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Alternating sums of binomial quotients

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Abstract. By combining telescoping and the linearization method, a class of alternating sums of binomial quotients is investigated. Several summation and transformation formulae are established. Asymptotic behavior for these sums is also examined.

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1. Introduction and outline

Binomial identities appear often in mathematics, physics and computer sciences. Evaluating binomial sums sometimes becomes an entertaining and challenging activity (cf. [7, Chapter 5]). Gould [6] made a comprehensive coverage of 500 classified binomial identities. Riordan [10] and Graham–Knuth–Patashnik [7] recorded several classical methods to treat binomial sums. A modern technological approach was presented by Petkovšek–Wilf–Zeilberger [9] through computer algebra.

In this paper, we shall examine the following alternating sums of binomial quotients:

$$\Omega_{\lambda,\delta}(m) := \sum_{k=0}^{m}(-1)^k \frac{\binom{m+\lambda}{2k+\delta}}{\binom{m}{k}}, \tag{1}$$

where $\lambda, m \in \mathbb{N}_0$ and $\delta \in \{0,1\}$. They are not on the list collected by Gould [6].

In the next section, we shall prove a general summation theorem for $\lambda \geq 2$ that expresses $\Omega_{\lambda,\delta}(m)$ as a finite sum of $\lambda - 1$ terms and contains several elegant closed formulae as consequences. Then in Section 3, the exceptional cases for $\lambda \in \{0,1\}$ will be investigated. Even though in these cases $\Omega_{\lambda,\delta}(m)$ do not admit closed formulae, they can be expressed as reciprocal sums of binomial coefficients. Finally, the asymptotic values of $\Omega_{\lambda,\delta}(m)$ will be determined as $m \to \infty$. The main results can be summarized as follows: For $\lambda \geq 0$, the limit of $\Omega_{\lambda,\delta}(m)$ as $m \to \infty$ results in 0 and 2, for $\delta = 0$ and $\delta = 1$, respectively.

Throughout the paper, we shall make use of the following notations for shifted factorials. For an indeterminate x and a nonnegative integer n, they are defined by $(x)_0 = \langle x\rangle_0 = 1$ and

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$$\left.\begin{aligned}(x)_n &= x(x+1)\cdots(x+n-1)\\ \langle x\rangle_n &= x(x-1)\cdots(x-n+1)\end{aligned}\right\}\quad n=1,2,\cdots.$$

2. Summation formulae by the linearization method

For $\lambda \geq 2$, we shall evaluate $\Omega_{\lambda,\delta}(m)$ in closed forms by means of the linearization method [1–3]. Let us start with an easier case.

Proposition 1.

$$\Omega_{2,1}(m) = \sum_{k=0}^{m}(-1)^k \frac{\binom{m+2}{2k+1}}{\binom{m}{k}} = 2.$$

Proof. Writing the summand as

$$\frac{\binom{m+2}{2k+1}}{\binom{m}{k}} = \mathrm{U}_k + \mathrm{U}_{k+1}, \quad \text{where} \quad \mathrm{U}_k = 2\frac{\binom{m+1}{2k}}{\binom{m+1}{k}}.$$

Then by telescoping, we can evaluate

$$\sum_{k=0}^{m}(-1)^k \frac{\binom{m+2}{2k+1}}{\binom{m}{k}} = \mathrm{U}_0 + (-1)^m \mathrm{U}_{m+1}.$$

The formula in Proposition 1 follows from the facts that $\mathrm{U}_0 = 2$ and $\mathrm{U}_{m+1} = 0$. □

Proposition 2.

$$\Omega_{2,0}(m) = \sum_{k=0}^{m}(-1)^k \frac{\binom{m+2}{2k}}{\binom{m}{k}} = -\frac{2}{m}.$$

Proof. Writing the summand as

$$\frac{\binom{m+2}{2k}}{\binom{m}{k}} = \mathrm{V}_k + \mathrm{V}_{k+1}, \quad \text{where} \quad \mathrm{V}_k = 2\frac{(2k-1)\binom{m+2}{2k}}{m\binom{m+1}{k}}.$$

Then by telescoping, we can evaluate

$$\sum_{k=0}^{m}(-1)^k \frac{\binom{m+2}{2k}}{\binom{m}{k}} = \mathrm{V}_0 + (-1)^m \mathrm{V}_{m+1}.$$

The formula in Proposition 2 follows since $\mathrm{V}_0 = -\frac{2}{m}$ and $\mathrm{V}_{m+1} = 0$. □

Observing that

$$\binom{m+3}{2k+1} = \binom{m+2}{2k+1} + \binom{m+2}{2k}$$

and then adding the two equations in propositions 1 and 2 together, we deduce the next identity.

