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AN ENVELOPE ENERGY MODEL FOR HIGH FREQUENCY DYNAMIC STRUCTURES

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High frequency structural and acoustic problems require prohibitive computational efforts. The tendency, nowadays, is to find a solution in statistical terms (SEA) through an average of the field variables on the space domain. A limitation of SEA is the loss of any local information. In contrast with SEA, a power flow method [1] can describe a trend of the energy density along the structure, thus improving the quality of the solution. However, in dealing with flexural waves, the power flow neglects the near field contribution: the related solution can sometimes differ considerably from the expected trend. In this paper a field trend is obtained in a totally different manner. An envelope energy is used that describes well the exact solution: specifically, only the decaying fields, obtained from the projection on the real axis of the damped bending wavenumbers are accounted for, while the propagating components are omitted. Simulated results are presented and compared with exact and approximate solutions.

1. INTRODUCTION

A local solution for acoustic and structural system in the high frequency range is generally not feasible because of the unacceptable dimension of the related numerical problem. An approach often adopted in practice is the evaluation of a space-average description of the field variables obtained through energy balance equations (SEA). The SEA solution yields a unique value of the field variables of interest, such as the means square velocity of the structure or the average pressure in an acoustic cavity. However, SEA provides a reliable result only if, in the frequency range of interest, the structural (and/or acoustic) subsystem is dominated by a high modal density. When this condition fails, a solution with large confidence limits is obtained.

To give a more detailed local description, Nefske and Sung [1] developed a power flow finite element analysis, capable of providing a spectral-average trend of the energy density along the structure. The method is also known as the "thermal analogy" because of the similarity of the differential energy equation with that for heat conduction in thermal problems. Subsequently other authors have reconsidered the thermal analogy and an analogous equation for the time-averaged energy density has been obtained [2].

By considering the physics of power transmission in three-dimensional structures, it was shown [3] that an exact time-averaged energy density equation can be obtained only for particular structures such as beams and plates. Yet, even in these simple cases, the power flow does not have a thermal-like behaviour and the equations depend non-linearly on the energy density. The analysis developed in reference [3] gives some insight into the mechanism of power transmission in mechanical structures and permits one to understand the limits of the thermal analogy. However, the complex form of the energy equations obtained suggests their rejection for practical applications. Drawbacks are represented by

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non-linear terms and by the oscillating behaviour of the energy density, which is the same serious limitation possessed by the much more simple equations of motion in terms of displacement.

Therefore the use of approximate solutions having a smooth non-oscillatory trend, similar to the one obtained by the thermal analogy [1], is desirable. This trend, in fact, provides a useful and convenient description of dynamic problems at high frequencies, avoids numerical pitfalls and can be numerically solved by standard finite element codes, with the use of a simple coarse mesh.

Attention here will be addressed to one-dimensional damped flexural structures.

Wohlever and Bernhard [2] discussed the approximations under which the thermal behaviour is valid for flexural beams: i.e., neglect of the near field contribution; neglect of the harmonic parts in the time-average energy expression.

Although the results obtained in references [1, 2] are satisfactory, they refer to particular cases, and do not provide indications on the behaviour in more complex situations or on the level of approximation introduced.

A theoretical investigation is presented here of an envelope energy, defined through the Hilbert transform, which yields a smooth description of the system's dynamic. Then the effect of damping on the flexural wavenumbers is considered. Through this analysis, a satisfactory trend for the energy distribution is obtained by the envelope energy, with use only of the decaying components of the whole damped solution.

To check the validity of the procedure, simulated experiments are performed on differently loaded and constrained beams. The results are compared with both the theoretical and power flow solutions presented in reference [1].

2. ENVELOPE ENERGY MODEL FOR FLEXURAL BEAMS

The relevant merit of the power flow method is that it has shown the possibility of approaching a trend solution for high frequency structural problems and has provided indications of the direction which can be taken in further investigations. In this context, a natural idea is the use of a suitable envelope concept. An appropriate envelope kinetic energy $\mathscr{T}$, in fact, can be defined through the Hilbert transform, that produces a space-average trend of the energy itself.

