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Complete solution of the exponential Diophantine equation $P_n^x + P_{n+1}^x = P_m^y$

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Abstract. In this paper, we find all the solutions of the title Diophantine equation in positive integer variables (m, n, x, y), where for a positive integer k, P_k is the kth term of the Pell sequence.

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1. Introduction

Let $(P_n)_{n\geq 0}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$, for all $n \geq 0$. The Diophantine equation

$$P_n^x + P_{n+1}^x = P_m \tag{1}$$

in nonnegative integers (m, n, x) was recently studied by some of us in [18]. We proved that all the solutions in positive integers (m, n, x) come from the identity

$$P_n^2 + P_{n+1}^2 = P_{2n+1},$$

and so we have $x = 2$ and $m = 2n + 1$, for all positive integers n. A more general equation $P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = P_m$, in positive integers (n, k, m, x) was studied in [13]. Other exponential Diophantine equations involving members of the Pell sequence appear in [10]. In this paper, we study a different extension of the

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equation (1). Namely, we allow an extra exponent y on its right–hand side, so we look at the equation

$$P_n^x + P_{n+1}^x = P_m^y \tag{2}$$

in positive integers m, n, x and y. Following the argument from [11], we prove the following theorem.

Theorem 1. *All solutions (m, n, x, y) in positive integers of the Diophantine equation (2) have $y = 1$. In particular, they also have $x = 2$ and $m = 2n + 1$.*

The proof of Theorem 1 uses linear forms in logarithms of algebraic numbers, a reduction algorithm originally introduced by Baker and Davenport in [1], as well as the LLL algorithm. In Section 2, we recall some properties of the Pell sequence, state the variant version of Matveev's theorem due to Bugeaud-Mignotte-Siksek [5], and two results on reduction methods (the Baker-Davenport reduction method and the LLL algorithm). The last section is devoted to the proof of our main result. This is done in eight steps. For the first step, we consider the very small values of the parameters and determine the solutions. For the remaining steps, we take $n \geq 2,\ x \geq 3,\ y \geq 2,\ m \geq 3$. The second step consists in using the properties of the Pell sequence to prove that $|(n + 1)x - my| < 2\max\{x, y\}$. In the next step, we denote $M := \min\{m, n + 1\}$, $N := \max\{m, n + 1\}$ and use Baker's method to bound x, y in terms of M, N. These bounds are very high. The fourth step consists in considering $N \leq 1000$ and using continued fractions we prove that there is no solution with $y \geq 2$ in this range. For the remaining steps, we work under the assumption that $N > 1000$. We use again Baker's method to bound x, y, N in terms of M. For the sixth step, we consider $M \leq 1000$ and use the LLL algorithm to prove that the Diophantine equation (2) has no solution with $n \geq 2,\ x \geq 3,\ y \geq 2,\ m \geq 3$ in this range. In the next step, another application of Baker's method allows us to obtain an absolute bound of m, n, x, y; that is, $\max\{x, y\} < 3 \times 10^{111}$. For the last step, we use the Baker-Davenport reduction method to obtain a contradiction to the condition $M > 1000$.

2. Preliminary results

2.1. The Pell sequence

Let $(\alpha, \beta) := (1+\sqrt{2}, 1-\sqrt{2})$ be the roots of the characteristic equation $x^2-2x-1 = 0$ of the Pell sequence $(P_n)_{n\geq 0}$. The Binet's formula

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \quad \text{holds for all } n \geq 0. \tag{3}$$

This implies easily that the inequality

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{4}$$

holds for all positive integers n. The Pell sequence has a companion $(Q_n)_{n\geq 0}$ given by $Q_0 = 2,\ Q_1 = 2$ an $Q_{n+2} = 2Q_{n+1} + Q_n$, for all $n \geq 0$. Its Binet's formula is

$$Q_n = \alpha^n + \beta^n, \quad \text{for all } n \geq 0.$$

There are several relations among members of the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ such as $P_{2n} = P_n Q_n$ valid for all $n \geq 0$. We will freely use such relations whenever we need them. We also have that the inequality

$$\frac{P_n}{P_{n+1}} \leq \frac{3}{7} \quad \text{holds for all } n \geq 2, \tag{5}$$

a fact which can be easily verified by induction.

2.2. Linear forms in logarithms

For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{j=1}^{d} \left(X - \gamma^{(j)}\right)$ with $a \geq 1$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log a + \sum_{j=1}^{d} \log \max\left\{1, \left|\gamma^{(j)}\right|\right\}\right)$$

the usual absolute logarithmic height of γ.

We start by recalling Theorem 9.4 of [5], which is a modified version of a result of Matveev [15].

Theorem 2. *Let $s \geq 1$, $\gamma_1, \ldots, \gamma_s$ be nonzero real algebraic numbers and let $b_1, \ldots, b_s$ be integers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1 \neq 0$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_s).$$

2.3. Reduction methods

The following result is a slightly modified version of Lemma 5 (a) in [9].

Lemma 1. *Let T be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6T$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon := ||\mu q|| - T||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |m\gamma - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$m \leq T \quad \textit{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

On one occasion, we need to find a lower bound for a linear form with bounded integer coefficients in three variables for which methods based on continued fractions are not applicable. Instead, we will use the LLL algorithm which we now briefly describe. The following notations can be found in [7, Section 2.3.5].

Let $\gamma_1, \ldots, \gamma_s \in \mathbb{R}$ and $X_1, \ldots, X_s$ be positive real numbers. We consider the linear form

$$x_1\gamma_1 + x_2\gamma_2 + \cdots + x_s\gamma_s \quad \text{with} \quad x_i \in [-X_i, X_i] \cap \mathbb{Z}.$$

We set $X := \max\{X_i\}$, $C > (sX)^s$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\gamma_j \rceil \, \mathbf{e}_s \qquad \text{for} \qquad 1 \le j \le s-1 \qquad \text{and} \qquad \mathbf{b}_s := \lfloor C\gamma_s \rceil \, \mathbf{e}_s,$$

where C is a sufficiently large positive constant. The next lemma is the main part of Proposition 2.3.20 in [7, Section 2.3.5].

Lemma 2. *Let $X_1, \ldots, X_s$ be positive real numbers such that $X := \max\{X_i\}$ and $C > (sX)^s$ is a fixed constant. With the above notation on Ω, we consider a reduced basis $\{\mathbf{b}_i\}$ to Ω and its associated Gram–Schmidt $\{\mathbf{b}_i^*\}$ basis. We set*

$$c_1 := \max_{1 \le i \le s} \frac{||\mathbf{b}_1||}{||\mathbf{b}_i^*||}, \quad \mathbf{m}_\Omega := \frac{||\mathbf{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{s-1} X_i^2, \qquad T := \left(1 + \sum_{i=1}^{s} X_i\right)/2.$$

If the integers x_i satisfy $|x_i| \le X_i$ for $i = 1, \ldots, s$ and ${\mathbf{m}_\Omega}^2 \ge T^2 + Q$, then we have

$$\left|\sum_{i=1}^{s} x_i\gamma_i\right| \ge \frac{\sqrt{\mathbf{m}_\Omega^2 - Q} - T}{C}.$$

3. The proof of Theorem 1

3.1. The case when one of m, n, x, y is small

We may assume that $y \ge 2$, since the case $y = 1$ was treated in [18]. For technical reasons we would like to work with $n \ge 2$ and $x \ge 3$. Well, assume that $n = 1$. Then the equation is

$$1 + 2^x = P_m^y.$$

Since $y \ge 2$, it follows that $x \ge 3$ so the above is a particular case of Catalan's conjecture solved by Preda Mihăilescu [17]. Its only solution $1 + 2^3 = 3^2$ is not convenient for us since 3 is not a member of the Pell sequence. This takes care of the case when $n = 1$.

