PERGAMON Mathematical and Computer Modelling 37 (2003) 261–281

MATHEMATICAL AND COMPUTER MODELLING

www.elsevier.com/locate/mcm

The Modelling of Political Dynamics by Generalized Kinetic (Boltzmann) Models

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(Received and accepted October 2002)

Abstract—This paper deals with the development of mathematical structures suitable to include a variety of Boltzmann type models for social competition. The first part is devoted to derive the framework suitable to design general models. The second part deals with some applications and simulations in the field of social dynamics, and specifically about the free choice of each individual about one among several families which partition the whole population. Finally, a critical analysis towards research perspectives is proposed.

Keywords—Generalized Boltzmann equation, Complexity, Kinetic theory, Nonlinearity, Population dynamics, Political dynamics, Social behaviours.

1. INTRODUCTION

The development of the methods of mathematical kinetic theory to modelling large systems of interacting individuals is systematically documented in [1], and in the review paper [2], which report about kinetic type models in several field of applied sciences. Recent developments and applications may be found in the collection of surveys edited in [3,4].

A common feature of these models is that the evolution, as in the kinetic theory of gases, refers to the statistical distributions over the microscopic states of large populations of interacting objects, or individuals, however, here the microscopic state is defined not only by mechanical variables, such as position and velocity, but also by additional internal variables which in some cases (as in the one which is dealt with in this paper) refer to an *intelligent*, or at least *organized*, microscopic structure. Further, interactions related to the mechanical variables do not necessarily obey the laws of classical mechanics since the organized behaviour may modify these rules and directly exert their influence on the dynamics. The mathematical framework is a new class of kinetic equations proposed in [5] as natural evolution of the model proposed in [6].

Mathematical results on the above class of equations are limited to particular models, and often in the spatially homogeneous case. Results on the analysis of the Cauchy problem are documented, among others, in the review paper [7], and in [8] concerning a mathematical model

Partially supported by GNFM-INDAM.
The author is thankful to N. Bellomo for interesting discussions, and to B. Firmani and E. Bertolazzi for helpful advice.

0895-7177/03/$ - see front matter Typeset by AMS-TEX
PII: S0895-7177(03)00005-0

that describes the immune competition between tumour and immune cells [9]. An analysis of the asymptotic theory suitable to obtain macroscopic equations from microscopic description is documented in [10,11].

Although the qualitative analysis of mathematical problems can take advantage of the valuable background of the mathematical literature on the Boltzmann equation, which may be found in specialized books [12–14], it is still limited to very special models and problems rather than being opened to the very general framework offered in [5]. This topic will be critically analysed in the last section of this paper.

Referring to the applications developed within the above framework, the additional microscopic structure can either be of mechanical nature or of more general origin. In the first case, it yields some mechanical variable, e.g., the size of the particles in granular media [15,16], the size of clusters in aggregation-fragmentation models [17], flow with dispersed particles [18,19], etc. In the second case, as the one considered in this paper, models are developed such that the microscopic state directly refers to organized behaviours. Specific models of this nature are already available in the literature in different fields of applied sciences. For instance, Boltzmann type kinetic models of traffic flow have been developed such that the mechanics of the microscopic interactions is related to the individual behaviours, see [20] and some specific models recently appeared in the literature, e.g., [21,22]. Another interesting field of application refers to the dynamics of cell populations interacting in a vertebrate in the framework of the immune competition, as documented in the review paper [23].

The contents of this paper refer to the modelling of the evolution of interacting individuals whose socio-biological state is not only modified by (microscopical) interactions between individuals, but also and in addition, by external mass actions. The idea of developing kinetic population dynamics for social interactions was already contained in Chapter 3 of [1], and this idea was also developed in [24] where a solver based on finite differences schemes was proposed. Independently, Carbonaro and Serra [25] have developed, following a technically analogous approach, models of evolution of personal feelings in pair interactions.