2.1. BASIC CONCEPTS AND DEFINITIONS

The Hilbert transform [4] is an integral operator the kernel of which is $1/\pi x$:

$$\mathscr{H}\{f(x)\} = \int_{-\infty}^{\infty} \frac{f(\xi)}{\pi(x-\xi)}\,\mathrm{d}\xi = f(x) * \frac{1}{\pi x}.$$

It exists under the hypothesis of existence of the Fourier transform: i.e., it is required that $f(x)$ is absolutely integrable over the definition domain. The Fourier transform F of the Hilbert transform is

$$F[\mathscr{H}\{f(x)\}] = F\{f(x)\}F\{1/\pi x\},$$

where

$$F\left\{\frac{1}{\pi x}\right\} = \begin{Bmatrix} -j & \text{for the wavenumber } k>0 \\ j & \text{for the wavenumber } k<0 \end{Bmatrix}.$$

Therefore the Hilbert transform is an operator that shifts all the harmonic components of $f(x)$ by $\pi/2$. Then, through the Hilbert transform, an envelope $\mathscr{F}(x)$ of a function $f(x)$ can be defined as [4, 5]

$$\mathscr{F}(x)=[f^2(x)+\mathscr{H}^2\{f(x)\}]^{1/2}.$$

Because of the shifting property of $\mathscr{H}$, the envelope tends to cancel the harmonic components of $f(x)$, performing a smoothing operation.

In the rest of this paper, the modulation property of the Hilbert transform will be used: i.e.,

$$\mathscr{H}[p(x)\,e^{-jkx}]=jp(x)\,e^{-jkx}, \qquad \mathscr{H}[p(x)\,e^{jkx}]=-jp(x)\,e^{jkx}, \tag{1}$$

where $p(x)$ is an arbitrary function that must satisfy only a band-limited condition [5].

Besides the smoothing property, the envelope has the following energetic property, that will be useful for further considerations:

$$\int_{-\infty}^{\infty} f^2(x)\,dx=\int_{-\infty}^{\infty}\mathscr{H}^2\{f(x)\}\,dx. \tag{2}$$

If a stationary wave $\hat{w}(x, t)=\mathrm{Im}\,[w(x)\,e^{j\omega t}]$ is forced into a flexural beam, the local kinetic energy per unit length can be written as

$$T(x, t)=\tfrac{1}{2}\rho S\dot{w}^2(x, t),$$

S being the cross-section of the beam. The time-average energy is then

$$\langle T\rangle=\kappa_T w^2(x),$$

with $\kappa_T=\frac{1}{4}\rho S\omega^2$. According to the previous proposals, the envelope kinetic energy of an undamped structure can be defined as

$$\mathscr{T}=(\kappa_T/2)[w^2+\mathscr{H}^2(w)].$$

By using this definition, property (2) provides an energetic equivalence between the total time-averaged kinetic energy of the system and the total envelope energy. In fact, from property (2),

$$\int_{-\infty}^{\infty}\mathscr{T}\,dx=\frac{\kappa_T}{2}\left[\int_{-\infty}^{\infty}w^2\,dx+\int_{-\infty}^{\infty}\mathscr{H}^2(w)\,dx\right]=2\,\frac{\kappa_T}{2}\int_{-\infty}^{\infty}w^2\,dx.$$

Consequently, the integrals of the two energies are equal: i.e.,

$$\int_{-\infty}^{\infty}\mathscr{T}\,dx=\int_{-\infty}^{\infty}\langle T\rangle\,dx. \tag{3}$$

For finite structures the previous equivalence is no longer strictly valid, because the Hilbert transform of a windowed signal is not usually defined within the same limits of the window. However, the difference is in general not appreciable, so that the areas under the two curves are very similar.

If a damped beam is considered, as in the rest of this paper, the envelope energy must be defined as

$$\mathscr{T}=(\kappa_T/2)[ww^*+\mathscr{H}(w)\mathscr{H}^*(w)], \tag{4}$$

where the asterisk denotes the complex conjugate.

2.2. ENVELOPE ENERGY EQUATIONS

The Euler–Bernoulli equation for damped harmonic flexural waves in a uniform beam is written as

$$E(1+\mathrm{j}\eta)I\,\mathrm{d}^4w/\mathrm{d}x^4-\omega^2\rho Sw=0,$$

or

$$\mathrm{d}^4w/\mathrm{d}x^4-k_d^4w=0,$$

with

$$k_d^4=\omega^2\rho S/E(1+\mathrm{j}\eta)I \quad\Rightarrow\quad k_d^4\simeq(1-\mathrm{j}\eta)k_B^4.$$

η is the system loss factor and k_d is the damped flexural wavenumber, with $k_B=\omega/c_B$, c_B being the speed of flexural waves. Upon dividing the Euler–Bernouilli equation into its far and near field operators, the damped elemental equations become

$$\mathrm{d}^2w/\mathrm{d}x^2+k_d^2w=0 \quad \text{for the far field,} \qquad \mathrm{d}^2w/\mathrm{d}x^2-k_d^2w=0 \quad \text{for the near field,}$$

with $k_d^2\simeq(1-\mathrm{j}\eta/2)k_B^2$. The solutions of these equations are given, respectively, by

$$w_{ff}(x)=A\,\mathrm{e}^{-kd_{ff}x}+B\,\mathrm{e}^{kd_{ff}x} \qquad \text{and} \qquad w_{nf}(x)=C\,\mathrm{e}^{-kd_{nf}x}+D\,\mathrm{e}^{kd_{nf}x},$$

with A, B, C and D being complex amplitudes, $k_{d_{ff}}=\mathrm{j}(1-\mathrm{j}\eta/4)k_B$ and $k_{d_{nf}}=(1-\mathrm{j}\eta/4)k_B$.

