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MEAN FIELD GAMES: NUMERICAL METHODS*

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Abstract. Mean field type models describing the limiting behavior, as the number of players tends to $+\infty$, of stochastic differential game problems, have been recently introduced by J.-M. Lasry and P.-L. Lions [*C. R. Math. Acad. Sci. Paris*, 343 (2006), pp. 619–625; *C. R. Math. Acad. Sci. Paris*, 343 (2006), pp. 679–684; *Jpn. J. Math.*, 2 (2007), pp. 229–260]. Numerical methods for the approximation of the stationary and evolutive versions of such models are proposed here. In particular, existence and uniqueness properties as well as bounds for the solutions of the discrete schemes are investigated. Numerical experiments are carried out.

Key words. mean field games, finite difference schemes

AMS subject classifications. 65M06, 65M012, 91-08, 91A23, 49L25

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1. Introduction.

Mean field type models describing the limiting behavior of stochastic differential game problems as the number of players tends to $+\infty$ have recently been introduced by J.-M. Lasry and P.-L. Lions [11, 12, 13]. In the stationary setting, a typical model of this kind comprises the following system:

$$-\nu\Delta u + H(x, \nabla u) + \lambda = V[m] \quad \text{in } \mathbb{T}^2, \tag{1}$$

$$-\nu\Delta m - \operatorname{div}\left(m\frac{\partial H}{\partial p}(x, \nabla u)\right) = 0 \quad \text{in } \mathbb{T}^2, \tag{2}$$

$$\int_{\mathbb{T}^2} u = 0, \quad \int_{\mathbb{T}^2} m = 1, \quad m > 0. \tag{3}$$

The unknowns are the scalar functions u, m defined on the two-dimensional torus $\mathbb{T}^2$ and the real number λ. We consider bidimensional problems for the sake of simplicity, but the results below hold for any space dimension. The data are a positive number ν, the Hamiltonian $H : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$, convex with respect to p and the (nonlinear) mapping V associating to a probability density m a Lipschitz function $V[m]$ on $\mathbb{T}^2$. Typical examples for V include nonlocal smoothing operators.

The time-dependent analogue of system (1)–(3), also considered in [11, 12, 13], is

$$\frac{\partial u}{\partial t} - \nu\Delta u + H(x, \nabla u) = V[m] \quad \text{in } \mathbb{T}^2 \times (0, T), \tag{4}$$

$$\frac{\partial m}{\partial t} + \nu\Delta m + \operatorname{div}\left(m\frac{\partial H}{\partial p}(x, \nabla u)\right) = 0 \quad \text{in } \mathbb{T}^2 \times (0, T), \tag{5}$$

$$\int_{\mathbb{T}^2} m(x, t)dx = 1, \quad m > 0, \tag{6}$$

$$u(t = 0) = V_0[m(t = 0)], \quad m(t = T) = m_0\,. \tag{7}$$

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We refer to the mentioned papers of J.-M. Lasry and P.-L. Lions for analytical results concerning system (1)–(3) as well as for its interpretation in stochastic game theory. Let us mention here only that a very important feature of the mean field model above is that uniqueness and stability may be obtained under reasonable assumptions (see [11, 12, 13]), in contrast with the Nash system describing the individual behavior of each player, for which uniqueness hardly occurs.

The aim of the present work is to propose discrete approximations by finite difference methods of the mean field model, both in the stationary case (1)–(3) or the evolutive one (4)–(7). The numerical schemes that we use rely basically on monotone approximations of the Hamiltonian and on a suitable weak formulation of the equation for m.

These schemes have several important features:

• Existence and uniqueness for the discretized problems can be obtained by similar arguments as those used in the continuous case.

• They are robust when $\nu \to 0$ (the deterministic limit of the models).

• Bounds on the solutions, which are uniform in the grid step, can be proved under reasonable assumptions on the data.

Let us mention in this respect that an important research activity is currently going on about approximation procedures for mean field games. Quite recently, we learned about a different numerical approach, based on the reformulation of the model as an optimization problem, which is restricted, however, to the case when $V[m](x) = g(m(x))$; see [10]. See also [5] for a recent work on discrete time, finite state space mean field games.

In section 2, we present the approximation of the nonlinear operators involved in, e.g., (1)–(3) and the main assumptions that are going to be made. The finite difference scheme for the stationary model is discussed in section 3. Emphasis is put on existence and uniqueness and on bounds on the solution; the main difficulty faced there is to obtain bounds on the solution which are uniform in the discretization parameters. An example of a convergence result is also supplied in section 3.

In section 4 we discuss an implicit in time finite difference method for the following evolution problem comprising two forward parabolic equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m] \quad \text{in } \mathbb{T}^2 \times (0, T), \tag{8}$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla u) \right) = 0 \quad \text{in } \mathbb{T}^2 \times (0, T), \tag{9}$$

$$\int_{\mathbb{T}^2} m(x,t) dx = 1, \quad m > 0, \tag{10}$$

$$u(x, 0) = u_0(x), \quad m(x, 0) = m_0(x). \tag{11}$$

By analogy with known results long time approximations for the cell problem in homogenization (see [14]), we expect that there exists some $\overline{\lambda} \in \mathbb{R}$ such that $u(x,t) - \overline{\lambda} t$ and $m(x,t)$ converge, as $t \to +\infty$, to the solution $(\overline{u}(x), \overline{m}(x), \overline{\lambda})$ of the stationary system (1)–(3). The main result in section 4 is Theorem 5 on the existence for the discrete system. section 5 deals with a semi-implicit scheme for the evolution system (4)–(7) and contains results on existence and uniqueness. Finally, the long time strategy mentioned above and the numerical experiments for the stationary models are described in section 6.

A Newton method for the evolutive problem (4)–(7) will be discussed in a forthcoming work (see [1]).

2. The finite difference operators.

2.1. The proposed schemes. For simplicity, we discuss the approximation of (1)–(3). Let $\mathbb{T}_h^2$ be a uniform grid on the torus with mesh step h (assuming that $1/h$ is an integer N_h) and x_{ij} denote a generic point in $\mathbb{T}_h^2$. The values of u and m at $x_{i,j}$ will be approximated by $U_{i,j}$ and $M_{i,j}$, respectively.

Hereafter, we will often make the following assumptions on the operator V:

(A_1): $V : m \to V[m]$ maps the set of probability measures into a bounded set of Lipschitz functions on $\mathbb{T}^2$.

(A_2): If m_n converges weakly to m, then $V[m_n]$ converges to $V[m]$ uniformly on $\mathbb{T}^2$.

A possible approximation of $V[m](x_{i,j})$ is

$$(V_h[M])_{i,j} = V[m_h](x_{i,j}), \tag{12}$$

calling m_h the piecewise constant function taking the value $M_{i,j}$ in the square $|x - x_{i,j}|_\infty \le h/2$, and assuming that $V[m_h]$ can be computed in practice.

We introduce the finite difference operators

$$(D_1^+U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h} \quad \text{and} \quad (D_2^+U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h} \tag{13}$$

and the numerical Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$. The finite difference approximation of $H(x, \nabla u)$ is $g(x_{i,j}, (D_1^+U^{n+1})_{i,j}, (D_1^+U^{n+1})_{i-1,j}, (D_2^+U^{n+1})_{i,j}, (D_2^+U^{n+1})_{i,j-1})$. Classically, we choose the discrete version of (1) as

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j} \tag{14}$$

with the notations

$$(\Delta_h W)_{i,j} = -\frac{1}{h^2}(4W_{i,j} - W_{i+1,j} - W_{i-1,j} - W_{i,j+1} - W_{i,j-1}), \tag{15}$$

$$[D_h W]_{i,j} = \left((D_1^+W)_{i,j}, (D_1^+W)_{i-1,j}, (D_2^+W)_{i,j}, (D_2^+W)_{i,j-1}\right)^T. \tag{16}$$

We make the following assumptions on the discrete Hamiltonian g: $(q_1, q_2, q_3, q_4) \to g\,(x, q_1, q_2, q_3, q_4)$:

(H_1): Monotonicity: g is nonincreasing w.r.t. q_1 and q_3 and nondecreasing w.r.t. q_2 and q_4.

(H_2): Consistency:

$$g\,(x, q_1, q_1, q_2, q_2) = H(x, q) \quad \forall x \in \mathbb{T}^2 \forall q = (q_1, q_2) \in \mathbb{R}^2. \tag{17}$$

(H_3): Differentiability: g is of class $\mathcal{C}^1$.

The discrete version of (2) is chosen according to the following heuristics:

- When u is fixed, (2) is a linear elliptic equation for m. Therefore, when U is fixed, the discrete version of (2) should yield a matrix with positive diagonal entries and nonpositive off-diagonal entries so that hopefully a discrete maximum principle may be used.
- The argument used in [13, 11, 12] for the uniqueness of (1)–(3) and (4)–(7) should hold in the discrete cases. For this reason, the discrete Hamiltonian g introduced above should be used in the discrete version of (2) as well, and we will often make another assumption on g.

(H_4): Convexity: the function $(q_1, q_2, q_3, q_4) \to g\,(x, q_1, q_2, q_3, q_4)$ is convex.

The main idea is to consider the weak form of (2). It involves the term

$$-\int_{\mathbb{T}^2} \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}^2} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot\nabla w$$

which will be approximated by

$$h^2\sum_{i,j} m_{i,j}\nabla_q g(x_{i,j},[D_hU]_{i,j})\cdot[D_hW]_{i,j}.$$

This yields the following discrete version of (9):

$$\left.\begin{array}{l}
-\nu(\Delta_h M)_{i,j}\\
-\dfrac{1}{h}\left(\begin{array}{l} M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j},[D_hU]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j},[D_hU]_{i-1,j})\\ +M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j},[D_hU]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j},[D_hU]_{i,j})\end{array}\right)\\
-\dfrac{1}{h}\left(\begin{array}{l} M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j},[D_hU]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1},[D_hU]_{i,j-1})\\ +M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1},[D_hU]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j},[D_hU]_{i,j})\end{array}\right)
\end{array}\right\} = 0. \tag{18}$$

We will also use the more compact but less explicit notation

$$\begin{aligned}
&\mathcal{B}_{i,j}(U,M)\\
&= \frac{1}{h}\left(\begin{array}{l}\left(\begin{array}{l} M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j},[D_hU]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j},[D_hU]_{i-1,j})\\ +M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j},[D_hU]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j},[D_hU]_{i,j})\end{array}\right)\\
+\left(\begin{array}{l} M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j},[D_hU]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1},[D_hU]_{i,j-1})\\ +M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1},[D_hU]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j},[D_hU]_{i,j})\end{array}\right)\end{array}\right),
\end{aligned} \tag{19}$$

which makes it possible to write (18) in a shorter way:

$$-\nu(\Delta_h M)_{i,j} - \mathcal{B}_{i,j}(U,M) = 0. \tag{20}$$

Remark 1. It is important to realize that the operator $M \mapsto -\nu(\Delta_h M)_{i,j} - \mathcal{B}_{i,j}(U,M)$ is the adjoint of the linearized version of the operator $U \mapsto -\nu(\Delta_h U)_{i,j} + g(x_{i,j},[D_hU]_{i,j})$.

(H_5): A further consistency assumption: One has to assume that the discrete operator in (20) is a consistent approximation of the differential operator in (2), i.e., that there exists a positive integer ℓ, a real number δ_0, $\delta_0 \in (0,1)$, and some positive real number r such that for every $v, m \in \mathcal{C}^{\ell,\delta_0}(\mathbb{T}^2)$, there is a constant $\overline{K}$ depending on the norms of v and m in the previously mentioned Schauder spaces such that for all $h<1$, calling $\widetilde{V}$ and $\widetilde{M}$ the grid functions defined by $\widetilde{V}_{i,j} = (1/h^2)\int_{|x-x_{i,j}|_\infty<h/2} v dx$ and $\widetilde{M}_{i,j} = (1/h^2)\int_{|x-x_{i,j}|_\infty<h/2} m dx$, we have for all i,j

$$\left|\mathcal{B}_{i,j}(\widetilde{V},\widetilde{M}) - \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla v)\right)(x_{i,j})\right| \le \overline{K}h^r. \tag{21}$$

This assumption is clearly fulfilled if g satisfies (17) and if g and H are smooth enough.