Corollary 1.

$$\Omega_{3,1}(m)=\sum_{k=0}^{m}(-1)^k\frac{\binom{m+3}{2k+1}}{\binom{m}{k}}=2-\frac{2}{m}.$$

In order to treat the $\lambda>2$ case, we have to invoke the following linear representation lemma whose equivalent form was found by the first author [1, Eq. (8)].

Lemma 1. *Let a and b be two indeterminates. There exist $\{\mathcal{X}_\rho^i\}_{i=0}^{\rho}$ such that*

$$1=\sum_{i=0}^{\rho}\langle a-k\rangle_i\,\langle b-k\rangle_{\rho-i}\,\mathcal{X}_\rho^i,$$

where the coefficient $\mathcal{X}_\rho^i$ is independent of the variable k and given explicitly by

$$\mathcal{X}_\rho^i=(-1)^{i+\rho}\binom{\rho}{i}\frac{a-b+\rho-2i}{(a-b-i)_{\rho+1}}.$$

Observe that

$$\langle m+\lambda\rangle_{2k+\delta}=4^k\,\langle m+\lambda\rangle_\delta\left\langle\frac{m+\lambda-\delta}{2}\right\rangle_k\left\langle\frac{m+\lambda-\delta-1}{2}\right\rangle_k.$$

By specifying in Lemma 1

$$\rho\to\lambda-2,\ a\to\frac{m+\lambda-\delta}{2}\quad\text{and}\quad b\to\frac{m+\lambda-\delta-1}{2},$$

we have a linear equation

$$1=\sum_{i=0}^{\lambda-2}\mathcal{Y}_\lambda^i\left\langle\frac{m+\lambda-\delta}{2}-k\right\rangle_i\left\langle\frac{m+\lambda-\delta-1}{2}-k\right\rangle_{\lambda-2-i},$$

where

$$\mathcal{Y}_\lambda^i=(-1)^{\lambda-i}\binom{\lambda-2}{i}\frac{\lambda-\frac{3}{2}-2i}{(\frac{1}{2}-i)_{\lambda-1}}. \tag{2}$$

Therefore, we can rewrite

$$\begin{aligned}\langle m+\lambda\rangle_{2k+\delta}&=\langle m+\lambda\rangle_{2k+\delta}\sum_{i=0}^{\lambda-2}\mathcal{Y}_\lambda^i\left\langle\frac{m+\lambda-\delta}{2}-k\right\rangle_i\left\langle\frac{m+\lambda-\delta-1}{2}-k\right\rangle_{\lambda-2-i}\\&=4^k\,\langle m+\lambda\rangle_\delta\sum_{i=0}^{\lambda-2}\mathcal{Y}_\lambda^i\left\langle\frac{m+\lambda-\delta}{2}\right\rangle_{k+i}\left\langle\frac{m+\lambda-\delta-1}{2}\right\rangle_{k+\lambda-2-i}.\end{aligned}$$

Now we are ready to examine the sum $\Omega_{\lambda,\delta}(m)$ for $2<\lambda\le 1+\delta+m$, where

$\delta \in \{0,1\}$. Express it first as a double sum below:

$$\begin{aligned}\Omega_{\lambda,\delta}(m) &= \sum_{k=0}^{m}(-1)^k \frac{k!\,\langle m+\lambda\rangle_{2k+\delta}}{(2k+\delta)!\,\langle m\rangle_k}\\ &= \binom{m+\lambda}{\delta}\sum_{i=0}^{\lambda-2}\mathcal{Y}_\lambda^i \left\langle \frac{m+\lambda-\delta}{2}\right\rangle_i \left\langle \frac{m+\lambda-\delta-1}{2}\right\rangle_{\lambda-2-i}\\ &\times \sum_{k=0}^{m}(-1)^k \frac{\left\langle \frac{m+\lambda-\delta}{2}-i\right\rangle_k \left\langle \frac{3+m-\lambda-\delta}{2}+i\right\rangle_k}{(\frac{1}{2}+\delta)_k\,\langle m\rangle_k}.\end{aligned}$$

