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On the weak maximum principle for fully nonlinear elliptic pde's in general unbounded domains

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Abstract[1]. The aim of this Note is to review some recent research on viscosity solutions of fully nonlinear equations of the form

$$F\left(x, u(x), Du(x), D^2u(x)\right) = 0\ , \quad x \in \Omega$$

where Ω is an open set in $\mathbb{R}^N$ and F is a nonlinear function of its entries which is elliptic with respect to the Hessian matrix D^2u of the unknown function u and satisfies some suitable structure condition. The main issues touched here are the Alexandrov-Bakelman-Pucci estimate, the weak Maximum Principle for bounded solutions in general unbounded domains and qualitative Phragmen-Lindelöf type theorems.

1. Introduction

The paper focuses on some global and local properties of continuous functions u satisfying fully nonlinear elliptic equations of the form

$$F\left(x, u(x), Du(x), D^2u(x)\right) = 0 \tag{1.1}$$

in the viscosity sense in an open set $\Omega \subset \mathbb{R}^N$. The main topics discussed here are the validity of Alexandrov-Bakelman-Pucci estimates, the Weak Maximum Principle (**wMP** in short) and qualitative Phragmen-Lindelöf type theorems in cylindrical and conical domains. The results presented apply to a wide class of unbounded domains, perhaps of infinite measure, which may have a quite irregular boundary and generalize in several aspects a number of well-known results for smooth or strong solution of linear elliptic equation see, for example, [19], [13], [12], [16], [17], [2], [4], [17].
The content of this note is mostly taken from the recent papers [6], [7], [8], [9]. We refer to these papers for the detailed proofs of the result presented here.

2. Viscosity solutions

We report for the convenience of the reader a few facts about viscosity solutions.

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An upper semicontinuous function $u \in USC(\Omega)$ is a viscosity subsolution of equation (1.1) if the inequality

$$F\left(x_0, \Phi(x_0), D\Phi(x_0), D^2\Phi(x_0)\right) \geq 0$$

holds at any point $x_0 \in \Omega$ and for all quadratic polynomials Φ touching from above the graph of u at x_0. Observe that u is a viscosity solution of $\Delta u \geq 0$ if and only if u is subharmonic in the sense of potential theory:

$$\text{for any ball } B \subset \Omega \text{ and for any function } h \text{ such that } \Delta h = 0 \text{ in } B$$

$$\text{the inequality } u \leq h \text{ on } \partial B \text{ implies } u \leq h \text{ in } B\,.$$

Viscosity supersolutions are defined in a symmetric fashion: a lower semicontinuous function $u \in LSC(\Omega)$ is a viscosity supersolution of (1.1) if

$$F\left(x_0, \Phi(x_0), D\Phi(x_0), D^2\Phi(x_0)\right) \leq 0$$

at any point $x_0 \in \Omega$ and for all for all quadratic polynomials Φ touching from below the graph of u at x_0.
A viscosity solution of (1.1) is a function $u \in C(\Omega)$ which is simultaneously a sub and a supersolution.
Most of the theory of strong solutions for linear elliptic equations in non-divergence form

$$\operatorname{Tr}\left(A(x)\, D^2 u\right) + b(x)\cdot Du + c(x)\, u = 0 \tag{2.1}$$

has been carried on to viscosity solutions of (1.1) under the leading assumption of ellipticity of F:

$$\lambda \operatorname{Tr} Y \leq F(x,t,p,X+Y) - F(x,t,p,X) \leq \Lambda \operatorname{Tr} Y \tag{2.2}$$

for some constants $0 < \lambda \leq \Lambda$ and for all $X, Y \in \mathcal{S}^N$ with $Y \geq 0$, where $\mathcal{S}^N$ and Tr denote, respectively, the space of real symmetric $N \times N$ matrices endowed with the partial ordering induced by non-negative definiteness and the trace of such a matrix.
We refer to [14] and [10] for existence, uniqueness and stability viscosity solutions of (1.1), to [3] for regularity theory.
Fundamental model examples of elliptic operators F are given by the Pucci extremal operators $\mathcal{P}^-_{\lambda,\Lambda}$ and $\mathcal{P}^+_{\lambda,\Lambda}$ defined for $X \in S^N$ and given parameters $0 < \lambda \leq \Lambda$ by

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \inf_{A \in \mathcal{A}} \operatorname{Tr}(AX)\,, \qquad \mathcal{P}^+_{\lambda,\Lambda}(X) = \sup_{A \in \mathcal{A}} \operatorname{Tr}(AX) \tag{2.3}$$

where

$$\mathcal{A} = \mathcal{A}(\lambda, \Lambda) = \left\{A \in \mathcal{S}^N \;:\; \lambda I \leq A \leq \Lambda I\right\}$$

see [20], [3]. Other important examples are the Isaac's operators

$$\sup_{j \in K} \inf_{k \in K} \operatorname{Tr}(A_{k,j}\, X)$$

with $A_{k,j} \in \mathcal{A}\,, k, j \in K$, arising in stochastic differential game theory [11].