Specifically, this paper deals with the derivation of a Boltzmann-type description of a population of individuals that are assumed free to adhere to one among several families that partition a population. Interactions between pairs of individuals are considered with reference to their action over the individuals' physical (internal) state. This base-model is then extended by considering mass interactions, able not only to act over all the individuals of a family, but also to influence the membership choice. That is, transitions from one party to the other are possible when interactions of the external world over all the supporters of a party are superimposed.

In fact, under the applicative point of view, the approach that is developed here is still abstract and paradigmatic. The representation is achieved in a qualitative way, and in this sense, and only to ease the interpretation of the various terms and controls that compose the model, the political terminology is used and the real world is addressed. However, in principle, predictions delivered by simulations, or possibly by an analytic asymptotic analysis, can be used to properly modify the microscopic and the mass interactions in order to obtain the desired trend. On the other hand, to actually proceed to a realistic validation of the model by means of effective data, some of the simplificative assumptions that are used in the present picture may no longer be sustained, as will be explained in the following sections. All the same, this introductory approach is an unavoidable step to reach a correct identification of the terms in the model and to analyse their relative impact on the whole system. Indeed, differently from a physical system of, e.g., identical particles of a gas in a vessel, for the system under consideration, no physical laws nor well-defined state variables are available, and the various terms need to be *a posteriori* justified as a whole.

The contents of this paper are organized into five sections as follows. Section 1 has given the above introduction with the aim to introduce the subject of the paper and its organization. Section 2 deals with the design of a mathematical framework that constitute the general modelling of social interaction and competition among different families. As already mentioned, a first base framework is developed, and then it is extended to a second more powerful one which adds to the base microscopic terms also global mass-interaction terms. Section 3 deals with the description of specific models derived within the above framework and applied to the political dynamics case. Section 4 develops some simulations also meant to anticipate aspects of the qualitative analysis such that identification of equilibrium configurations and their stability properties. Finally, Section 5 is devoted to a critical analysis on the contents of the paper, starting with an overview of some specific models which can be possibly cast (and developed) into the above-mentioned structures.

2. MATHEMATICAL STRUCTURES FOR POPULATION DYNAMICS

In what follows, and because of the specific model that will be described and simulated in the next sections, the terminology will be borrowed by the politics. It has to be remarked that the mathematical structure is quite general and that no quantitative reference has been made to any real political situation because, and in particular, of the assumptions concerning the identical nature of the parties and the autonomous character of the model (see later sections).

2.1. The Base Framework

Consider mathematical models corresponding to a system of several families within a certain population, each family labelled with the subscript $i = 1, \dots, p$ and $p \in \mathbb{N}$. Each family consists of several interacting individuals homogenously distributed in space. In other words, the space variable does not play a relevant role in the description of the system. The sum of individuals of the whole population is constant in time. Microscopic interactions modify the state of the interacting individuals.

Let, for such a system, $u \in I_u = [-1, 1]$ be a variable suitable to define the *state* of the individuals of the population. To take into account the choice of a certain party, and possible changes from a party to another, this state variable is meant to describe the intimate degree of satisfaction that each individual proves for the whole social system. Not related to a specific party, the state variable summarizes the internal feeling of the individual about the social state and his desire to remain in his present situation or to change it. This state variable will be called *happiness*, and it has to be remarked that it is only loosely connected to the wealth of the individual, and not only to it, but also to his personal aims and satisfaction in the complete (political and social) situation. In this sense, negative values of u correspond to unsatisfied individuals, whereas positive values correspond to happy ones. The value $u = 0$ refers to indifferent individuals and, hence, scarcely active and eager to be involved.