Then defining the sequence

$$T_k = \frac{\left\langle \frac{m+\lambda-\delta}{2}-i\right\rangle_k \left\langle \frac{3+m-\lambda-\delta}{2}+i\right\rangle_k}{(\delta-\frac{1}{2})_k\,\langle m+1\rangle_k},$$

we can compute

$$T_k + T_{k+1} = \Delta_m^i \frac{\left\langle \frac{m+\lambda-\delta}{2}-i\right\rangle_k \left\langle \frac{3+m-\lambda-\delta}{2}+i\right\rangle_k}{(\frac{1}{2}+\delta)_k\,\langle m\rangle_k},$$

where Δ_m^i is a constant independent of k

$$\Delta_m^i = \frac{(\delta+2+m-\lambda+2i)(\delta-1+m+\lambda-2i)}{2(2\delta-1)(m+1)}. \tag{3}$$

By telescoping, we can evaluate the sum

$$\begin{aligned}\sum_{k=0}^{m}(-1)^k \frac{\left\langle \frac{m+\lambda-\delta}{2}-i\right\rangle_k \left\langle \frac{3+m-\lambda-\delta}{2}+i\right\rangle_k}{(\frac{1}{2}+\delta)_k\,\langle m\rangle_k} &= \sum_{k=0}^{m}(-1)^k \frac{T_k+T_{k+1}}{\Delta_m^i}\\ &= \frac{T_0}{\Delta_m^i} + (-1)^{m+1}\frac{T_{m+1}}{\Delta_m^i} = \frac{1}{\Delta_m^i},\end{aligned}$$

where T_{m+1} vanishes since among the two factors appearing in the numerator of T_{m+1}, there is one falling factorial with its parameter inside $\langle\cdots\rangle$ being an integer between 0 and m.

Therefore, after substitution, we find the following general summation formula.

Theorem 1 ($2 < \lambda \le 1+\delta+m$ with $\delta \in \{0,1\}$)**.** *For $\lambda \in \mathbb{N}$ with $\lambda > 2$, let $\mathcal{Y}_\lambda^i$ and Δ_m^i be defined by* (2) *and* (3)*, respectively. Then the following formula holds:*

$$\Omega_{\lambda,\delta}(m) = \binom{m+\lambda}{\delta}\sum_{i=0}^{\lambda-2}\frac{\mathcal{Y}_\lambda^i}{\Delta_m^i}\left\langle \frac{m+\lambda-\delta}{2}\right\rangle_i \left\langle \frac{m+\lambda-\delta-1}{2}\right\rangle_{\lambda-2-i}.$$

When λ is a small integer, we can compute $\Omega_{\lambda,\delta}(m)$ by this theorem in a few terms, for example, those displayed in propositions 1, 2 and Corollary 1. Further summation formulae are recorded below.

$$\boxed{\Omega_{3,0}(m)} \quad \sum_{k=0}^{m}(-1)^k\frac{\binom{m+3}{2k}}{\binom{m}{k}} = \frac{2(3-m)}{m(m-1)}.$$

$$\boxed{\Omega_{4,0}(m)} \quad \sum_{k=0}^{m}(-1)^k\frac{\binom{m+4}{2k}}{\binom{m}{k}} = \frac{2(m^2-7m+16)}{m(m-1)(2-m)}.$$

$$\boxed{\Omega_{4,1}(m)} \quad \sum_{k=0}^{m}(-1)^k\frac{\binom{m+4}{2k+1}}{\binom{m}{k}} = \frac{2(m^2-3m+4)}{m(m-1)}.$$

$$\boxed{\Omega_{5,0}(m)} \quad \sum_{k=0}^{m}(-1)^k\frac{\binom{m+5}{2k}}{\binom{m}{k}} = \frac{2(5-m)(m^2-7m+24)}{\langle m\rangle_4}.$$

$$\boxed{\Omega_{5,1}(m)} \quad \sum_{k=0}^{m}(-1)^k\frac{\binom{m+5}{2k+1}}{\binom{m}{k}} = \frac{2(m-3)(m^2-3m+8)}{\langle m\rangle_3}.$$

3. Four binomial transformation formulae

When $\lambda \in \{0,1\}$, the corresponding binomial sums $\Omega_{\lambda,\delta}(m)$ have no closed formulae. However, they can be expressed in this case as finite reciprocal sums of binomial coefficients. For this purpose, we need the following crucial lemma.