Two fundamental tools in deriving the Alexandrov-Bakelman-Pucci estimates for viscosity solutions of equation (1.1) are the weak Harnack inequality and its so-called boundary version for nonnegative supersolutions of Pucci type differential inequalities.

Proposition 2.1 (the weak Harnack inequality). *Let A be an open bounded domain of $\mathbb{R}^N$. If $w \in LSC(A)$ satisfies*

$$w \geq 0\ , \quad \mathcal{P}^-_{\lambda,\Lambda}(D^2 w) - b(x)|Dw| \leq g(x) \tag{2.4}$$

with $b, g \in C(A) \cap L^\infty(A)$,in the viscosity sense, then there exist positive numbers C, p depending on $N, \lambda, \Lambda, \|b\|_{L^\infty(B_4)}$ such that

$$\left(\frac{1}{|B_1|}\int_{B_1} w^p\right)^{1/p} \leq C\left(\inf_{B_2} w + \|g\|_{L^N(B_4)}\right) \tag{2.5}$$

where $B_1 \subset B_2 \subset B_4 \subset A$ are concentric balls of radii $1, 2$ and 4, respectively.

Let B_R, $B_{R/\tau}$ with $\tau \in (0,1)$ be concentric balls such that

$$A \cap B_R \neq \emptyset\ , \quad B_{R/\tau} \backslash A \neq \emptyset\ .$$

For $w \in LSC(\bar{A})$, $w \geq 0$, consider the following lower semicontinuous extension w_m^- of w

$$w_m^-(x) = \begin{cases} \min(w(x); m) & \text{if } x \in A \\ m & \text{if } x \notin A \end{cases}$$

where $m = \inf_{x \in \partial A \cap B_{R/\tau}} w(x)$.

Proposition 2.2 (the boundary weak Harnack inequality). *Let A be an open bounded domain of $\mathbb{R}^N$. If $w \in LSC(A)$ satisfies (2.4) in the viscosity sense, with $b, g \in C(A) \cap L^\infty(A)$ in the viscosity sense, then*

$$\left(\frac{1}{|B_R|}\int_{B_R} (w_m^-)^p\right)^{1/p} \leq C^*\left(\inf_{A \cap B_R} w + R\,\|g^+\|_{L^N(A \cap B_{R/\tau})}\right) \tag{2.6}$$

where p and C^* depend on $N, \lambda, \Lambda, \tau, R\|b\|_{L^\infty(A)}$.

See [3] for the case $b \equiv 0$ and [6] for the proof in the (slightly) more general case $b \neq 0$.

3. A general class of unbounded domains in $\mathbb{R}^N$

As mentioned in the Introduction, the aim of this Note is to present some results about the validity of the **wMP** for equation (1.1) in general domains. Let us consider then domains in $\mathbb{R}^N$ satisfying the following measure/geometric condition **wG**:
there exist constants $\sigma, \tau \in (0,1)$ such that for all $y \in \Omega$ there is a ball B_{R_y} of radius R_y containing y such that

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y.

The above condition, proposed first in [5], [22], generalizes the notion of **G** domains previously introduced by X. Cabrè: any domain Ω fulfilling condition **wG** with

$$R_y = O(1) \quad \text{as} \quad |y| \to \infty$$

is in fact a **G** domain in the sense of [4].
Condition **wG** is therefore satisfied by any **G** domain, for example:

- bounded domains: in this case, $R_y \equiv \operatorname{diam}(\Omega)$
- domains with finite measure: in this case, $R_y \equiv C(N)|\Omega|^{1/N}$

- unbounded cylinders with bounded cross-section:
$$\Omega = \mathbb{R}^k \times \omega \subset \mathbb{R}^{N-k} \ , \quad k \geq 1 \ .$$
In this case, $R_y \equiv \operatorname{diam}(\omega)$
- periodically perforated domains with period $\rho > 2$:
$$\Omega = \mathbb{R}^N \setminus \sum_{k \in \mathbb{Z}^N} (\rho k + B_1(0)) \ .$$
In this case $R_y \equiv \rho$
- the complement of a plane spiral with constant step k, represented in polar cohordinates as
$$\Omega = \mathbb{R}^2 \setminus \left\{ \rho = \frac{k}{2\pi}\, \theta \right\} \ .$$
In this case $R_y \equiv k$

Note that condition **G** implies in particular $\sup_{y \in \Omega} \operatorname{dist}(y, \partial\Omega) < +\infty$; on the other hand **wG** domains can have points at arbitrarily large distance from the boundary. Typical examples of unbounded domains satisfying **wG** but not **G** are:

- non-degenerate cones of $\mathbb{R}^N$ (and their unbounded subsets); for such a set **wG** holds with $R_y = O(|y|)$ as $|y| \to \infty$
- parabolically shaped domains, defined for $k > 1$ by the inequalities
$$|x'| \equiv \sqrt{\sum_{i=1}^{N-1} x_i^2} < x_N^{1/k} \ , \quad x_N > 0 \ .$$
Condition **wG** holds in this case with $R = O(x_N^{1/k})$
- the complement of the logarithmic spiral: $\Omega = \mathbb{R}^2 \setminus \{ \varrho = \mathrm{e}^\theta, \ \theta \geq 0 \}$. Condition **wG** is satisfied with $R_y = O(|y|)$ as $|y| \to \infty$

To conclude this section, let us point out explicitly that exterior domains such as $\mathbb{R}^N \setminus B_R$ are not **wG**.