Assume next that $x < 3$. Then $x \in \{1, 2\}$. Suppose that $x = 2$. In this case, we get $P_{2n+1} = P_n^2 + P_{n+1}^2 = P_m^y$ with $y \ge 2$. In particular, P_{2n+1} is a perfect power and $2n+1 \ge 5$. It is known (see [8]) that the only perfect power which is a member of the Pell sequence of index at least 5 is $13^2 = 169 = P_7$, but 13 is not a member of the Pell sequence. This shows that the case $x = 2$ is not possible. Suppose next that $x = 1$. Then we get

$$P_n + P_{n+1} = P_m^y.$$

Since $P_n \equiv n \pmod 2$, the left–hand side above is odd. Hence, P_m is odd, therefore m is odd. On the left, we have $P_n + P_{n+1} = Q_{n+1}/2$. Let p be any prime factor of $Q_{n+1}/2$ which exists since $n > 1$, so $Q_{n+1} > 2$. Clearly, p is odd. We also have that $p \mid P_m$, thus $p \mid \gcd(Q_{n+1}, P_m)$. By a result of McDaniel (see [16]), it follows that if we put $d := \gcd(n+1, m)$, then m/d is even. In particular, m is even, contradicting the fact that m is odd. Hence, there are no solutions to our equation for $x = 1$ and $y \geq 2$.

3.2. An inequality among the variables m, n, x, y

In what follows, we assume that $n \geq 2,\ x \geq 3,\ y \geq 2$. Note also that $m > 1$ is odd, so $m \geq 3$.

Lemma 3. *If (m, n, x, y) is a solution of (2) with $n \geq 2$, $x \geq 3$ and $y \geq 2$, then the inequality*

$$|(n+1)x - my| < 2\max\{x, y\} \tag{6}$$

holds.

Proof. Equation (2) and inequality (4) imply that

$$\alpha^{y(m-2)} < P_m^y = P_n^x + P_{n+1}^x < (P_n + P_{n+1})^x < P_{n+2}^x < \alpha^{x(n+1)}$$

and

$$\alpha^{y(m-1)} > P_m^y = P_n^x + P_{n+1}^x > P_{n+1}^x > \alpha^{x(n-1)},$$

which lead to

$$-2y < (n+1)x - my \quad \text{and} \quad my - (n+1)x > -2x + y.$$

Therefore, we obtain the desired inequality

$$|(n+1)x - my| < 2\max\{x, y\}.$$

□

3.3. Bounds on x and y in terms of N and M

From now on, we set

$$M := \min\{m, n+1\} \quad \text{and} \quad N := \max\{m, n+1\}.$$

Note that $M \geq 3$.

Lemma 4. *If (m, n, x, y) is a solution in positive integers of equation (2), with $n \geq 2$, $x \geq 3$ and $y \geq 2$, then both inequalities*

$$x < 1.4 \times 10^{11} MN \log N \quad \textit{and} \quad y < 1.4 \times 10^{11} MN^2 \log N$$

hold.

Proof. The equation (2) can be rewritten as

$$P_m^y - P_{n+1}^x = P_n^x.$$

By using inequality (5), we have

$$P_m^y P_{n+1}^{-x} - 1 = \left(\frac{P_n}{P_{n+1}}\right)^x \leq \frac{1}{2.33^x}. \tag{7}$$

Let

$$\Lambda_1 := P_m^y P_{n+1}^{-x} - 1$$

be the expression appearing on the left-hand side of inequality (7). Observe that Λ_1 is positive. To get a lower bound for Λ_1, we apply Theorem 2 with the data

$$s := 2, \quad \gamma_1 := P_m, \quad \gamma_2 := P_{n+1}, \quad b_1 := y, \quad b_2 := -x.$$

Since γ_1 and γ_2 are integers, we have $D := 1$. We can take $A_1 := m \log \alpha$ and $A_2 := n \log \alpha$. Now Matveev's theorem gives that

$$\begin{aligned} |\Lambda_1| &> \exp(-1.4 \times 30^5 \times 2^{4.5}(m \log \alpha)(n \log \alpha)(1 + \log \max\{x, y\})) \\ &> \exp(-1.2 \times 10^9 \times m \times n \times \log(\max\{x, y\})), \end{aligned} \tag{8}$$

where we used the fact that $1+\log \max\{x, y\} < 2 \log \max\{x, y\}$, which holds because $\max\{x, y\} \geq x \geq 3$. Comparing inequalities (7) and (8), we obtain

$$x < 1.42 \times 10^9 mn \log \max\{x, y\}. \tag{9}$$

If $x > y$, the above inequality gives

$$x < 1.42 \times 10^9 mn \log x. \tag{10}$$

If $y > x$, Lemma 3 implies that

$$|my - (n + 1)x| < 2y,$$

therefore, since $m \geq 3$, we obtain

$$y \leq (m - 2)y < (n + 1)x \leq Nx. \tag{11}$$

Note that the case $x = y$ does not occur by Fermat's Last Theorem proved by Wiles. So, inequality (9) shows that

$$x < 1.42 \times 10^9 mn \log(Nx). \tag{12}$$

If

$$x \leq N, \tag{13}$$

we already have a bound on x. Otherwise, $x > N$ and inequality (12) gives that

$$x < 1.42 \times 10^9 mn \log(Nx) < 3 \times 10^9 mn \log x. \tag{14}$$

Comparing (10), (13) and (14), we conclude that inequality (14) holds in all cases. So, we have

$$\frac{x}{\log x} < 3 \times 10^9 mn,$$

which gives us

$$x < 6 \times 10^9 mn \log(3 \times 10^9 N^2) < 1.4 \times 10^{11} mn \log N,$$

where we used the fact that $\log(3 \times 10^9 N^2) < 22 \log N$ for all $N \geq 3$. From estimate (11), we also deduce that

$$y < Nx < 1.4 \times 10^{11} MN^2 \log N.$$

This finishes the proof of the lemma. □

3.4. Solutions with $N \leq 1000$

Lemma 5. *There are no solutions to the Diophantine equation* (2) *with* $y \geq 2$ *and* $N \leq 1000$.