2.2. Summary. Finally, the finite difference approximation of (1)–(3) is to look for two grid functions U, M on $\mathbb{T}_h^2$ and for a scalar λ such that

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j} \quad \forall i,j:\ 0 \le i,j < N_h, \tag{22}$$

$$-\nu(\Delta_h M)_{i,j} - \mathcal{B}_{i,j}(U, M) = 0 \quad \forall i,j:\ 0 \le i,j < N_h, \tag{23}$$

$$\sum_{i,j} U_{i,j} = 0, \tag{24}$$

$$h^2 \sum_{i,j} M_{i,j} = 1 \quad \text{and} \quad M_{i,j} \ge 0 \text{ for } 0 \le i,j < N_h \tag{25}$$

with $(V_h[M])_{i,j}$ defined in (12), $\mathcal{B}_{i,j}(U,M)$ defined in (19), and the numerical Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$ satisfying at least (H_1)–(H_3) above.

The same strategy will be used to approximate the evolutive problems (4)–(7) and (8)–(11) with an implicit scheme; see, respectively, (75)–(79) and (61)–(62) below. For two grid functions W and Z, we define the inner product $(W,Z)_2 = \sum_{i,j} W_{i,j} Z_{i,j}$.

2.3. A useful lemma. We recall a useful lemma, which can be found in, e.g., [4]. We give its proof for completeness.

LEMMA 1. *Let V be a grid function on $\mathbb{T}_h^2$ and ρ be a positive parameter. Assume that g satisfies (H_1)–(H_3). There exists a unique grid function U such that*

$$\rho U_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) - \nu(\Delta_h U)_{i,j} = V_{i,j}. \tag{26}$$

Proof. Existence for (26) is proved by using the Leray–Schauder fixed point theorem. Indeed, we consider the mapping $\mathcal{F} : \mathbb{R}^{N_h^2} \mapsto \mathbb{R}^{N_h^2}$,

$$(\mathcal{F}(U))_{i,j} = \frac{1}{\rho}\left(\nu(\Delta_h U)_{i,j} - g(x_{i,j}, [D_h U]_{i,j}) + V_{i,j}\right),$$

and the real number $r = \max_{(i,j)} |H(x_{i,j}, 0) - V_{i,j}| / \rho$. From the continuity of g, $\mathcal{F}$ is continuous from $B_r = \{U \in \mathbb{R}^{N_h^2} : \|U\|_\infty \le r\}$ to $\mathbb{R}^{N_h^2}$.

Assuming that $U \in \partial B_r$, there must exist at least one pair of indices (i_0, j_0) such that $U_{i_0,j_0} = \pm r$. Assuming that $U_{i_0,j_0} = r$, we have

$$\nu(\Delta_h U)_{i_0,j_0} - g(x_{i_0,j_0}, [D_h U]_{i_0,j_0}) \le -H(x_{i_0,j_0}, 0)$$

from the monotonicity and the consistency of g. Hence,

$$(\mathcal{F}(U))_{i_0,j_0} \le \frac{1}{\rho}\left(-H(x_{i_0,j_0}, 0) + V_{i_0,j_0}\right) \le r$$

and $(\mathcal{F}(U))_{i_0,j_0} \ne \lambda U_{i_0,j_0}$ whenever $\lambda > 1$. Similarly, if $U_{i_0,j_0} = -r$, then $(\mathcal{F}(U))_{i_0,j_0} \ge -r$ which implies that $(\mathcal{F}(U))_{i_0,j_0} \ne \lambda U_{i_0,j_0}$. Therefore, $\mathcal{F}(U) \ne \lambda U$ for all $\lambda > 1$ and $U \in \partial B_r$. The Leray–Schauder fixed point theorem can be used; there exists a solution of (26) in B_r. Uniqueness for (26) stems from the monotonicity of g. ∎

3. Numerical analysis of the stationary problem (22)–(25). Existence results for (22)–(25) can be proved under additional assumptions on g and V_h. The strategy will be to apply the Brouwer theorem to a map χ defined on the compact and convex set

$$\mathcal{K} = \left\{ (M_{i,j})_{0 \le i,j < N_h} : h^2 \sum_{i,j} M_{i,j} = 1, M_{i,j} \ge 0 \right\} \tag{27}$$

which can be viewed as the set of the discrete probability measures.

We will see below that existence can be proved without bounds on U uniform with respect to h since the problem is finite dimensional. However, when possible, we will insist much on obtaining such bounds, for example, equicontinuity with respect to h, because they are important for passing to the limit when $h \to 0$.

We first define a map $\Phi : M \in \mathcal{K} \to U$, where (U, λ) is the unique solution of (22) subject to the constraint in (24). The map $M \to \chi(M)$ is then obtained by solving a perturbation of (23) with $U = \Phi(M)$, subject to the constraints in (25).

The discrete function $U = \Phi(M)$ will be obtained by passing to the limit in the following Hamilton–Jacobi–Bellman equation (hereafter the HJB equation)

$$\rho U^{(\rho)}_{i,j} + g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) - \nu(\Delta_h U^{(\rho)})_{i,j} = (V_h[M])_{i,j} \tag{28}$$

when the positive parameter ρ tends to 0. Such a strategy is reminiscent of those used for solving the cell problems in the homogenization of HJB equations; see, e.g., [14, 2, 3]. We first need to study (28) and obtain some bounds on U^ρ uniform w.r.t. ρ and M (and possibly uniform w.r.t. h); these will yield bounds on U uniform w.r.t. M (and possibly uniform w.r.t. h).

3.1. Preliminary results. Concerning the continuous problem, one of the assumptions made in [13, 11] was that there exists $\theta \in (0, 1)$ such that for $|p|$ large,

$$\inf_{x \in \mathbb{T}^2} \left(\nabla_x H \cdot p + \frac{\theta}{2\nu} H^2 \right) > 0. \tag{29}$$

It was useful in order to apply Bernstein's method to (1) and get a bound on $\|\nabla u\|_\infty$. With assumption (29), we were not able to extend the method of Bernstein to the discrete level. Several other assumptions on H and g can be made. Assumption 1 below will make it possible to use the results of Kuo and Trudinger [9] and [8] on Hölder estimates for the solution of (28). The slightly stronger Assumption 2 will make it possible to apply the recent results of Krylov [7].

We will use the following notation:

$$\begin{aligned}
(D_1^c U)_{i,j} &= \frac{U_{i+1,j} - U_{i-1,j}}{2h} = \frac{1}{2}\left((D_1^+ U)_{i,j} + (D_1^+ U)_{i-1,j} \right), \\
(D_2^c U)_{i,j} &= \frac{U_{i,j+1} - U_{i,j-1}}{2h} = \frac{1}{2}\left((D_2^+ U)_{i,j} + (D_2^+ U)_{i,j-1} \right), \\
(D_1^2 U)_{i,j} &= \frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{h^2} = \frac{1}{h}\left((D_1^+ U)_{i,j} - (D_1^+ U)_{i-1,j} \right), \\
(D_2^2 U)_{i,j} &= \frac{U_{i,j+1} + U_{i,j-1} - 2U_{i,j}}{h^2} = \frac{1}{h}\left((D_2^+ U)_{i,j} - (D_2^+ U)_{i,j-1} \right).
\end{aligned}$$

Assumption 1. (a) The Hamiltonian H is of the form

$$H(x, p) = \max_{\alpha \in \mathcal{A}} \left(p \cdot \alpha - L(x, \alpha) \right), \tag{30}$$

where $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$ and L is a $\mathcal{C}^0$ function on $\mathbb{T}^2 \times \mathcal{A}$. The function H is continuous with respect to x and of class $\mathcal{C}^1$ with respect to p.

(b) The discrete Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$, $(x, q) \mapsto g(x, q)$, is continuous with respect to x uniformly in h. For all $h \le h_0$, it satisfies (H_1), (H_2), and (H_3).

(c) Defining the function $\mathcal{F}:\mathbb{T}^2\times\mathbb{R}^4\to\mathbb{R}$, $(x,q_1,q_2,s_1,s_2)\mapsto\mathcal{F}(x,q_1,q_2,s_1,s_2)$ by

$$-\nu(\Delta_h U)_{i,j}+g(x_{i,j},[D_hU]_{i,j})=\mathcal{F}(x_{i,j},(D_1^cU)_{i,j},(D_2^cU)_{i,j},(D_1^2U)_{i,j},(D_2^2U)_{i,j}), \tag{31}$$

we assume that there exist positive constants a_0, a_1, and b_0 such that for $h=1/N_h\le h_0$,

$$a_0\le-\frac{\partial\mathcal{F}}{\partial s_i}\le a_1 \quad\text{and}\quad \left|\frac{\partial\mathcal{F}}{\partial q_i}\right|\le b_0. \tag{32}$$

(d) There exists a function $g^\infty:\mathbb{R}^4\to\mathbb{R}$ such that

- $g^\infty(x,0)=0$,
- for all $q\in\mathbb{R}^4$, $\lim_{\epsilon\to0}\sup_{x\in\mathbb{T}^2}|\epsilon g(x,\frac{q}{\epsilon})-g^\infty(q)|=0$,
- g^∞ is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to q_2 and q_4.

Example 1. Let H be given by (30) with $\mathcal{A}=\{\alpha\in\mathbb{R}^2,|\alpha|\le1\}$ and $L(x,\alpha)=L(\alpha)=|\alpha|^\gamma/\gamma$ with $\gamma>1$. It can be seen that

$$H(x,p)=H(p)=\begin{cases}\dfrac{\gamma-1}{\gamma}|p|^{\frac{\gamma}{\gamma-1}}, & |p|\le1,\\[2mm] |p|-\dfrac{1}{\gamma}, & |p|\ge1\end{cases}$$

and that with the Godunov scheme

$$g(x,q_1,q_2,q_3,q_4)=H\left(\sqrt{(q_1^-)^2+(q_3^-)^2+(q_2^+)^2+(q_4^+)^2}\right),$$

Assumption 1 holds with $g^\infty(x,q_1,q_2,q_3,q_4)=\sqrt{(q_1^-)^2+(q_3^-)^2+(q_2^+)^2+(q_4^+)^2}$ in point (d).

Assumption 2. (a) The Hamiltonian H is of the form (30), where $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$ and L is a $\mathcal{C}^0$ function on $\mathbb{T}^2\times\mathcal{A}$. The function H is Lipschitz continuous with respect to x uniformly in p and of class $\mathcal{C}^1$ with respect to p.

(b) The discrete Hamiltonian g satisfies point (b) in Assumption 1.

(c) The discrete Hamiltonian g is of the form

$$g(x,q_1,q_2,q_3,q_4)=\sup_{\beta\in\mathcal{B}}\left(\sum_{\ell=1}^4(-a_\ell(x,\beta)s_\ell+b_\ell(x,\beta)q_\ell)-f(x,\beta)\right), \tag{33}$$

where

- $\mathcal{B}$ is a compact set,
- $s_1=s_2=(q_1-q_2)/h$, $s_3=s_4=(q_3-q_4)/h$,
- $a_1=a_2\ge0$ and $a_3=a_4\ge0$,
- the functions $a_\ell,b_\ell:\mathbb{T}^2\times\mathcal{B}\to\mathbb{R}$ are continuous w.r.t. β (uniformly w.r.t. h), and b_ℓ and $\sqrt{\frac{\nu}{2}+a_\ell}$ are Lipschitz continuous w.r.t. x (uniformly w.r.t. h),
- the function $f:\mathbb{T}^2\times\mathcal{B}\to\mathbb{R}$ is continuous w.r.t. β and Lipschitz continuous w.r.t. x,
- for all $h\le h_0$, $(x,\beta)\in\mathbb{T}^2\times\mathcal{B}$,

$$\max\begin{pmatrix} hb_1^+(x,\beta)-a_1(x,\beta), hb_2^-(x,\beta)-a_2(x,\beta),\\ hb_3^+(x,\beta)-a_3(x,\beta), hb_4^-(x,\beta)-a_4(x,\beta)\end{pmatrix}\le0.$$

(d) The discrete Hamiltonian g satisfies point (d) in Assumption 1.