Lemma 2. *For the reciprocal sum of binomial coefficients*

$$\mathcal{S}_m = \sum_{k=0}^{m}\binom{m}{k}^{-1},$$

we have the following identity:

$$\mathcal{S}_m = \frac{m+1}{2^m}\sum_{k=0}^{m}\frac{2^k}{k+1}.$$

Furthermore, $\mathcal{S}_m$ satisfies the recurrence relation

$$\mathcal{S}_m = 1 + \frac{m+1}{2m}\mathcal{S}_{m-1}.$$

The results in this lemma were the subject of problem 1 in the afternoon session of the 1958 Putnam Exam and then recorded first by Comtet [4, Exercise 15, p. 294] (see also [7, Exercise 5.100, pp. 542–543]). Different proofs for the identity in the middle can be found in [8, 11–13]. To make the paper self–contained, we produce their proofs below.

Proof. Writing the inverse binomial coefficient in terms of the Beta integral

$$\binom{m}{k}^{-1} = (m+1)B(m-k+1,k+1) = (m+1)\int_0^1 x^{m-k}(1-x)^k dx,$$

then interchanging the order between the sum and the integral, we can proceed with

$$\begin{aligned}
\mathcal{S}_m &= \sum_{k=0}^{m} \frac{1}{\binom{m}{k}} = (m+1)\int_0^1 \sum_{k=0}^{m} x^{m-k}(1-x)^k dx \\
&= (m+1)\int_0^1 \frac{x^{m+1}-(1-x)^{m+1}}{2x-1} dx \\
&= \frac{m+1}{2}\int_0^1 \frac{x^{m+1}-2^{-m-1}}{x-1/2} dx + \frac{m+1}{2}\int_0^1 \frac{2^{-m-1}-(1-x)^{m+1}}{x-1/2} dx \\
&= \frac{m+1}{2}\sum_{k=0}^{m}\int_0^1 \frac{x^k}{2^{m-k}} dx + \frac{m+1}{2}\sum_{k=0}^{m}\int_0^1 \frac{(1-x)^k}{2^{m-k}} dx \\
&= (m+1)\sum_{k=0}^{m}\int_0^1 \frac{x^k}{2^{m-k}} dx = \frac{m+1}{2^m}\sum_{k=0}^{m}\frac{2^k}{k+1}.
\end{aligned}$$

This proves the identity in the lemma. The recurrence relation follows easily by putting aside the end term from the above expression:

$$\begin{aligned}
\mathcal{S}_m &= \frac{m+1}{2^m}\sum_{k=0}^{m}\frac{2^k}{k+1} = 1 + \frac{m+1}{2^m}\sum_{k=0}^{m-1}\frac{2^k}{k+1} \\
&= 1 + \frac{m+1}{2^m}\left\{\mathcal{S}_{m-1}\times\frac{2^{m-1}}{m}\right\} = 1 + \frac{m+1}{2m}\mathcal{S}_{m-1}.
\end{aligned}$$

□

Alternative proof. Thanks to the anonymous referee, who offers the following elegant and independent proof for the above recurrence relation. Firstly, by taking out the initial term with $k = 0$, write

$$\mathcal{S}_m = 1 + \sum_{k=1}^{m}\binom{m}{k}^{-1} = 1 + \sum_{k=0}^{m-1}\frac{k+1}{m\binom{m-1}{k}}.$$

Then by putting aside the end term $k = m$, we can also write

$$\mathcal{S}_m = 1 + \sum_{k=0}^{m-1}\binom{m}{k}^{-1} = 1 + \sum_{k=0}^{m-1}\frac{m-k}{m\binom{m-1}{k}}.$$

Now adding the two equations, we confirm the recurrence relation

$$2\mathcal{S}_m = 2 + \sum_{k=0}^{m+1}\binom{m}{k}^{-1} = 2 + \frac{m+1}{m}\mathcal{S}_{m-1}.$$

□

Proposition 3.