4. The structure conditions on F

We shall assume that $F = F(x, t, p, X)$ is continuous with respect to all variables and (degenerate) elliptic, that is

$$F(x, t, p, X) \geq F(x, t, p, Y) \tag{4.1}$$

for all $x \in \Omega$, $t \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $X, Y \in \mathcal{S}^N$ with $X - Y \geq O$.
Moreover, we assume that the following structure condition holds for all $x \in \Omega$, $t \geq 0$, $p \in \mathbb{R}^N$ and $X \in \mathcal{S}^N$:

$$F(x, t, p, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + b(x)\,|p| \tag{4.2}$$

for some non-negative function $b \in C(\Omega) \cap L^\infty(\Omega)$ and for all $x \in \Omega$, $t \geq 0$, $p \in \mathbb{R}^N$, $X \in \mathcal{S}^N$. Here, $\mathcal{P}^+_{\lambda,\Lambda}$ is the Pucci maximal operator.
Assumptions (4.1) and (4.2) are satisfied by any uniformly elliptic F, see (2.2), such that

$$t \to F(x, t, p, X) \quad \text{non increasing,} \ \ F(x, 0, p, O) \leq b(x)\,|p| \ .$$

Note, however, that some nonlinear degenerate elliptic operators fulfill our assumptions. An example is

$$F(X) = \Lambda \left(\sum_{i=1}^{N} \varphi(\mu_i^+) \right) - \lambda \left(\sum_{i=1}^{N} \psi(\mu_i^-) \right) .$$

Here, $\mu_i^\pm$, $i = 1, \dots N$, are the positive and negative eigenvalues of the matrix $X \in \mathcal{S}^N$ and $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ are continuous and nondecreasing functions such that $\varphi(s) \leq s \leq \psi(s)$.
Observe, finally, that while $X \to \mathcal{P}^+_{\lambda,\Lambda}(X)$ is convex, the structure condition does not require convexity nor concavity of $X \to F(x, t, p, X)$.
We will also consider the case of F having quadratic growth in the gradient variable. In order to treat this case we will employ the structure condition

$$F(x, t, p, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + b(x)|p| + b_2|p|^2 \tag{4.3}$$

for $t \geq 0$, where b_2 is a positive constant.

5. The Weak Maximum Principle in unbounded domains

We present in this section some recent result concerning the validity of the **wMP** for upper semicontinuous viscosity solutions of the partial differential inequality

$$F\left(x, u(x), Du(x), D^2u(x)\right) \geq 0 \ , \ x \in \Omega \ , \tag{5.1}$$

in unbounded domains Ω of type **wG** and for degenerate elliptic functions F satisfying the structure condition (4.2) or (4.3)) . We will consider in the next subsections a few different quite general situations in which the validity of the **wMP** can be established:

- bounded above solutions, linear growth in Du
- bounded above solutions, quadratic growth in Du
- bounded above solutions in domains of small measure and/or for operators with a small zero-order term
- exponentially growing solutions in narrow domains
- Phragmèn-Lindelöf theorems in cylindrical and conical domains

5.1. **Bounded above solutions, linear growth in Du.** Our first result is an Alexandrov-Bakelman-Pucci type estimate for solutions of

$$F\left(x, u(x), Du(x), D^2u(x)\right) \geq f(x) \ , \ x \in \Omega \ . \tag{5.2}$$

Theorem 5.1. *Let $u \in USC(\overline{\Omega})$ with $\sup_\Omega u < +\infty$ be a viscosity solution of* (5.2) *where $f \in C(\Omega) \cap L^\infty(\Omega)$. Assume that F is continuous and satisfies* (4.1) *and* (4.2)*. Assume, moreover, that Ω satisfies* **wG** *for some R_y such that*

$$Rb := \sup_{y \in \Omega} R_y \, \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty \ . \tag{5.3}$$

Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \sup_{y \in \Omega} R_y \, \|f^-\|_{L^N(\Omega_{y,\tau})} \tag{5.4}$$

for some positive constant C depending on N, λ, Λ, σ, τ and Rb.