Example 2. We take H as in Example 1. The Lax–Friedrichs scheme with a large enough artificial viscosity parameter θ satisfies Assumption 1. It also satisfies Assumption 2 because

$$\begin{aligned} g(x, q_1, q_2, q_3, q_4) &= H\left(x, \left(\frac{q_1+q_2}{2}, \frac{q_3+q_4}{2}\right)\right) - \theta(q_1 - q_2 + q_3 - q_4) \\ &= \sup_{\alpha\in\mathcal{A}} \left(\alpha_1 \frac{q_1+q_2}{2} + \alpha_2 \frac{q_3+q_4}{2} - \theta(q_1 - q_2 + q_3 - q_4) - L(\alpha)\right). \end{aligned}$$

Example 3. We give a simple example where H depends only on p_1. Let H be given by (30) with $\mathcal{A} = \{(\alpha, 0), |\alpha| \le 1\}$ and $L(x, (\alpha, 0)) = |\alpha|^\theta/\theta$ with $\theta > 1$. It can be seen that $H(x, p) = H(p_1)$ with $H(p_1) = \frac{\theta-1}{\theta}|p_1|^{\frac{\theta}{\theta-1}}$ if $|p_1| \le 1$ and $H(p_1) = |p_1| - \frac{1}{\theta}$ if $|p_1| \ge 1$. Consider the discrete Hamiltonian

$$g(x, q_1, q_2, q_3, q_4) = \max_{|\beta_1|\le 1, |\beta_2|\le 1} \left(-\beta_1^- q_1 + \beta_2^+ q_2 - \frac{1}{\theta}|\beta_1|^\theta - \frac{1}{\theta}|\beta_2|^\theta\right).$$

Assumption 2 holds. In particular, $g^\infty(x, q_1, q_2, q_3, q_4) = \max(q_2, -q_1, q_2 - q_1, 0)$.

PROPOSITION 1. *Assume that H and g satisfy Assumptions* 1 *or* 2. *Let V be a grid function on $\mathbb{T}_h^2$ (we agree to write V instead of V_h) and ρ be a nonnegative real number.*

For $h \le h_0$, let the grid function U on $\mathbb{T}_h^2$ be the solution of

$$\rho U_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) - \nu(\Delta_h U)_{i,j} = V_{i,j}; \tag{34}$$

see Lemma 1.

If Assumption 1 *holds and if $\|V\|_\infty$ is bounded uniformly w.r.t. $h \le h_0$ by a constant c_0, then there exist two constants δ, $\delta \in (0, 1)$ and C, $C > 0$, both depending on a_0, a_1, b_0, c_0 and on $\|U\|_\infty$ but not on h and ρ such that for all $h \le h_0$, $0 < \rho \le 1$,*

$$|U(\xi) - U(\xi')| \le C|\xi - \xi'|^\delta \qquad \forall \xi, \xi' \in \mathbb{T}_h^2. \tag{35}$$

If Assumption 2 *holds and if*

$$\|V\|_\infty + \|D_h V\|_\infty \text{ is bounded uniformly w.r.t. } h \le h_0 \text{ by a constant } c_0, \tag{36}$$

there exists a constant C depending on $\|U\|_\infty$ but not on h and ρ such that for all $h \le h_0$, $0 < \rho \le 1$,

$$|U(\xi) - U(\xi')| \le C|\xi - \xi'| \qquad \forall \xi, \xi' \in \mathbb{T}_h^2. \tag{37}$$

Proof. In the first case, the result is a consequence of a theorem due to Kuo and Trudinger; see formula (3.10) in [9] and also [8] (which makes use of (32)).

In the second case, (37) is a particular case of a discrete Lipschitz estimate recently proved by Krylov with a very clever discrete Bernstein method (see [7], Theorem 2.5 and Remark 4.5). □

Remark 2. To cast the discrete quasi-linear operator into the setting of Theorem 2.5 in [7], one needs to consider the grid function $W_{i,j} = -U_{i,j}$.

In Proposition 1 the constants C depend on $\|U^\rho\|_\infty$. It is possible to improve this result by realizing that the constants actually depend on $\|U^\rho - U^\rho_{0,0}\|_\infty$ and by showing that this quantity is bounded uniformly w.r.t. ρ and h. The following proposition is due to Camilli and Marchi; see [3]. We give its proof for completeness.

PROPOSITION 2.

1. *If Assumption* 1 *holds and if* $\|V\|_\infty$ *is bounded uniformly w.r.t.* $h \le h_0$ *by a constant* c_0*, then there exist two constants* δ*,* $\delta \in (0,1)$ *and* $C > 0$*, both independent of* $h \le h_0$ *and* ρ *such that for all* ρ*,* $1 \ge \rho > 0$*, the solution of* (34) *satisfies*

$$\max_{\xi \neq \xi' \in \mathbb{T}_h^2} \frac{|U^{(\rho)}(\xi) - U^{(\rho)}(\xi')|}{|\xi - \xi'|^\delta} \le C. \tag{38}$$

2. *If Assumption* 2 *and* (36) *hold, then there exists a constant* C *independent of* $h \le h_0$ *and* ρ *such that for all* ρ*,* $1 \ge \rho > 0$*, the solution of* (34) *satisfies*

$$\max_{\xi \neq \xi' \in \mathbb{T}_h^2} \frac{|U^{(\rho)}(\xi) - U^{(\rho)}(\xi')|}{|\xi - \xi'|} \le C. \tag{39}$$

Proof. We give the proof in only the first case since the second case is done similarly.

Lemma 1 yields existence for (34). We also easily obtain a bound on $\|\rho U^{(\rho)}\|_\infty$, namely, that

$$\|\rho U^{(\rho)}\|_\infty \le \max_{i,j} \left(|H(x_{i,j}, 0)| + |V_{i,j}| \right) \tag{40}$$

so there exists a positive constant C_1 independent of h and ρ such that $\|\rho U^{(\rho)}\|_\infty \le C_1$. Let us have ρ tend to 0. We set $W^{(\rho)} = U^{(\rho)} - U^{(\rho)}_{0,0}$. Fixing h, assume that there exists a sequence ρ_k such that $\lim_{k\to\infty} \|W^{(\rho_k)}\|_\infty = +\infty$. We use the notation $\epsilon_k = 1/\|W^{(\rho_k)}\|_\infty$. The grid function $Z^{(k)} = \epsilon_k W^{(\rho_k)}$ satisfies

$$Z^{(k)}_{0,0} = 0, \qquad \|Z^{(k)}\|_\infty = 1, \tag{41}$$

$$\frac{\rho_k}{\epsilon_k} Z^{(k)}_{i,j} - \frac{\nu}{\epsilon_k} (\Delta_h Z^{(k)})_{i,j} + g\left(x_{i,j}, \frac{1}{\epsilon_k} [D_h Z^{(k)}]_{i,j} \right) + \rho_k U^{(\rho_k)}_{0,0} = V_{i,j}. \tag{42}$$

But (42) is equivalent to

$$\rho_k Z^{(k)}_{i,j} - \nu (\Delta_h Z^{(k)})_{i,j} + \epsilon_k g\left(x_{i,j}, \frac{1}{\epsilon_k} [D_h Z^{(k)}]_{i,j} \right) + \rho_k \epsilon_k U^{(\rho_k)}_{0,0} = \epsilon_k V_{i,j}. \tag{43}$$

Note that

$$\begin{aligned} &- \nu (\Delta_h Z^{(k)})_{i,j} + \epsilon_k g\left(x_{i,j}, \frac{1}{\epsilon_k} [D_h Z^{(k)}]_{i,j} \right) \\ &\qquad\qquad = \mathcal{G}(x_{i,j}, (D_1^c Z)_{i,j}, (D_2^c Z)_{i,j}, (D_1^2 Z)_{i,j}, (D_2^2 Z)_{i,j}), \end{aligned}$$

where $\mathcal{G}(x, q_1, q_2, s_1, s_2) = \epsilon \mathcal{F}(x, q_1/\epsilon, q_2/\epsilon, s_1/\epsilon, s_2/\epsilon)$, so $\mathcal{G}$ also satisfies estimate (32).

From this and (40), we can apply estimate (35) to $Z^{(k)}$; we get that the grid functions $Z^{(k)}$ are equibounded and equicontinuous. Up to a subsequence, $Z^{(k)}$ converges to Z as k tends to infinity and Z satisfies

$$Z_{0,0} = 0 \quad \text{ and } \quad - \nu (\Delta_h Z)_{i,j} + g^\infty (x_{i,j}, [D_h Z]_{i,j}) = 0 \quad \forall i,j.$$

The assumptions made above on g^∞ and the discrete maximum principle yield that $Z = 0$, which contradicts (41). We have proved that $\|U^{(\rho)} - U^{(\rho)}_{0,0}\|_\infty \le C$ for a constant C independent of ρ, and (38) is a consequence of Proposition 1.

It can be shown by a similar contradiction argument using the Ascoli–Arzelà theorem that the constant C in the bound $\|U^{(\rho)} - U^{(\rho)}_{0,0}\|_\infty \le C$ does not depend on h because the unique viscosity solution of $-\nu\Delta z + \sup_{\alpha\in\mathcal{A}} \alpha.\nabla z = 0$ with $z(0) = 0$ is 0. $\square$

3.2. Existence for the discrete problem.

THEOREM 1. *If Assumption* 1 *is satisfied and if the operator V maps the probability measures to a bounded set of continuous functions on $\mathbb{T}^2$ and satisfies (A_2), then the discrete problem* (22)–(25) *has at least a solution and there exist two constants δ, $\delta \in (0,1)$ and $C > 0$ such that (s.t.) for all $h = 1/N_h < h_0$,*

$$\|U\|_\infty + \max_{\xi\neq\xi'\in\mathbb{T}_h^2} \frac{|U(\xi) - U(\xi')|}{|\xi - \xi'|^\delta} \le C. \tag{44}$$

Proof. Step 1. We consider a mapping $\Phi : M \in \mathcal{K} \to U$, where U is part of the solution of the problem. Find a grid function U and a scalar λ such that

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j} \tag{45}$$

with $\sum_{i,j} U_{i,j} = 0$. Indeed, it can be proved that if g satisfies the assumptions mentioned above, then there exist a unique $\lambda \in \mathbb{R}$ and a unique grid function U satisfying (45). To do it, we pass to the limit in (28) as $\rho \to 0$. Existence and uniqueness for (28) stem from Lemma 1. We may apply Proposition 2 since $V_h[M]$ is bounded uniformly w.r.t. h and M. Proposition 2 implies that there exist two constants $C > 0$ and $\delta \in (0,1)$ independent of h, M, and ρ such that (38) holds. Thus, there exists a constant $c(h) \sim h^{\delta-1}$ independent of M and ρ such that

$$\|D_h U^{(\rho)}\|_\infty \le c(h). \tag{46}$$

Up to the extraction of a subsequence, we may say that as ρ tends to 0, $U^{(\rho)} - h^2 \sum_{i,j} U^{(\rho)}_{i,j}$ converges to a grid function U such that $\sum_{i,j} U_{i,j} = 0$ and that $\rho h^2 \sum_{i,j} U^{(\rho)}_{i,j}$ converges to $\lambda \in \mathbb{R}$. It is an easy matter to check that (U, λ) satisfies (45) and that the bounds (38) and (46) hold for U.

Uniqueness for λ stems from the following comparison principle: If U is a subsolution of (45) with $\lambda = \lambda_1$ and W is a supersolution of (45) with $\lambda = \lambda_2$, then $\lambda_2 \le \lambda_1$. Uniqueness for U is obtained by repeatedly applying the discrete maximum principle from the monotonicity of g. We have defined the map $\Phi : M \in \mathcal{K} \to U$, where (U, λ) solve (45) and $\sum_{i,j} U_{i,j} = 0$.

Step 2: *continuity of* Φ. Consider a sequence of grid functions $M^{(k)}$ in $\mathcal{K}$ which tends to $M \in \mathcal{K}$ as k tends to infinity. From the assumptions on V and V_h, $V_h[M^{(k)}]$ converges to $V_h[M]$. Consider λ and U a solution of (45), and call $\lambda^{(k)}$, $U^{(k)}$ a solution of (45) with $M = M^{(k)}$. From the estimates above, the sequences $(\lambda^k)_k$ and $(\|U^{(k)}\|_\infty)_k$ are bounded. One can extract a subsequence k' such that $\lambda^{(k)}$ tends to $\widetilde{\lambda}$ and $U^{(k')}$ tends to $\widetilde{U}$ and such that

$$-\nu(\Delta_h \widetilde{U})_{i,j} + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \widetilde{\lambda} = (V_h[M])_{i,j} \ \forall i,j, \quad \text{and} \quad \sum_{i,j} \widetilde{U}_{i,j} = 0.$$

Uniqueness for (45) implies that $\widetilde{\lambda} = \lambda$ and $\widetilde{U} = U$. The whole sequences $(\lambda^{(k)})_k$, $(U^{(k)})_k$ therefore tend to λ, U.