$$\Omega_{1,1}(m) = \sum_{k=0}^{m}(-1)^k\frac{\binom{m+1}{2k+1}}{\binom{m}{k}} = \sum_{k=0}^{m}\frac{1}{\binom{m}{k}}.$$

Proof. This can be done by showing that the sum $\Omega_{1,1}(m)$ satisfies the same recurrence relation as $\mathcal{S}_m$ in Lemma 2. Recalling that

$$\binom{m}{k} = \binom{m-1}{k}\frac{m}{m-k} \quad \text{and} \quad \frac{m-k}{m} = \frac{1}{2} + \frac{m-2k}{2m},$$

we can reformulate the sum

$$\begin{aligned}\Omega_{1,1}(m) &= \sum_{k=0}^{m}(-1)^k \frac{\binom{m+1}{2k+1}}{\binom{m}{k}} = \sum_{k=0}^{m-1}(-1)^k \frac{\binom{m+1}{2k+1}}{\binom{m-1}{k}}\frac{m-k}{m} \\ &= \frac{1}{2}\sum_{k=0}^{m-1}(-1)^k \frac{\binom{m+1}{2k+1}}{\binom{m-1}{k}} + \frac{m+1}{2m}\sum_{k=0}^{m-1}(-1)^k \frac{\binom{m}{2k+1}}{\binom{m-1}{k}} \\ &= \frac{1}{2}\Omega_{2,1}(m-1) + \frac{m+1}{2m}\Omega_{1,1}(m-1).\end{aligned}$$

According to Proposition 1, we get the recurrence relation

$$\Omega_{1,1}(m) = 1 + \frac{m+1}{2m}\Omega_{1,1}(m-1).$$

Keeping in mind the initial value $\Omega_{1,1}(0) = 1$, we conclude that $\Omega_{1,1}(m) = \mathcal{S}_m$. □

Proposition 4.

$$\Omega_{1,0}(m) = \sum_{k=0}^{m}(-1)^k \frac{\binom{m+1}{2k}}{\binom{m}{k}} = 2 - \sum_{k=0}^{m}\frac{1}{\binom{m}{k}}.$$

Proof. According to the binomial recurrence

$$\binom{m+2}{2k+1} = \binom{m+1}{2k+1} + \binom{m+1}{2k},$$

we get the following expression

$$\Omega_{1,0}(m) = \Omega_{2,1}(m) - \Omega_{1,1}(m).$$

Then the desired identity follows immediately from propositions 1 and 3. □

Proposition 5.

$$\Omega_{0,0}(m) = \sum_{k=0}^{m}(-1)^k \frac{\binom{m}{2k}}{\binom{m}{k}} = m + 1 - \frac{m}{2}\sum_{k=0}^{m}\frac{1}{\binom{m}{k}}.$$

Proof. Similarly to the proof of Proposition 3, we can reformulate

$$\begin{aligned}\Omega_{0,0}(m) &= \sum_{k=0}^{m}(-1)^k \frac{\binom{m}{2k}}{\binom{m}{k}} = \sum_{k=0}^{m-1}\frac{\binom{m}{2k}}{\binom{m-1}{k}}\frac{m-k}{m} \\ &= \frac{1}{2}\sum_{k=0}^{m-1}(-1)^k \frac{\binom{m}{2k}}{\binom{m-1}{k}} + \frac{1}{2}\sum_{k=0}^{m-1}(-1)^k \frac{\binom{m-1}{2k}}{\binom{m-1}{k}} \\ &= \frac{1}{2}\Omega_{1,0}(m-1) + \frac{1}{2}\Omega_{0,0}(m-1).\end{aligned}$$

Evaluating the above $\Omega_{1,0}(m-1)$ by Proposition 4 and then multiplying the resulting equation by 2^m, we get the recurrence relation below:

$$2^m\Omega_{0,0}(m) - 2^{m-1}\Omega_{0,0}(m-1) = 2^m - 2^{m-1}\mathcal{S}_{m-1}.$$

Summing the above equation over m from 1 to m by telescoping and taking into account that $\Omega_{0,0}(0) = 1$, we get the equality

$$2^m\Omega_{0,0}(m) = 2^{m+1} - 1 - \sum_{n=1}^{m} 2^{n-1}\mathcal{S}_{n-1}. \tag{4}$$