At any fixed time t, the statistical distribution over the *state* of the i^{th}-party individuals is given by the number density function

$$f_i : (t, u) \in [0, T] \times [-1, 1] \longmapsto [0, \infty), \qquad f_i \in C^0[-1, +1], \quad i = 1, \dots, p, \tag{2.1}$$

which defines the number of individuals of the i^{th}-family, at the time $t \in [0, T]$, in the domain $[u_1, u_2]$ of the social state, as follows:

$$N_i(t; u_1, u_2) = \int_{u_1}^{u_2} f_i(t, u)\, du. \tag{2.2}$$

Values at $t = 0$ will be denoted by

$$f_i^0(u) = f_i(t = 0, u). \tag{2.3}$$

The above densities assign at each time t the expected total number of votes of each party

$$N_i(t) = \int_{-1}^{+1} f_i(t, u)\, du. \tag{2.4}$$

The total number of individuals in the population is assumed to be constant in time

$$N(t) = \sum_{i=1}^{p} N_i(t) = N^0 = \sum_{i=1}^{p} N_i(t=0). \tag{2.5}$$

The mathematical model is an evolution equation for the above distribution functions (2.1) suitable to describe the evolution of given initial conditions. Higher-order moments can be recovered under appropriate integrability conditions

$$E_i\left[u^q\right](t) = \int_{I_u} u^q f_i(t,u)\,du, \qquad q \in \mathbb{N}. \tag{2.6}$$

According to [6], the base mathematical model can be derived assuming that after a suitable phenomenological and experimental analysis, it is possible to model the following functions.

ENCOUNTER RATE.

$$\eta_{i,j} : (v,w) \in [-1,+1]^2 \longmapsto [0,+\infty), \qquad i,j = 1,\dots,p, \tag{2.7}$$

which denotes the number of encounters per unit time between individuals of the i^{th} and j^{th}-parties in the states v and w, respectively.

INTERACTION TRANSITION FUNCTION.

$$\zeta_{j,k}^{(i)} : (u,v,w) \in \; [-1,+1]^3 \longmapsto [0,+\infty), \qquad i,j,k = 1,\dots,p, \tag{2.8}$$

which denotes the probability density that an individual in the party j and status v, because of a personal encounter with an individual of the k^{th} party in the state w, changes his party and status into i and u, respectively.

However, due to the specific meaning of the state variable of the model, for function ζ, the following special, although powerful, structure will be here assumed:

$$\zeta_{j,k}^{(i)}(u,v,w) \equiv p_{i,j}(u) \cdot \psi_{j,k}(u,v,w), \tag{2.9}$$

where the following two quantities have been introduced.

- The state transition density

$$\psi_{j,k} : (u,v,w) \in [-1,+1]^3 \longmapsto [0,+\infty), \qquad j,k = 1,\dots,p, \tag{2.10}$$

  which denotes the probability density that an individual in the family j and status v, because of a personal encounter with an individual in the family k and state w, changes his feelings about the (political) system, i.e., his satisfaction status v into u *without modifying* his current family j.

- The change of party probabilities

$$p_{i,j} : (u) \in [-1,+1] \longmapsto [0,1), \qquad i,j = 1,\dots,p, \tag{2.11}$$

  which denotes the conditional probability that an individual in the party j, having reached the satisfaction level summarized by u, autonomously decides to change his party into i.

In fact, it may be argued that, in a pairwise individual interaction, i.e., when two individuals meet and exchange their ideas and feelings about the political situation, at most a change of happiness may be produced, and not a straight and direct change of party which, on the contrary, is an intimate and deeper decision that each individual keeps for himself.

Using the above expressions and following [6], the formal structure of the base model writes

$$\begin{aligned}\frac{\partial f_i}{\partial t}(t,u) &= G_i[\mathbf{f}](t,u) - L_i[\mathbf{f}](t,u)\\ &= \sum_{j,k=1}^{p}\left\{\int_{-1}^{+1}\int_{-1}^{+1}\eta_{j,k}(v,w)p_{i,j}(u)\ \psi_{j,k}(u,v,w)\ f_j(t,v)f_k(t,w)\,dv\,dw\right.\\ &\quad\left. -f_i(t,u)\sum_{j=1}^{p}\int_{-1}^{+1}\eta_{i,j}(u,w)\ f_j(t,w)\,dw\right\},\end{aligned} \tag{2.12}$$

where $\mathbf{f} = \{f_i\}_{i=1}^{p}$.