Proof. Assume that $N \leq 1000$. By Lemma 4, we have

$$\begin{aligned} x &< 1.4 \times 10^{11} \times (10^3)^2 \log(10^3) < 10^{18}, \\ y &< 1.4 \times 10^{11} \times (10^3)^3 \log(10^3) < 10^{21}. \end{aligned}$$

Put

$$\Gamma_1 := y \log P_m - x \log P_{n+1},$$

and observe that $\Gamma_1 > 0$ since $\Lambda_1 = e^{\Gamma_1} - 1 > 0$. Hence, from (7), we get

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \Lambda_1 < \frac{1}{2.33^x}.$$

Dividing both sides of the last inequality by $x \log P_m$, we get

$$0 < \frac{y}{x} - \frac{\log P_{n+1}}{\log P_m} < \frac{1}{x 2.33^x \log P_m}. \tag{15}$$

Observe that

$$2.33^x \log P_m \geq 2.33^x \log 5 > 2x \quad \text{for all } x \geq 3 \quad \text{and} \quad m \geq 3.$$

Therefore, inequality (15) implies

$$0 < \frac{y}{x} - \frac{\log P_{n+1}}{\log P_m} < \frac{1}{2x^2}.$$

By Legendre's criterion, we infer that y/x is a convergent to the continued fraction of $\log P_{n+1}/\log P_m$. Let $d := \gcd(x, y)$. By Fermat's Last Theorem once again, it follows that $d \in \{1, 2\}$, for otherwise the triple $(X, Y, Z) = (P_n^{x/d}, P_{n+1}^{x/d}, P_m^{y/d})$ is a positive integer solution to the Fermat equation $X^d + Y^d = Z^d$ with integer exponent

$d \geq 3$ and coprime positive integers X and Y, which we know does not exist. Further, since the convergent p_k/q_k of any irrational number γ satisfies $q_k \geq F_k$, where F_k is the kth Fibonacci number, and since $F_{91} > 10^{18}$, it follows that $(x, y) = (p_k, q_k)$ or $(2p_k, 2q_k)$ for some $k \leq 90$. Here, p_k/q_k is the kth convergent to $\log P_{n+1}/\log P_m$ for some odd $m \geq 3$, $n \geq 3$, m coprime to $n(n+1)$ and $N \leq 1000$. Indeed, all the above assertions are clear except perhaps the assertion of m being coprime to $n(n+1)$. But if say p divides both m and n, then P_p divides both P_n^x and P_m^y but not P_{n+1}^x, so our equation is impossible. A similar argument shows that m and $n+1$ are coprime. So, for all pairs (m, n) such that $m \geq 2$, $n \geq 3$, m coprime to $n(n+1)$ and $N \leq 1000$, we generated the first 90 convergents of $\log P_{n+1}/\log P_m$ and checked whether for one of the pairs $(x, y) \in \{(p_k, q_k), (2p_k, 2q_k)\}$ with $x \geq 3$, the congruence $P_n^x + P_{n+1}^x \equiv P_m^y \pmod{p_{10^{10}}}$ holds, where $p_{10^{10}} = 252097800623$ is the 10^{10}th prime. The computations were carried on in Mathematica and the `PowerMod`$[A, u, B]$ feature of Mathematica was used to compute $A^u \pmod B$ for $B := p_{10^{10}}$ and $(A, u) \in \{(P_n, x),\ (P_{n+1}, x),\ (P_m, y)\}$. This computation took a couple of hours and we got no new solutions. $\square$

From now on we assume that $N > 1000$.

3.5. Bounds for x, y and N in terms of M

Let $\{w, z\} = \{x, y\}$ be such that (w, M), (z, N) are the two pairs $(x, n+1)$, (y, m). We will prove the following lemma.

Lemma 6. *If (m, n, x, y) is a solution to (2) with $n \geq 2$, $x \geq 3$ and $y \geq 2$, then*

$$\begin{aligned} N &< 2 \times 10^{31} M^2 (\log M)^2, \\ x &< 1.3 \times 10^{28} M^2 (\log M)^2, \\ y &< 10^{47} \times M^2 (\log M)^3, \\ \max\{Mw, Nz\} &< 10^{47} \times M^3 (\log M)^3. \end{aligned}$$

Proof. From lemmas 4 and 5, we have

$$\max\{x, y\} < 1.4 \times 10^{11} N^3 \log N < \alpha^N. \tag{16}$$

The right-most inequality above holds in fact for all $N \geq 44$. The inequality (16) implies that

$$\frac{z}{\alpha^{2N}} < \frac{1}{\alpha^N}. \tag{17}$$

By the Binet's formula (3) and the fact that $\beta = -\alpha^{-1}$, we have

$$P_N^z = \frac{\alpha^{Nz}}{8^{z/2}} \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z = \frac{\alpha^{Nz}}{8^{z/2}} \exp\left(z \log\left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)\right).$$

Following the argument in [11] and distinguishing between the cases N odd and N even, we deduce that if we put

$$\varepsilon_{N,z} := \left(1 - \frac{(-1)^N}{\alpha^{2N}}\right)^z - 1,$$

then

$$P_N^z = \frac{\alpha^{Nz}}{8^{z/2}}\left(1+\varepsilon_{N,z}\right), \quad \text{and} \quad |\varepsilon_{N,z}| < \frac{2}{\alpha^N}. \tag{18}$$

Since $x \geq 3$ and $N > 1000$, we deduce easily from (7) and (18) that

$$\frac{P_m^y}{P_{n+1}^x}, \frac{P_N^z}{\alpha^{Nz}/8^{z/2}} \in \left(\frac{1}{2}, \sqrt{2}\right). \tag{19}$$

Suppose now that $N = n+1$. Then $z = x$ and

$$P_m^y = P_{n+1}^x + P_n^x = \frac{\alpha^{(n+1)x}}{8^{x/2}} + \left(\frac{\alpha^{(n+1)x}}{8^{x/2}}\right)\varepsilon_{n+1,x} + P_n^x.$$

So, we get

$$\begin{aligned} \left|P_m^y \alpha^{-(n+1)x} 8^{x/2} - 1\right| &= \left|\varepsilon_{n+1,x} + \frac{P_n^x}{\alpha^{(n+1)x}/8^{x/2}}\right| \\ &< |\varepsilon_{n+1,x}| + \left(\frac{P_n}{P_{n+1}}\right)^x \left(\frac{P_{n+1}^x}{\alpha^{(n+1)x}/8^{x/2}}\right) \\ &< \frac{2}{\alpha^{n+1}} + \frac{2}{2.3^x} \leq \frac{4}{2.3^\lambda}, \end{aligned} \tag{20}$$

where

$$\lambda := \min\{x, N\}. \tag{21}$$

Here we used, in addition to (19), the fact that $\alpha > 2.3$. The same inequality is obtained when $N = m$, because in this case $z = y$ and

$$P_{n+1}^x = P_m^y - P_n^x = \frac{\alpha^{my}}{8^{y/2}} + \left(\frac{\alpha^{my}}{8^{y/2}}\right)\varepsilon_{m,y} - P_n^x.$$

Thus, we have

$$\begin{aligned} \left|P_{n+1}^x \alpha^{-my} 8^{y/2} - 1\right| &= \left|\varepsilon_{m,y} - \frac{P_n^x}{\alpha^{my}/8^{y/2}}\right| \\ &< |\varepsilon_{m,y}| + \left(\frac{P_n}{P_{n+1}}\right)^x \left(\frac{P_{n+1}^x}{P_m^y}\right)\left(\frac{P_m^y}{\alpha^{my}/8^{y/2}}\right) \\ &< \frac{2}{\alpha^m} + \frac{2}{2.3^x} \leq \frac{4}{2.3^\lambda}. \end{aligned} \tag{22}$$

From (20) and (22), we summarize that the inequality

$$\left|P_M^w \alpha^{-Nz} 8^{z/2} - 1\right| < \frac{4}{2.3^\lambda} \tag{23}$$

holds, where λ is given by formula (21). We will use (23) and Matveev's theorem to get an upper bound on x and N in terms of M. We continue by getting a lower bound on the left-hand side of inequality (23). For this, we take

$$s := 3, \quad \gamma_1 := P_M, \quad \gamma_2 := \alpha, \quad \gamma_3 := 2\sqrt{2}.$$