After some mathematics and using the modulation property of the Hilbert transform, it can be shown that the envelope kinetic energies, defined in equation (4), are given by

$$\begin{aligned}
&\mathscr{T}_{ff}=\kappa_T(|A|^2\,\mathrm{e}^{-k_B\eta/2x}+|B|^2\,\mathrm{e}^{k_B\eta/2x}) \qquad \text{for the far field flexural component,}\\
&\mathscr{T}_{nf}=\kappa_T(|C|^2\,\mathrm{e}^{-2k_Bx}+|D|^2\,\mathrm{e}^{2k_Bx}) \qquad \text{for the near field flexural component.}
\end{aligned} \tag{5}$$

These envelopes are solutions, respectively, of the equations

$$\mathscr{T}_{ff}''-(\eta\omega/2c_B)^2\mathscr{T}_{ff}=0, \qquad \mathscr{T}_{nf}''-(2\omega/c_B)^2\mathscr{T}_{nf}=0. \tag{6}$$

They can be interpreted as follows. The envelope kinetic energies of elemental flexural components in one-dimensional systems have a non-oscillating trend, which is formally (only formally) equivalent to the Nefske and Sung solution [1]. It is not difficult to show that the potential energy and total energy envelopes of the far and near fields also have similar behaviours.

For complete flexural waves, the envelope energy is not so smooth. In fact, $\mathscr{T}$ is not simply the sum of $\mathscr{T}_{ff}$ and $\mathscr{T}_{nf}$ but mixed terms appear that, when developed, give rise to the following oscillatory component:

$$\begin{aligned}
&\kappa_T[(\mathrm{Re}\,[AD^*]\,\mathrm{e}^{\mu k_Bx}+\mathrm{Re}\,[BC^*]\,\mathrm{e}^{-\mu k_Bx})\cos\mu k_Bx+(\mathrm{Im}\,[AD^*]\,\mathrm{e}^{\mu k_Bx}\\
&\quad-\mathrm{Im}\,[BC^*]\,\mathrm{e}^{-\mu k_Bx})\sin\mu k_Bx].
\end{aligned}$$

Here $\mu=1-\eta/4\simeq1$. It is worth noting, however, that this oscillatory component is absolutely negligible: thus, it can be conveniently omitted.

With this position, the envelope $\mathscr{T}$ of the energy density is

$$\mathscr{T}\cong\kappa_T(|A|^2\,\mathrm{e}^{-k_B(\eta/2)x}+|B|^2\,\mathrm{e}^{k_B(\eta/2)x}+|C|^2\,\mathrm{e}^{-2k_Bx}+|D|^2\,\mathrm{e}^{2k_Bx}). \tag{7}$$

The corresponding envelope energy equation can be determined by noting that the characteristic polynomial of the equation has the following four roots: $\pm k_B(\eta/2)$; $\pm2k_B$. Simple operations lead to the fourth order differential equation of the envelope, which is

$$\mathscr{T}^{\mathrm{IV}}-4k_B^2v^2\mathscr{T}''+16k_B^4(v^2-1)\mathscr{T}=0, \tag{8}$$

with $v^2 = 1 + \eta^2/16$. Equation (8) satisfies the requirements of our approximate analysis, in that the dimension of the eventual numerical problem is independent of frequency, as in the thermal analogy. Note that the solution of equation (8) provides the complete trend of the exact solution.

It is now possible to introduce the power flow associated with the envelope energy, provided that the relations that hold for the physical energy can be extended to the envelope. Assuming the structural dissipated envelope power is proportional to the local envelope energy, i.e. $\mathscr{P}_{diss} = 2\eta\omega\mathscr{T}$, and assuming that the envelope power flow is given by $\phi = \mathscr{P}_{diss}$, from equation (8) one obtains

$$\phi = -(2\omega/k_B^4\eta)[\mathscr{T}^{IV} - 4k_B^2 v^2 \mathscr{T}'']. \tag{9}$$

Finally, an envelope transmission potential ψ, that can be useful to express some boundary conditions of the problem, can be defined as [3]

$$\phi = -\mathscr{P}' = \partial^2\psi/\partial x^2 \quad \Rightarrow \quad \psi = -(2\omega/k_B^4\eta)[\mathscr{T}'' - 4k_B^2 v^2 \mathscr{T}]. \tag{10}$$

The main result of this section can be summarized as follows. An envelope energy can be defined which provides the desired local average trend one looks for when analyzing high frequency problems. In fact, by neglecting the oscillatory term, which is certainly acceptable, an interesting linear equation is obtained, the solution of which yields the trend of the exact energy density, and can be successfully used for the analysis at high frequencies.