REMARK 2.1. The total number of individuals is assumed to be a constant, and the following conditions are set on the above functions:

$$p_{j,j}(u) = 1 - \sum_{i\neq j} p_{i,j}(u), \qquad \forall j \in \{1,\dots,p\}, \qquad u \in [-1,+1], \tag{2.13}$$

$$\int_{-1}^{1}\psi_{j,h}(u;v,w)\,du = 1, \qquad \forall j,h \in \{1,\dots,p\}, \quad v,w \in [-1,+1]. \tag{2.14}$$

REMARK 2.2. A technical difference, with respect to the model proposed in [24] is that the domain of the variable u is now bounded. This has to be regarded as a consequence of the meaning here ascribed to the state variable. This choice, which on the one hand seems quite natural in this case, on the other creates the need of *a priori* modelling border effects or boundary conditions. In what follows, the assumption that the total number of individual is a constant will reduce this problem, in that no flux will be allowed in any case through the boundaries of the state space although, there, the density functions may have nonzero values.

REMARK 2.3. Although the system is constituted by several families, no direct interactions among them is considered. Transitions from one party to the other may happen only on an individual basis, and are ruled by the terms $p_{i,j}$ which in this base model at most may depend on the social state.

REMARK 2.4. Several fundamental assumptions have already been made on this base model that may be considered too restrictive for a realistic political application and, hence, rejected or at least rediscussed. A first assumption refers to the identical character of the parties that compose the population. In reality, the *abstention party*, the party of the nonvoting or void-voting electors, *may not be compared* to any kind of well-organized and directly sustained party. A second assumption is to disregard the time dependency of all the terms and controls of the model. In particular, no electoral period is considered that would introduce a superimposed delay time and nonautonomous periodic structure on the model. In the third assumption, no sources are present, no immigrants or new voters, or natural death is introduced, which often may produce appreciable differences in the global dynamics. Further, no state variables are introduced that may be directly referred to a party to testify the quality of the politics played by it, or that may be used to model the *appeal* of the party on the individuals. All the individuals are indistinguishable with respect to their ability to interact one with the other and to produce a change on their counterpart status. Finally, but of sufficient importance to motivate an extension of the base approach, all the interactions are at an individual (microscopic) level and no mass interactions are considered. In the following section, interactions of a global nature are introduced that act on each individual of a certain party, and not on a pairwise internal basis but on an average mass level.

2.2. The External (Average) Contribution

In this section, the basis framework is extended to introduce global effects and interactions on the system to be thought of as the mass media, externally driven, propaganda terms of collective character.

At first, the global variables of mean field nature that will be referred to in the following are here summarized.

- The party power, i.e., the number of votes of the party

$$N_i(t) = \int_{-1}^{+1} f_i(t,u)\,du, \qquad i = 1,\dots,p. \tag{2.15}$$

- The party spirit, i.e., the average satisfaction degree of the adherents to the party

$$U_i(t) = \int_{-1}^{+1} u\ f_i(t,u)\,du, \qquad i = 1,\dots,p. \tag{2.16}$$

- The vox-populi, i.e., the global average satisfaction

$$U(t) = \sum_{i=1}^{p} \int_{-1}^{+1} u\ f_i(t,u)\,du. \tag{2.17}$$

- The relative political power of the party i (with respect to the party j)

$$F_{i,j} := \frac{N_i - N_j}{N}. \tag{2.18}$$

These variables play a twofold role in the model. First, they control the internal probabilities $p_{i,j}$ in that their dependence on the family entries (i,j) is acquired via the above collective variables. Second, an overall term is added to the left-hand side of equation (2.12) which now becomes