We also take

$$b_1 := w, \quad b_2 := -Nz, \quad b_3 := z.$$

Hence,

$$\Lambda_2 := P_M^w \alpha^{-Nz} 8^{z/2} - 1$$

is the expression which appears under the absolute value on the left-hand side of inequality (23). If $\Lambda_2 = 0$, then we get $\alpha^{2Nz} = P_M^{2w} 8^z \in \mathbb{Z}$, which is impossible since no power of α of a positive integer exponent can be an integer. Thus, $\Lambda_2 \neq 0$. Observe next that $\gamma_1, \gamma_2, \gamma_3$ are all real and belong to the field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, so we can take $D = 2$. Next, since $P_M < \alpha^M$, it follows that we can take

$$A_1 = 2M \log \alpha > D \log P_M = Dh(\gamma_1).$$

Next, since $h(\gamma_2) = (\log \alpha)/2$ and $h(\gamma_3) = (\log 8)/2$, it follows that we can take $A_2 = \log \alpha$ and $A_3 = \log 8$. Finally, Lemma 4 and that fact that $N > 1000$ tell us that we have

$$\begin{aligned} \max\{Nz, z, w\} &< (1.4 \times 10^{11} M N^2 \log N) \times N \\ &< (10^3)^4 \times \log N \times M N^2 \times N \\ &< N^4 \times N \times N^4 = N^9. \end{aligned}$$

Hence, we can take $B := N^9$. Matveev's theorem tells us that

$$\begin{aligned} \log |\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + 9 \log N)(2M \log \alpha)(\log \alpha)(\log 8) \\ &> -3.2 \times 10^{13} \times M \log N, \end{aligned} \tag{24}$$

where we used the fact that $1 + 9 \log N < 10 \log N$ holds for all $N \geq 3$. Comparing (23) and (24), we get

$$\lambda < 4 \times 10^{13} M \log N. \tag{25}$$

At this point, we remark the following about Mw and Nz: if $z > w$, by inequality (6), we have

$$Mw \leq (N + 2)z \leq 2Nz,$$

while if $z < w$, then, again by (6) and the fact that $M \geq 3$, we have

$$\frac{Mw}{3} \leq (M - 2)w \leq Nz.$$

Therefore, $Mw \leq 3Nz$. So, it is always the case that $Mw \leq 3Nz$. A similar argument shows that $Nz \leq 2Mw$. Thus, we obtain

$$\frac{Nz}{Mw} \in \left(\frac{1}{3}, 2 \right). \tag{26}$$

Next, we distinguish several cases.

Case 1. $\lambda = N$. In this case, by inequality (25), we get

$$N < 4 \times 10^{13} M \log N.$$

Hence, we have

$$\begin{aligned} N &< 2 \times 4 \times 10^{13} M \log(4 \times 10^{13} M) \\ &< 8 \times 10^{13} M (30 \log M) \\ &< 2.4 \times 10^{15} M \log M. \end{aligned} \tag{27}$$

From Lemma 4, we get

$$\begin{aligned} x &< 1.4 \times 10^{11} MN \log N \\ &< 1.4 \times 10^{11} M (2.4 \times 10^{15} M \log M) \log(2.4 \times 10^{15} M \log M) \\ &< 3.4 \times 10^{26} M^2 \log M (36 + 2 \log M) \\ &< 3.4 \times 10^{26} M^2 \log M (38 \log M) \\ &< 1.3 \times 10^{28} M^2 (\log M)^2. \end{aligned} \tag{28}$$

Thus, if $w = x$, then

$$Mw = Mx < 1.3 \times 10^{28} M^3 (\log M)^2, \tag{29}$$

while if $w = y$, then $z = x$ and

$$\begin{aligned} Nz = Nx &< (2.4 \times 10^{15} M \log M)(1.3 \times 10^{28} M^2 (\log M)^2) \\ &< 3.2 \times 10^{43} M^3 (\log M)^3. \end{aligned} \tag{30}$$

Using (26) and from estimates (29) and (30), we deduce that

$$\max\{Nz, Mw\} < 10^{44} M^3 (\log M)^3. \tag{31}$$

Since $My \le \max\{Mw, Nz\}$, by inequality (31) we get

$$y < 10^{44} M^2 (\log M)^3. \tag{32}$$

Case 2. $\lambda = x$. In this case, from inequality (25), we have

$$x = \lambda < 4 \times 10^{13} M \log N. \tag{33}$$

We now distinguish two subcases.

Case 2.1. $N = m$. Then $M = n + 1$. Further, if $x > y$, then by inequality (6), the fact that $y \ge 2$ and (33), we have

$$\begin{aligned} N = m &\le \frac{my}{2} < \frac{(n+3)x}{2} < (n+1)x \\ &= Mx < 4 \times 10^{13} M^2 \log N, \end{aligned}$$

while if $x < y$, then by inequality (6), the fact that $y \ge 3$ in this case and (33), we have

$$N = m \le my - 2y < (n+1)x = Mx < 4 \times 10^{13} M^2 \log N.$$

So, in this case we always have

$$N < 4 \times 10^{13} M^2 \log N. \tag{34}$$

Case 2.2. $N = n+1$. First, note that if $y > x$, then, by (6), we have

$$y < (m-2)y < (n+1)x = Nx,$$

while if $x > y$, then

$$y \le \frac{my}{2} < \frac{(n+3)x}{2} < (n+1)x = Nx.$$

Hence, the inequality

$$y < Nx \tag{35}$$

also holds in this case.

Further, observe that $z = x$. Thus, we also have

$$P_n^x = \frac{\alpha^{nx}}{8^{x/2}} \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^x.$$

Then, by (17), we have

$$\frac{x}{\alpha^{2n}} = \frac{x}{\alpha^{2N-2}} < \frac{\alpha^2}{\alpha^N}.$$

The argument preceding (18) now shows that

$$P_n^x = \frac{\alpha^{nx}}{8^{x/2}}(1+\varepsilon_{n,x}), \quad \text{where} \quad |\varepsilon_{n,x}| < \frac{2\alpha^2}{\alpha^N}.$$

Thus we get

$$P_m^y = P_{n+1}^x + P_n^x = \frac{\alpha^{nx}(\alpha^x+1)}{8^{x/2}} + \left(\frac{\alpha^{(n+1)x}}{8^{x/2}}\right)\varepsilon_{n+1,x} + \left(\frac{\alpha^{nx}}{8^{x/2}}\right)\varepsilon_{n,x}.$$

So, we have

$$\left| P_m^y \alpha^{-nx} \left(\frac{8^{x/2}}{\alpha^x+1}\right) - 1 \right| < |\varepsilon_{n+1,x}| \left(\frac{\alpha^x}{\alpha^x+1}\right) + |\varepsilon_{n,x}| \left(\frac{1}{\alpha^x+1}\right) < \frac{4}{\alpha^N}, \tag{36}$$

where we used the facts that

$$|\varepsilon_{n+1,x}| < \frac{2}{\alpha^N}, \quad |\varepsilon_{n,x}| < \frac{2\alpha^2}{\alpha^N}, \quad \frac{\alpha^x}{\alpha^x+1} < 1, \quad \frac{1}{\alpha^x+1} < \frac{1}{\alpha^2},$$

since $x \ge 3$.

