Math. Commun. **27**(2022), 241–255

Periodic solutions for a higher-order parabolic equation set on a singular domain in $\mathbb{R}^{N+1}$

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Received April 6, 2021; accepted July 11, 2022

Abstract. The present work is devoted to the study of a higher-order parabolic equation set on a singular domain in $\mathbb{R}^{N+1}$. The existence and uniqueness of a periodic strict solution are discussed in the framework of Hölder spaces. The techniques used here are essentially based on the Dunford functional calculus and the methods applied in [1] and [2].

AMS subject classifications: 34G10, 34K30, 35J25, 47D0

Keywords: abstract differential equations, periodic boundary conditions, Hölder spaces

1. Introduction of the problem

The solvability and regularity of the boundary value problems (BVP) on certain non-smooth domains is a popular subject of research of many mathematicians. In [2] and [3], some regularity results for a particular BVP model set on cusp domains were obtained in the framework of little Hölder spaces. The authors opted for the use of the theory of abstract differential equations. Using the same strategy, a detailed studies of a BVP for a second order linear differential equation set on a singular cylindrical domain and a BVP for an elliptic system on a conical domain can be found in [4] and [5], respectively (see also references therein). Here, we will show that a similar approach can be exploited in order to give a complete study of an initial BVP set on a cylindrical domain in the framework of Hölder continuous functions. More precisely, we consider a cylindrical domain Π defined by

$$\Pi = \mathbb{R}^+ \times \Omega,$$

where its base $\Omega \subset \mathbb{R}^N$ is the singular domain given by

$$\Omega = \left\{ (x_1, x_2, \dots, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^+ : \sqrt{x_1^2 + \cdots + x_{N-1}^2} \leq \varphi(x_N) \right\},$$

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and φ denotes a non-negative function of parametrization, which is supposed to be of class C^{θ}, $0<\theta<1$, satisfying $\varphi(0)=0$.
In the cylindrical domain Π, we consider the equation:

$$D_t^4 u(t,x) - \sum_{i=1}^{N} D_{x_i}^{2m} u(t,x) - \lambda u(t,x) = h(t,x), \quad m \in \mathbb{N}^*, \tag{1}$$

where $x=(x_1,x_2,\ldots,x_N)$ denotes an arbitrary generic point of $\mathbb{R}^N$, λ is a positive spectral parameter and $D_t u = \partial u/\partial t$.

As a special case of (1), we can consider the fourth order differential equations, which are often encountered in several concrete situations. Problems of this kind actually arise in the context of linear elasticity, deformation of structures, or soil settlement. Equation (1) can also be viewed, in some sense, as a fourth order version of the following equation:

$$D_t u = \mu \Delta^2 u + \kappa \Delta |\nabla|^2 u + h,$$

known in the literature as the equation of Kardar-Parisi-Zhang, which often appears in crystallography; here μ and κ are some suitable physical parameters, see [9]. Another motivation for this work comes from the fact that the initial BVP, which is studied in the present paper, appears as a model linear version for a large class of higher order equations arising in statistical mechanics, phase field models, hydrodynamics and suspension bridge models, see [7], [12], and references therein.
Suppose first that $h \in BUC(\mathbb{R}^+; C(\Omega))$, where $BUC(\mathbb{R}^+; C(\Omega))$, denotes the space of vector-valued functions $h : \mathbb{R}^+ \to C(\Omega)$, uniformly continuous and bounded in $[0,+\infty[$, while $BUC^l(\mathbb{R}^+; C(\Omega))$ denotes the space of vector-valued functions $h : \mathbb{R}^+ \to C(\Omega)$, uniformly continuous and bounded derivatives up to order l in $[0,+\infty[$. The introduction of these spaces is necessary in order to be able to state our main results. We also introduce the Banach space of bounded and θ-Hölder continuous functions $C^{\theta}(\mathbb{R}^+; C(\Omega))$, $0<\theta<1$, consisting of functions $h : \mathbb{R}^+ \to C(\Omega)$ such that $\sup_{t\geq 0} \|h(t)\|_{C(\Omega)} < \infty$ and there exists $C>0$ such that for every $t',\ t \in \mathbb{R}^+$, we have

$$\|h(t') - h(t)\|_{C(\Omega)} \leq C |t'-t|^{\theta}.$$

This space is endowed with the norm

$$\|h\|_{C^{\theta}(\mathbb{R}^+;C(\Omega))} := \sup_{t\geq 0} \|h(t)\|_{C(\Omega)} + \sup_{t'>t>0} \frac{\|h(t') - h(t)\|_{C(\Omega)}}{|t'-t|^{\theta}}.$$

Furthermore, h is supposed to be periodic in the variable t with period $T>0$. We also impose the following condition:

$$h|_{\partial\Omega} = 0, \tag{2}$$

and consider equation (1) under homogenous Dirichlet conditions:

$$u|_{\mathbb{R}^+ \times \partial\Omega} = 0. \tag{3}$$

The main aim of this paper is to investigate the solvability of problem (1)-(3) associated with the following $T-$periodic initial conditions:

$$u|_{\{t\}\times\Omega} = 0,\, u|_{\{t+T\}\times\Omega} = 0,\ t \in \mathbb{R}^+. \tag{4}$$

In this work, we develop a new abstract approach to discuss the solvability of (1)-(4). The choice of this problem is justified by the fact that the framework of Hölder spaces and the singular character of domain Π make the usage of the potential theory or variational techniques or other classical methods a hard task.