It is easy to show that this result can be extended to the envelope energy of longitudinal waves.

2.3. BOUNDARY CONDITIONS

The differential equation (8) of the envelope energy needs four suitable boundary conditions.

To deal with this problem, an approach commonly used in the wave train closure principle [6] is considered. It consists in assuming that, close to any constraint, the displacement wave of the damped flexural equation $w^{IV} - k_d^4 w = 0$ can be represented by

$$w(x) \simeq A\, e^{-jk_d x} + B\, e^{jk_d x} + C\, e^{-k_d x}, \tag{11}$$

the near field arriving from the other end being neglected.

The procedure to determine the boundary conditions of the envelope will be shown here with reference to the general constraint shown in Figure 1. In this case the boundary conditions are

$$E_d I w'''(0) + K_w w(0) = 0, \qquad E_d I w''(0) + K_\theta w'(0) = 0, \tag{12}$$

where $E_d = E(1 + j\eta)$, and K_w and K_θ are generally complex if the translational and rotational springs include a structural damping.

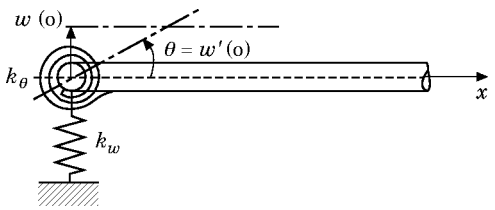


Figure 1. Scheme of a general constraint.

By defining the quantities

$$h_w = k_d^3 E_d I / K_w, \qquad h_\theta = k_d E_d I / K_\theta,$$

and substituting equation (11) into the boundary conditions, one obtains

$$A(1+\mathrm{j}h_w) + B(1-\mathrm{j}h_w) + C(1-h_w) = 0, \qquad -A(1+\mathrm{j}h_\theta) + B(1-\mathrm{j}h_\theta) - C(1-h_\theta) = 0.$$

For the sake of simplicity, let

$$\alpha_{11} = (1+\mathrm{j}h_w)/(1-h_w), \qquad \alpha_{12} = (1-\mathrm{j}h_w)/(1-h_w),$$
$$\alpha_{21} = -(1+\mathrm{j}h_\theta)/(1-h_\theta), \qquad \alpha_{22} = (1-\mathrm{j}h_\theta)/(1-h_\theta).$$

By simple manipulation, C can be eliminated from the previous equations and the ratio between A and B can be determined as a parameter γ, which is a function of h_w and h_θ:

$$A/B = -(\alpha_{12}+\alpha_{22})/(\alpha_{11}+\alpha_{21}) = \gamma.$$

Thus A and B can be expressed through C as follows:

$$A = \gamma C/(\alpha_{21}\gamma + \alpha_{22}), \qquad B = C/(\alpha_{21}\gamma + \alpha_{22}).$$

Substitution of A and B into the displacement expression gives

$$w(x) = C\left[\frac{1}{\alpha_{21}\gamma + \alpha_{22}}(\gamma\, \mathrm{e}^{-\mathrm{j}k_d x} + \mathrm{e}^{\mathrm{j}k_d x}) + \mathrm{e}^{-k_d x}\right].$$

The displacement field can now be associated with the envelope energy, by using equation (7), to obtain

$$\mathscr{T}(x) = \kappa_T |C|^2 \left[\frac{1}{|\alpha_{21}\gamma + \alpha_{22}|^2}(|\gamma|^2\, \mathrm{e}^{-k_B(\eta/2)x} + \mathrm{e}^{k_B(\eta/2)x}) + \mathrm{e}^{-2k_B x}\right]. \tag{13}$$

From this expression the values of the envelope energy and its first two derivatives at the constraint ($x = 0$) can be easily computed, showing that, whatever the boundary conditions, it is possible to write down two linear relationship between them, as follows:

$$\mathscr{T}_0' - k_B f_1(h_w, h_\theta, \eta)\mathscr{T}_0 = 0, \qquad \mathscr{T}_0'' - k_B^2 f_2(h_w, h_\theta, \eta)\mathscr{T}_0 = 0. \tag{14}$$

Here two new parameters, called envelope constraint factors (ECF) (f_1 and f_2), are introduced, that depend on the particular boundary condition, the beam parameters, the frequency of excitation and the dissipation mechanism. The previous relationships completely solve the boundary conditions for the envelope energy. Very simple expressions for f_1 and f_2 can be determined for standard boundary conditions. In Table 1 the values of h_w and h_θ for some constraints and the correspondent envelope constraints factors are provided.

