and H is a semidirect product of a group of odd order* $n+1$ *with a group of order two. Moreover, H is transitive on the points of* $\mathcal{O}$, *fixes an exterior line to* $\mathcal{O}$ *and G does not contain Baer involutions.*

(B) $n = 2^h$, Π *is desarguesian*, $H \cong \mathrm{SL}(2,n)$ *and* $\mathcal{O}$ *is a conic.*

(C) $n = q^2$, *where* $q = 2^{2d+1}$ *with* $d \geqslant 1$, Π *is dual to the Lüneburg plane of order* q^2, $H \cong \mathrm{Sz}(q)$ *and* H *acts on* $\mathcal{O}$ *as* $\mathrm{Sz}(q)$ *in its natural* 2*-transitive permutation representation.*

All three cases occur. Apart from the trivial case $n = 2$, the only known example for case (A) is $n = 4$. It is conjectured that indeed case (A) for $n > 4$ cannot happen. The conjecture has been shown true for all projective planes of order a power of 2 (see [16] and [17]). Case (A) is also investigated in [4] and [3]. Case (C) is a consequence of Theorem 1.2.

In conclusion:

Theorem 1.4. *Let* Π *be a projective plane of even order* $n = 2^h$ *and let* $\mathcal{O}$ *be an oval. Then* Π *admits a collineation group fixing* $\mathcal{O}$ *and acting* 2*-transitively on its points, if and only if either* Π *is desarguesian,* G *contains a subgroup isomorphic to* $\mathrm{SL}(2,n)$ *and* $\mathcal{O}$ *is a conic, or* $n = 2^{2(2e+1)}$, *with* $e \geqslant 1$, Π *is the dual Lüneburg plane of order* n *and* G *contains a subgroup isomorphic to* $\mathrm{Sz}(2^{2e+1})$.

Case (A) of Theorem 1.3 remains open, and it seems evident that *ad hoc* geometric methods are needed for its solution. This is a recurrent theme following theorems using properties of some simple groups, as already remarked by other authors (see for example Kantor [11]).

The paper is structured as follows. In Section 2 we fix notation and recall some results necessary for the proof of Theorem 1.2. Section 3 is devoted to the proof of this theorem.

2 Background

Let Π be a projective plane of even order $q \geqslant 8$, and let $\mathcal{O}$ be an oval in Π. Any line of Π meets $\mathcal{O}$ in either 0, 1 or 2 points and is called *exterior*, *tangent* or *secant*, respectively. Since the order is even, all the tangent lines to $\mathcal{O}$ concur at the same point N, called the *nucleus* (or also the *knot*) of $\mathcal{O}$. Moreover, on each point not in $\mathcal{O} \cup \{N\}$ there is one tangent line, $q/2$ secant lines and $q/2$ exterior lines. We denote by $\mathbf{P}$ the set of points of Π which are not in $\mathcal{O} \cup \{N\}$. If ℓ is a line of Π, the same symbol ℓ will also denote the set of points incident with ℓ. In particular, if P and Q are distinct points, the line through them is denoted by PQ.

Definition 2.1. Let s be a tangent line to $\mathcal{O}$. We say that $\mathcal{O}$ is *s-regular* if, for every pair of distinct points $X, Y \in s \cap \mathbf{P}$, there is a third point $Z \in s \cap \mathbf{P}$ such that, for every point $P \neq N$, at least one of the lines PX, PY or PZ is secant to $\mathcal{O}$. If $\mathcal{O}$ is s-regular for every tangent line s, then $\mathcal{O}$ is called *completely regular*.

s-regular ovals are investigated in [20], [21], [23].

For $P \in \mathbf{P}$ let

$$S_P := \{Q \in \mathbf{P} \mid Q \neq P \text{ and } PQ \text{ is a secant line}\}. \tag{2.1.1}$$

It is easy to verify that

$$|S_P| = \frac{q^2}{2} - q. \tag{2.1.2}$$

Fix a tangent line s to $\mathcal{O}$, and let O be the point of tangency of s. For every $P \in \mathbf{P}$, the set $S_P \cap s$ consists of $q/2 - 1$ distinct points. Let $\sim$ be the following equivalence relation on the set $\mathbf{P} \setminus s$:

$$P \sim Q \Leftrightarrow S_P \cap s = S_Q \cap s. \tag{2.1.3}$$

Denote by Λ_i, $i = 1, \ldots, b$, the equivalence classes.