As an immediate consequence of the above result, the **wMP** holds:
if $u \in USC(\overline{\Omega})$ is bounded above and $F(x,u,Du,D^2u) \geq 0,\ x \in \Omega$, in the viscosity sense then

$$u \leq 0 \quad \text{on } \partial\Omega \quad \text{implies } u \leq 0 \text{ in } \Omega\ .$$

Remark 5.2. For $F = F(X)$ and Ω bounded, the estimate (5.4) has been established in [3]. When F does not depend on Du, then $b \equiv 0$ and condition (5.3) is trivially satisfied in any **wG** domain. In general, however, some condition relating the size of the domain with the size of first order terms at infinity is required for the validity of the **wMP** in unbounded domains. Indeed,

$$u(x) = u(x_1, x_2) = \left(1 - \mathrm{e}^{1-x_1^\alpha}\right)\left(1 - \mathrm{e}^{1-x_2^\alpha}\right)$$

with $0 < \alpha < 1$, is bounded and satisfies

$$u_{|\partial\Omega} = 0\ , \quad u > 0\ , \quad \Delta u + B(x) \cdot Du = 0 \quad \text{in } \Omega$$

in the cone

$$\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2\ :\ x_1 > 1,\ x_2 > 1\right\}$$

with B given by

$$B(x) = B(x_1, x_2) = \left(\frac{\alpha}{x_1^{1-\alpha}} + \frac{1-\alpha}{x_1}\ ,\ \frac{\alpha}{x_2^{1-\alpha}} + \frac{1-\alpha}{x_2}\right)\ .$$

Since Ω satisfies **wG** with $R_y = O(|y|)$ as $|y| \to \infty$ and the structure condition (4.2) holds with $b(x) = |B(x)|$, condition (5.3) fails in this example.

Remark 5.3. For bounded b, in order to enforce (5.3) the requirement on Ω amounts to

$$\sup_{y\in\Omega} R_y \leq R_0 < +\infty$$

meaning that Ω should be in fact a **G** domain. For **G** domains, the result of our Theorem 5.1 can be regarded essentially as a nonlinear version of the Alexandrov-Bakelman-Pucci estimate for linear elliptic equations with bounded coefficients proved in [4].
In the case of parabolically shaped domains, defined for $k > 1$ by the inequalities

$$|x'| \equiv \sqrt{\sum_{i=1}^{N-1} x_i} < x_N^{1/k}, x_N > 0\ ,$$

for which **wG** holds with $R = O(x_N^{1/k})$, one can show that (5.3) holds provided

$$b(x) = O(1/x_N^{1/k}) \quad \text{as } |x| \to \infty\ .$$

Note that cylindrical and conical domains can be seen as limiting cases of above situation when, respectively, $k \to +\infty$ and $k \to 1$.

The detailed proof of Theorem 5.1 can be found in [6]. The first step is to observe that $w(x) = M - u^+(x)$ with $M := \sup_{x\in\Omega} u^+(x) < +\infty$ satisfies

$$w \geq 0\ , \quad \mathcal{P}^-(D^2w) - b(x)\,|Dw| \leq f^-(x) \quad \text{in } \Omega$$

in the viscosity sense. By the boundary weak Harnack inequality (2.6)

$$\left(\frac{1}{|B_{R_y}|}\int_{B_{R_y}} (w_m^-)^p\right)^{1/p} \leq C_y^*\left(\inf_{\Omega_{y,\tau}\cap B_{R_y}} w + R_y\,\|f^-\|_{L^N(\Omega_{y,\tau})}\right) \tag{5.5}$$

for positive constants p and C_y^* depending on N, λ, Λ, τ and R_y, $\|b\|_{L^\infty(\Omega_{y,\tau})}$.
Using the **wG** condition it is not hard to show that the left-hand side of the above inequality can be estimated from below as follows

$$\left(\frac{1}{|B_{R_y}|}\int_{B_{R_y}}(w_m^-)^p\right)^{1/p} \geq m\,\sigma^{1/p}\ . \tag{5.6}$$

From (5.5), (5.6) we deduce that

$$m\,\sigma^{1/p} \leq C_y^*\ \left(M - u^+(y) + R_y\,\|f^-\|_{L^N(\Omega_{y,\tau})}\right)$$

at any point $y \in \Omega$. Observing that $m \geq M - \sup_{z\in\partial\Omega} u^+(z)$, after some simple computations we are led to the pointwise estimate

$$u^+(y) \leq \left(1 - \frac{\sigma^{1/p}}{C_y^*}\right)\sup_\Omega u^+ + \frac{\sigma^{1/p}}{C_y^*}\ \sup_{\partial\Omega} u^+ + R_y\,\|f^-\|_{L^N(\Omega_{y,\tau})} \tag{5.7}$$

Thanks to the assumption (5.3), the constant $C_y^*/\sigma^{1/p}$ can be bounded above by some $\theta \in (0,1)$, independently on y. Taking the supremum on both sides of (5.7), we obtain (5.4).