We have proved that the map Φ is continuous.

Step 3. For $M \in \mathcal{K}$ and $U = \Phi(M)$, consider the following linear problem. Find $\widetilde{M}$ such that

$$\mu \widetilde{M}_{i,j} - \nu(\Delta_h \widetilde{M})_{i,j} - \mathcal{B}_{i,j}(U, \widetilde{M}) = \mu M_{i,j}, \tag{47}$$

where μ is a sufficiently large positive number which will be chosen later. This linear problem may be written

$$\mu \widetilde{M} + A\widetilde{M} = \mu M, \tag{48}$$

where A is a linear operator depending on U.

The assumptions of the monotonicity of g imply that $\frac{\partial g}{\partial q_1} \le 0$, $\frac{\partial g}{\partial q_2} \ge 0$, $\frac{\partial g}{\partial q_3} \le 0$, and $\frac{\partial g}{\partial q_4} \ge 0$. This yields that the matrix corresponding to A has positive diagonal entries and nonpositive off-diagonal entries. Furthermore, since g is $\mathcal{C}^1$, (46) implies that there exists a constant C independent of M (but possibly on h) such that for all i,j, $0 \le i,j \le N_h$, and for all $\ell = 1,2,3,4$,

$$\left| \frac{\partial g}{\partial q_\ell}(x_{i,j}, [D_h U]_{i,j}) \right| \le C. \tag{49}$$

From this, we see that for μ large enough depending possibly on h but not on M, the matrix corresponding to $\mu Id + A$ is an M-matrix and is therefore invertible. The system of linear equations (47) has a unique solution $\widetilde{M}$, and $\widetilde{M}$ is nonnegative since M is nonnegative.

We are left with proving that $h^2(\widetilde{M},1)_2 = h^2(M,1)_2 = 1$. For two grid functions W and Z, let us compute $(AW,Z)_2$. Discrete integrations by part lead to

$$\begin{aligned}(AW,Z)_2 = {} & \nu \sum_{i,j} \left((D_1^+ W)_{i,j} (D_1^+ Z)_{i,j} + (D_2^+ W)_{i,j} (D_2^+ Z)_{i,j} \right) \\ & + \sum_{i,j} W_{i,j} [D_h Z]_{i,j} \cdot \nabla_q g \left(x_{i,j}, [D_h U]_{i,j} \right).\end{aligned} \tag{50}$$

It is easy to check that for all grid functions W, $(AW,1)_2 = 0$. Therefore, taking the inner product of (48) with the function $Z = 1$, we obtain that $h^2(\widetilde{M},1)_2 = h^2(M,1)_2 = 1$, so $\widetilde{M} \in \mathcal{K}$.

We call $\chi : \mathcal{K} \mapsto \mathcal{K}$ the mapping defined by $\chi : M \to \widetilde{M}$.

Step 4: *existence of a fixed point of* χ. From the boundedness and continuity of the mapping Φ, and from the fact that g is $\mathcal{C}^1$ in the variable q , we obtain that χ is continuous. Therefore, we can apply Brouwer's fixed point theorem and obtain that χ has a fixed point. ◻

We obtain a better result under Assumption 2 and stronger assumptions on V.

THEOREM 2. *If Assumption* 2 *holds and if* V *satisfies* (A_1) *and* (A_2), *then the discrete problem* (22)–(25) *has at least a solution and there exists a constant* C *such that for all* $h = 1/N_h < h_0$,

$$\|U\|_\infty + \max_{\xi \ne \xi' \in \mathbb{T}_h^2} \frac{|U(\xi) - U(\xi')|}{|\xi - \xi'|} \le C. \tag{51}$$

Proof. The proof is similar to that of Theorem 1, using now the second part of Proposition 2. ◻

Since the discrete problem is finite dimensional, existence for (22)–(25) can be proved without a bound on U uniform with respect to h. Different assumptions on the structure of g can be made; for example, see the following theorem.

THEOREM 3. *Assume that*

- g *satisfies* (H_1), (H_2) *and* (H_3),
- *there exist two positive constants* $\alpha > 0$ *and* $\gamma > 1$ *and a nonnegative constant* C *such that*

$$g(x, q_1, q_2, q_3, q_4) \geq \alpha((q_1^-)^2 + (q_2^+)^2 + (q_3^-)^2 + (q_4^+)^2)^{\gamma/2} - C \quad \forall x \in \mathbb{T}^2, \tag{52}$$

- V *maps the probability measures to a bounded set of continuous functions on* $\mathbb{T}^2$ *and satisfies* (A_2).

The discrete problem (22)–(25) *has at least a solution.*

Proof. The proof follows the same steps as that of Theorem 1. Only the first step of the proof is modified as follows.

Existence and uniqueness for (28) follow from Lemma 1. We also easily obtain a bound on $\|\rho U^{(\rho)}\|_\infty$, namely, that

$$\|\rho U^{(\rho)}\|_\infty \leq \max_{i,j} \left| H(x_{i,j}, 0) + (V_h[M])_{i,j} \right|, \tag{53}$$

so there exists a positive constant C_1 independent of M and ρ such that $\|\rho U^{(\rho)}\|_\infty \leq C_1$. From this, we deduce that there exists a positive constant C_2 independent of M and ρ such that

$$g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) - \nu(\Delta_h U^{(\rho)})_{i,j} \leq C_2 \qquad \forall i, j. \tag{54}$$

Using (52), we see that

$$\begin{aligned} &g(x_{i,j}, [D_h W]_{i,j}) - \nu(\Delta_h W)_{i,j} \\ &\quad \geq \alpha \left(((D_1^+ W)_{i,j}^-)^2 + ((D_1^+ W)_{i-1,j}^+)^2 + ((D_2^+ W)_{i,j}^-)^2 + ((D_2^+ W)_{i,j-1}^+)^2 \right)^{\gamma/2} - C \\ &\qquad\qquad - \frac{\nu}{h} \left((D_1^+ W)_{i,j}^+ + (D_1^+ W)_{i-1,j}^- + (D_2^+ W)_{i,j}^+ + (D_2^+ W)_{i,j-1}^- \right). \end{aligned}$$

Calling $P_\rho = \|D_h U^{(\rho)}\|_\infty = \max_{i,j} \max(|(D_1^+ U^{(\rho)})_{i,j}|, |(D_2^+ U^{(\rho)})_{i,j}|)$, we deduce from (54) and the previous estimate that there exists a constant C_3 independent of M and ρ such that

$$\alpha P_\rho^\gamma - 4\frac{\nu}{h} P_\rho \leq C_3.$$

This yields (46) for a constant $c(h)$ independent of M and ρ. Up to the extraction of a subsequence, we may say that as ρ tends to 0, $U^{(\rho)} - h^2 \sum_{i,j} U_{i,j}^{(\rho)}$ converges to a grid function U such that $\sum_{i,j} U_{i,j} = 0$ and that $\rho h^2 \sum_{i,j} U_{i,j}^{(\rho)}$ converges to $\lambda \in \mathbb{R}$. The limits U and λ satisfy (45). Uniqueness for (45) is proved as above, so Φ is well defined. ∎

Remark 3. In Theorem 3, we were not able to obtain an estimate on U uniform w.r.t. h.

Remark 4. Note that the assumption (A_2) can be relaxed in the discrete case. Indeed, Theorem 1 holds if we replace (A_2) with the assumption that V_h is a continuous map from $\mathcal{K}$ defined in (27) to grid functions bounded by a constant independent of h. Theorem 3 holds if V_h is a continuous map from $\mathcal{K}$ to grid functions.

These observations lead to existence results when V_h is a local operator; see section 3.4.

3.3. Uniqueness.

PROPOSITION 3. *If g satisfies (H_1)–(H_4) and if the operator V_h is strictly monotone, i.e.,*

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 \le 0 \Rightarrow M = \widetilde{M},$$

then (22)–(25) *has at most a solution.*

Proof. Let (U, M, λ) and $\widetilde{U}, \widetilde{M}, \widetilde{\lambda}$ be two solutions of (22)–(25). We have

$$-\nu(\Delta_h(U - \widetilde{U}))_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \lambda - \widetilde{\lambda} = \left(V_h[M] - V_h[\widetilde{M}]\right)_{i,j}. \tag{55}$$

Take a grid function such that $\sum_{i,j} Z_{i,j} = 0$. Multiplying by $Z_{i,j}$ and summing over all i, j yields

$$\begin{aligned} -\nu(\Delta_h(U - \widetilde{U}), Z)_2 + \sum_{i,j} (g(x_{i,j}, [D_h U]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}]_{i,j})) Z_{i,j} \\ = \left(V_h[M] - V_h[\widetilde{M}], Z\right)_2. \end{aligned} \tag{56}$$

On the other hand, multiplying (23) by $-W_{i,j}$ and summing over all i, j leads to

$$\nu(M, \Delta_h W)_2 - \sum_{i,j} M_{i,j} [D_h W]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U]_{i,j}\right) = 0. \tag{57}$$

From this and the similar equation satisfied by $\widetilde{M}$, we obtain

$$\begin{aligned} 0 = \nu((M - \widetilde{M}), \Delta_h W)_2 - \sum_{i,j} M_{i,j} [D_h W]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U]_{i,j}\right) \\ + \sum_{i,j} \widetilde{M}_{i,j} [D_h W]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right). \end{aligned} \tag{58}$$

Taking $Z = M - \widetilde{M}$ in (56) and $W = U - \widetilde{U}$ in (58) and adding the resulting equations leads to

$$\begin{aligned} 0 = \sum_{i,j} M_{i,j} &\left(\begin{array}{c} g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) - g\left(x_{i,j}, [D_h U]_{i,j}\right) \\ -D_h(\widetilde{U} - U)_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U]_{i,j}\right) \end{array} \right) \\ + \sum_{i,j} \widetilde{M}_{i,j} &\left(\begin{array}{c} g\left(x_{i,j}, [D_h U]_{i,j}\right) - g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) \\ -D_h(U - \widetilde{U})_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) \end{array} \right) \\ + &\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2. \end{aligned}$$

From the convexity of g and the monotonicity of F, the three terms in the left-hand side must vanish. The strong monotonicity of V_h implies that $M = \widetilde{M}$. A comparison argument similar to that used in the first step of the proof of Theorem 1 yields that $\lambda = \widetilde{\lambda}$ and that $U = \widetilde{U}$. ∎

3.4. The case when V is a local operator. We now aim at relaxing the assumptions on V. We assume that V is a local operator, i.e., $V[m](x) = F(m(x), x)$, where F is a bounded and $\mathcal{C}^0$ function defined on $\mathbb{R} \times \mathbb{T}^2$.

3.4.1. Existence. From Remark 4, we have the analogues of Theorems 1 and 3 in the following proposition.

PROPOSITION 4. *Take V as above. If Assumption 1 holds, then the discrete problem (22)–(25) has at least a solution and there exist two constants δ, $\delta \in (0,1)$ and $C > 0$ such that (44) is satisfied for all $h = 1/N_h < h_0$.*

PROPOSITION 5. *If g satisfies the same assumptions as in Theorem 3 and if $V[m](x) = F(m(x), x)$ with F is a $\mathcal{C}^0$ function defined on $\mathbb{R} \times \mathbb{T}^2$, then the problem (22)–(25) has at least a solution.*

Remark 5. In Proposition 5, there is no bound on U uniform w.r.t. h.

3.4.2. Uniqueness. We have the following corollary of Proposition 3.

COROLLARY 1. *If g satisfies assumptions (H_1)–(H_4) and if F is strictly monotone, i.e.,*

$$(F(m,x) - F(\tilde{m},x))(m - \tilde{m}) \leq 0 \Rightarrow m = \tilde{m},$$

then (22)–(25) has at most a solution.