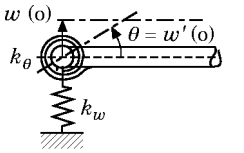
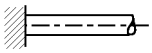
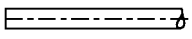
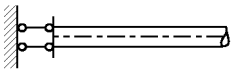
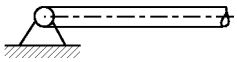
For zero damping ($\eta = 0$), K_w and K_θ as well as h_w and h_θ are real. Moreover, in this case, it is $f_1 = f_2 = ECF(h_w, h_\theta)$. In Figure 2 the envelope constraint factor ECF is plotted as a function of the two constraint parameter h_w and h_θ. Note that in the h_w–h_θ plane all the boundary conditions can be represented.

It is interesting to show, by using equation (13), that the third derivative of the envelope energy at the constraint can be linked to $\mathscr{T}'(0)$, to give

$$\mathscr{T}_0''' - 4k_B^2 \mathscr{T}_0' = 0.$$

TABLE 1

Envelope constraint factors

Constraint	h_w	h_θ	f_1	f_2
	$\frac{E_d I k_d^3}{K_w}$	$\frac{E_d I k_d}{K_\theta}$	$\frac{\frac{\eta}{2}(\lvert\gamma\rvert^2-1)-2\lvert\alpha_{21}\gamma+\alpha_{22}\rvert^2}{\lvert\gamma\rvert^2+1+\lvert\alpha_{21}\gamma+\alpha_{22}\rvert^2}$	$\frac{\frac{\eta^2}{4}(\lvert\gamma\rvert^2+1)+4\lvert\alpha_{21}\gamma+\alpha_{22}\rvert^2}{\lvert\gamma\rvert^2+1+\lvert\alpha_{21}\gamma+\alpha_{22}\rvert^2}$
	0	0	1	$\frac{\eta^2+16}{8}$
	∞	∞		
	∞	0	0	$\frac{\eta^2}{4}$
	0	∞		

This is equivalent to stating that, at the boundary, the first derivative of the transmission potential is zero: i.e., the envelope output power is zero,

$$\mathscr{P}_{out} = -\psi_0' = 0.$$

This condition can be used instead of either of the conditions (12).

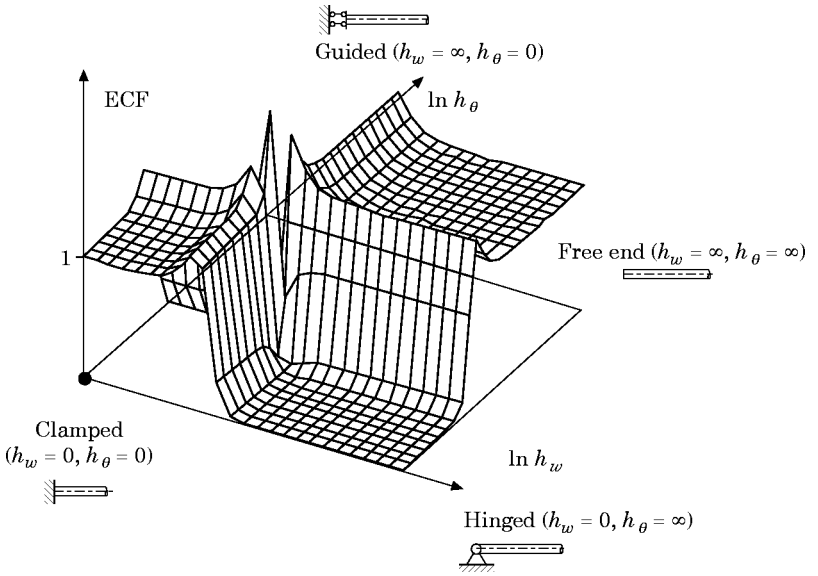


Figure 2. Envelope constraint factors versus h_w and h_θ.

By using the definition of envelope power, given in section 2.2 as the derivative of the transmission potential, the envelope equation for a beam forced by a lumped force at $x = x_F$ can be written as

$$\mathscr{P}' + 2\eta\omega\mathscr{T} = \mathscr{P}_{in}\,\delta(x - x_F).$$

By integrating along the beam, one obtains

$$\int_0^l \mathscr{P}'\,\mathrm{d}x + 2\eta\omega \int_0^L \mathscr{T}\,\mathrm{d}x = \int_0^l \mathscr{P}_{in}\,\delta(x - x_F)\,\mathrm{d}x:$$

that is,

$$\mathscr{P}_{out}(l) - \mathscr{P}_{out}(0) + 2\eta\omega \int_0^l \mathscr{T}\,\mathrm{d}x = \mathscr{P}_{in}.$$

But it was previously shown that $\mathscr{P}_{out} = 0$ at the boundaries. Therefore, recalling equation (3), one finally obtains

$$\mathscr{P}_{in} = 2\eta\omega \int_0^l \langle T \rangle\,\mathrm{d}x = P_{in},$$

showing that the envelope input power is equal to the physical input power.