We continue by getting a lower bound on the left-hand side of inequality (36) using Matveev's theorem again. For this, we take $s := 3$, $\gamma_1 := P_m$, $\gamma_2 := \alpha$, $\gamma_3 := (\alpha^x+1)/8^{x/2}$. We also take $b_1 := y$, $b_2 := -nx$, $b_3 := -1$. Hence,

$$\Lambda_3 := P_m^y \alpha^{-nx} \left(\frac{8^{x/2}}{\alpha^x+1}\right) - 1$$

is the expression which appears under the absolute value on the left-hand side of inequality (36). We first check that $\Lambda_3 \neq 0$. If $\Lambda_3 = 0$, then

$$\alpha^{2nx}(\alpha^x + 1)^2 = P_m^{2y} 8^x \in \mathbb{Z}.$$

Conjugating the above expression in $\mathbb{Q}(\sqrt{2})$, we get that

$$\alpha^{2nx}(\alpha^x + 1)^2 = \beta^{2nx}(\beta^x + 1)^2,$$

which is impossible because its left-hand side is very large (at least α^{2000}), while its right-hand side is less than 2 for $x \geq 3$. Observe that $\gamma_1, \gamma_2, \gamma_3$ belong to $\mathbb{Q}(\sqrt{2})$, so we can take $D := 2$. Next, as in the previous application, we can take $A_1 := 2M \log \alpha$ and $A_2 := \log \alpha$. For γ_3, its conjugate in $\mathbb{K}$ is $(-1)^x(\beta^x + 1)/8^{x/2}$, so its minimal polynomial over the integers is a divisor of

$$\begin{aligned} a_0 X^2 &+ a_1 X + a_2 \\ &= 8^x \left(X - \frac{\alpha^x + 1}{8^{x/2}}\right)\left(X - (-1)^x \frac{\beta^x + 1}{8^{x/2}}\right) \\ &= 8^x X^2 - 8^{x/2}(\alpha^x + (-1)^x\beta^x + 1 + (-1)^x)X + (-1)^x(\alpha^x + \beta^x + 1 + (-1)^x). \end{aligned}$$

Thus, we have $a_0 \leq 8^x$,

$$\left|\gamma_3^{(1)}\right| = |\gamma_3| = \frac{\alpha^x + 1}{8^{x/2}} < 1.1 \left(\frac{\alpha}{2\sqrt{2}}\right)^x < 1,$$

because $x \geq 3$, and

$$\left|\gamma_3^{(2)}\right| = \frac{|\beta^x + 1|}{8^{x/2}} < \frac{2}{8^{x/2}} < 1.$$

So, we can take

$$A_3 = \log 8^x = \frac{D \log 8^x}{2} \geq \frac{D \log a_0}{2} \geq Dh(\gamma_3).$$

Finally, inequalities (33) and (35), and the fact that $N > 1000$, tell us that we can take

$$\begin{aligned} \max\{|b_1|, |b_2|, |b_3|\} &= \max\{Nx, y, 1\} = Nx \\ &< N \times 4 \times 10^{13} \times M \log N \\ &= (10^3)^4 \times MN \times 40 \log N \\ &< N^4 \times N^2 \times N = N^7. \end{aligned}$$

In the above, we used the fact that $N > 40 \log N$ holds for all $N > 1000$. Thus we get

$$\begin{aligned} \log|\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + 7\log N)(2M \log \alpha)(\log \alpha)(x \log 8) \\ &> -2.6 \times 10^{13} Mx \log N. \end{aligned} \tag{37}$$

Inserting (33) into (37), we get

$$\begin{aligned} \log|\Lambda_3| &> -2.6 \times 10^{13} M(4 \times 10^{13} M \log N) \log N \\ &= -1.1 \times 10^{27} M^2 (\log N)^2. \end{aligned} \tag{38}$$

From inequalities (36) and (38), we get

$$N < 1.3 \times 10^{27} M^2 (\log N)^2. \tag{39}$$

Taking the worst possibility between (34) and (39), we get that

$$N < 1.3 \times 10^{27} M^2 (\log N)^2.$$

Now we use the following fact: if $A > 100$, then the inequality

$$\frac{t}{(\log t)^2} < A \quad \text{implies} \quad t < 4A(\log A)^2.$$

From this inequality, if we take $A := 1.3 \times 10^{27} M^2$ and $t := N$, we get

$$\begin{aligned} N &< 4 \times 1.3 \times 10^{27} M^2 (\log(1.3 \times 10^{27} M^2))^2 \\ &< 5.2 \times 10^{27} M^2 (2\log M)^2 \left(\frac{\log(1.3 \times 10^{27})}{2\log 3} + 1\right)^2 \\ &< 2 \times 10^{31} M^2 (\log M)^2. \end{aligned} \tag{40}$$

Using also inequality (33), we get

$$\begin{aligned} x &< 4 \times 10^{13} M \log N \\ &< 4 \times 10^{13} M \log(2 \times 10^{31} M^2 (\log M)^2) \\ &< 4 \times 10^{13} M (4 \log M) \left(\frac{\log(2 \times 10^{31})}{4\log 3} + 1\right) \\ &< 2.8 \times 10^{15} M \log M. \end{aligned} \tag{41}$$

So, as in Case 1, we deduce that if $w = x$, then

$$Mw = Mx < 2.8 \times 10^{15} M^2 \log M, \tag{42}$$

while if $w = y$, then $z = x$ and

$$\begin{aligned} Nz = Nx &< (2 \times 10^{31} M^2 (\log M)^2 (2.8 \times 10^{15} M \log M) \\ &< 6 \times 10^{46} M^3 (\log M)^3. \end{aligned} \tag{43}$$

Using condition (26), we deduce from estimates (42) and (43)

$$\max\{Nz, Mw\} < 10^{47} M^3 (\log M)^3. \tag{44}$$

In particular, since $My \le \max\{Mz, Mw\}$ and from (44), we also get

$$y < 10^{47} M^2 (\log M)^3. \tag{45}$$

From (27), (28), (31), (32), (40), (41), (44) and (45), we get the bounds appearing in Lemma 6. $\square$

Remark. For computational purposes, it might be interesting to make some remarks. First, x must be even. Indeed, if x is odd, then

$$Q_{n+1}/2 = P_{n+1} + P_n \mid P_{n+1}^x + P_n^x.$$

In particular, every odd prime factor of Q_{n+1} divides P_m. But $Q_{n+1} = P_{2(n+1)}/P_n$, so every primitive prime factor of $P_{2(n+1)}$ (namely prime factor of $P_{2(n+1)}$ which does not divide P_k for any $1 \le k < 2(n+1)$) is a prime factor of Q_{n+1}. By the Carmichael's version of the Primitive Divisor Theorem [6], $P_{2(n+1)}$ has a primitive prime factor for all $n \ge 5$ and by inspection one observes that it also has a primitive prime factor for $n \in \{1, 2, 3, 4, 5\}$ as well. Thus, since for us $n \ge 2$, there is a primitive p prime factor of $P_{2(n+1)}$ dividing Q_{n+1}. Since $Q_{n+1}/2 \mid P_m^y$, we get that $p \mid P_m$, so $2(n+1) \mid m$, a contradiction since m is odd. This shows that x is even. Since $x \ge 3$, we get that in fact $x \ge 4$. Now the left–hand side of our equation is u_{2x}/u_x, where $\{u_k\}_{k\ge 0}$ is the kth term of the Lucas sequence with roots P_{n+1} and P_n. By the Primitive Divisor Theorem one more time, since $x \ge 4$, the number u_{2x} has a primitive prime factor q (see Table 1 in [2] for the "exceptional" members of the Lucas sequence of index greater than 6 which lack primitive divisors). In particular, the multiplicative order of P_{n+1}/P_n modulo q is exactly $2x$. Thus, the multiplicative order of P_{n+1}/P_n modulo P_m is exactly $2x$ as well. The main result of [3] shows that

$$m \le 20000(2x)^2 \le 80000x^2.$$

One may combine the above bound with the bound on x given by Lemma 6 to get some bound on m, which unfortunately ends up being worse than the bound on N provided by the same Lemma 6.