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial u}\left(f_i\ K_i[\mathbf{f}]\right) = G_i[\mathbf{f}] - L_i[\mathbf{f}], \qquad i = 1,\dots,p, \tag{2.19}$$

$$K_i[\mathbf{f}](t,u) = k_i(u) + \sum_{h=1}^{p} \int_{-1}^{1} \varphi_{i,h}(u,w)\ f_h(t,w)\,dw, \tag{2.20}$$

$$G_i[\mathbf{f}](t,u) = \sum_{j,h=1}^{p} \int_{-1}^{1}\int_{-1}^{1} p_{i,j}(u)\eta_{j,h}(v,w)\psi_{j,h}(u,v,w) \times f_j(t,v) f_h(t,w)\,dv\,dw, \tag{2.21}$$

$$L_i[\mathbf{f}](t,u) = f_i(t,u) \sum_{h=1}^{p} \int_{-1}^{1} \eta_{i,h}(u,w)\ f_h(t,w)\,dw. \tag{2.22}$$

As well as in the purely internal case, an experimentally driven modelling is assumed to be possible to depict the following functions of social character.

THE INDIVIDUAL TENDENCY FUNCTION.

$$k_i : u \in [-1,+1] \longmapsto \mathbb{R}, \qquad i = 1,\dots,p, \tag{2.23}$$

which denotes the drift coefficient which is globally produced, on the individuals of party i in the status u, by the ensemble of external factors.

THE PROPAGANDA COEFFICIENTS.

$$\varphi_{i,h} : (u,w) \in [-1,+1]^2 \longmapsto \mathbb{R}, \qquad i,h = 1,\dots,p, \tag{2.24}$$

which assign the drift weight, of mass nature, that may be directly ascribed to the above mean field variables.

As already mentioned, the nature of the variables used here is such that a null net-flow condition is necessary

$$[f_1K_1 + f_2K_2]_{u=+1} - [f_1K_1 + f_2K_2]_{u=-1} = 0. \tag{2.25}$$

On the other hand, arbitrariness of the distribution function values at the borders suggests the following more restrictive although quite natural assumption:

$$\begin{aligned} k_i(-1) = k_i(+1) = 0, &\qquad \forall\, i \in \{1,\dots,p\}, \\ \varphi_{i,h}(-1,w) = \varphi_{i,h}(+1,w) = 0, &\qquad \forall\, i,h \in \{1,\dots,p\}, \quad w \in [-1,+1]. \end{aligned} \tag{2.26}$$

In the next section, this structure is specialized into a detailed model that may take advantage of the special character of the various terms when applied to a realistic political example.

3. APPLICATIONS TO A BIPARTITE POLITICAL SYSTEM

As already done in the preceding section, here the base model will be at first developed by specializing its terms, and only afterwards the extensions mentioned in Section 2.2 superimposed on it. In this way, two explicit models will be obtained and, in Section 4, discussed on the basis of a convenient set of simulations. For simplicity reasons, the application will be restricted to the easiest case, i.e., to a bipartite system, $p = 2$.

3.1. Specializing the Base Model

The present section is reserved to introduce and motivate the main assumptions that lead to specific choices of the terms in (2.12).

THE STATE TRANSITION PROBABILITY DENSITY. Due to the intrinsic meaning of function ψ which only produces changes in happiness caused by personal pairwise contacts between indistinguishable identical voters, it is not difficult to accept that the functional form of $\psi_{i,j}$ does not depend on the party indices the two interacting individuals belong to. Nevertheless, the results of a discussion held with a supporter of the opposite party may, and generally are, different from those that arise from a contact with a partner fellow. Specifically, when two individuals of the same party analyse the political system, they presumably end up with a satisfaction degree that is midway between their former convincements. On the contrary, due to the opposite way of looking, the happiness of an individual of the opposite side is seen *with the opposite sign*, and drives a small way in that opposite direction, of its actual value. Moreover, all normal individuals share a common, and small, capacity of convincing everybody else. Finally, only one type of interaction between the individuals is considered. These *a priori* rather simple considerations are modelled as follows.