In the first example presented in section 5, we also use the transmission potential to express one of the boundary conditions at the end of the beam excited by a lumped force. In fact the envelope power $\mathscr{P}$ across a section S_x of the beam is written as

$$\mathscr{P} = -\psi'|_{S_x} = (2\omega/k_B^4\eta)(\mathscr{T}''' - 4\nu^2 k_B^2 \mathscr{T}')|_{S_x}.$$

If P_{in} is the known physical power entering the system, this condition is simply obtained as $\mathscr{P} = -\psi' = P_{in}$. In the second example of section 5, the lumped force is applied at the center of a symmetric system. Therefore the input power is exactly known on each side of the beam ($= P_{in}/2$) and a condition analogous to the previous one can be used. For more complex situations, such as forces acting on any point of the beam or axisymmetric systems, the continuity conditions are much more complex to determine: this point is currently being investigated.

The second continuity condition at the point of application of the lumped force can be determined by stating that, at the excitation point, the local energy $\langle T \rangle$ is equal to the envelope energy $\mathscr{T}$.

3. SOME NOTES ON THE ENVELOPE MODEL

3.1. DAMPED FLEXURAL WAVENUMBERS

So far the main results related to the envelope energy have been presented. To achieve a deeper insight into the envelope, the damped behaviour of one-dimensional flexural structures can be considered. Flexural waves in finite beams present a twofold contribution from a propagating and a decaying near field. For undamped systems, the wavenumber corresponding to these fields lie on the positive and negative axes of the complex plane, being all equal in modulus (k). For damped structures, the elasticity modulus becomes complex ($E_c = E(1 + j\eta)$), and it can be shown that the corresponding wavenumbers rotate in the complex plane by an equal angle, thus all presenting a real and an imaginary component (see Figure 3). Their modulus is not changed. The real components of the propagating fields

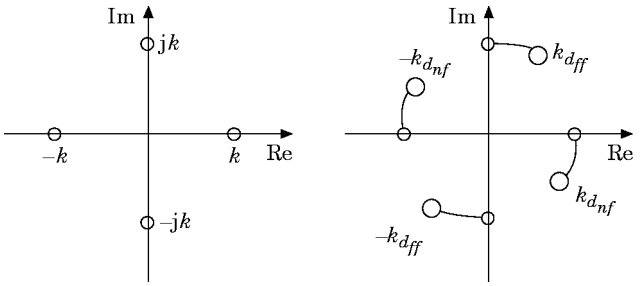


Figure 3. Pure bending waves: wavenumber position without and with damping.

introduce a decaying behaviour, while the imaginary parts of the near fields generate a travelling wave, which is much faster than the undamped propagating wave.

The projection on the real axis of the complex wavenumbers ($\pm k_{d_{ff}}$, $\pm k_{d_{nf}}$) produce the four real numbers

$$\overline{k}_{d_{ff}} \simeq \pm k_B \eta/4, \qquad \overline{k}_{d_{nf}} \simeq \pm k_B.$$

At this point one can observe that, for a flexural beam, the energy envelope in which the oscillating term is neglected, is described by the exponential coefficients $\pm k_B\eta/2$ and $\pm 2k_B$. Comparing these coefficients with the real part of the complex wavenumbers of the flexural damped waves, one notes that they exactly correspond, the former being double than the latter. Thus knowledge of the real part of the complex wavenumbers is sufficient to determine the envelope trend. In particular, $k_B\eta/2$ is associated with the decaying component of the far field, and $2k_B$ is associated with the decaying component of the near field.

With the concept of wavenumber projection on the real axis in mind, one can now observe that for $\eta \to 0$ ($v^2 \to 1$), equation (8) transforms into

$$\mathscr{T}^{\mathrm{IV}} - 4k_B^2 \mathscr{T}'' = 0,$$

and, after integration, into

$$\mathscr{T}'' - 4k_B^2 \mathscr{T} = 0.$$

This exactly corresponds to the equation of the envelope energy for the near field. In fact, for $\eta \to 0$, the projections on the real axis of the far field components are so small that they collapse into the origin, and only the near field components are kept.

A similar result is obtained when correcting terms for shear and rotary inertia is considered. When no damping is present the four wavenumbers still lie on the complex axes, but the imaginary components of the corrected wavenumbers (far field components) become bigger than the real ones (near fields). For damped structures, the rotation

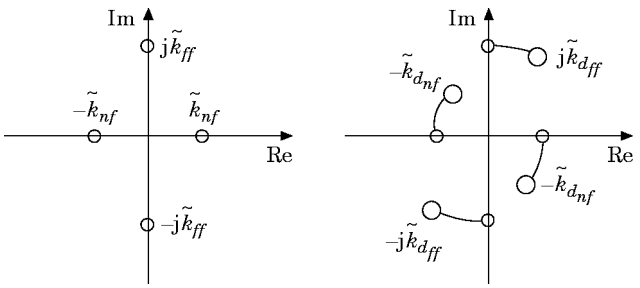


Figure 4. Corrected bending waves: wavenumber position without and with damping.