Recalling Lemma 2 and then exchanging the summation order, we can express the last sum with respect to n as follows:

$$\begin{aligned}\sum_{n=1}^{m} 2^{n-1}\mathcal{S}_{n-1} &= \sum_{n=1}^{m} n\sum_{k=1}^{n}\frac{2^{k-1}}{k} = \sum_{k=1}^{m}\frac{2^{k-1}}{k}\sum_{n=k}^{m} n\\ &= \sum_{k=1}^{m}\frac{2^{k-1}}{k}\left\{\binom{m+1}{2} - \binom{k}{2}\right\}\\ &= \binom{m+1}{2}\sum_{k=1}^{m}\frac{2^{k-1}}{k} - \sum_{k=1}^{m}(k-1)2^{k-2}.\end{aligned}$$

Evaluating further

$$\sum_{k=1}^{m}(k-1)2^{k-2} = 1 - 2^m + 2^{m-1}m,$$

we find the closed form expression

$$\begin{aligned}\sum_{n=1}^{m} 2^{n-1}\mathcal{S}_{n-1} &= 2^{m-1}m(\mathcal{S}_m - 1) - \Big(1 - 2^m + 2^{m-1}m\Big)\\ &= 2^{m+1} - 1 + 2^{m-1}m\mathcal{S}_m - 2^m(m+1).\end{aligned}$$

By making substitution in (4), we finally arrive at

$$2^m\Omega_{0,0}(m) = 2^m(m+1) - 2^{m-1}m\mathcal{S}_m.$$

Dividing across by 2^m gives rise to the identity in Proposition 5. □

Proposition 6.

$$\Omega_{0,1}(m) = \sum_{k=0}^{m}(-1)^k\frac{\binom{m}{2k+1}}{\binom{m}{k}} = \frac{m+2}{2}\sum_{k=0}^{m}\frac{1}{\binom{m}{k}} - m - 1.$$

Proof. By making use of the binomial recurrence

$$\binom{m+1}{2k+1} = \binom{m}{2k} + \binom{m}{2k+1}$$

we get the following expression

$$\Omega_{0,1}(m) = \Omega_{1,1}(m) - \Omega_{0,0}(m).$$

Then the desired identity follows immediately from propositions 3 and 5. □

4. Asymptotic values

Farmer and Leth [5] investigated asymptotic behaviors of some binomial sums. When $m \to \infty$, asymptotic values of the $\Omega_{\lambda,\delta}(m)$ sums can be determined. We start from the following limiting relations.

Lemma 3. *There is the limiting value*

$$\lim_{m\to\infty} \mathcal{S}(m) = \lim_{m\to\infty} \sum_{k=0}^{m} \binom{m}{k}^{-1} = 2.$$

More precisely, we have the asymptotic estimation with the subdominant term

$$\mathcal{S}(m) = 2\Big\{1 + \frac{1}{m} + \mathcal{O}\Big(\frac{1}{m^2}\Big)\Big\}.$$

Proof. For $m \geq 4$, by pulling out the two initial and the two end terms from the sum, write

$$\mathcal{S}(m) = 2 + \frac{2}{m} + \sum_{k=2}^{m-2} \binom{m}{k}^{-1}.$$

Since the binomial coefficients $\binom{m}{k}$ are unimodal with respect to k, we have the inequalities

$$2 + \frac{2}{m} < \mathcal{S}(m) = 2 + \frac{2}{m} + \sum_{k=2}^{m-2} \binom{m}{k}^{-1} \leq 2 + \frac{2}{m} + \frac{m-3}{\binom{m}{2}} < 2 + \frac{4}{m}.$$

Letting $m \to \infty$, we confirm the limiting value of $\mathcal{S}(m)$.

The asymptotic estimation is similarly determined by

$$\mathcal{S}(m) = 2 + \frac{2}{m} + \frac{2}{\binom{m}{2}} + \sum_{k=3}^{m-3} \binom{m}{k}^{-1} \leq 2 + \frac{2}{m} + \frac{2}{\binom{m}{2}} + \frac{m-5}{\binom{m}{3}}.$$

□

Applying this lemma to propositions 3–6, we can easily deduce the following interesting asymptotic relations.