5.2. **Bounded above solutions, quadratic growth in Du.** Let us briefly describe how the results of the previous section can be extended to the case of an F having at most quadratic growth in the gradient variable.
Observe at this purpose that if v, $0 \leq v \leq M = \sup v$ is a viscosity solution of

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2 v) - b(x)|Dv| - b_2|Dv|^2 \leq g(x) \tag{5.8}$$

then the Hopf-Cole type transform

$$w = h^{-1}(v)$$

where h is smooth, non-negative, increasing and convex, satisfies

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2 w) + \lambda\frac{h''(w)}{h'(w)};|Dw|^2 - b(x)|Dw| - b_2 h'(w)|Dw|^2 \leq \frac{g(x)}{h'(w)}$$

in the viscosity sense. The proof of this fact requires some viscosity calculus together with the superadditivity and the ellipticity of $\mathcal{P}^-_{\lambda,\Lambda}$.
Solving the ordinary differential equation

$$\lambda h^{\prime'}(t) - b_2(h')^2(t) = 0$$

one finds

$$h(t) = \frac{\lambda}{b_2}\ \log\left(1 - \frac{b_2 t}{\lambda}\right)^{-1}$$

which satisfies the required properties for $t \in [0, \frac{\lambda}{b_2})$. Correspondingly, the function

$$w = \frac{\lambda}{b_2}\left(1 - e^{-b_2 v/\lambda}\right)$$

is a solution of

$$w \geq 0\ ,\quad \mathcal{P}^-_{\lambda,\Lambda}(D^2 w) - b(x)|Dw| \leq g(x)\left(1 - \frac{1}{\lambda}\ b_2 w\right)\ .$$

Applying inequality (2.5) of Section 2 to w and observing that

$$\frac{1 - e^{-b_2 M/\lambda}}{\frac{b_2 M}{\lambda}}\ v \leq w \leq v$$

we conclude that the weak Harnack inequality

$$\left(\frac{1}{|B_1|}\int_{B_1} v^p\right)^{1/p} \le C\left(\inf_{B_2} v + \|g\|_{L^N(B_4)}\right) \tag{5.9}$$

holds for solutions of (5.8).

Remark 5.4. Observe that the constant C depends on b_2M. The dependence on the upper bound M in the estimate seems to be unavoidable, see [21], [15].

A boundary version of inequality (5.9) can be easily obtained in the present setting much in the same way as in Section 2:

$$\left(\frac{1}{|B_R|}\int_{B_R} (v_m^-)^p\right)^{1/p} \le C^*\left(\inf_{A\cap B_R} v + R\,\|g^+\|_{L^N(A\cap B_{R/\tau})}\right)$$

where p and C^* are positive constants depending on N, λ, Λ, τ, b_2M and $R\|b\|_{L^\infty(A\cap B_{R/\tau})}$.

The Alexandrov-Bakelman-Pucci estimate and the **wMP** continue therefore to hold true in the quadratic case under consideration:

Theorem 5.5. *Let $u \in USC(\overline{\Omega})$ with $\sup_\Omega u < +\infty$ be a viscosity solution of* (5.2) *where $f \in C(\Omega) \cap L^\infty(\Omega)$.*
Assume that F is continuous and satisfies (4.1) *and* (4.3). *Assume, moreover, that Ω satisfies* **wG** *for some R_y such that*

$$Rb := \sup_{y\in\Omega} R_y\,\|b\|_{L^\infty(\Omega_{y,\tau})} < \infty\ .$$

Then

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+ + C\,\sup_{y\in\Omega} R_y\,\|f^-\|_{L^N(\Omega_{y,\tau})}$$

for some positive constant C depending on N, λ, Λ, σ, τ and Rb.

As an immediate consequence of the above result, the **wMP** holds:
if $u \in USC(\overline{\Omega})$ is bounded above and $F(x,u,Du,D^2u) \ge 0,\ x \in \Omega$, in the viscosity sense then

$$u \le 0 \text{ on } \partial\Omega \quad \text{implies } u \le 0 \text{ in } \Omega\ .$$

5.3. **Bounded above solutions for the perturbed operator** $F + c(x)$**.** The next result, see [2] for the linear case, establishes the validity of a qualitative version of the **wMP** for the perturbed operator $F + c(x)$ under a condition relating the radii R_y in condition **wG** with the size of function c^+. Note that the case $c \le 0$ is trivially included in Theorem 5.1.

Theorem 5.6. *Let $u \in USC(\overline{\Omega})$ with $\sup_\Omega u < +\infty$ and $u \le 0$ on $\partial\Omega$ be a viscosity solution of*

$$F(x,u,Du,D^2u) + c(x)\,u \ge f(x)$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$. Assume that F is continuous and satisfies (4.1) *and* (4.2). *Assume, moreover, that Ω satisfies* **wG** *for some R_y such that*

$$Rb := \sup_{y\in\Omega} R_y\,\|b\|_{L^\infty(\Omega_{y,\tau})} < \infty$$

and that

$$\sup_{y\in\Omega} R_y^2\,\|c^+\|_{L^\infty(\Omega_{y,\tau})} \quad \textit{is sufficiently small.}$$

Then

$$\sup_{\Omega} u \leq C \sup_{y\in\Omega} R_y \, \|f^-\|_{L^N(\Omega_{y,\tau})}$$

for some positive constant C depending on N, λ, Λ, σ, τ and Rb.