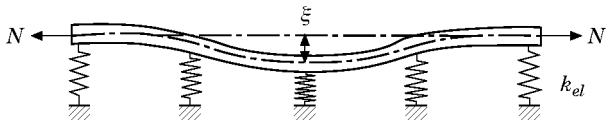


Figure 5. Static analogy of the envelope model.

involved introduces, for any wavenumber, a real and an imaginary contribution, namely, again, a far field decaying trend and a near field tranvelling contribution (see Figure 4).

3.2. THE STATIC ANALOGY

The analytical structure of equation (8) for the envelope energy suggests an interesting physical interpretation. Let us write equation (8) again as

$$EI\mathscr{T}^{\mathrm{IV}} - 4EIk_B^2 v^2 \mathscr{T}'' + 16EIk_B^4 (v^2 - 1)\mathscr{T} = 0.$$

This equation is formally equivalent to the static equation for a beam on a spring layer of stiffness K_{el}, subjected to a constant, tensile, longitudinal force N (see Figure 5), the equation of which is

$$EI\xi^{\mathrm{IV}} - N\xi'' + K_{el}\xi = 0.$$

Comparing these two equations, a correspondence among the physical variables can be determined as $\xi = \mathscr{T}$, $N = 4EIk_B^2 v^2$ and $K_{el} = 16EIk_B^4(v^2 - 1)$. Therefore a static analogy is established between the kinetic envelope energy and the static displacement of the beam on a spring layer subjected to a constant, tensile, longitudinal force.

It can be easily verified that, in this analogy, the envelope power through any section of the beam ($\mathscr{P} = -\psi'|_{S_x}$) corresponds to the shear force of the static beam. Finally, the physical constraints of the dynamic beam can be transformed into suitable constraints in the static analogy. Generally, they do not correspond (for example, a hinged end is transformed into a guided end in the static analogy). The transformation laws can be derived from equations (14) of the boundary conditions, by using the appropriate ECF.

As a matter of fact, the static analogy makes clear the numerical advantage obtained by using the envelope energy, because the dynamic problem is transformed into an analogous static one.

4. DISPLACEMENT APPROACH TO THE ENVELOPE ENERGY

By using the projection of the complex wavenumbers, another approximate solution, with the same character as the envelope can be determined.

For a given frequency, the exponential coefficients of the envelope energy depend on the elastic, geometric, inertial and damping characteristics of the system (E, I, A, ρ, η). Therefore the previous analysis suggests that an envelope trend can be determined by the knowledge of these parameters.

The displacement field of the beam is given by

$$w(x) = A\, \mathrm{e}^{-k_{d_{ff}}x} + B\, \mathrm{e}^{k_{d_{ff}}x} + C\, \mathrm{e}^{-k_{d_{nf}}x} + D\, \mathrm{e}^{k_{d_{nf}}x}, \tag{15}$$

where the four complex wavenumbers are shown in Figure 3. Writing down an envelope displacement using the real contributions $\overline{k}_{d_{ff}}$ and $\overline{k}_{d_{nf}}$, instead of the complex wavenumbers $k_{d_{ff}}$ and $k_{d_{nf}}$, one obtains

$$\hat{w}(x) = \hat{A}\, \mathrm{e}^{-k_B(\eta/4)x} + \hat{B}\, \mathrm{e}^{k_B(\eta/4)x} + \hat{C}\, \mathrm{e}^{-k_B x} + \hat{D}\, \mathrm{e}^{k_B x}. \tag{16}$$

The envelope energy associated with this displacement field is

$$\hat{\mathscr{T}}(x) = \kappa_T \hat{w}^2(x).$$

By simple substitution, the exponential coefficients of $\hat{\mathscr{T}}(x)$ are given by $\pm k_B(\eta/2)$, $\pm 2k_B$ and $\pm k_B(1 \pm \eta/4) \simeq \pm k_B$. One can observe that the first two terms are exactly the same coefficients obtained for the energy envelope, while the last one is the wavenumber related to the mixed terms. One can thus conclude that using $\hat{w}(x)$, determined from the real parts of the complex wavenumbers, yields an envelope energy $\hat{\mathscr{T}}(x)$ the character of which is analogous to the energy envelope in equation (7). Of course, an envelope displacement equation, analogous to equation (8), can be easily determined, provided that one uses only the first two wavenumbers and omits the wavenumber of the oscillating component. By operating as in section 2.2, a fourth order differential equation for the displacement envelope is obtained as follows:

$$\hat{w}^{\mathrm{IV}} - v^2 k_B^2 \hat{w}'' + k_B^4 (v^2 - 1)\hat{w} = 0. \tag{17}$$

It should be evident that the displacement envelope is directly related to the energy envelope through the square root operation. Therefore either the energy or the displacement envelopes can be successfully used to obtain the approximate field solution.