Result 2.2 ([23, Theorem 2.17]). *$\mathcal{O}$ is s-regular if and only if $|\Lambda_i| = q$, for all $i = 1, \ldots, b$.*

The following observation will be used in the proof of Theorem 1.2.

Lemma 2.3. *Let $X \in s \setminus \{O, N\}$. Then S_X is union of equivalence classes.*

Proof. Let $P \in S_X$. Let $Q \sim P$. Then, from $S_P \cap s = S_Q \cap s$ and $X \in S_P \cap s$, it follows that QX is a secant line to $\mathcal{O}$. Therefore $Q \in S_X$.

We refer to [19] for the theory of translation planes. A translation plane is called *symplectic* if it is defined by a spread consisting of maximal totally isotropic subspaces with respect to a nondegenerate alternating bilinear form on the underlying vector space. Classical examples of symplectic planes are the desarguesian planes and the Lüneburg planes. Many families of non-classical symplectic planes have been constructed by Kantor [10], [12] and by Kantor and Williams [13], [14].

Result 2.4 ([23, Theorem 4.7]). *Let Π be a projective plane of even order. Then Π admits a completely regular oval if and only if the plane is dual to a symplectic translation plane.*

3 Proof of Theorem 1.2

In this section Π is a projective plane of even order q^2, where $q = 2^{2d+1}$ with $d \geqslant 1$, $\mathcal{O}$ is an oval with nucleus N and G is a collineation group of Π isomorphic to $\mathrm{Sz}(q)$ and acting 2-transitively on $\mathcal{O}$. This situation corresponds to Case (2) of Theorem 1.1.

We outline the main steps of the proof of Theorem 1.2. First, we prove that $\mathcal{O}$ is completely regular. This is the main step, which involves Result 2.2. Then, from Result 2.4, it follows that the plane is dual to a symplectic translation plane of order q^2.

The final step uses Liebler's characterization of Lüneburg planes [18] or [19, Theorem 31.1].

Before we begin the proof, we need a summary of some known facts about the action of G.

Lemma 3.1 ([19, Lemma 28.3]). *All involutions of G are elations.*

Lemma 3.2. (1) *G fixes the nucleus N of $\mathcal{O}$ and acts in its natural 2-transitive representation on the set of tangent lines to $\mathcal{O}$.*

(2) *Let s be a tangent line to $\mathcal{O}$. Then G_s is a Frobenius group of order $q^2(q-1)$. The Frobenius kernel is a Sylow 2-subgroup Σ and each Frobenius complement is cyclic of order $q-1$ and coincides with the stabilizer $G_{s,m}$, for some tangent line $m \neq s$. Moreover $G_s = N_G(\Sigma)$ (the normalizer of Σ in G) and Σ is the unique Sylow 2-subgroup of G fixing s.*

(3) *Let Σ be a Sylow 2-subgroup of G. Then Σ has exponent 4, its centre is $Z(\Sigma) = \{g \in \Sigma \mid g^2 = 1\}$ and $|Z(\Sigma)| = q$. Finally, each Frobenius complement acts transitively on $Z(\Sigma)\backslash\{1\}$.*

Proof. (1) is Lemma 28.4 of [19]. The rest of the lemma is a consequence of (1) and [19, Theorem 24.2], where the properties of Sylow 2-subgroups of $\mathrm{Sz}(q)$ are described.

Lemma 3.3. *Let s be a tangent line to $\mathcal{O}$, and let Σ be the Sylow 2-subgroup of G fixing s. The following hold.*

(1) *Σ acts faithfully and semiregularly on the set of points off s, and has q^2 point orbits of length q^2. One of these orbits is $\mathcal{O}\backslash\{O\}$, where O is the point of tangency of s.*

(2) *Σ splits the points of s in $q+1$ orbits of length 1 and $q-1$ orbits of length q. The orbits of length 1 are $\{O\}$, $\{N\}$ and the $q-1$ centres of the non-trivial elations of Σ.*

Proof. This result is inside the proof of [19, Lemma 28.8].

Now we prove the first step of Theorem 1.2.

Theorem 3.4. *$\mathcal{O}$ is a completely regular oval.*

Proof. Fix a tangent line s. We prove that $\mathcal{O}$ is s-regular. Denote by O the point of tangency of s and let $\mathbf{P}$ be the set of points of Π not in $\mathcal{O} \cup \{N\}$. Let $\sim$ be the equivalence relation on $\mathbf{P}\backslash s$

$$P \sim Q \Leftrightarrow S_P \cap s = S_Q \cap s,$$

see (2.1.3). Since the order of the plane is q^2, we have

$$|S_P| = \frac{q^4}{2} - q^2 \quad \text{and} \quad |S_P \cap s| = \frac{q^2}{2} - 1.$$

Denote by Λ_i, $i = 1, \dots, b$, the equivalence classes. Let Σ be the Sylow 2-subgroup fixing s. By Lemma 3.3, Σ has $q^2 - 1$ orbits of length q^2 on the points of $\mathbf{P} \backslash s$.