The paper is organized as follows. The first section of the paper is devoted to the statement of the transformed version of our problem (1)-(4). Section 2 contains some regularity results for the transformed problem. Finally, in the last section, we go back to our original domain and deduce our main result.

2. Change of variables and statement of the transformed problem

Our strategy is mainly based on the approximation of the singular domain Π by a sequence of domains of the form:

$$\Pi_n = \mathbb{R}^+ \times \Omega_n,$$

where

$$\Omega_n = \left\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{N-1} \times [x_{N,n}, +\infty[\, : \sqrt{x_1^2 + \cdots + x_{N-1}^2} \leq \varphi\left(x_N\right) \right\},$$

and $\left(x_{N,n}\right)_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers such that

$$\lim_{n\to+\infty} x_{N,n} = 0.$$

We put

$$u_n = u|_{\Pi_n}.$$

Consequently, the solution u of (1) will be defined using the approximate solutions u_n of

$$D_t^4 u_n\left(t,x\right) - \sum_{i=1}^{N} D_{x_i}^{2m} u_n\left(t,x\right) - \lambda u_n\left(t,x\right) = h\left(t,x\right),\ \ (t,x) \in \Pi_n, \tag{5}$$

accompanied with the initial conditions

$$u|_{\{t\}\times\Omega_n} = 0,\, u|_{\{t+T\}\times\Omega_n} = 0,$$

and

$$u|_{\mathbb{R}^+\times\partial\Omega_n} = 0. \tag{6}$$

3. Some regularity results for the transformed problem

We start with the abstract setting of the transformed problem (5)-(6). First, we consider the following change of variables:

$$\begin{aligned}&\Upsilon : \Pi_n \to Q_n,\\ &(t, x_1, x_2, \dots, x_N) \mapsto (t, \xi_1, \dots, \xi_N) = \left(t, \frac{x_1}{\varphi(x_N)}, \dots, \frac{x_{N-1}}{\varphi(x_N)}, x_N\right),\end{aligned} \tag{7}$$

where

$$\Pi_n := \mathbb{R}^+ \times \Omega_n \text{ and } Q_n := \mathbb{R}^+ \times \mathcal{D}_n.$$

Here,

$$\mathcal{D}_n := D \times [\xi_{N,n}, +\infty[= D \times [x_{N,n}, +\infty[$$

and

$$D := D(0,1) = \left\{(\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1} : \sqrt{\xi_1^2 + \cdots + \xi_{N-1}^2} \leq 1\right\}.$$

Now, we introduce the following change of functions:

$$\begin{cases} u_n(t,x) = v_n(t,\xi), \\ h(t,x) \quad = f(t,\xi), \end{cases} \tag{8}$$

where arbitrary generic points in $\mathbb{R}^N$ will be denoted by $x = (x_1, \dots, x_N)$ and $\xi = (\xi_1, \dots, \xi_N)$. Using this change of variables and functions, equation (5) is reduced to the following one:

$$D_t^4 v_n(t,\xi) - \mathcal{A}(\xi, D_{\xi_i}) v_n(t,\xi) - \lambda v_n(t,x) = f(t,\xi), \quad t \in \mathbb{R}^+, \tag{9}$$

where

$$\mathcal{A}(\xi, D_{\xi_i}) = D_{\xi_N}^{2m} + \psi_1(\xi_N) \sum_{i=1}^{N-1} D_{\xi_i}^{2m} + \psi_2(\xi_N) \sum_{i=1}^{N-1} \xi_i D_{\xi_i}, \;\; t \in \mathbb{R}^+. \tag{10}$$

Here,

$$\psi_1(\xi_N) = \frac{1}{\varphi^{2m}(\xi_N)} \text{ and } \psi_2(\xi_N) = \frac{\varphi'(\xi_N)}{\varphi(\xi_N)}. \tag{11}$$

The following lemma is stated to describe the effect of the change of variables in the functional framework.

Lemma 1. *Let* $0 < \theta < 1$. *Then*

$$h \in C^\theta\left(\mathbb{R}^+; C(\Omega_n)\right) \Leftrightarrow f \in C^\theta\left(\mathbb{R}^+; C(\mathcal{D}_n)\right).$$

Proof. We prove the implication

$$h \in C^\theta\left(\mathbb{R}^+; C(\Omega_n)\right) \Rightarrow f \in C^\theta\left(\mathbb{R}^+; C(\mathcal{D}_n)\right).$$

Given a small $\delta > 0$, let (t',ξ), $(t,\xi) \in Q_n$ such that $t' \neq t$ and $|t'-t| \leq \delta$. Assume, for instance, that $t \leq t'$. Then it is easy to see that

$$\sup_{(t,\xi)\in Q_n} |f(t,\xi)| < \infty.$$

Furthermore,

$$\frac{|f(t',\xi)-f(t,\xi)|}{|t'-t|^{\theta}} = \frac{\left|h\circ\Upsilon^{-1}(t',\xi')-h\circ\Upsilon^{-1}(t,\xi)\right|}{|t'-t|^{\theta}}.$$

Consequently,

$$\frac{|f(t',\xi)-f(t,\xi)|}{|t'-t|^{\theta}} = \frac{\left|h\circ\Upsilon^{-1}(t',\xi)-h\circ\Upsilon^{-1}(t,\xi)\right|}{|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)|}\,\frac{\left|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)\right|}{|t'-t|^{\theta}}.$$