Proposition 7 (Limiting values)**.**

$$\boxed{\Omega_{1,1}} \quad \lim_{m\to\infty} \sum_{k=0}^{m} (-1)^k \frac{\binom{m+1}{2k+1}}{\binom{m}{k}} = 2,$$

$$\boxed{\Omega_{1,0}} \quad \lim_{m\to\infty} \sum_{k=0}^{m} (-1)^k \frac{\binom{m+1}{2k}}{\binom{m}{k}} = 0,$$

$$\boxed{\Omega_{0,0}} \quad \lim_{m\to\infty} \sum_{k=0}^{m} (-1)^k \frac{\binom{m}{2k}}{\binom{m}{k}} = 0,$$

$$\boxed{\Omega_{0,1}} \quad \lim_{m\to\infty} \sum_{k=0}^{m} (-1)^k \frac{\binom{m}{2k+1}}{\binom{m}{k}} = 2.$$

Instead, the explicit formulae obtained in Section 2 for $\Omega_{\lambda,\delta}(m)$ with $\lambda \geq 2$ suggest the following remarkable results that are not deducible directly from Theorem 1.

Theorem 2 (Limiting values: $\lambda \geq 0$)**.**

$$\boxed{\Omega_{\lambda,0}} \qquad \lim_{m\to\infty}\sum_{k=0}^{m}(-1)^k\frac{\binom{m+\lambda}{2k}}{\binom{m}{k}}=0,$$

$$\boxed{\Omega_{\lambda,1}} \qquad \lim_{m\to\infty}\sum_{k=0}^{m}(-1)^k\frac{\binom{m+\lambda}{2k+1}}{\binom{m}{k}}=2.$$

Proof. The two limiting values can be shown by means of the induction principle on λ. For $\lambda = 0, 1$, the limiting values are already given in Proposition 7. When $\lambda = 2$, both values are confirmed by the explicit formulae given in propositions 1 and 2. Suppose that they are true for all λ with $2 \leq \lambda < n$. Then we have to validate them for $\lambda = n$. For $\delta \in \{0, 1\}$, recall the binomial recurrence relation

$$\binom{m+n}{2k+\delta}=\binom{m+n-1}{2k+\delta}+\binom{m+n-1}{2k+\delta-1}. \tag{5}$$

When $\delta = 1$, we can write

$$\Omega_{n,1}(m)=\Omega_{n-1,1}(m)+\Omega_{n-1,0}(m).$$

Then according to the induction hypothesis, we get

$$\lim_{m\to\infty}\Omega_{n,1}(m)=\lim_{m\to\infty}\Omega_{n-1,1}(m)+\lim_{m\to\infty}\Omega_{n-1,0}(m)=2+0=2.$$

Instead, when $\delta = 0$, we have from (5) that

$$\begin{aligned}\Omega_{n,0}(m)&=\Omega_{n-1,0}(m)+\sum_{k=0}^{m}(-1)^k\frac{\binom{m+n-1}{2k-1}}{\binom{m}{k}}\\&=\Omega_{n-1,0}(m)-\sum_{k=0}^{m-1}(-1)^k\frac{\binom{m+n-1}{2k+1}}{\binom{m}{k+1}}\\&=\Omega_{n-1,0}(m)-\sum_{k=0}^{m-1}(-1)^k\frac{\binom{m+n-1}{2k+1}}{\binom{m-1}{k}}\times\frac{k+1}{m},\end{aligned}$$

where the summation index k has been shifted to $k+1$ in the middle line. In view of the linear equation

$$k+1=\frac{(2k+1)(m+n)}{2(m+n-1)}+\frac{m+n-2-2k}{2(m+n-1)},$$

we have the corresponding binomial relation:

$$\frac{k+1}{m}\binom{m+n-1}{2k+1}=\frac{m+n}{2m}\binom{m+n-2}{2k}+\frac{1}{2m}\binom{m+n-2}{2k+1}.$$

Therefore, we can express further

$$\Omega_{n,0}(m) = \Omega_{n-1,0}(m) - \frac{m+n}{2m}\Omega_{n-1,0}(m-1) - \frac{1}{2m}\Omega_{n-1,1}(m-1).$$

Letting $m \to \infty$ across the last equation and then appealing to the induction hypothesis, we deduce that

$$\Omega_{n,0}(m) = 0 - 0 \times \frac{1}{2} - 0 \times 2 = 0.$$

This completes the proof for the limiting values in Theorem 2. □

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