3.6. The case when $M \le 1000$

Lemma 7. *If (m, n, x, y) is a solution of equation (2) with $x \ge 3$, $n \ge 2$ and $y \ge 2$, then $N > M > 1000$.*

Proof. Assume $M \le 1000$. By Lemma 6, we have that

$$X := \max\{w, Nz, z\} < 10^{47} \times (10^3)^3 (\log(10^3))^3 < 10^{57}.$$

Note that by Lemma 3 we have that $\lambda \ge 3$. Then the right-hand side of (23) is at most 1/2 so by a classical argument it follows that

$$\left| w \log P_M - Nz \log \alpha + z \log(2\sqrt{2}) \right| < \frac{8}{2.3^{\lambda}}. \tag{46}$$

We calculate a lower bound for the above absolute value through Lemma 2. We take $C := (3X)^3$ and consider the lattice Ω spanned by

$$\gamma_1 := (1, 0, \lfloor C \log P_M \rfloor), \quad \gamma_2 := (0, 1, \lfloor C \log \alpha \rfloor), \quad \gamma_3 := \left(0, 0, \left\lfloor C \log(2\sqrt{2}) \right\rfloor\right).$$

Using Mathematica, we estimate a reduced basis $\{\mathbf{b}_i\}$ (LLL–algorithm) for Ω and its associated Gram–Schmidt $\{\mathbf{b}_i^*\}$ basis. So, for each $M \in [3, 1000]$ we calculated the parameters Q, T, c_1 and $\mathbf{m}_\Omega = ||\mathbf{b}_1||/c_1$. By the conclusion of Lemma 2 together with inequality (46), we obtain that

$$7.7 \times 10^{-500} < \left| w \log P_M - Nz \log \alpha + z \log(2\sqrt{2}) \right| < \frac{8}{2.3^\lambda},$$

which leads to

$$\lambda < 1410.$$

Suppose $\lambda = N$, so $N < 1410$ and $X = \max\{w, Nz, z\} < 2.8 \times 10^{24}$. A new reduction cycle on inequality (46) yields 5.2×10^{-218} as a lower bound for the above absolute value, so $N = \lambda < 610$, contrary to the assumption $N > 1000$. Hence, $\lambda = x$, so $x < 1410$.

Assume $(N, z) = (n+1, x)$, so $(M, w) = (m, y)$. By inequality (36), it follows that

$$\left| w \log P_M - (N-1) z \log \alpha + \log(8^{x/2}/(\alpha^x + 1)) \right| < \frac{4}{\alpha^N}.$$

Fixing M and x, we are in a suitable position to apply the Baker-Davenport reduction method. We have

$$|u\gamma - v + \mu| < AB^{-N},$$

where we take

$$\gamma := \frac{\log P_M}{\log \alpha}, \quad \mu := \frac{\log(8^{x/2}/(\alpha^x + 1))}{\log \alpha}, \quad A := 4.6 \left(> \frac{4}{\log \alpha} \right), \quad B := \alpha,$$

and $(u, v) = (w, (N-1)z)$. By estimate (35), Lemma 6 and the fact that $x < 1410$, we note that $u = y < Nx < T := 1.4 \times 10^{42}$. We loop over all even values $x \in [4, 1410]$ and all odd values $m = M \in [3, 1000]$. In all cases $\varepsilon > 1.1 \times 10^{-249}$ and $\log(Aq/\varepsilon)/\log B < 1405$, which is a lower bound for N. The same second reduction cycle on inequality (46) leads to a contradiction on N.

Assume $(N, z) = (m, y)$, so $(M, w) = (n+1, x)$. Then by inequality (6),

$$\max\{N, y\} < (n+1)x = Mx < 1.5 \times 10^6.$$

Returning to inequality (46), we note that according to the above inequalities $X < 2.3 \times 10^{12}$. A reduction cycle on inequality (46) once more leads to the conclusion that $\lambda = x < 320$. Repeating the same reduction argument, we get $x < 285$. Thus, $(x, n) \in [3, 285] \times [2, 999]$. Then, by inequality (6) and the fact that $m = N > 1000$, we get

$$998y < y(m-2) < x(n+1) < 285000.$$

Hence, $(y, m) \in [3, 285] \times [1001, 2 + \lfloor 285000/y \rfloor]$. Then, we check the congruence $P_n^x + P_{n+1}^x \equiv P_m^y \pmod{10^{20}}$. For the pairs (x, y) we used once again the $\texttt{PowerMod}[A, u, B]$ feature of Mathematica to compute

$$S = \{P_n^x + P_{n+1}^x \pmod{10^{20}} : (x, y) \in [3, 285] \times [2, 999]\},$$

while for the ring-hand side of (2) we used

$$\texttt{Mod}[\texttt{Drop}[\texttt{LinearRecurrence}[\{2,1\},\{1,2\},\#1],1000]^\wedge\#2,10^{20}]\&;$$

to compute $R_y := \{P_m^y \pmod{10^{20}} : m \in [1001, 2 + \lfloor 285000/y \rfloor]\}$. Then we verify that $S \cap R_y = \emptyset$ for each $y \in [3, 285]$. This computation took four minutes and showed that no new solutions exist when $M \leq 1000$. $\square$

3.7. An absolute bound on all the variables m, n, x, y

Lemma 8. *If (m, n, x, y) is a solution of equation (2) with $n \geq 2$, $x \geq 3$ and $y \geq 2$, then*

$$\max\{x, y\} < 3 \times 10^{111}.$$

Proof. Since $N > M > 1000$, it follows from Lemma 6 that

$$\max\{x, y\} < 10^{47} M^2 (\log M)^3 < \alpha^{M-2} \leq \min\{\alpha^{n-1}, \alpha^m\}.$$

The above middle inequality holds for all $M \geq 142$. Hence, all three inequalities

$$\frac{x}{\alpha^{2n}} \leq \frac{1}{\alpha^{n+1}}, \quad \frac{x}{\alpha^{2n+2}} \leq \frac{1}{\alpha^{n+1}}, \quad \frac{y}{\alpha^{2m}} \leq \frac{1}{\alpha^m}$$

hold, so we may write

$$P_n^x = \frac{\alpha^{nx}}{8^{x/2}}(1 + \zeta_{n,x}), \tag{47a}$$

$$P_{n+1}^x = \frac{\alpha^{(n+1)x}}{8^{x/2}}(1 + \zeta_{n+1,x}), \tag{47b}$$

$$P_m^y = \frac{\alpha^{my}}{8^{y/2}}(1 + \zeta_{m,y}), \tag{47c}$$

where

$$\max\{|\zeta_{n,x}|, |\zeta_{n+1,x}|\} \leq \frac{2}{\alpha^{n+1}}, \quad |\zeta_{m,x}| \leq \frac{2}{\alpha^m}.$$

We also have the analogs of condition (19), namely,

$$\frac{P_n^x}{\alpha^{nx}/8^{x/2}}, \quad \frac{P_{n+1}^x}{\alpha^{(n+1)x}/8^{x/2}}, \quad \frac{P_m^y}{\alpha^{my}/8^{y/2}} \quad \text{belong to} \quad \left(\frac{1}{2}, \sqrt{2}\right). \tag{48}$$