First, observe that

$$\begin{aligned}\left\|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)\right\| &= \left((t'-t)^2+\left(\sum_{i=1}^{N-1}(\varphi(t')\,\xi_i-\varphi(t)\,\xi_i)\right)^2\right)^{1/2}\\ &\leq \sup_{\Omega_n}|\xi_i|\left((t'-t)^2+(\varphi(t)-\varphi(t'))^2\right)^{1/2}.\end{aligned}$$

Since φ is of class C^{θ}, then there exists $C > 0$ such that

$$\begin{aligned}\left\|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)\right\| &\leq \sup_{\Omega_n}|\xi_i|\left((t'-t)^2+|t'-t|^{2\theta}\right)^{1/2}\\ &\leq C\left((t'-t)^2+|t'-t|^{2\theta}\right)^{1/2}\\ &\leq C\left((t'-t)^{2\theta}(t'-t)^{1-2\theta}+|t'-t|^{2\theta}\right)^{1/2}\\ &\leq C(\delta)\,|t'-t|^{\theta}.\end{aligned}$$

Summing up, we deduce that

$$\frac{\left\|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)\right\|}{|t'-t|^{\theta}} \leq C.$$

Taking into account the regularity of the function h, we conclude that

$$\frac{|f(t',\xi)-f(t,\xi)|}{|t'-t|^{\theta}} = \frac{\left|h\circ\Upsilon^{-1}(t',\xi)-h\circ\Upsilon^{-1}(t,\xi)\right|}{|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)|}\,\frac{\left|\Upsilon^{-1}(t',\xi)-\Upsilon^{-1}(t,\xi)\right|}{|t'-t|^{\theta}} < +\infty.$$

We can prove the opposite implication in the same way. □

A direct computation allows us to deduce the smoothness of coefficients of the operator $\mathcal{A}$ given by (10).

Lemma 2. *Let $\psi_i(.)$, $i=1,2$ be the real valued functions given by* (11)*. Then, $\psi_i(.)$, $i=1,2$ are of class C^θ, with $0<\theta<1$.*

We consider the following vector-valued functions:

$$\begin{aligned} v_n &: \mathbb{R}^+ \to E;\ t \longrightarrow v_n(t); \quad v_n(t)(\xi) = v_n(t,\xi), \\ f &: \mathbb{R}^+ \to E;\ t \longrightarrow f(t); \quad f(t)(\xi) = f(t,\xi), \end{aligned}$$

where $E = C(\mathcal{D}_n)$ and problem (6)-(9) is formulated in its abstract setting by

$$v_n^{(4)}(t) + Av_n(t) - \lambda v_n(t) = f(t),\ t \in \mathbb{R}^+,$$

with

$$v_n(t) = v_n(t+T).$$

Here, A is a closed linear operator given by

$$\begin{cases} D(A) := \Big\{ w \in W_0^{2m,p}(\mathcal{D}_n) \cap C(\mathcal{D}_n),\ p>2,\ m \in \mathbb{N} - \{0\} : \\ \qquad\qquad \mathcal{A}(\xi_i, D_{\xi_i})\, w \in C(\mathcal{D}_n) \Big\}, \\ Aw(\xi) := \mathcal{A}\left(\xi_i, D_{\xi_j}\right) w(\xi),\ 2 \le i \le N-1. \end{cases}$$

Set

$$Q := A - \lambda I. \tag{12}$$

After doing this, we obtain the following problem:

$$v_n^{(4)}(t) - Qv_n(t) = f(t),\ \ t \in \mathbb{R}^+, \tag{13}$$

with

$$v_n(t) = v_n(t+T). \tag{14}$$

In the following proposition, we clarify interesting spectral properties of the operator Q given by (12), see also [8]:

Proposition 1. *For $t \in \mathbb{R}^+$, the closed linear operator Q with domain $D(Q)$, not necessarily dense in E, satisfies the Krein ellipticity property, namely,*

$$\exists M>0,\ \forall z \geqslant 0 : \left\| (Q-zI)^{-1} \right\|_{L(E)} \leqslant \frac{M}{1+z}. \tag{15}$$

Proof. Keeping in mind Lemma 2, estimate (15) is handled by exploiting the results of [10]. □

Remark 1. *The use of a classical argument of analytic continuation on the resolvent allows us to conclude that the previous statement holds in the sector:*

$$\Pi_{\theta_0,r_0} = \{ z : |z| \geqslant r_0 \text{ and } \theta_0 \leqslant \arg z \leqslant 2\pi - \theta_0 \}, \tag{16}$$

with some small $\theta_0 > 0$, and $r_0 > 0$. Here, $\rho(Q)$ denotes the resolvent set of Q. Furthermore, we can replace z by $z+\lambda$.

In the sequel, our goal will be to provide a complete study for problem (13)-(14) by building an explicit representation of the solution v_n and studying its regularity. We note here that by virtue of (15), it is possible to make use of square roots

$$-\left(-Q\right)^{1/2}, \quad t \in \mathbb{R}^{+},$$

which are well defined and generate analytic semigroups (not strongly continuous at zero). The operator Q is not densely defined. So, we prefer the natural use of Green's kernels. The techniques used here are essentially based on the use of the Dunford functional calculus. The calculus is very cumbersome, so one must be careful. The main lines of the proof are as follows:

1. First, our representation formula for the solution of (13)-(14) can be heuristically derived by the following argument: taking into account the scalar case, our solution is given in the following form:

$$v_n\left(t\right) = -\frac{1}{2\pi i}\int_{\gamma}\int_{t}^{t+T} K_{\omega}\left(t,s\right)\left(Q-zI\right)^{-1} f(s)\,ds\,dz, \tag{17}$$

where

$$K_{\omega}\left(t,s\right) = \frac{e^{-\omega(s-t)}+e^{\omega(s-t-T)}}{4\omega^{3}\left(1-e^{-\omega T}\right)} + \frac{e^{-\omega(t-s)}+e^{\omega(t-s+T)}}{4\omega^{3}\left(1-e^{-\omega T}\right)},\ t \le s \le t+T.$$

Here, $\omega := \sqrt[4]{z}$ with $\mathrm{Re}(\omega)$ is positive. The curve γ is the retrograde oriented boundary of the sector Π_{θ_0,r_0} defined by (16).