An analytic form for function ψ is selected among those which are completely known when the mean value $m_{j,h}$, and variance $\sigma_{j,h}$ are given

$$\psi_{j,h}(u,v,w) = \psi\left(u; m_{j,h}(v,w), \sigma_{j,h}(v,w)\right), \qquad j,h \in \{1,2\}, \tag{3.1}$$

in this way, to proceed in specializing the model, it will be sufficient to properly assign the two functions

$$m = m_{j,h}(v,w), \qquad m,u,v \in [-1,1], \tag{3.2}$$

$$\sigma = \sigma_{j,h}(v,w), \qquad \sigma > 0. \tag{3.3}$$

An easy choice that may produce smooth and unpeaked distributions, with possibly nonnull border values, and that may enhance the mixing in the population features, relies on the standard density function, normalized on the interval $[-1, +1]$

$$\psi(u; m, \sigma) = \frac{\left(\exp - (u-m)^2/2\sigma\right)}{\left(\int_{-1}^{+1} \exp(-(u-m)^2/2\sigma)\, du\right)}. \tag{3.4}$$

Further, the function $m_{j,h}(v, w)$ is modelled by

$$\begin{aligned} m_{j,h}(v,w) &= (1-\alpha)v + \alpha w, \qquad \text{if } j = h, \quad \alpha \in [0,1], \\ m_{j,h}(v,w) &= (1-\beta)v - \beta w, \qquad \text{if } j \neq h, \quad \beta \in [0,1]). \end{aligned} \tag{3.5}$$

Finally, the undistinguishable character of all the individuals yields

$$\sigma_{j,h}(v, w) = \sigma = \text{const}. \tag{3.6}$$

The Change-of-Party Procedure. In what follows, only the terms $p_{i,j}(u) \in [-1, +1]$ for $i \neq j$ will be illustrated, since in all cases, condition (2.13) has to be verified which assigns $p_{j,j}(u) = 1 - \sum_{i\neq j} p_{i,j}(u)$.

The reasons why an individual, with a satisfaction degree of u, decides to change his current party are, in fact, obscure. It is scarcely possible, and it would be quite unrealistic, to presume at this scale level that a *direct connection* exists between the unsatisfaction of a voter and his vote, neither on a threshold basis nor on a proportional one. At least for this base model, it seems more acceptable to restrict the picture to a simple aleatoric behaviour, which may be at most *modulated* by the individual happiness. Therefore, it is assumed that

$$p_{i,j}(u) = \tilde{p}_{i,j}, \qquad \text{for } \tilde{p}_{i,j} \in [0,1], \quad i,j \in \{1,2\}, \quad i \neq j. \tag{3.7}$$

In assigning the numbers $\tilde{p}_{i,j}$, the following quite natural remarks have been made. Two main independent reasons (of average nature) may motivate a change of side: the one possibly related with the parties' powers, the other with the average satisfaction. On the other hand, everybody is spontaneously driven to sustain his current decision unless both the above factors concur to change it. For the base model, this leads to

$$\tilde{p}_{i,j} = \tilde{p}_0^{(N)} \cdot \tilde{p}_0^{(U)}, \qquad \tilde{p}_0^{(N)}, \tilde{p}_0^{(U)} \in [0,1], \quad i,j \in \{1,2\}, \quad i \neq j, \tag{3.8}$$

where the factors $\tilde{p}_0^{(N)}$ and $\tilde{p}_0^{(U)}$ are separately related to the mass variables N and U, respectively.

The Interaction Frequencies. All the assumptions made in modelling the probabilistic term of equation (2.12) are quite general and improper to characterize different qualitative behaviours. This task is reserved to the frequency term. The idea that has been followed is that opportunities, even more than predetermined tactics, are suitable to produce distinct results. Hence, the change of happiness procedure, which resides on the microscopical pairwise scale and as such is not easily controllable on an *a priori* basis, is ruled by the unbiased function ψ, whereas the collective behaviour of the group plays its role on a global basis and is controlled by functions η.

To realize a set of behaviours which is sufficiently various and representative, three different and in a sense extreme situations are now introduced concerning the interactions between (pairs of) individuals. The pictures so obtained will be analysed and compared one with the others, also on the simulations basis.