Since

$$F(x,u,Du,D^2u) - c^-(x)\,u \geq -c^+(x)\,u + f(x)$$

a direct application of Theorem 5.1 yields

$$\begin{aligned}\sup_{\Omega} u &\leq C \sup_{y\in\Omega} R_y \, \left(\|(-c^+u)^-\|_{L^N(\Omega_{y,\tau})} + \|f^-\|_{L^N(\Omega_{y,\tau})}\right) \\ &\leq C' \sup_{y\in\Omega} R_y^2 \, \|c^+\|_{L^\infty(\Omega_{y,\tau})} \sup_{y\in\Omega} u^+(y) + C \sup_{y\in\Omega} R_y \, \|f^-\|_{L^N(\Omega_{y,\tau})} \ .\end{aligned}$$

If $\sup_{y\in\Omega} R_y^2 \|c^+\|_{L^\infty(\Omega_{y,\tau})}$ is sufficiently small, we conclude that $\sup_\Omega u \leq \theta \sup_\Omega u^+$ for some $\theta < 1$ and the statement follows.

Remark 5.7. Theorem 5.6 applies of course when either $\sup_{y\in\Omega} R_y$ or $\|c^+\|_{L^\infty(\Omega)}$ is small enough, e.g. if $|\Omega|$ is finite and sufficiently small. A more interesting case is when Ω a strictly convex cone with sufficiently small opening and $c^+ = O(1/|y|^2)$ as $|y| \to \infty$. Indeed, in this case we can apply Theorem 5.6 taking $R_y \leq \epsilon\,|y|$, for sufficiently small ϵ, in condition **wG** and $\|c^+\|_{L^\infty(\Omega_{y,\tau})} = O(1/|y|^2)$.

5.4. **Exponentially growing solutions in narrow domains.** The next result shows that a qualitative version of the **wMP** holds even for unbounded above solutions of (5.2) provided the unbounded domain satisfies an appropriate narrowness condition related to the admissible rate of growth at infinity of the solution. More precisely, consider the unbounded cylinder

$$\Omega = \mathbb{R}^k \times \omega \quad \text{with } k + h = N, \ h, k \geq 1 \ ,$$

where ω is a bounded domain of $\mathbb{R}^h$. As pointed out in Section 3 this is typical example of **G** domain.

Theorem 5.8. *For F as in Theorem 5.1 and Ω as above, suppose $\|b\|_{L^\infty(\Omega)} \leq b_1$ and let*

$$u \in USC(\overline{\Omega}) \ , \ F\left(x, u(x), Du(x), D^2u(x)\right) \geq 0 \ , \quad x \in \Omega \ ,$$

with

$$u \leq 0 \quad on\ \partial\Omega \ , \quad u^+(x) = o(e^{\beta|x|}) \ as\ |x| \to +\infty \ .$$

Then for any $\beta > 0$ there exists a positive number $d = d(N, \lambda, \Lambda, b_1, \beta)$ such that, if $\operatorname{diam}(\omega) < d$, *then $u \leq 0$ in Ω.*

Conversely, for any fixed $d > 0$ there exists $\beta = \beta(N, \lambda, \Lambda, b_1, d)$ such that if $\operatorname{diam}(\omega) < d$, *then $u \leq 0$ in Ω.*

Qualitative results of this type for general linear uniformly elliptic operators can be found in [1], for semilinear equations on cylinders.
In the special case of subharmonic functions on the 2-dimensional strip $\Omega = \mathbb{R} \times (0, d)$ there is a precise quantitative relationship between the diameter d and the growth exponent β, namely $\beta = \pi/d$, see [12]. The proof of Theorem 5.8 is based on the construction of a suitable sequence of smooth barrier functions Φ_k on finite cylinders $\bar{C}_k = \bar{B}_k(0) \times \bar{\omega}$, $k \in \mathbb{N}$, such that

$$\mathcal{P}^+\left(D^2\Phi_k(x)\right) + b_1\,|D\Phi_k(x)| \leq 0 \quad \text{in } C_k \ ,$$

$$\Phi_k \geq 0 \quad \text{in } \bar{C}_k \ , \quad \Phi_k \geq u^+ \quad \text{on } \partial\bar{C}_k \backslash \partial\Omega$$

and for each fixed $x \in \Omega$

$$\lim_{k\to\infty} \Phi_k(x) = 0 \; .$$

The barriers have the form

$$\Phi_k(x,y) = K_k/(e^{\beta R}\cos^h(\alpha d/2)) \exp(\beta|x|) \prod_{j=1}^{h} \sin \alpha y_j$$

for suitable choices of the parameters. It is a familiar technique in the case of a linear operator to use the **wMP** in bounded domains C_k, considering differences $u - \Phi_k$, and then passing to the limit as $k \to \infty$. The difficulty in implementing this procedure in the present nonlinear setting is overcome by the use of the structure condition (4.2), together with the superadditivity of the maximal Pucci operator, since standard calculus rules apply due to the fact that Φ_k is twice continuously differentiable, see [6] for details.