2. Next, by the periodic character of the function f and adapting the same argument applied in [3], [4] and [5], we show that

$$\begin{aligned} v_n^{(4)}(t) = {} & -\frac{1}{2i\pi}\int_{\gamma}\int_{t}^{t+T}\left(\frac{e^{-\omega(s-t)}+e^{\omega(s-t-T)}}{4\omega^{3}\left(1-e^{-\omega T}\right)} + \frac{e^{-\omega(t-s)}+e^{\omega(t-s+T)}}{4\omega^{3}\left(1-e^{-\omega T}\right)}\right) \\ & \times Q(Q-zI)^{-1} f(s)\,ds\,dz + \frac{1}{\pi i}\int_{\gamma}\frac{Q(Q-zI)^{-1}}{2} f(t)\,dz. \end{aligned}$$

Since

$$Qv_n\left(t\right) = -\frac{1}{2\pi i}\int_{\gamma}\int_{t}^{t+T} K_{\omega}\left(t,s\right) Q(Q-zI)^{-1} f(s)\,ds\,dz,$$

we have

$$v_n^{(4)}(t) - Qv_n\left(t\right) = \frac{1}{\pi i}\int_{\gamma}\frac{Q(Q-zI)^{-1}}{2} f(t)dz.$$

At this level, the Cauchy theorem allows us to conclude that

$$v_n^{(4)}(t) - Qv_n\left(t\right) = f(t).$$

In this study, we are dealing with the notion of a strict solution for (13)-(14), i.e., any vector-valued function v_n satisfying

$$v_n \in BUC^{4}\left(\mathbb{R}^{+},E\right),\ v_n \in D\left(Q\right),\ Qv_n \in BUC\left(\mathbb{R}^{+},E\right),$$

and the initial periodic boundary conditions (14). Let us start with the following technical result:

Lemma 3. *There exists $C_{\theta_0} > 0$ such that for all $z \in \Pi_{\theta_0,r_0}$ one has:*

$$|(1-\exp(-\omega T))| \geqslant C_{\theta_0} = \min\left(1 - e^{-T\frac{\pi}{2\tan\left(\frac{\pi}{2}-\frac{\theta_0}{4}\right)}}, 1 - e^{-Tr_0^{1/4}\cos\left(\frac{\pi}{2}-\frac{\theta_0}{4}\right)}\right) > 0.$$

Proof. Let $z \in \Pi_{\theta_0,r_0}$ satisfy $|z| = r_0$. Then

$$\left|1 - e^{-T\omega}\right| \geqslant 1 - e^{-T\operatorname{Re}\omega} = 1 - e^{-Tr_0^{1/4}\cos\nu} \geqslant 1 - e^{-Tr_0^{1/4}\cos\left(\frac{\pi}{4}-\frac{\theta_0}{4}\right)},$$

with $\nu \in \left[-\frac{\pi}{4}+\frac{\theta_0}{4}, \frac{\pi}{4}-\frac{\theta_0}{4}\right]$. For $z \in \Pi_{\theta_0,r_0}$ and $|z| > r_0$, one has

$$\begin{aligned}\left|1 - e^{-T\omega}\right|^2 &= \left|1 - e^{-T\operatorname{Re}\omega - iT\operatorname{Im}\omega}\right|^2 \\ &= \left|e^{-T\operatorname{Re}\omega}\left(e^{T\operatorname{Re}\omega} - e^{-iT\operatorname{Im}\omega}\right)\right|^2 \\ &= e^{-2T\operatorname{Re}\omega}\left(e^{2T\operatorname{Re}\omega} - 2e^{T\operatorname{Re}\omega}\cos\left(T\operatorname{Im}\omega\right) + 1\right) \\ &= \left(e^{-2T\operatorname{Re}\omega} + 1\right) - 2e^{-T\operatorname{Re}\omega}\cos\left(T\operatorname{Im}\omega\right).\end{aligned}$$

If $\operatorname{Re}\omega \geqslant \pi/\left(2\tan\left(\frac{\pi}{4}-\frac{\theta_0}{4}\right)\right)$, then

$$\cos\left(T\operatorname{Im}\sqrt{-z}\right) \leqslant 0,$$

and

$$\begin{aligned}\left|1 - e^{-T\omega}\right|^2 &\geqslant \left(1 + e^{-2T\operatorname{Re}\omega}\right) - 2e^{-T\operatorname{Re}\omega} + 2e^{-T\operatorname{Re}\omega} \\ &\geqslant \left(1 - e^{-T\operatorname{Re}\omega}\right)^2 + 2e^{-T\operatorname{Re}\omega} \\ &\geqslant \left(1 - e^{-T\operatorname{Re}\omega}\right)^2 \\ &\geqslant \left(1 - e^{-T\frac{\pi}{2\tan\left(\frac{\pi}{4}-\frac{\theta_0}{4}\right)}}\right)^2.\end{aligned}$$