Put $\Lambda = \Lambda_i$, and let Σ_Λ be the setwise stabilizer of Λ within Σ. Set $Z(\Sigma) = Z$.

Lemma 3.5. $Z \trianglelefteq \Sigma_\Lambda$.

Proof. Clearly $(S_P \cap s)^g = S_{P^g} \cap s$ for all $g \in \Sigma$. Let $P \in \Lambda$. Then $S_P \cap s = S_{P^z} \cap s$, for all $z \in Z$, since the elements of Z are elations with axis s, because of Lemma 3.1 and Lemma 3.2 (3).

From Lemma 3.2 (3), $g^2 \in Z$ for all $g \in \Sigma$. Therefore Σ/Z is an elementary abelian group. Since $Z \trianglelefteq \Sigma_\Lambda$, then $\Sigma_\Lambda/Z \trianglelefteq \Sigma/Z$. Hence $\Sigma_\Lambda \trianglelefteq \Sigma$.

Let $|\Sigma_\Lambda| = 2^a$, where $q \leqslant 2^a \leqslant q^2$, because of Lemma 3.5.

Lemma 3.6. *Either* $|\Sigma_\Lambda| = q$ *or* $|\Sigma_\Lambda| = q^2$.

Proof. Assume that $Z \neq \Sigma_\Lambda \neq \Sigma$. Since Σ/Z is an elementary abelian group of order $q = 2^{2d+1}$, it can be viewed as a vector space of dimension $2d+1$ over GF(2). Then Σ_Λ/Z is a proper subspace of Σ/Z. Let T/Z be a complement of Σ_Λ/Z in Σ/Z. Then

$$\Sigma/Z = (\Sigma_\Lambda/Z)(T/Z) \quad \text{and} \quad \Sigma_\Lambda/Z \cap T/Z = \{1\}.$$

Moreover

$$(\Sigma/Z)/(\Sigma_\Lambda/Z) \cong T/Z \quad \text{and} \quad |T/Z| = \frac{q^2}{2^a}.$$

Consider the subgroup $\Sigma_\Lambda T \leqslant \Sigma$. Since $\Sigma_\Lambda \trianglelefteq \Sigma_\Lambda T$ and $T \trianglelefteq \Sigma_\Lambda T$, we get

$$T/(T \cap Z) = T/Z \cong \Sigma_\Lambda T/Z$$

and

$$\Sigma_\Lambda/(\Sigma_\Lambda \cap Z) = \Sigma_\Lambda/Z \cong \Sigma_\Lambda T/Z.$$

Therefore $\Sigma_\Lambda/Z \cong T/Z$, and so

$$|T/Z| = \frac{q^2}{2^a} = |\Sigma_\Lambda/Z| = \frac{2^a}{q}.$$

From this, $2^{2a} = q^3$ follows; this is absurd, as $q^3 = 2^{3(2d+1)}$ is not a square.

Because of this lemma, the equivalence classes are partitioned into two classes: Z-*classes*: classes whose stabilizer is Z; and Σ-*classes*: classes whose stabilizer is Σ. Clearly, Σ-classes are point orbits of Σ.

Let t and u be the number of the Z-classes and of the Σ-classes, respectively. Let C be a Frobenius complement of G_s (see Lemma 3.2 (2)).

If Λ is a Z-class, then Λ^{Σ} consists of q distinct Z-classes, and $(\Lambda^{\Sigma})^{C} = \Lambda^{G_s}$ consists of $q(q-1)$ distinct Z-classes. Therefore

$$t = \mu q(q-1), \quad \text{for some integer } \mu \geqslant 0. \tag{3.6.1}$$

The Σ-classes are split by C into v subsets of size $q-1$. Therefore

$$u = v(q-1), \quad \text{for some integer } v \geqslant 0. \tag{3.6.2}$$

From (3.6.1) and (3.6.2), the number b of the equivalence classes is

$$b = t + u = \mu q(q-1) + v(q-1) = (q-1)(\mu q + v). \tag{3.6.3}$$

Let $m \neq s$ be a line through N. The line m intersects each point orbit in exactly one point. Therefore the Σ-classes have $v(q-1)$ points on m. The remaining $q^2 - 1 - v(q-1)$ points of m distinct from N and the point of tangency belong to Z-classes. Therefore the number of Z-classes equals at most the number of point orbits that are not Σ-classes. Hence

$$\mu q(q-1) \leqslant q^2 - 1 - v(q-1). \tag{3.6.4}$$

From (3.6.4)

$$\mu q + v \leqslant q + 1. \tag{3.6.5}$$

From (3.6.5), either $\mu = 0$ or $\mu = 1$.