Inserting approximation (47) into equation (2) and shuffling some terms, we get

$$\frac{\alpha^{my}}{8^{y/2}} - \frac{\alpha^{(n+1)x}}{8^{x/2}} - \frac{\alpha^{nx}}{8^{x/2}} = \left(\frac{\alpha^{(n+1)x}}{8^{x/2}}\right)\zeta_{n+1,x} + \left(\frac{\alpha^{nx}}{8^{x/2}}\right)\zeta_{n,x} - \left(\frac{\alpha^{my}}{8^{y/2}}\right)\zeta_{m,y},$$

which, together with (48), yields the following string of inequalities:

$$\begin{aligned}
\left|\alpha^{my-(n+1)x} 8^{(x-y)/2} - 1\right| &< \frac{1}{\alpha^x} + |\zeta_{n+1,x}| + \frac{|\zeta_{n,x}|}{\alpha^x} + \left(\frac{\alpha^{my}/8^{y/2}}{\alpha^{(n+1)x}/8^{x/2}}\right)|\zeta_{m,y}| \\
&< \frac{1}{\alpha^x} + \frac{3}{\alpha^{n+1}} \\
&\quad + \left(\frac{\alpha^{my}/8^{y/2}}{P_m^y}\right)\left(\frac{P_m^y}{P_{n+1}^x}\right)\left(\frac{P_{n+1}^x}{\alpha^{(n+1)x}/8^{x/2}}\right)|\zeta_{m,y}| \\
&< \frac{1}{\alpha^x} + \frac{2}{\alpha^{n+1}} + \frac{2\sqrt{2}}{\alpha^m} < \frac{6}{\alpha^{\lambda_1}},
\end{aligned} \tag{49}$$

where $\lambda_1 := \min\{x, M\}$ and also

$$\begin{aligned}\left|\alpha^{my-nx}8^{(x-y)/2}(\alpha^x+1)^{-1}-1\right| &< \frac{\alpha^x}{\alpha^x+1}\left|\zeta_{n+1,x}\right| + \frac{|\zeta_{n,x}|}{\alpha^x+1}\\ &\quad + \left(\frac{\alpha^x}{\alpha^x+1}\right)\left(\frac{\alpha^{my}/8^{y/2}}{\alpha^{(n+1)x}/8^{x/2}}\right)|\zeta_{m,y}|\\ &< \frac{2}{\alpha^{n+1}} + \frac{2\sqrt{2}}{\alpha^m} < \frac{6}{\alpha^M}.\end{aligned} \tag{50}$$

We apply Matveev's theorem to the left-hand side of inequality (49) with $s := 2$, $\gamma_1 := \alpha$, $\gamma_2 := 2\sqrt{2}$, $b_1 := my-(n+1)x$, $b_2 := x-y$ and $D = 2$. Thus,

$$\Lambda_4 := \alpha^{my-(n+1)x}8^{(x-y)/2} - 1.$$

Observe that $\Lambda_4 \neq 0$ since otherwise we would get that $\alpha^{2my-2(n+1)x} = 8^{x-y} \in \mathbb{Z}$, and this is possible only if $my = (n+1)x$ and $y = x$, but this last equality is not allowed. We take as in the prior application of this theorem $A_1 := \log\alpha$ and $A_2 := \log 8$. Further, since $x \geq 4$, the right-hand side in (49) is at most $6/\alpha^3 < 1/2$. Therefore, we obtain

$$\frac{\alpha^{|my-(n+1)x|}}{8^{|x-y|/2}} < 2.$$

So, we have

$$\begin{aligned}|b_1| = |my-(n+1)x| &< \frac{\log(2\times 8^{|y-x|/2})}{\log\alpha}\\ &= \left(\frac{\log 8}{2\log\alpha}\right)|y-x| + \frac{\log 2}{\log\alpha} < 2|y-x|+1\\ &< 4|y-x|.\end{aligned} \tag{51}$$

Thus, using lemmas 6 and 7, we can take

$$\begin{aligned}B := M^{19} = M^{16}\times M^2\times M &> (10^3)^{16}M^2(\log M)^3\\ &> 4\times 10^{47}M^2(\log M)^3 > 4\max\{x,y\} > \max\{|b_1|,|b_2|\}.\end{aligned} \tag{52}$$

Matveev's theorem tells us that

$$\begin{aligned}\log|\Lambda_4| &> -1.4\times 30^5\times 2^{4.5}\times 2^2(1+\log 2)(1+19\log M)(\log\alpha)(\log 8)\\ &> -2\times 10^{11}\log M.\end{aligned} \tag{53}$$

Comparing estimates (49) and (53), we get

$$\lambda_1 < 3\times 10^{11}\log M. \tag{54}$$

We now distinguish two cases.

Case 1. $\lambda_1 = M$. In this case, from (54), we get

$$M < 3\times 10^{11}\log M.$$

Therefore, we obtain

$$M < 2 \times 3 \times 10^{11} \log(3 \times 10^{11}) < 2 \times 10^{13}. \tag{55}$$

Case 2. $\lambda_1 = x$. In this case, from (54), we get

$$x < 3 \times 10^{11} \log M. \tag{56}$$

We apply Matveev's theorem to the left-hand side of the inequality (50) with $s := 3$, $\gamma_1 := \alpha$, $\gamma_2 := 2\sqrt{2}$, $\gamma_3 := \alpha^x + 1$, $b_1 := my - nx$, $b_2 := x - y$, $b_3 := -1$ and $D = 2$. Thus, we take

$$\Lambda_5 := \alpha^{my-nx} 8^{(x-y)/2} (\alpha^x + 1)^{-1} - 1.$$

Let us check that $\Lambda_5 \neq 0$. If $\Lambda_5 = 0$, we get that

$$\alpha^x + 1 = 8^{(x-y)/2} \alpha^{mx-ny}.$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{2})$, we get

$$\beta^x + 1 = \pm 8^{(x-y)/2} \beta^{mx-ny}.$$

Multiplying the two above relations, we get

$$\begin{aligned} \alpha^x + \beta^x + (-1)^x + 1 &= (\alpha^x + 1)(\beta^x + 1) \\ &= \pm(\alpha\beta)^{my-nx} 8^{x-y} \\ &= \pm 8^{x-y}. \end{aligned} \tag{57}$$

Since the left-hand side of equation (57) is larger than 1 for $x \geq 4$, the sign in the right-hand side is plus and $x > y$. Since x is even (see the remark at the end of the previous section), the equation (57) becomes

$$\alpha^x + \beta^x + 2 = 8^{x-y}.$$

If $4|x$, the above equation gives $Q_{x/2}^2 = 2^{3(x-y)}$, which is again impossible because Q_k is never divisible by 4 for any k. So $2||x$, which in turn yields the relation $(Q_{x/2}/2)^2 = 2^{3(x-y)-2} - 1 \equiv -1 \pmod 8$, which is again impossible. Thus, $\Lambda_5 \neq 0$.