If $\operatorname{Re}\omega \leqslant \pi/\left(2\tan\left(\frac{\pi}{4}-\frac{\theta_0}{4}\right)\right)$, then

$$\cos\left(T\operatorname{Im}\omega\right) \geqslant 0,$$

and

$$\begin{aligned}\left|1 + e^{-T\omega}\right|^2 &\geqslant \left(e^{-2T\operatorname{Re}\omega} + 1\right) - 2e^{-T\operatorname{Re}\omega} \\ &\geqslant \left(1 - e^{-T\operatorname{Re}\omega}\right)^2 \\ &\geqslant \left(1 - e^{-T\frac{\pi}{2\tan\left(\frac{\pi}{4}-\frac{\theta_0}{4}\right)}}\right)^2.\end{aligned}$$

The case $z \in \Pi_{\theta_0,r_0}$ with $\arg(z) = \theta \in]\theta_0, 2\pi - \theta_0[$ can be treated by the same argument. □

Remark 2. *By the previous lemma and estimate (15), the integrals appearing in formula (17) converge. This implies that our formal solution given above is well defined.*

Next, we give our first result concerning the formal solution (17).

Proposition 2. *Suppose that* $f \in C^{\theta}(\mathbb{R}^{+}, E)$, $0 < \theta < 1$ *and (15) holds. Then*

$$v_n(\cdot) \in D(Q).$$

Proof. First, one has

$$f(s) = f(s) - f(t) + f(t).$$

Then we can write

$$\begin{aligned} v_n(t) &= -\frac{1}{2i\pi} \int_{\gamma_1} \int_{t}^{t+T} k_\omega(t,s)(Q - zI)^{-1} f(s) ds dz \\ &= -\frac{1}{2i\pi} \int_{\gamma} \int_{t}^{t+T} k_\omega(t,s)(Q - zI)^{-1} \left(f(s) - f(t)\right) ds dz \\ &\quad -\frac{1}{2i\pi} \int_{\gamma} \int_{t}^{t+T} k_\omega(t,s)(Q - zI)^{-1} f(t) ds dz \\ &= I + J. \end{aligned}$$

Let us start with the second integral. One has

$$\begin{aligned} J &= -\frac{1}{2i\pi} \int_{\gamma_1} \int_{t}^{t+T} k_\omega(t,s)(Q - zI)^{-1} f(t) ds dz \\ &= -\frac{1}{2i\pi} \int_{\gamma_1} \left(\int_{t}^{t+T} k_\omega(t,s) ds \right) (Q - zI)^{-1} f(t) dz \\ &= -\frac{1}{2i\pi} \int_{\gamma_1} \left(\int_{t}^{t+T} \frac{e^{-\omega(s-t))} + e^{\omega(s-t-T))}}{4\omega^3 \left(1 - \exp e^{-\omega T}\right)} ds \right) (Q - zI)^{-1} f(t) dz \\ &\quad -\frac{1}{2i\pi} \int_{\gamma_1} \left(\int_{t}^{t+T} \frac{e^{-\omega(t-s)} + e^{\omega(t-s+T)}}{4\omega^3 \left(1 - e^{-\omega T}\right)} ds \right) (Q - zI)^{-1} f(t) dz \end{aligned}$$

A direct computation yields

$$\begin{aligned} &-\frac{1}{2i\pi} \int_{\gamma_1} \left(\int_{t}^{t+T} \frac{e^{-\omega(s-t))} + e^{\omega(s-t-T))}}{4\omega^3 \left(1 - \exp e^{-\omega T}\right)} ds \right) (Q - zI)^{-1} f(t) dz \\ &\quad = -\frac{1}{4i\pi} \int_{\gamma_1} \frac{(Q - zI)^{-1}}{z} f(t) dz \end{aligned}$$

and

$$\begin{aligned} &-\frac{1}{2i\pi} \int_{\gamma_1} \frac{1}{4\omega^3 \left(1 - \exp e^{-\omega T}\right)} \left(\int_{t}^{t+T} e^{-\omega(t-s)} + e^{\omega(t-s+T)} ds \right) (Q - zI)^{-1} f(t) dz \\ &= -\frac{1}{4i\pi} \int_{\gamma_1} e^{\omega T} \frac{(Q - zI)^{-1}}{z} f(t) dz, \end{aligned}$$

from which we conclude that

$$J=-\frac{1}{4i\pi}\int_{\gamma_1}\frac{(Q-zI)^{-1}}{z}f(t)dz-\frac{1}{4i\pi}\int_{\gamma_1}\frac{e^{\omega T}(Q-zI)^{-1}}{z}f(t)dz.$$

Now, using the well-known Cauchy theorem, we deduce that

$$J=Q^{-1}f(t).$$

Regarding the integral I, the direct use of Hölder's inequality shows that

$$\begin{aligned}\left\|\int_t^{t+T}k_\omega(t,s)\left(f(s)-f(t)\right)ds\right\|_E &\leqslant C\left(\int_0^{+\infty}\left|k_\omega(t,s)\right|\left|t-s\right|^\theta ds\right)\\ &\leqslant C\left(\sup_{t>0}\int_0^{+\infty}\left|k_\omega(t,s)\right|\left|t-s\right|^\theta ds\right)\\ &\leqslant \frac{C}{\left|z\right|^{1+\theta}}.\end{aligned}$$