3.5. A convergence result. It is possible to prove several convergence results. We give the simplest one as an example.

THEOREM 4. *We make the same assumptions as in Theorem 1, and we suppose furthermore that (H_4)–(H_5) hold and that there exist real numbers c, s, $c > 0$, $s > 0$, such that for all $h < 1$ for all grid functions M and $\widetilde{M}$,*

$$h^2 \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \geq c \| V_h[M] - V_h[\widetilde{M}] \|_\infty^s. \tag{59}$$

Assume that (1)–(3) has a unique solution such that u and m belong to $\mathcal{C}^{\ell,\delta_0}(\mathbb{T}^2) \cap \mathcal{C}^2(\mathbb{T}^2)$; see (21). Calling (U, M, λ_h) the solution of the discrete problem (22)–(25), we have

$$\lim_{h\to 0} \sup_{i,j} |u(x_{i,j}) - U_{i,j}| = 0 \quad \textit{and} \quad \lim_{h\to 0} |\lambda - \lambda_h| = 0.$$

Proof. We call $\widetilde{U}$ and $\widetilde{M}$ the grid functions s.t. $\widetilde{U}_{i,j} = h^{-2} \int_{|x - x_{i,j}|_\infty < h/2} u(x)$ and $\widetilde{M}_{i,j} = h^{-2} \int_{|x - x_{i,j}|_\infty < h/2} m(x)$. From the consistency assumptions, we have that

$$\begin{cases} -\nu(\Delta_h \widetilde{U})_{i,j} + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \lambda = \left(V_h[\widetilde{M}] \right)_{i,j} + o(1), \\ -\nu(\Delta_h \widetilde{M})_{i,j} - \mathcal{B}_{i,j}(\widetilde{U}, \widetilde{M}) = o(1), \\ \widetilde{M}_{i,j} \geq 0, \\ h^2 \sum_{i,j} \widetilde{M}_{i,j} = 1, \quad \text{and} \sum_{i,j} \widetilde{U}_{i,j} = 0, \end{cases} \tag{60}$$

where $o(1)$ means a grid function whose maximum norm tends to 0 as h tends to 0. On the other hand, from Theorem 1, $\|U\|_\infty$ is bounded by a constant. Therefore, with the same argument as the one used for uniqueness in section 3.3, we obtain that

$$\begin{aligned} o(h^{-2}) = & \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \\ & + \sum_{i,j} M_{i,j} \begin{pmatrix} g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) - g\left(x_{i,j}, [D_h U]_{i,j}\right) \\ -D_h(\widetilde{U} - U)_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U]_{i,j}\right) \end{pmatrix} \\ & + \sum_{i,j} \widetilde{M}_{i,j} \begin{pmatrix} g\left(x_{i,j}, [D_h U]_{i,j}\right) - g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) \\ -D_h(U - \widetilde{U})_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h \widetilde{U}]_{i,j}\right) \end{pmatrix}. \end{aligned}$$

From this, the convexity of g, and (59), we obtain that $\lim_{h\to 0} \|V_h[M]-V_h[\widetilde{M}]\|_\infty = 0$. Thus,

$$\begin{cases} -\nu(\Delta_h \widetilde{U})_{i,j} + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + \lambda = (V_h[M])_{i,j} + o(1), \\ -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda_h = (V_h[M])_{i,j} . \end{cases}$$

A comparison argument at the maximum of $\widetilde{U}-U$ yields that $\lambda-\lambda_h \le o(1)$. The same argument at the maximum of $U-\widetilde{U}$ yields $\lambda_h-\lambda \le o(1)$. Therefore, $\lim_{h\to 0} |\lambda-\lambda_h| = 0$.

We know that the family of grid functions u_h is equibounded and equicontinuous. There exists a function u' and a subsequence $u_{h_n} = (U^{(n)}_{i,j})_n \in \mathbb{N}$ such that $\lim_{h\to 0} \sup_{i,j} |u'(x_{i,j}) - U^{(n)}_{i,j}| = 0$. The function u' is a viscosity solution of (1) and is such that $\int_{\mathbb{T}^2} u' dx = 0$. This implies that $u = u'$. Therefore, the whole sequence u_h converges to u. ∎

4. Approximation of the evolution system (8)–(11). Let N_T be a positive integer and $\Delta t = T/N_T$, $t_n = n\Delta t$, and $n = 0, \dots, N_T$. The values of u and m at $(x_{i,j}, t_n)$ are approximated by $U^n_{i,j}$ and $M^n_{i,j}$, respectively. Given $M^0 \in \mathcal{K}$ (the compact set $\mathcal{K}$ is defined in (27)) and U^0, the discrete problem is to look for (U^n, M^n), $n = 1, \dots, N_T$, s.t.

$$\begin{cases} \dfrac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = \left(V_h[M^{n+1}]\right)_{i,j}, \\ \dfrac{M^{n+1}_{i,j} - M^n_{i,j}}{\Delta t} - \nu(\Delta_h M^{n+1})_{i,j} - \mathcal{B}_{i,j}(U^{n+1}, M^{n+1}) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M^{n+1}_{i,j} \ge 0, \end{cases} \tag{61}$$

for all $n, i, j : 0 \le n < N_T$, $0 \le i, j < N_h$, with the notations introduced above (in particular, $\mathcal{B}_{i,j}$ is defined in (19)) and

$$h^2 \sum_{i,j} M^{n+1}_{i,j} = 1 \quad \text{for } n = 0, \dots N_T - 1. \tag{62}$$

4.1. The main theorem on existence.

THEOREM 5. *Assume that*

- *g satisfies (H_1)–(H_3) and there exists a constant C such that*

$$\left|\frac{\partial g}{\partial x}(x, (q_1, q_2, q_3, q_4)\right| \le C(1 + |q_1| + |q_2| + |q_3| + |q_4|) \quad \forall x \in \mathbb{T}^2\ \forall q_1, q_2, q_3, q_4, \tag{63}$$

- *V satisfies (A_1) and (A_2).*

If $M^0 \in \mathcal{K}$, then (61)–(62) has a solution. If there exists a constant C independent of h such that $\|D_h U^0\|_\infty \le C$, then for all n, $\|D_h U^n\|_\infty \le c$ for a constant c independent of h and δt.

Proof. The strategy of the proof is similar to that used for Theorem 1. We are going to construct a continuous mapping $\chi : \mathcal{K}^{N_T} \to \mathcal{K}^{N_T}$ and use Brouwer's fixed point theorem. We proceed in several steps.

Step 1: *a mapping $M \to U$.* Given $(U^0_{i,j})_{0\le i,j<N_h}$, consider the map $\Phi : (M^n)_{1\le n\le N_T} \in \mathcal{K}^{N_T} \to (U^n)_{1\le n\le N_T}$, a solution of the first equation in (61),

i.e.,

$$\frac{U^{n+1}_{i,j}-U^n_{i,j}}{\Delta t}-\nu(\Delta_h U^{n+1})_{i,j}+g(x_{i,j},[D_hU^{n+1}]_{i,j})=\left(V_h[M^{n+1}]\right)_{i,j} \tag{64}$$

for $n=0,\dots,N_T-1$ and $0\le i,j<N_h$. The existence and uniqueness of U^{n+1}, $n=0,\dots,N_T-1$ are obtained by induction. At each step, we use Lemma 1 with $\rho=1/\Delta t$ and $V_{i,j}=U^n_{i,j}/\Delta t+\left(V_h[M^{n+1}]\right)_{i,j}$.

Step 2: *boundedness and continuity of* Φ. Looking at the proof of Lemma 1, we see that

$$\|U^{n+1}\|_\infty\le\max_{(i,j)}\left|\Delta t\left(H(x_{i,j},0)-\left(V_h[M^{n+1}]\right)_{i,j}\right)-U^n_{i,j}\right|,$$

which implies, from the uniform boundedness assumption on V and of $H(\cdot,0)$, that there exists a constant C depending on $\|U^0\|$ but independent of (M^n) such that $\|U^n\|_\infty\le C(1+T)$. Therefore, Φ maps $\mathcal{K}^{N_T}$ to a bounded subset of $(\mathbb{R}^{N_h^2})^{N_T}$. Moreover, by using the assumption on the continuity of V and well-known results on continuous dependence on the data for monotone schemes (see, e.g., [4]), we see that the mapping Φ is continuous from $\mathcal{K}^{N_T}$ to $(\mathbb{R}^{N_h^2})^{N_T}$.

Step 3: *discrete Lipschitz continuity estimates on* $\Phi((M^n)_{n=1,\dots,N_T})$. The solution of (64) is noted

$$U^{n+1}=\Psi(U^n,M^{n+1}).$$

Standard arguments on monotone schemes yield that for all $M\in\mathcal{K}$, $U,W\in\mathbb{R}^{N_h^2}$,

$$\|\left(\Psi(U,M)-\Psi(W,M)\right)^+\|_\infty\le\|(U-W)^+\|_\infty, \tag{65}$$

$$\|\Psi(U,M)-\Psi(W,M)\|_\infty\le\|U-W\|_\infty. \tag{66}$$

For $(\ell,m)\in\mathbb{Z}^2$, call $\tau_{\ell,m}U$ the discrete function defined by

$$(\tau_{\ell,m}U)_{i,j}=U_{\ell+i,m+j}.$$

It is a simple matter to check that

$$\begin{aligned}\frac{(\tau_{\ell,m}U)^{n+1}_{i,j}-(\tau_{\ell,m}U)^n_{i,j}}{\Delta t}&-\nu(\Delta_h(\tau_{\ell,m}U^{n+1}))_{i,j}+g(x_{i,j},[D_h(\tau_{\ell,m}U^{n+1})]_{i,j})\\&=\left(V_h[M^{n+1}]\right)_{i,j}+\left(V_h[M^{n+1}]\right)_{i+\ell,j+m}-\left(V_h[M^{n+1}]\right)_{i,j}\\&\quad-g(x_{i+\ell,j+m},[D_h(\tau_{\ell,m}U^{n+1})]_{i,j})+g(x_{i,j},[D_h(\tau_{\ell,m}U^{n+1})]_{i,j})\end{aligned}$$

and therefore

$$\begin{aligned}&\tau_{\ell,m}U^{n+1}=\Psi(\tau_{\ell,m}U^n+\Delta tE,M^{n+1}),\\&E_{i,j}=\begin{pmatrix}\left(V_h[M^{n+1}]\right)_{i+\ell,j+m}-\left(V_h[M^{n+1}]\right)_{i,j}\\-g(x_{i+\ell,j+m},[D_h(\tau_{\ell,m}U^{n+1})]_{i,j})+g(x_{i,j},[D_h(\tau_{\ell,m}U^{n+1})]_{i,j})\end{pmatrix}.\end{aligned}$$

From the assumptions on V and on g (in particular, (63)), there exists a constant C (independent of n, (M^n), h, and Δt) such that

$$\|E\|_\infty\le C\left(1+\|D_hU^{n+1}\|_\infty\right)h\sqrt{\ell^2+m^2}.$$

We conclude from (66) that

$$\|\tau_{\ell,m}U^{n+1}-U^{n+1}\|_\infty \le \|\tau_{\ell,m}U^n-U^n\|_\infty + Ch\Delta t\sqrt{\ell^2+m^2}\left(1+\|D_hU^{n+1}\|_\infty\right). \tag{67}$$

Thanks to (67),

$$(1-C\Delta t)\|D_hU^{n+1}\|_\infty \le \|D_hU^n\|_\infty + C\Delta t.$$

A discrete version of Gronwall's lemma yields that there exists a constant L which depends only on C, T and the initial condition $\|D_hU^0\|_\infty$ such that for all n, $1\le n\le N_T$,

$$\|D_hU^{n+1}\|_\infty \le L, \tag{68}$$

which is a discrete Lipschitz continuity estimate, uniform with respect to $(M^n)_{1\le n\le N_T}$.

Step 4: *a fixed point problem for* $(M^n)_{1\le n\le N_T}$. For $(M^n)_{1\le n\le N_T}\in\mathcal{K}^{N_T}$ and $(U^n)_{1\le n\le N_T}=\Phi((M^n)_{1\le n\le N_T})$ and a positive real number μ, consider the following linear problem. Find $(\widetilde{M}^n)_{1\le n\le N_T}$ such that

$$\frac{\widetilde{M}^{n+1}_{i,j}-\widetilde{M}^n_{i,j}}{\Delta t}+\mu\widetilde{M}^{n+1}_{i,j}-\nu(\Delta_h\widetilde{M}^{n+1})_{i,j}-\mathcal{B}_{i,j}(U^{n+1},\widetilde{M}^{n+1})=\mu M^{n+1}_{i,j} \tag{69}$$

with the initial condition $\widetilde{M}^0=M^0$ with $h^2\sum_{i,j}M^0_{i,j}=1$ and $M^0\ge 0$.

We are going to prove first that for μ large enough, (69) has a unique solution $(\widetilde{M}^n)_{1\le n\le N_T}\in\mathcal{K}^{N_T}$ and then that the mapping $(M^n)_{1\le n\le N_T}\to(\widetilde{M}^n)_{1\le n\le N_T}$ has a fixed point. Existence for (61)–(62) will then be proved.