A similar result holds for viscosity subsolutions with polynomial growth $u(x) = O(|x|^\alpha)$ in angular sectors $\Omega = \mathbb{R}^k \times \omega$, where ω is a cone in $\mathbb{R}^h$ and $h + k = N$, provided (4.2) holds true with $b(x) = O(1/|x|)$ as $|x| \to \infty$. In this case, in order to deduce the validity of the **wMP** the opening of the cone has to be sufficiently small depending on the exponent α and the various structural parameters. We refer to [18] for previous results in this direction.

5.5. **Phragmen-Lindelöf type theorems in general domains.** In Subsection 5.4 we proposed some Phragmèn-Lindelöf type results for viscosity solutions in cylinders and narrow cones. On this basis, one may expect that **wMP** should hold in more general **wG** domains of cylindrical type (that is, **wG** holds with $R_y \le R < +\infty$) or conical type (that is, **wG** holds with $R_y = O(|y|)$ as $|y| \to \infty$) under a suitable exponential, respectively, polynomial growth of subsolutions at infinity. However, the explicit constructions of the barrier functions used in the proofs of the above mentioned results, see [6] for more details, relies heavily on the simple geometry of cylinders and cones and cannot be easily carried over to geometrically more complex general **G** or **wG** domains.
An alternative way to obtain qualitative Phragmèn-Lindelöf type results in general cylindrical or conical **wG** domains relies instead on the validity of the Maximum Principle for bounded above viscosity solutions of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 w(x)) + \gamma_1(x)|Dw(x)| + \gamma_2(x) w^+(x) \ge 0$$

where the coefficient γ_2 is allowed to be positive but suitably small.
Indeed, by Theorem 5.6, if $\gamma_2^+(x) \le c_1$ (in the case of cylindrical domains) and $\gamma_2^+(x) \le c_1/|x|^2$ as $|x| \to \infty$ (in the case of conical domains), then the **wMP** holds provided c_1, a positive number depending on the structure of F and on the geometric parameters occurring in the **G** or **wG** conditions, is small enough. Two model results in this direction are the following:

Theorem 5.9. *Assume that Ω is a* **wG** *domain of $\mathbb{R}^N$ of cylindrical type and that F satisfies*

$$F(x,t,p,X) \le \mathcal{P}^+_{\lambda,\Lambda}(X) + b(x)|p| + c(x)t$$

with

$$|b(x)| \le b_0 \; , \quad c(x) \le c_1 \; ,$$

$c_1 > 0$ small enough. Then there exists $\alpha > 0$, depending on F and Ω, such that if $u \in USC(\overline{\Omega})$ is a viscosity solution of

$$F\left(x, u(x), Du(x), D^2u(x)\right) \geq 0 \quad \text{in } \Omega$$

with $u \leq 0$ on $\partial\Omega$ and $u(x) = O(e^{\alpha|x|})$ as $|x| \to +\infty$, then $u \leq 0$ in Ω.

Theorem 5.10. *Assume that Ω is a* **wG** *domain of conical type and that F satisfies*

$$F(x, t, p, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + b(x)|p| + c(x)t$$

with

$$|b(x)| \leq \frac{b_0}{(1+|x|^2)^{\frac{1}{2}}}\ , \quad c(x) \leq \frac{c_1}{1+|x|^2}\ ,$$

$c_1 > 0$ small enough. Then there exists $\alpha > 0$, depending on F and Ω, such that if $u \in USC(\overline{\Omega})$ is a viscosity solution of

$$F\left(x, u(x), Du(x), D^2u(x)\right) \geq 0 \quad \text{in } \Omega$$

with $u \leq 0$ on $\partial\Omega$ and $u(x) = O(|x|^\alpha)$ as $|x| \to +\infty$, then $u \leq 0$ in Ω.

A sketchy proof of Theorem 5.10 starts with the consideration of the smooth positive function

$$\xi(x) = (1+|x|^2)^{\alpha/2}$$

where $\alpha > 0$ is a parameter. If $u(x) = O(|x|^\alpha)$ then

$$w(x) = \frac{u(x)}{\xi(x)}$$

is bounded above and obviously $w(x) \leq 0$ on $\partial\Omega$. A straightforward calculation shows now that

$$\frac{|D\xi|}{\xi} \leq \frac{\alpha}{2(1+|x|^2)^{1/2}}\ , \quad \frac{|D^2\xi|}{\xi} \leq \frac{2N\,\alpha}{1+|x|^2}\ .$$

By some viscosity calculus and using the decay condition on b we deduce that

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w(x)) + \gamma_1(x)|Dw(x)| + \gamma_2(x)w^+(x) \geq 0$$

with

$$\gamma_1(x) = \frac{CN\Lambda\alpha + b_0}{2(1+|x|^2)^{1/2}}\ , \quad \gamma_2(x) = \frac{\alpha(CN^2\Lambda + b_0) + c_1}{1+|x|^2}$$

for some positive constant C. The zero order coefficient γ_2 in the above inequality can be made are arbitrarily small by choosing suitably small values of α. From Theorem 5.6 it follows then that w and u are non positive on Ω.
The proof of Theorem 5.9 goes the same way, modulo the use of the function

$$\zeta(x) = e^{\alpha(1+|x|^2)^{1/2}}$$

instead of ξ in the above computations.