Consequently, it is easy to see that the quantity

$$\left\|QI\right\|_E=\left\|\frac{1}{2i\pi}\int_{\gamma_1}\int_t^{t+T}k_\omega(t,s)Q(Q-zI)^{-1}\left(f(s)-f(t)\right)ds\right\|_E$$

is bounded. Summing up, we are able to conclude that

$$v_n(t)\in D\left(Q\right).$$

□

Furthermore, we have:

Proposition 3. *Let* $f\in BUC(\mathbb{R}^+,E)$. *Then,*

$$Qv_n\left(\cdot\right)\ \text{ and } v_n^{(4)}\left(\cdot\right)\in BUC(\mathbb{R}^+;E).$$

Proof. First, observe that

$$v_n^4\left(\cdot\right)=f(\cdot)+Qv_n\left(\cdot\right).$$

Keeping this in mind, it suffices to prove that

$$Qv_n\left(\cdot\right)\in BUC(\mathbb{R}^+;E).$$

Let $t_2>t_1\in\mathbb{R}^+$ such that

$$t_1<t_2<t_1+T<t_2+T.$$

First, we write

$$\begin{aligned}Qv\left(t_2\right)-Qv\left(t_1\right)=&-\frac{1}{2\pi i}\int_\gamma\int_{t_2}^{t_2+T}K_\omega\left(t_2,s\right)Q(Q-zI)^{-1}f(s)\,ds\,dz\\ &+\frac{1}{2\pi i}\int_\gamma\int_{t_1}^{t_1+T}K_\omega\left(t_1,s\right)Q(Q-zI)^{-1}f(s)\,ds\,dz.\end{aligned}$$

Then

$$
\begin{aligned}
Qv\,(t_2) - Qv\,(t_1) = & -\frac{1}{2\pi i}\int_\gamma \int_{t_2}^{t_1+T} K_\omega\,(t_2,s)\,Q(Q-zI)^{-1}f(s)\,ds\,dz \\
& -\frac{1}{2\pi i}\int_\gamma \int_{t_1+T}^{t_2+T} K_\omega\,(t_2,s)\,Q(Q-zI)^{-1}f(s)\,ds\,dz \\
& +\frac{1}{2\pi i}\int_\gamma \int_{t_1}^{t_2} K_\omega\,(t_1,s)\,Q(Q-zI)^{-1}f(s)\,ds\,dz \\
& +\frac{1}{2\pi i}\int_\gamma \int_{t_2}^{t_1+T} K_\omega\,(t_1,s)\,Q(Q-zI)^{-1}f(s)\,ds\,dz,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
Qv\,(t_2) - Qv\,(t_1) = & -\frac{1}{2\pi i}\int_\gamma \int_{t_2}^{t_1+T} \left(K_\omega\,(t_2,s) - K_\omega\,(t_1,s)\right)Q(Q-zI)^{-1}f(s)\,ds\,dz \\
& +\frac{1}{2\pi i}\int_\gamma \int_{t_1}^{t_2} K_\omega\,(t_1,s)\,Q(Q-zI)^{-1}f(s)\,ds\,dz \\
& -\frac{1}{2\pi i}\int_\gamma \int_{t_1+T}^{t_2+T} K_\omega\,(t_2,s)\,Q(Q-zI)^{-1}f(s)\,ds\,dz \\
:\ = & \sum_{i=1}^{3} I_i.
\end{aligned}
$$

All these three integrals can be treated similarly. So, we restrict ourselves to the first one and have

$$
I_1 = \frac{1}{2\pi i}\int_\gamma \int_{t_2}^{t_1+T} \left(K_\omega\,(t_2,s) - K_\omega\,(t_1,s)\right)Q(Q-zI)^{-1}f(s)\,ds\,dz,
$$

so

$$
\begin{aligned}
I_1 = & \frac{1}{2\pi i}\int_\gamma \frac{\left(e^{\omega t_2} - e^{\omega t_1}\right)}{4\left(1-e^{-\omega T}\right)}\left(\int_{t_2}^{t_1+T} \frac{e^{-\omega s}}{\omega^3}Q(Q-zI)^{-1}f(s)\,ds\right)dz \\
& +\frac{1}{2\pi i}\int_\gamma \frac{e^{\omega(t_2+T)} - e^{\omega(t_1+T)}}{4\left(1-e^{-\omega T}\right)}\left(\int_{t_2}^{t_1+T} \frac{e^{-\omega s}}{\omega^3}Q(Q-zI)^{-1}f(s)\,ds\right)dz \\
& +\frac{1}{2\pi i}\int_\gamma \frac{\left(1+e^{-\omega T}\right)\left(e^{-\omega t_2} - e^{-\omega t_1}\right)}{4\left(1-e^{-\omega T}\right)}\left(\int_{t_2}^{t_1+T} \frac{e^{\omega s}}{\omega^3}Q(Q-zI)^{-1}f(s)\,ds\right)dz \\
= & \sum_{k=1}^{3} I_{1k}.
\end{aligned}
$$

Let us start with the quantity I_{11}. We have

$$
I_{11} = \frac{1}{2\pi i}\int_\gamma \frac{\left(e^{\omega t_2} - e^{\omega t_1}\right)}{4\left(1-e^{-\omega T}\right)}\left(\int_{t_2}^{t_1+T} \frac{e^{-\omega s}}{\omega^3}Q(Q-zI)^{-1}f(s)\,ds\right)dz,
$$