Step 5: *existence for* (69). Clearly (69) is a discrete version of a linear parabolic initial value problem. It can be written as

$$\widetilde{M}^{n+1}+\Delta t(\mu\widetilde{M}^{n+1}+A^{n+1}\widetilde{M}^{n+1})=\widetilde{M}^n+\mu\Delta t M^{n+1}, \tag{70}$$

where A^{n+1} is a linear operator depending on U^{n+1}.

As in the proof of Theorem 1, the assumptions on the monotonicity of g imply that the matrix corresponding to $Id+\Delta tA^{n+1}$ has positive diagonal entries and nonpositive off-diagonal entries. Furthermore, since g is $\mathcal{C}^1$, (68) implies that there exists a constant C depending only on $\|D_hU^0\|$ (in particular, independent of (M^n)) such that for all n, $1\le n\le N_T$, for all i,j, $0\le i,j\le N_h$, and for all $\ell=1,2,3,4$,

$$\left|\frac{\partial g}{\partial q_\ell}(x_{i,j},[D_hU^n]_{i,j})\right|\le C. \tag{71}$$

From this, we see that for μ large enough but independent of (M^n), the matrix corresponding to $Id+\Delta t(\mu Id+A^{n+1})$ is an M-matrix and is therefore invertible. The system of linear equations (70) has a unique solution.

Moreover, since $M^0\ge 0$ for all $n=0,\dots,N_T$ and since $Id+\Delta t(\mu Id+A^{n+1})$ is an M-matrix for all n, $1\le n\le N_T$, $\widetilde{M}^n\ge 0$ for all $n=0,\dots,N_T$.

We are left with proving that $h^2\sum_{i,j}\widetilde{M}^n_{i,j}=1$ for all n, $1\le n\le N_T$. As in the proof of Theorem 1, we see that for two grid functions W and Z, we have

$$\begin{aligned}(A^nW,Z)_2&=\nu\sum_{i,j}(D_1^+W)_{i,j}(D_1^+Z)_{i,j}+\nu\sum_{i,j}(D_2^+W)_{i,j}(D_2^+Z)_{i,j}\\&\quad+\sum_{i,j}W_{i,j}[D_hZ]_{i,j}\cdot\nabla_q g\left(x_{i,j},[D_hU^n]_{i,j}\right).\end{aligned} \tag{72}$$

From (72) and (70), it can be proved by induction that if $h^2(M^0,1)_2 = 1$, then the condition $h^2(\widetilde{M}^n,1)_2 = 1$ holds for all n, $1 \le n \le N_T$.

Step 6: *existence of a fixed point of* χ. From the boundedness and continuity of the mapping Φ and from the fact that g is $\mathcal{C}^1$, we obtain that χ is continuous. Therefore, we can apply Brouwer's fixed point theorem and obtain that χ has a fixed point.

Conclusion. Assuming that M^0 is such that $M^0 > 0$ and $h^2(M^0,1)_2 = 1$, we have proved that the mapping χ has a fixed point that we call $(M^n)_{1\le n\le N_T}$. Calling $(U^n)_{1\le n\le N_t} = \Phi((M^n)_{1\le n\le N_t})$, $(M^n)_{n=1\dots,N_T}$ and $(U^n)_{n=1\dots,N_T}$ satisfy (61) and (62). □

Remark 6. Existence for problem (61)–(62) can also be obtained without (63) and when V is a local operator (see section 3.4).

Remark 7. The second equation in (61) can be written in the form

$$M^{n+1} + \Delta t A^{n+1} M^{n+1} = M^n, \tag{73}$$

where A^{n+1} is a linear operator depending on U^{n+1}. From the monotonicity of g and Remark 1, we know that $(I + \Delta t A^{n+1})^T$ is an M-matrix. Therefore, it is invertible and so is $I + \Delta t A^{n+1}$. This implies that (73) has a unique solution.

Remark 8. From Remark 7, keeping the strategy of the latter proof unchanged up to Step 3, it is tempting to take $\mu = 0$ in (69), so Step 4 consists of solving the sequence of linear problems

$$\frac{\widetilde{M}^{n+1}_{i,j} - \widetilde{M}^n_{i,j}}{\Delta t} - \nu(\Delta_h \widetilde{M}^{n+1})_{i,j} - \mathcal{B}_{i,j}(U^{n+1}, \widetilde{M}^{n+1}) = 0. \tag{74}$$

We must then check that if $M^0 \in \mathcal{K}$, then $\widetilde{M}^n \in \mathcal{K}$ for $n = 1,\dots,N_T$. This can be achieved via a fixed point argument for the map $M \in \mathcal{K} \to \check{M} \in \mathcal{K}$, where

$$\frac{\check{M}_{i,j} - \widetilde{M}^n_{i,j}}{\Delta t} + \mu \check{M}_{i,j} - \nu(\Delta_h \check{M})_{i,j} - \mathcal{B}_{i,j}(U^{n+1}, \check{M}) = \mu M_{i,j}$$

and $\mu > 0$ is large enough so the matrix of the problem is an M-matrix.

Remark 9. Note also that under a mild restriction on the time step, it is possible to prove a discrete Gårding's inequality from the uniform Lipschitz bound (68) on U^n. There exists a nonnegative constant σ depending only on $\|D_h U^0\|_\infty$ s.t. for all grid functions W,

$$(A^n W, W)_2 \ge \frac{\nu}{2} |||W|||^2 - \sigma \|W\|_2^2 \quad \text{with} \quad |||W|||^2 = \sum_{i,j} \left((D_1^+ W)^2_{i,j} + (D_2^+ W)^2_{i,j} \right).$$

Remark 10. Using the discrete Gårding's inequality above and assuming that V is a Lipschitz map from $L^1(\mathbb{T}^2)$ to $\mathcal{C}^0(\mathbb{T}^2)$, it is possible to prove that if $2\sigma\Delta t < 1$, then there exists a constant C such that $\max_n(h\|M^{n+1} - M^n\|_2 + \|U^{n+1} - U^n\|_\infty) \le C\sqrt{\Delta t}$, i.e., Hölder in time estimates for the solution of (61)–(62).

5. Approximation of the evolution system (4)–(7).

5.1. Description of the scheme. Although an implicit scheme for (4)–(7) is quite possible, we rather describe a semi-implicit scheme because uniqueness is easier to prove.

Given M^{N_T}, we consider the semi-implicit scheme

$$\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{\Delta t}-\nu(\Delta_h U^{n+1})_{i,j}+g(x_{i,j},[D_hU^{n+1}]_{i,j})=(V_h[M^n])_{i,j}\,, \tag{75}$$

$$\frac{M_{i,j}^{n+1}-M_{i,j}^{n}}{\Delta t}+\nu(\Delta_h M^{n})_{i,j}+\mathcal{B}_{i,j}(U^{n+1},M^n)=0, \tag{76}$$

$$M_{i,j}^n\geq 0, \tag{77}$$

for $n=0,\ldots,N_T-1$ and $0\leq i,j<N_h$ with

$$h^2\sum_{i,j}M_{i,j}^n=1 \quad \text{for } n=0,\ldots,N_T-1 \tag{78}$$

and

$$U_{i,j}^0=\left(V_{0,h}(M^0)\right)_{i,j}\equiv V_0[m_h^0](x_{i,j}), \tag{79}$$

where m_h^0 is the piecewise constant function taking the value $M_{i,j}^0$ in the square $|x-x_{i,j}|_\infty\leq h/2$. We have the analogue of Theorem 5.

THEOREM 6. *We make the same assumptions as in Theorem 5, and we also assume that V_0 satisfies (A_1) and (A_2). If $M^{N_T}\geq 0$ and $\sum_{i,j}M_{i,j}^{N_T}=1$, then (75)–(79) has a solution. There exists a constant C independent of h and Δt such that $\|D_hU^n\|_\infty\leq C$ for all n.*

Proof. The proof is similar to that of Theorem 5. □

5.2. Uniqueness.

THEOREM 7. *We make the same assumptions as in Theorem 6. We assume furthermore that g is convex w.r.t. to (q_1,q_2,q_3,q_4), i.e., (H_4), and that the operators V_h and $V_{0,h}$ are strictly monotone, i.e.,*

$$\left(V_h[M]-V_h[\widetilde{M}],M-\widetilde{M}\right)_2\leq 0\Rightarrow V_h[M]=V_h[\widetilde{M}],$$
$$\left(V_{0,h}[M]-V_{0,h}[\widetilde{M}],M-\widetilde{M}\right)_2\leq 0\Rightarrow V_{0,h}[M]=V_{0,h}[\widetilde{M}].$$

Problem (75)–(79) has a unique solution.

Proof. Let $(U^n,M^n)_{n=0,\ldots,N_T}$ and $(\widetilde{U}^n,\widetilde{M}^n)_{n=0,\ldots,N_T}$ be two solutions of (75)–(79). Multiplying (75) satisfied by $(\widetilde{U}_{i,j}^n,\widetilde{M}_{i,j}^n)_{n,i,j}$ by $M_{i,j}^n-\widetilde{M}_{i,j}^n$, doing the same thing with (75) satisfied by $(U_{i,j}^n,M_{i,j}^n)_{n,i,j}$ and subtracting, then summing the results for all $n=0,\ldots,N_T-1$ and all (i,j), we obtain

$$\begin{aligned}&\sum_{n=0}^{N_T-1}\frac{((U^{n+1}-\widetilde{U}^{n+1})-(U^n-\widetilde{U}^n),M^n-\widetilde{M}^n)_2}{\Delta t}\\&+\sum_{n=0}^{N_T-1}\sum_{i,j}(g(x_{i,j},[D_hU^{n+1}]_{i,j})-g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j}))(M_{i,j}^n-\widetilde{M}_{i,j}^n)\\&-\nu(\Delta_h(U^{n+1}-\widetilde{U}^{n+1}),M^n-\widetilde{M}^n)_2=\sum_{n=0}^{N_T-1}\left(V_h[M^n]-V_h[\widetilde{M}^n],M^n-\widetilde{M}^n\right)_2.\end{aligned} \tag{80}$$

Similarly, subtracting (76) satisfied by $(\widetilde{U}^n_{i,j}, \widetilde{M}^n_{i,j})_{n,i,j}$ from the same equation satisfied by $(U^n_{i,j}, M^n_{i,j})_{n,i,j}$ and multiplying the result by $U^{n+1}_{i,j} - \widetilde{U}^{n+1}_{i,j}$, summing for all $n = 0, \dots, N_T - 1$ and all (i,j) leads to

$$\begin{aligned}
&\frac{1}{\Delta t} \sum_{n=0}^{N_T-1} ((M^{n+1} - M^n) - (\widetilde{M}^{n+1} - \widetilde{M}^n), (U^{n+1} - \widetilde{U}^{n+1}))_2 \\
&+ \nu((M^n - \widetilde{M}^n), \Delta_h(U^{n+1} - \widetilde{U}^{n+1}))_2 \\
&- \sum_{n=0}^{N_T-1} \sum_{i,j} M^n_{i,j} [D_h(U^{n+1} - \widetilde{U}^{n+1})]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U^{n+1}]_{i,j}\right) \\
&+ \sum_{n=0}^{N_T-1} \sum_{i,j} \widetilde{M}^n_{i,j} [D_h(U^{n+1} - \widetilde{U}^{n+1})]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j}\right) = 0.
\end{aligned} \tag{81}$$

Adding (80) and (81) leads to

$$\begin{aligned}
0 = &\sum_{n=0}^{N_T-1} \sum_{i,j} M^n_{i,j} \left(\begin{array}{l} g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j}) - g\left(x_{i,j}, [D_h U^{n+1}]_{i,j}\right) \\ -[D_h(\widetilde{U}^{n+1} - U^{n+1})]_{i,j} \cdot \nabla_q g\left(x_{i,j}, [D_h U^{n+1}]_{i,j}\right) \end{array} \right) \\
&+ \sum_{n=0}^{N_T-1} \sum_{i,j} \widetilde{M}^n_{i,j} \left(\begin{array}{l} g(x_{i,j}, [D_h U^{n+1}]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j}) \\ -[D_h(U^{n+1} - \widetilde{U}^{n+1})]_{i,j} \cdot \nabla_q g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j}) \end{array} \right) \\
&+ \sum_{n=0}^{N_T-1} \left(V_h[M^n] - V_h[\widetilde{M}^n], M^n - \widetilde{M}^n \right)_2 + \frac{1}{\Delta t} \left(V_{0,h}[M^0] - V_{0h}[\widetilde{M}^0], M^0 - \widetilde{M}^0 \right)_2
\end{aligned}$$

from (79) and because M and $\widetilde{M}$ satisfy the same terminal conditions.