We take, as in the prior application of Matveev's theorem, $A_1 := \log\alpha$, $A_2 := \log 8$. As for $\alpha_3 := \alpha^x + 1$, this is an algebraic integer whose conjugate is $\beta^x + 1$ whose absolute value is smaller than 2. Thus, one can see that

$$\begin{aligned} Dh(\alpha_3) &\leq \log(\alpha^x + 1) + \log 2 < \log(2\alpha^x) + \log 2 \\ &= x \log\alpha + \log 2 + \log(1 + \alpha^{-x}) \leq x \left(\log\alpha + \frac{\log 2}{x} + \frac{1}{x\alpha^x} \right) \\ &< 1.1x, \end{aligned}$$

since $x \geq 4$. So, we can take $A_3 := 1.1x$. Finally, observe that by the calculation (51), we have

$$\begin{aligned} |b_1| &= |my - nx| \leq |my - (n+1)x| + x < 2\,|y - x| + 2 + x \\ &< 4 \max\{x, y\}. \end{aligned}$$

Hence, using lemmas 6 and 7, we conclude, as at estimate (52), that we can take

$$\begin{aligned} B &= M^{19} = M^{16} \times M^2 \times M > (10^3)^{16} \times M^2 \times (\log M)^3 \\ &> 4 \times 10^{47} \times M^2 \times (\log M)^3 > 4 \max\{x, y\} > \max\{|b_1|, |b_2|\}. \end{aligned}$$

Matveev's theorem now implies that

$$\begin{aligned} \log|\Lambda_5| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log M^{19})(\log\alpha)(\log 8)(1.1x) \\ &> -4 \times 10^{13} x \log M. \end{aligned} \tag{58}$$

From estimates (50) and (58), we get

$$M < 5 \times 10^{13} x \log M. \tag{59}$$

Inserting estimate (56) into (59), we obtain

$$M < 5 \times 10^{13}(3 \times 10^{11} \times \log M)\log M < 2 \times 10^{25}(\log M)^2.$$

Thus, one can see that

$$M < 4 \times 2 \times 10^{25}(\log(2 \times 10^{25}))^2 < 3 \times 10^{29}. \tag{60}$$

Comparing (55) and (60) on M in the two cases, we conclude that the inequality (60) always holds. Inserting the above bound for M into the inequalities of Lemma 6, we get

$$\begin{aligned} N &< 2 \times 10^{31} \times (3 \times 10^{29})^2(\log(3 \times 10^{29}))^2 < 10^{94}, \\ x &< 1.3 \times 10^{28}(3 \times 10^{29})^2(\log(3 \times 10^{29}))^2 < 10^{91}, \\ y &< 10^{47} \times (3 \times 10^{29})^2(\log(3 \times 10^{29}))^3 < 3 \times 10^{111}. \end{aligned}$$

□

3.8. Reducing the bound

We work more on inequality (49). Assume that $\lambda_1 > 400$. Then $6/\alpha^{\lambda_1} < 1/2$, so by a classic argument we get

$$\left|(my-(n+1)x)\log\alpha - (y-x)\log(2\sqrt{2})\right| < \frac{12}{\alpha^{\lambda_1}}.$$

Thus, we see that

$$\begin{aligned} \left|\frac{my-(n+1)x}{x-y} - \frac{\log(2\sqrt{2})}{\log\alpha}\right| &< \frac{12}{(\log\alpha)\,|x-y|\,\alpha^{\lambda_1}} \\ &< \frac{13}{|x-y|\,\alpha^{\lambda_1}}. \end{aligned} \tag{61}$$

Since $\lambda_1 > 400$, by Lemma 8, we have

$$\alpha^{\lambda_1} > \alpha^{400} > 3 \times 10^{113} > 100\max\{x, y\} > 100\,|x-y|,$$

showing that the expression appearing on the right-hand side of (61) is smaller than $1/(2\,|x-y|)^2$, so by Legendre's criterion, $(my-(n+1)x)/(x-y)$ equals some convergent p_k/q_k of $\gamma := \log(2\sqrt{2})/\log\alpha$ for some nonnegative integer k. If $k < 100$, then

$$\frac{1}{10^{120}} < \frac{1}{1105q_{99}^2} < \min\left\{\left|\gamma - \frac{p_k}{q_k}\right| : k \in \{0,1,\ldots,99\}\right\} \leq \left|\gamma - \frac{my-(n+1)x}{x-y}\right| < \frac{13}{\alpha^{\lambda_1}}.$$

Therefore, we deduce that

$$\lambda_1 < \frac{\log(13 \times 10^{120})}{\log\alpha} < 317,$$

which is false since we assume that $\lambda_1 > 400$. In the above, we used that if $[a_0,\ldots,a_{99},\ldots]$ is the continued fraction of γ, then $\max\{a_k : 0 \leq k \leq 99\} = 1102$. Thus, $k \geq 100$, and since the 207th convergent p_{207}/q_{207} of γ satisfies

$$q > 3 \times 10^{111} > |x-y|\,,$$

we conclude that $k \in [100, 207]$. Since

$$\left|\gamma - \frac{p_{207}}{q_{207}}\right| > \frac{1}{10^{227}},$$

we get

$$\frac{1}{10^{227}} < \left|\gamma - \frac{p_k}{q_k}\right| < \frac{13}{|x-y|\,\alpha^{\lambda_1}} \leq \frac{13}{q_{100}\alpha^{\lambda_1}} \leq \frac{1}{10^{53}\alpha^{\lambda_1}},$$

giving

$$\lambda_1 \leq \frac{\log(10^{278})}{\log\alpha} < 732.$$

If $M \leq x$, then we have $M = \lambda_1 < 732$, a contradiction. Thus, $x = \lambda_1$, therefore $x < 732$. We get now a better bound for M. That is, using estimate (59) and comparing it also with estimate (55) according to the two cases distinguished in Subsection 3.7, we conclude that

$$M < 5 \times 10^{17} x \log M < 5 \times 732 \times 10^{17} \log M < 3.7 \times 10^{20} \log M,$$

giving

$$M < 2 \times 3.7 \times 10^{20} \log(3.7 \times 10^{20}) < 4 \times 10^{22},$$

which, via Lemma 6, yields

$$\begin{aligned} x &< 1.3 \times 10^{28}(4 \times 10^{22})^2(\log((4 \times 10^{22}))^2 < 6 \times 10^{76}, \\ y &< 10^{47}(4 \times 10^{22})^2(\log(4 \times 10^{22}))^3 < 3 \times 10^{97}. \end{aligned} \tag{62}$$

Now, the convergent p_{183}/q_{183} of γ has $q_{183} > 3 \times 10^{97} > |x-y|$ and

$$\left|\gamma - \frac{p_{183}}{q_{183}}\right| > \frac{1}{10^{198}}.$$

Therefore, by an argument previously used, we have

$$\lambda_1 < \frac{\log(13 \times 10^{198})}{\log \alpha} < 522.$$

Thus, $\lambda_1 \in [3, 521]$. Since $M > 1000$ and $\lambda_1 = \min\{x, M\}$, it follows that $\lambda_1 = x$. We now move on to inequality (50). Since $M > 1000$, we get

$$\left|(x-y)\log(2\sqrt{2}) - (nx - my)\log\alpha - \log(\alpha^x + 1)\right| < \frac{12}{\alpha^M}.$$

Here, we fix x and apply again the Baker-Davenport reduction method. We have

$$|u\gamma - v + \mu| < AB^{-M},$$

where we take

$$\gamma := \frac{\log(2\sqrt{2})}{\log\alpha}, \quad \mu := -\frac{\log(\alpha^x + 1)}{\log\alpha}, \quad A := 14 \left(> \frac{12}{\log\alpha}\right), \quad B := \alpha,$$

and $(u, v) = (x-y, nx-my)$. By estimate (62), we can take $T := 10^{100}$ as the bound on $|u|$. We loop over all even values $x \in [4, 520]$. In all cases we choose $q = q_{499}$ the denominator of the convergent of index 499 of γ. In all cases $\varepsilon > 0.001$ and the worst (largest) upper bound on M is 707. This took a few seconds. This is a contradiction since we assume that $M > 1000$. This finishes the proof of Theorem 1.

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