Keeping in mind (15) and using the Lagrange mean value theorem, we get:

$$\begin{aligned}\|I_{11}\|_E &\leq C\int_\gamma \frac{\operatorname{Re}\omega\ e^{\operatorname{Re}\omega\tau}}{4\left(1-e^{-\omega T}\right)}\int_{t_2}^{t_1+T}\frac{e^{-\operatorname{Re}\omega s}}{|\omega|^3}\,ds\ |dz|\ \ |t_2-t_1| \\ &\leq C\int_\gamma \frac{\operatorname{Re}\omega\ e^{\operatorname{Re}\omega\tau}}{4\left(1-e^{-\omega T}\right)}\int_{t_2}^{t_1+T}\frac{e^{-\operatorname{Re}\omega s}}{|\omega|^3}\,ds\ |dz|\ \ |t_2-t_1| \\ &\leq C\int_\gamma\left(\frac{e^{-\operatorname{Re}\omega(t_2-\tau)}-\ e^{-\operatorname{Re}\omega(t_1+T-\tau)}}{4\left|1-e^{-\omega T}\right|\ \ |\omega|^4}\ |dz|\right)\ \ |t_2-t_1|\,.\end{aligned}$$

Therefore,

$$\|I_{11}\|_E \leq C\,|t_2-t_1|\,.$$

Furthemore,

$$\begin{aligned}I_2 &= \frac{1}{2\pi i}\int_\gamma\int_{t_1}^{t_2}\frac{e^{-\omega(s-t_1)}+e^{\omega(s-t_1-T)}}{4\omega^3\left(1-e^{-\omega T}\right)}Q(Q-zI)^{-1}f(s)\,ds\,dz \\ &\quad+\frac{1}{2\pi i}\int_\gamma\int_{t_1}^{t_2}\frac{e^{-\omega(t_1-s)}+e^{\omega(t_1-s+T)}}{4\omega^3\left(1-e^{-\omega T}\right)}Q(Q-zI)^{-1}f(s)\,ds\,dz.\end{aligned}$$

Then

$$\begin{aligned}I_2 &= \frac{1}{2\pi i}\int_\gamma\int_{t_1}^{t_2}\frac{e^{-\omega(s-t_1)}+e^{-\omega(t_1-s)}}{4\omega^3\left(1-e^{-\omega T}\right)}Q(Q-zI)^{-1}f(s)\,ds\,dz \\ &\quad+\frac{1}{2\pi i}\int_\gamma\int_{t_1}^{t_2}\frac{e^{-\omega(t_1+T-s)}+e^{\omega(t_1-s+T)}}{4\omega^3\left(1-e^{-\omega T}\right)}Q(Q-zI)^{-1}f(s)\,ds\,dz \\ &= I_{21}+I_{22}.\end{aligned}$$

For I_{21}, the required estimates can be easily handled. Regarding I_{22}, we have

$$\begin{aligned}I_{22} &= \frac{1}{2\pi i}\int_\gamma\int_{t_1}^{t_2}\frac{e^{\omega(t_1+T-s)}+e^{-\omega(t_1+T-s)}}{4\omega^3\left(1-e^{-\omega T}\right)}Q(Q-zI)^{-1}f(s)\,ds\,dz \\ &= \frac{1}{2\pi i}\int_\gamma\frac{1}{2\omega^3\left(1-e^{-\omega T}\right)}\left(\int_{t_1}^{t_2}\cosh\left(\omega\left(t_1+T-s\right)\right)Q(Q-zI)^{-1}f(s)\,ds\right)\,dz.\end{aligned}$$

This implies that

$$\begin{aligned}\|I_{22}\|_E &\leq C\int_\gamma\frac{1}{2\,|\omega|^3\left|1-e^{-\omega T}\right|}\int_{t_1}^{t_2}\left|\cosh\left(t_1+T-s\right)\right|\left\|Q(Q-zI)^{-1}f(s)\right\|_E\,ds\ |dz| \\ &\leq\left(\int_\gamma\frac{\cosh\left(\operatorname{Re}\omega\left(t_1+T-s\right)\right)}{2\,|\omega|^3\left|1-e^{-\omega T}\right|}\ |dz|\right)|t_2-t_1| \\ &\leq C\,|t_2-t_1|\,.\end{aligned}$$

□

Reiterating the same techniques as above, we may deduce the following:

Proposition 4. *Let $f \in C^{\theta}(\mathbb{R}^{+}, E)$, $0 < \theta < 1$. Then the formal solution* (17) *satisfies equation* (13) *accompanied with the initial condition* (14)*. Furthermore,*

$$Qv_n(\cdot) \ \text{ and } v_n^{(4)}(\cdot) \in C^{\theta}(\mathbb{R}^{+}; E).$$

Immediately, we get the following:

Corollary 1. *Let $f \in C^{\theta}(\mathbb{R}^{+}, E)$, $0 < \theta < 1$. Then there exist $\lambda^* > 0$ and $C > 0$ such that for all $\lambda \geqslant \lambda^*$, the strict solution w given by* (17) *fulfills the estimate*

$$\max_t \|v_n(t)\|_E \leq C.$$

Proof. The result is a direct consequence of estimate (15). □

Next, we will describe the smoothness of (17), when the right-hand term of problem (13)-(14) has spatial smoothness; that is, for every $t \geq 0$, we have:

$$f \in L^{\infty}(\mathbb{R}^{+}, D_Q(\sigma, +\infty)),\ 0 < \sigma < \frac{1}{2m}.$$

where $D_Q(\sigma, +\infty)$ denotes a real Banach interpolation space between $D(Q)$ and E defined by the set

$$\left\{\zeta \in E : \sup_{r>0} \left\| r^{\sigma} Q\,(Q - rI)^{-1}\,\zeta \right\|_E < \infty \right\}.$$

More details about these spaces are given, for instance, in [10] and [11].