The four terms in the sum above being nonnegative, they must be zero. The strict monotonicity of V_h and $V_{0,h}$ implies that $V_h[M^n] = V_h[\widetilde{M}^n]$ for all $n = 0, \dots, N_T$ and $V_{0,h}[M^0] = V_{0,h}[\widetilde{M}^0]$. This implies that $U^0 = \widetilde{U}^0$ and then that $U^n = \widetilde{U}^n$ for all $n = 0, \dots, N_T$ since the scheme for U^n is monotone. Finally, uniqueness for (76) (given $(U^n)_n$) yields that $M^n = \widetilde{M}^n$ for all $n = 0, \dots, N_T$. ∎

5.3. A convergence result. A convergence result similar to Theorem 4 for the infinite horizon problem can be obtained. For brevity, we state it for smooth solutions of (4)–(7) which indeed exist with our assumptions; see [12, 13].

THEOREM 8. *We make the same assumptions as in Theorem* 6, *and we suppose furthermore that* (H_4)–(H_5) *hold and that there exist real numbers* c, s, $c > 0$, $s > 0$, *such that for all* $h < 1$ *for all grid functions* M *and* $\widetilde{M}$, (59) *holds.*

Assume that (4)–(7) *has a unique solution* u *and* m *in* $\mathcal{C}^\infty(\mathbb{T}^2 \times [0,T])$. *Calling* (U^n, M^n) *the solution of the discrete problem* (75)–(79), *we have*

$$\lim_{h,\Delta t \to 0} \sup_{i,j,n} |u(x_{i,j}, t_n) - U^n_{i,j}| = 0.$$

Proof. Since the proof is rather similar to that of Theorem 4, we just sketch it. We call $\widetilde{U}^n$ and $\widetilde{M}^n$ the grid functions such that $\widetilde{U}^n_{i,j} = h^{-2} \int_{|x - x_{i,j}|_\infty < h/2} u(x, n\Delta t) dx$ and $\widetilde{M}^n_{i,j} = h^{-2} \int_{|x - x_{i,j}|_\infty < h/2} m(x, n\Delta t) dx$. Note that $\widetilde{M}^0 = M^0$.

The scheme (75)–(79) is satisfied by $\widetilde{U}^n$ and $\widetilde{M}^n$ up to a consistency error, which tends to zero in maximum norm as h and Δt tend to zero.

On the other hand, from Theorem 6, $\|U^n\|_\infty$ is bounded by a constant independent of n. This makes it possible to carry out a similar argument as the one used for proving uniqueness. We obtain that

$$\lim_{h,\Delta t\to 0} h^2 \sum_{n=0}^{N_T-1} \Delta t \left(V_h[M^n] - V_h[\widetilde{M}^n], M^n - \widetilde{M}^n\right)_2 = 0$$

and from (59), $\lim_{h,\Delta t\to 0}\sum_{n=0}^{N_T-1}\Delta t\|V_h[M^n]-V_h[\widetilde{M}^n]\|_\infty^s=0$. From assumption (A_1), this implies that $\lim_{h,\Delta t\to 0}\sum_{n=0}^{N_T-1}\Delta t\|V_h[M^n]-V_h[\widetilde{M}^n]\|_\infty=0$.

Then, using the consistency of $V_{0,h}$ and applying inductively a maximum principle (using the monotonicity of g), we get that $\lim_{h,\Delta t\to 0}\max_n\|U^n-\widetilde{U}^n\|_\infty=0$. The conclusion follows easily. ∎

5.4. Solution procedure for (75)–(79). The system (75)–(79) can be seen as a forward discrete HJB equation for U with a Cauchy condition at $t=0$ (possibly involving M) coupled with a backward discrete Fokker–Planck equation for M with a Cauchy condition at final time. This structure prohibits the use of a straightforward time-marching solution procedure. In [1], in the context of planning problems, we discuss a strategy which consists of solving the whole coupled system (whose number of unknowns is large, of the order of $2N_TN^2$) by means of a Newton method; the systems of linear equations which arise are solved by means of an iterative method (for example, BiCGStab). Numerical experiments are presented, too.

6. Numerical simulations.

6.1. Long time approximation of the stationary problem. As mentioned in the introduction, we consider a solution $(\widetilde{u},\widetilde{m})$ of (8)–(11) with the Cauchy data $\widetilde{m}_0$ and $\widetilde{u}_0$ defined on $\mathbb{T}^2$, m_0 being a probability measure. We expect that there exist a $\mathcal{C}^2$ function u on $\mathbb{T}^2$, a function m in $W^{1,p}(\mathbb{T}^2)$, and a scalar λ such that

$$\lim_{t\to\infty}\widetilde{u}(t,x)-\lambda t=u(x), \qquad \lim_{t\to\infty}\widetilde{m}(t,x)=m(x),$$

and $\int_{\mathbb{T}^2}u=0$. If so, then (u,m,λ) is a solution of (1)–(3).

Such long time approximations have been justified for the cell problem in the homogenization of Hamilton–Jacobi or HJB equations; see, for example, [14, 2, 3]. This approach is close to the so-called *eductive strategy* in economy. In [6], Guéant studies the *eductive stability* on some examples where V has not the monotony property used in Proposition 3 and justifies the approach.

The same long time approximation method may be used at the discrete level.

In the results presented here, the discrete version of (8)–(11) is the implicit scheme (61)–(62). Each time step consists of solving a coupled system of nonlinear equations for (U^{n+1},M^{n+1}) (by means of a Newton method). The time step can be progressively increased; when the asymptotic regime is reached, very long time steps ($\Delta t\sim 1000$) can been used.

Remark 11. Alternatively it is possible to solve the coupled system of nonlinear equations for (U^{n+1},M^{n+1}) only approximatively by performing only one step of the Newton method. Indeed, we have observed that generally one Newton step is enough to reduce the residual by a factor smaller that 10^{-4}. Here, too, the time step can be progressively increased. On the other hand, if the condition $m^{n+1}\geq 0$ is violated, then it is possible to start back from (u^n,m^n) and to decrease the time step δt^m. This method gives similar resuts as the fully implicit scheme, which will not be reported here.

TABLE 1
The real number ν, the Hamiltonian H, and the operator V.

Test	ν	$H(x,p)$	$V[m](x)$
1	1	$\widetilde{H}(x)+\|p\|^2$	$m^2(x)$
2	0.01	$\widetilde{H}(x)+\|p\|^2$	$m^2(x)$
3	0.1	$\widetilde{H}(x)+\|p\|^\alpha$ $\alpha=1.5,3,6,9$	$m^2(x)$
4	0.1	$\widetilde{H}(x)+\|p\|^2$	$-\log(m(x))$
5	0.1	$\widetilde{H}(x)+\|p\|^{3/2}$	$200\left((1-\Delta)^{-1}(1-\Delta)^{-1}m\right)(x)$
6	0.001	$\widetilde{H}(x)+\|p\|^{3/2}$	$200\left((1-\Delta)^{-1}(1-\Delta)^{-1}m\right)(x)$

In Tests 1, 2, and 4, the Hamiltonian is of the form $H(x,p)=|p|^2+g(x)$. In such cases, as observed in [13], the system (1)–(3) is equivalent to a generalized Hartree equation. Indeed, introducing $\phi(x)=\phi_0\exp\left(-u(x)/\nu\right)$ and taking $m=\phi^2$, (1)–(3) becomes

$$-\nu^2\Delta\phi-g\phi+\phi V[\phi^2]=\lambda\phi \quad \text{in } \mathbb{T}^2 \quad \text{and} \quad \int_{\mathbb{T}^2}\phi^2=1, \tag{82}$$

and the constant ϕ_0 is fixed by the equation $\int_{\mathbb{T}^2}\log(\phi/\phi_0)=0$. As a consequence, m can be written as a function of u.

6.2. Results. In all the problems considered below, the Hamiltonian is of the form $H(x,p)=\psi(x,|p|)$, and the discrete Hamiltonian is obtained via a Godunov scheme, i.e,

$$g(x,q_1,q_2,q_3,q_4)=\psi\left(x,\sqrt{(q_1^-)^2+(q_3^-)^2+(q_2^+)^2+(q_4^+)^2}\right).$$

Table 1 contains the data of the problems simulated below, i.e., the real number ν, the Hamiltonian H, and the operator V. Hereafter, we note $\widetilde{H}$ the potential

$$\widetilde{H}(x)=\sin(2\pi x_2)+\sin(2\pi x_1)+\cos(4\pi x_1). \tag{83}$$

The contours of the potential $\widetilde{H}$ are displayed in Figure 1.

6.2.1. Test 1. See Table 1 for the data of the problem. We first check that the long time approximation yields the expected asymptotic behavior. In the left side of Figure 2, we plot the graph of $(h^2/T)\sum_{i,j}U_{i,j}(t=T)$ when the mesh step is $h=1/50$; as T tends to infinity, this quantity tends to a constant λ_h, as expected. Here we find that $\lambda_h\simeq 0.9784$.

Here $V[m](x)=F(m(x))$, where $F(y)=y^2$ is a nondecreasing function. Such a function is used to model repulsive cases when the players do not like to share their position with others. If ν is not too small, then the players' positions should be well distributed, i.e., the density m should not be strongly localized. In Figure 3, we plot the contours of u_h and m_h; we see, indeed, that m_h is supported in the whole domain $\mathbb{T}^2$. We have seen above that m is a smooth decreasing function of u since the Hamiltonian is quadratic. This explains why the contour plots of u_h and m_h have the same aspect.

In order to estimate the rate of convergence as h tends to 0, we compare the solutions with that computed by solving the Hartree equation (82) with a fourth

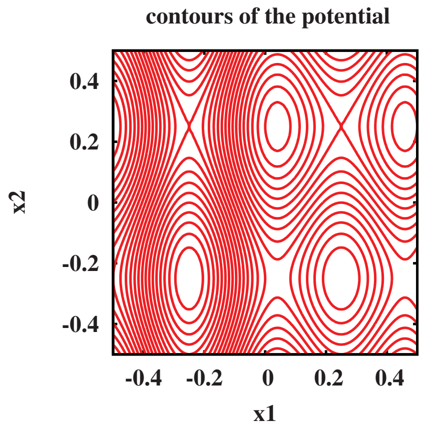

FIG. 1. *The contours of the potential* $\widetilde{H}$ *used in Tests* 1–6.

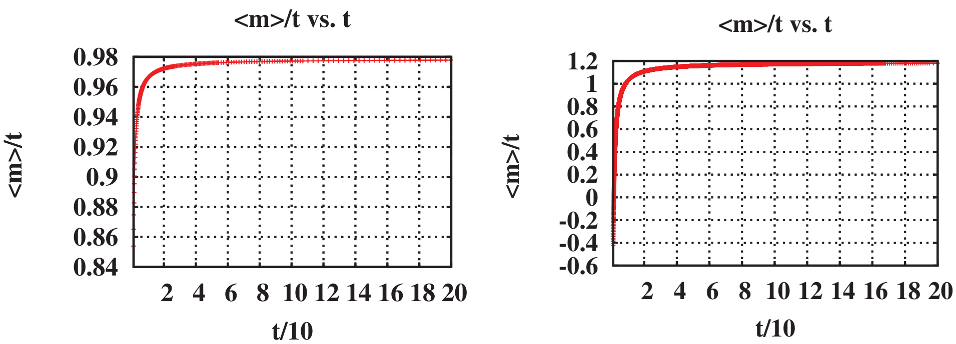

FIG. 2. $N_h = 50$. *Graph of* $(h^2/T)\sum_{i,j} U_{i,j}(t = T)$. *Left: Test* 1. *Right: Test* 2.