We prove that the solution $\mu = 0$ is impossible. This will be a consequence of the following lemma.

Lemma 3.7. *If the point orbit Γ is a Σ-class, then, for every $P \in \Gamma$, $S_P \cap s$ contains all the centres of the non-trivial elations of Z, and $q/2 - 1$ orbits of Σ on s of length q.*

Proof. Let u be the number of centres contained in $S_P \cap s$. Now

$$(S_P \cap s)^g = S_{P^g} \cap s = S_P \cap s, \quad \text{for all } g \in \Sigma.$$

Therefore, if $X \in S_P \cap s$ is a non-centre, then X^{Σ} is an orbit of Σ on s of length q. So the non-centres of $S_P \cap s$ are partitioned into v orbits of length q. Hence

$$|S_P \cap s| = \frac{q^2}{2} - 1 = u + vq. \tag{3.7.1}$$

From this equation it follows that q is a solution of the equation

$$x^2 - 2vx - 2(u+1) = 0.$$

Therefore q divides $-2(u+1)$. Since $u+1 \leqslant q$, only two cases are possible: (1) $u+1 = q/2$ and (2) $u+1 = q$. Case (1) leads to $q-1 = 2v$, which is clearly absurd. Therefore $u = q-1$ and $v = q/2 - 1$.

We can now prove that $\mu = 0$ is impossible. Clearly, $\mu = 0$ means that there is no Z-class. Therefore all the point orbits are Σ-classes. Each point orbit consists of q^2 points. So, from Result 2.2, $\mathcal{O}$ is s-regular (note that the plane has order q^2). Let $X \in s$ be a centre. Because of Lemma 3.7 and Lemma 2.3, S_X should contain all the Σ-classes, which is clearly absurd, since $|S_X| = q^4/2 - q^2$.

We have then $\mu = 1$. In this case, either $\nu = 0$ or $\nu = 1$.

We prove that $\nu = 0$ is impossible. Now $\nu = 0$ means that there are only Z-classes. Moreover, each Z-class has the same number of points. Let n be this number. Then

$$nq(q-1) = q^2(q^2-1);$$

hence

$$n = q(q+1).$$

Let $X \in s \backslash \{O, N\}$. From Lemma 2.3, the set S_X is the union of ρ Z-classes. Therefore

$$|S_X| = \frac{q^4}{2} - q^2 = \rho q(q+1),$$

which is clearly absurd, as $q+1$ is odd.

In conclusion, there are $q(q-1)$ Z-classes and $q-1$ Σ-classes. Each Σ-class contains q^2 points. Every Z-class also contains q^2 points. For, if n is the number of points that each Z-class contains, then from

$$nq(q-1) = q^2(q^2-1) - q^2(q-1)$$

we get $n = q^2$.

From Result 2.2, $\mathcal{O}$ is s-regular. Since the proof holds for every tangent line s, $\mathcal{O}$ is a completely regular oval.

Completion of the proof of Theorem 1.2. Since $\mathcal{O}$ is completely regular, by Result 2.4 the plane is dual to a symplectic translation plane Π^* of order q^2, and Π^* admits a collineation group isomorphic to the Suzuki group $\mathrm{Sz}(q)$. By Liebler's characterization of Lüneburg planes (see [18] or [19, Theorem 31.1]), Π^* is the Lüneburg plane of order q^2. Therefore Π is dual to the Lüneburg plane of order q^2.

Theorem 1.2 also provides a new characterization of Lüneburg planes.

Theorem 3.8. *Let $\mathfrak{A}$ be an affine plane of even order q^2, where $q = 2^{2d+1}$ with $d \geqslant 1$. Let ℓ_∞ be its line at infinity. Then $\mathfrak{A}$ is the Lüneburg plane of order q^2 if and only if it admits a collineation group G isomorphic to $\mathrm{Sz}(q)$ and a line-oval $\mathcal{O}$ with nucleus ℓ_∞, such that G acts 2-transitively on the set of lines of $\mathcal{O}$.*

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Received 19 July, 2004

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