Proposition 5. *Suppose that $f \in L^{\infty}(\mathbb{R}^{+}, D_Q(\sigma, +\infty))$, $0 < \sigma < \dfrac{1}{2m}$. Then assumptions* (15) *and* (2) *imply*

$$v_n(\cdot) \in L^{\infty}(\mathbb{R}^{+}, D_Q(\sigma, +\infty)).$$

Proof. The required result is obtained from the estimate

$$\sup_{t>0}\sup_{r>0} \left\| r^{\sigma} Q(Q - rI)^{-1} v_n(t) \right\|_E < +\infty.$$

Using the identity

$$Q(Q - rI)^{-1}(Q - zI)^{-1} = \frac{Q(Q - zI)^{-1}}{z - r} - \frac{Q(Q - rI)^{-1}}{z - r},$$

we get

$$\begin{aligned} r^{\sigma} Q(Q - rI)^{-1} v_n(t) = &-\frac{r^{\sigma}}{2\pi i} \int_{\gamma} \int_{t}^{t+T} K_{\omega}(t, s)\, \frac{Q(Q - zI)^{-1}}{z - r} f(s)\, ds\, dz \\ &- \frac{r^{\sigma}}{2\pi i} \int_{t}^{t+T} \int_{\gamma} K_{\omega}(t, s)\, \frac{Q(Q - rI)^{-1}}{z - r} f(s)\, ds\, dz = I_1 + I_2. \end{aligned}$$

It is easy to see that $I_2 = 0$ by using the Cauchy formula and integrating to the left of γ. For the quantity I_1, we have:

$$\|I_1\|_E \leqslant r^\sigma \int_\gamma \int_t^{t+T} \frac{|K_\omega(t,s)|\, ds}{|z-r|\,|z|^\sigma} |dz| \, \|f\|_{D_A(\sigma,+\infty)} .$$

Since

$$\int_\gamma \frac{|dz|}{|z-r|\,|z|^\sigma} = O\Big(r^{-\sigma}\Big),$$

we obtain

$$\|I_1\|_E \leqslant C \, \|f\|_{D_Q(\sigma,+\infty)} .$$

This concludes the proof. □

Keeping in mind assumption (2), we have the following characterization:

$$D_Q(\sigma,+\infty) = \left\{\psi \in C^{2m\sigma}(D_n) : \psi|_{\partial\mathcal{D}_n} = 0\right\}, \ 0 < \sigma < \frac{1}{2m}.$$

Summing up, our main results concerning the abstract problem (13)-(14) are formulated as follows:

Theorem 1. *Let*

$$f \in C^\theta(\mathbb{R}^+, E) \cap L^\infty(\mathbb{R}^+, D_Q(\sigma,+\infty)), \ 0 < \theta < 1, \ 0 < \sigma < \frac{1}{2m}.$$

Then there exist $\lambda^* > 0$ and $C > 0$ such that for all $\lambda \geqslant \lambda^*$ the strict solution v_n given by (17) *is bounded in the sense*

$$\max_t \|v_n(t)\|_E \leq C.$$

The following remark expresses the anisotropic character of the Hölder continuous spaces.

Remark 3. *We feel it is our duty to stress the distinction between $C^\upsilon(\mathbb{R}^+; C^\upsilon(\Omega))$ and $C^\upsilon(\Pi)$, $0 < \upsilon < 1$. In fact, it is well known that the space $C^\theta(\Pi)$ is decomposed as follows:*

$$C^\upsilon(\Pi) = L^\infty(\mathbb{R}^+; C^\upsilon(\Omega)) \cap C^\upsilon(\mathbb{R}^+; C(\Omega)).$$

For more details, we refer the reader to [6].

4. Return to the singular domain

Now, we are able to justify our main result concerning the specific transformation given by (9), where Π_n is transformed into $Q_n := \mathbb{R}^+ \times \mathcal{D}_n$. Recall that, by the change of variables (7), the transformed domain $\mathcal{D}_n = D \times [\xi_{N,n}, +\infty[$, can be identified with $\mathcal{D}_n = D \times [x_{N,n}, +\infty[$. Observe also that, as a direct consequence of previous considerations and using a classical argument, it is possible to extract a convergent subsequence (x_{N,n_i}) from $(x_{N,n})$ such that

$$\lim_{i\to+\infty} x_{N,n_i} = 0 \text{ and } \lim_{i\to+\infty} [x_{N,n_i}, +\infty[= [0, +\infty[\,.$$

It is easy to see that if we denote by $v(\cdot)$ the limit of (v_{n_i})

$$v_{n_i} = v_n\left(t, \xi_1, \ldots, \xi_{N,n_i}\right),$$

then

$$v_{n_i} = v|_{Q_{n_i}} \to v|_Q \,,$$

where $Q := \mathbb{R}^+ \times D \times \mathbb{R}^+$. Summing up all these facts and keeping in mind the inverse change of variables, we can show our main result for the specific problem (1)-(4) set in the singular cylindrical domain:

Theorem 2. *Let* $h \in C^{\min(\theta,\sigma)}(\Pi)$, *where* $0 < \theta,\ 2m\sigma < 1$, *satisfying condition (2). Then problem* (1)-(4) *has a unique strict solution* $u \in C^4(\Pi)$ *such that*

$$D_t^4 u \text{ and } \sum_{i=1}^{N} D_{x_i}^{2m} u \in C^{\min(\theta,\sigma)}(\Pi)\,.$$

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