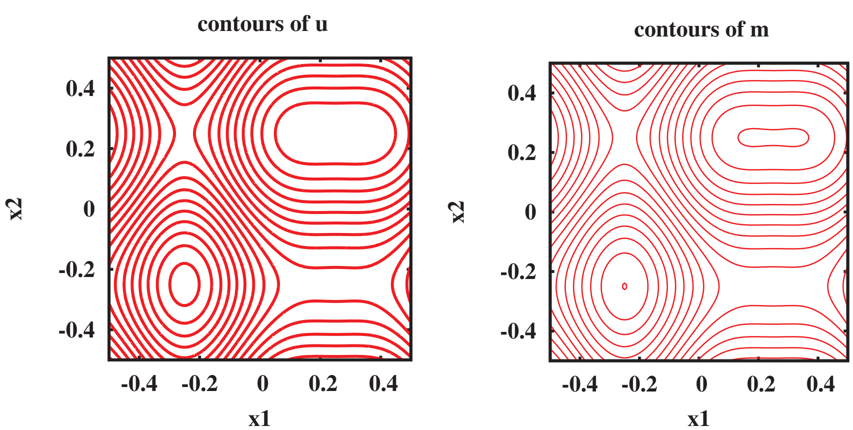

FIG. 3. *Test* 1*: the contours of* u_h *(left) and* m_h *(right) with* $N_h = 200$.

order scheme on a 400×400 grid. (A Newton solver is used for solving the system of nonlinear equations.) We consider the sum of the relative errors in the max norm

$$Err = \frac{\|U - U_{hartree}\|_\infty}{\|U_{hartree}\|_\infty} + \frac{\|M - M_{hartree}\|_\infty}{\|M_{hartree} - 1\|_\infty}. \tag{84}$$

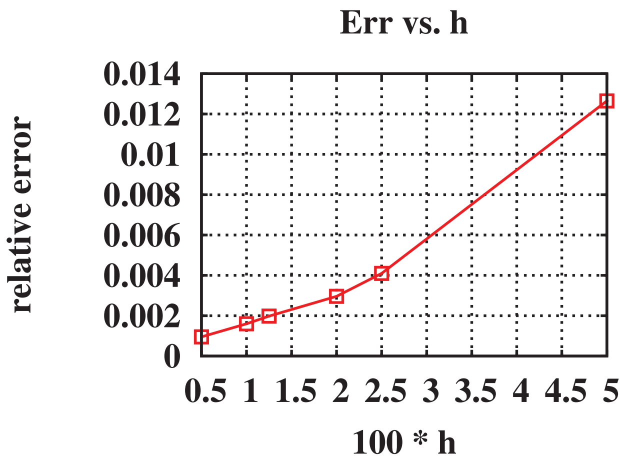

FIG. 4. *Test* 1*: relative error (see* (84)*) w.r.t. the solution of* (82) *computed with a fourth order scheme on a* 400×400 *grid as a function of* h*.*

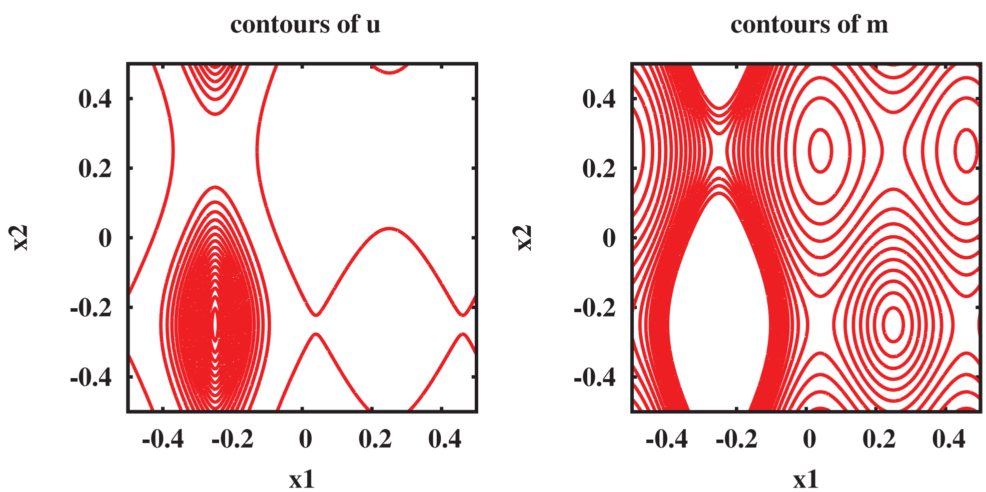

FIG. 5. *Test* 2*: the contours of* u_h *(left) and* m_h *(right) with* $N_h = 200$*.*

The graphs of the error are shown in Figure 4. The convergence looks linear (for h small enough).

6.2.2. Test 2. See Table 1 for the data of the problem. Here the value of $\nu = 0.01$ is small, so the case is close to the deterministic limit.

As in Test 1, the solution of the discrete evolution problem has the expected behavior for large times. In the right side of Figure 2, we plot the graph of $(h^2/T)\sum_{i,j} U_{i,j}(t = T)$ when the mesh step is $h = 1/50$; as T tends to infinity, this quantity tends to a constant λ_h, as expected. Here we find that $\lambda_h \simeq 1.187$.

In Figure 5, we plot the contours of u_h and m_h. Note that the supports of ∇u_h and of m_h tend to be disjoint for such a small value of ν. This is coherent with the results concerning the deterministic limit in [13]. This test shows that the method is robust for small values of ν.

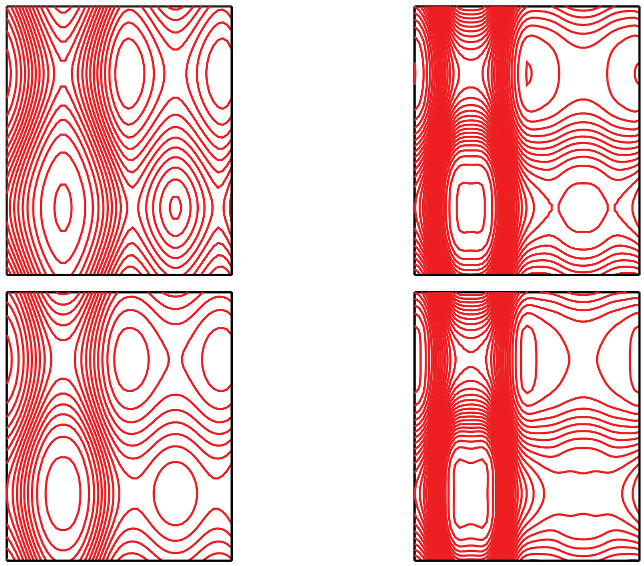

FIG. 6. *Test 3: contours of m_h: comparison for $\alpha = 1.5$ (top left), $\alpha = 3$ (bottom left), $\alpha = 6$ (top right), and $\alpha = 9$ (bottom right).*

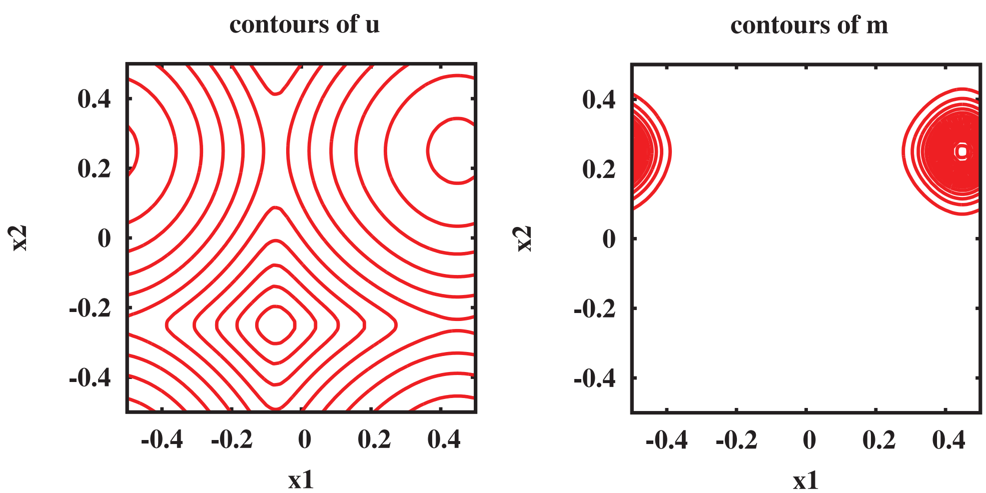

FIG. 7. *Test 4: the contours of u_h (left) and m_h (right) with $N_h = 100$.*

6.2.3. Test 3. In Figure 6, we compare m_h for different values of α (see Table 1). We see that the variations of m_h become stiffer as α grows.

6.2.4. Test 4. See Table 1 for the data of the problem. Here $V[m](x) = F(m(x))$ with $F(y) = -\log(y)$. By contrast with Tests 1 and 2, F is a decreasing function. Such a function F is used to describe situations when the agents are gregarious, i.e., they like to all be in the same position. Guéant proved results concerning the eductive stability in close cases; see [6]. Indeed, we observe that the solution of the discrete evolution problem has the expected behavior for large times.

In Figure 7, we plot the contours of u_h and m_h. Note that the measure m_h concentrates near the minimum of u_h, i.e., the players take positions close to each other.

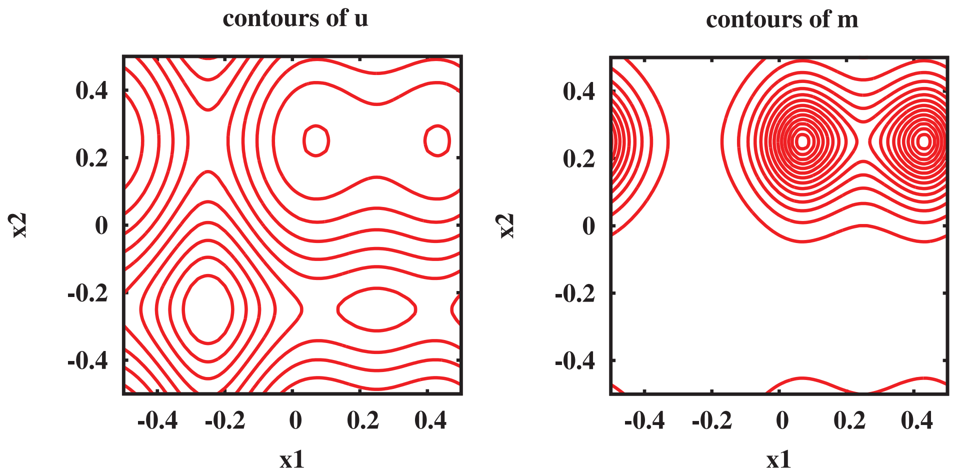

FIG. 8. *Test 5: the contours of u_h(left) and m_h(right) with $N_h = 100$.*

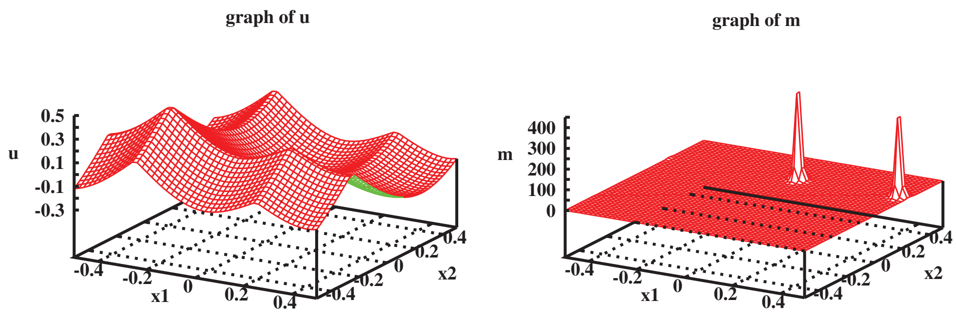

FIG. 9. *Test 6: the graphs of u_h(top) and m_h(bottom).*

6.2.5. Test 5. See Table 1 for the data of the problem. By contrast with the previous cases, the operator V is nonlocal. This example has been chosen because V satisfies the assumptions ($\mathbf{A_1}$) and ($\mathbf{A_2}$). At the discrete level, applying V is done by solving a system of linear equations. Alternatively a method based on fast Fourier transform could be used. In Figure 8, we plot the contours of u_h and m_h.

6.2.6. Test 6. See Table 1 for the data of the problem. Compared to Test 5, everything is kept unchanged except that $\nu = 0.001$. In Figure 9, we plot the graphs of u_h and m_h. We see that u_h is not better than Lipschitz continuous and that m_h is close to a sum of two Dirac masses located at the minima of u_h, which does not contradict the theoretical results in section 3. Since V is a smoothing operator, it is not surprising that singular measures m arise in the deterministic limit.

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