5. SIMULATION ON BEAMS AND COMPARISONS

To illustrate the envelope approach and emphasize the differences with the results obtained by the thermal analogy, two cases are presented. For both beams the envelope solution is compared with the exact solution, obtained by modal analysis, and with the power flow finite element method solution of reference [1].

The first simulation is performed on a clamped–free beam ($1 \times 0{\cdot}1 \times 0{\cdot}1$ m) with a lumped harmonic force applied at the free end. The excitation frequency is 10 000 Hz. The three results (exact, envelope approach and power flow [1]) are summarized in Figure 6. All the curves were obtained with shear deformation and rotary inertia neglected. A damping loss factor of 0·05 was assumed. The energy envelope solution fits the exact result better than the power flow method. Due to the neglect of the near field, the power flow solution is particulary inconsistent near the application force.

Very similar results are obtained for the second beam ($1 \times 0{\cdot}03 \times 0{\cdot}03$ m), clamped at both ends, with a concentrated force applied at its center. The excitation frequency is, in

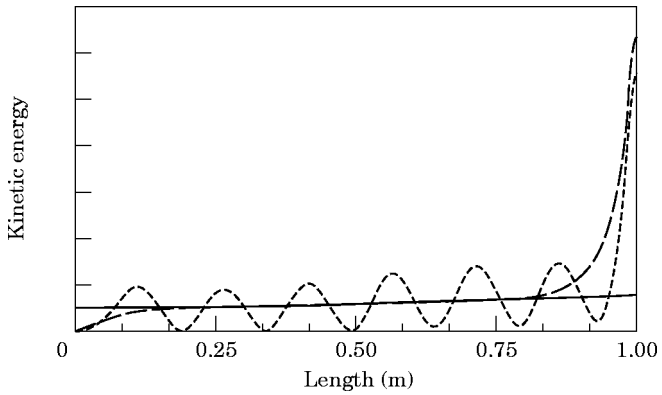


Figure 6. Clamped–free beam: comparison of results. - - - -, Exact solution; – – –, envelope solution; ——, PFFEM solution.

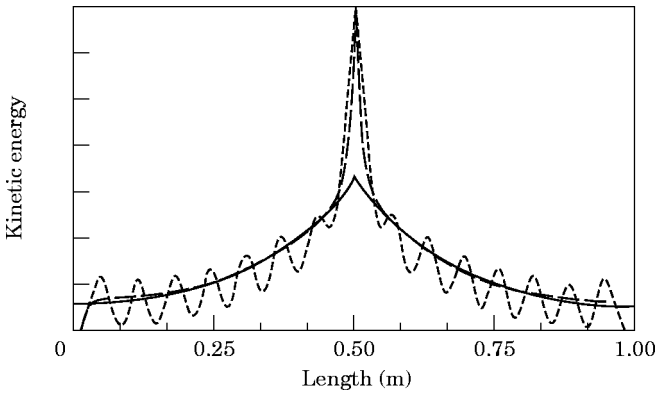


Figure 7. Clamped–clamped beam: comparison of results. Key as Figure 6.

this case, 17 000 Hz, and the damping loss factor is 0·2. As for the previous beam, the envelope solution, obtained by the direct energy envelope approach, matches very satisfactorily the exact trend. As in the previous case, it differs from the power flow solution especially at the excitation point (see Figure 7).

It is worthwhile to stress that both the envelope energy and power flow solutions differ considerably from a classical SEA solution, which would have a constant value along the whole structure.

6. CONCLUSIONS

In studying structural–acoustic problems at high frequencies, serious drawbacks, from a numerical point of view, are introduced by the oscillatory solution, the wavelength of which decreases with the increase of frequency. The possibility of using approximate solutions describing the trend of the energy density along a structure has been explored. It has been shown that an appropriate definition of an envelope energy permits one to obtain a very effective equation the solution of which is appropriate for the analysis of dynamic problems in the high frequency range. The smooth trend of the envelope can be determined with any finite element code, without requring too fine a mesh.

The envelope is characterized by the projection on the real axis of the complex damped wavenumbers, that can be easily estimated by the knowledge of the geometrical and physical properties of the structure.

Ongoing work is being devoted to the development of this approach to two- and three-dimensional as well as to joined structures. If the early promise of the method continues, the quality of the SEA solution will be considerably improved, and new advances in structural–acoustic problems will be obtained.

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