Remark 5.11. Theorems 5.9 and 5.10 above extend in particular the results of [16], [23] in the direction of more general unbounded domains as well as of viscosity solutions of (non necessarily uniformly) elliptic fully nonlinear differential inequalities containing lower order terms. Finally, let us point out that, in view of the discussion in Subsection 5.2, the Phragmèn-Lindelöf theorems above continue to hold true for operators with quadratic growth in the gradient variable.

References

[1] H. Berestycki, L. A. Caffarelli & L. Nirenberg, *Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash*, Duke Math. J., 81(2)(1996), 467–494.

[2] H. Berestycki. L. Nirenberg & S.R.S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math., 47(1)(1994), 47–92.

[3] L.A. Caffarelli, X. Cabrè, *Fully Nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, 43, American Mathematical Society, Providence, RI, 1995.

[4] X. Cabrè, *On the Alexandroff-Bakelman-Pucci estimate and reversed Holder inequalities for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math., 48(1995), 539–570.

[5] V. Cafagna & A. Vitolo, *On the maximum principle for second-order elliptic operators in unbounded domains*, C.R. Acad. Sci. Paris Ser. I, 334(5)(2002), 359–363.

[6] I. Capuzzo Dolcetta, F. Leoni & A. Vitolo, *The Alexandrov-Bakelman-Pucci weak maximum principle for fully nonlinear equations in unbounded domains*, Comm. Partial Differential Equations, 30(2005), 1863–1881.

[7] I. Capuzzo Dolcetta & A. Vitolo, *A Qualitative Phragmèn-Lindelöf theorem for fully nonlinear elliptic equations*, J. Differential Equations, 243(2)(2006), 578–592.

[8] I. Capuzzo Dolcetta & A. Vitolo, *Local and global estimates for viscosity solutions of fully nonlinear elliptic equations*, Dynamics of Continuous, Discrete & Impulsive Systems. Series A: mathematical Analysis, 14(S2)(2007), 11–16.

[9] I. Capuzzo Dolcetta & A. Vitolo, *On the maximum principle for viscosity solutions of fully nonlinear elliptic equations in general domains*, Le Matematiche, LXII(II)(2007), 69–91.

[10] M.G. Crandall, H. Ishii & P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27(1)(1992), 1–67.

[11] W.H. Fleming & H.M. Soner, *Controlled Markov processes and viscosity solutions*, Applications of Mathematics 25, Springer-Verlag, Berlin-New York, 1993.

[12] L.E. Fraenkel, *Introduction to maximum principles and symmetry in elliptic problems*, Cambridge University Press, 2000.

[13] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of second order*, II ed., Grundlehren der Mathematischen Wissenschaften 224, Springer-Verlag, Berlin-New York, 1983.

[14] H. Ishii, *On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's*, Comm. Pure Appl. Math., 42(1989), 14–45.

[15] S. Koike & A. Swiech, *Maximum principle and existence of L^p viscosity solutions for fully nonlinear uniformly elliptic equations with measurable and quadratic terms*, NoDEA, 11(2004), 491–509.

[16] V.A. Kondrate'v & E.M. Landis, *Qualitative theory of second order linear partial differential equations*, Partial Differential Equations III, Egorov, Yu.V. & Shubin M.A. eds., Encyclopedia of Mathematical Sciences, 32(1991), 87–192.

[17] E.M. Landis, *Second order equations of elliptic and parabolic type*, Translations of Mathematical Monographs 171, Amer. Math. Soc., Providence, R.I., 1998.

[18] K. Miller, *Extremal barriers on cones with Phragmèn-Lindelöf theorems and other applications*, Ann. Math. Pura Appl. (4), 90(1971), 297–329.

[19] M.H. Protter & H.F Weinberger, em Maximum principles in differential equations, II ed., Springer-Verlag, New York, 1984.

[20] C. Pucci, *Operatori ellittici estremanti*, Ann. Mat. Pura Appl. (4), 72(1966), 141–170 .

[21] N.S. Trudinger, *Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations*, NoDEA, 3-4(1988), 453–468.

[22] A. Vitolo, *On the maximum principle for complete second-order elliptic operators in general domains*, J. Differential Equations, 194(1)(2003), 166–184.

[23] A. Vitolo, *On the Phragmén-Lindelöf principle for second-order elliptic equations*, J. Math. Anal. Appl., 300(1)(2004), 244–259.