Experiment 1. Uniform Frequencies. The interaction frequencies do not depend on the status of the interacting individuals. Moreover, the individuals' parties play a role only in that it is much more common that two fellow partners meet to interact than it happens for individuals of different sides. This approach leads to

$$\eta_{j,h}(v,w) = \tilde{\eta}_{j,h} := \begin{cases} \tilde{e}_h > 0, & \text{if } j = h, \\ \varepsilon \tilde{e}_h, & \text{if } j \neq h, \end{cases} \tag{3.9}$$

for $0 < \varepsilon \ll 1$ and $i, j \in \{1, 2\}$.

EXPERIMENT 2. SHORT RANGE FREQUENCIES. The preceding frequency values are modulated by a factor which allows interactions to be possible only between individuals which are not too disparate in their status. This approach is the more natural the more the status variable is related to the individuals' wealth. In detail, this second picture is given by

$$\eta_{j,h}(v,w) = \tilde{\eta}_{j,h} \cdot \eta_d(v - w), \tag{3.10}$$

where $\tilde{\eta}_{j,h}$ is given by equation (3.9) and where

$$\begin{aligned} \eta_d(x) &= 1, && \text{if } |x| \leq d, \quad d \in [0,2], \\ \eta_d(x) &= 0, && \text{otherwise.} \end{aligned} \tag{3.11}$$

EXPERIMENT 3. LONG RANGE FREQUENCIES. Unlike Experiment 2, here the uniform frequencies are modulated by a factor which is meant to increase with the difference in the status variables. This is to take into account the global nature of a society wherein people of quite different happiness and satisfaction degree are not only more and more vital, but also motivated in interacting with each other. On the contrary, middle status, uninterested, and uninvolved people are supposed to almost completely refuse any direct interactions and contacts with anybody else. A possible model for this requirements is given by

$$\eta_{j,h}(v,w) = \tilde{\eta}_{j,h} \eta_c(v,w), \tag{3.12}$$

where $\tilde{\eta}_{j,h}$ is given by equation (3.9) and where

$$\eta_c(v,w) := 1 + |vw| \left(1 + c|v - w|\right), \qquad c > -1. \tag{3.13}$$

In Figure 1, the two tuning functions η_d and η_c are shown for comparison purposes.

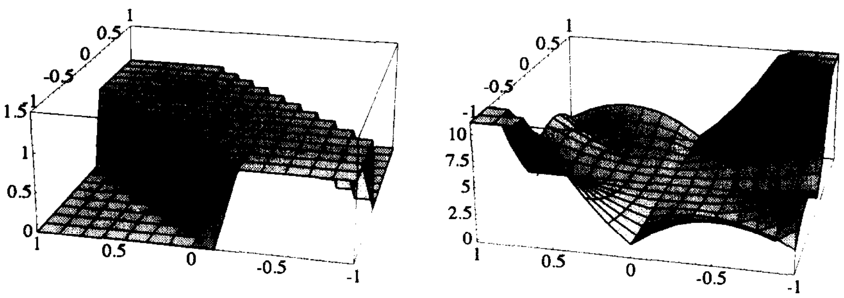

Figure 1. $\eta_d(v,w)$, for $d = 0.8$ and $\eta_c(v,w)$, for $c = 10.0$.

3.2. Effects of the Environment

All the terms in the base model are of individual nature. In particular, it may poorly describe a society wherein group interactions are as important as, or even more than, pairwise interactions. This is to emphasize once again that the global effects of (for instance) mass media and of mean field variables are of the utmost importance and may not be neglected. Their first action is seen as an overall drift acting on the distribution functions.

Tendency Terms of Personal Nature. Each individual of a certain party and personal status is influenced by external events and by the mass messages broadcasted on them, the final effects generally depending on the party the individual adheres to. In fact, he may be more or less influenced, starting from an unbiased ground level, because of not only of his interpretative key, but also of his happiness status. These remarks suggest the following form for the individual tendency term $k_i(u)$:

$$k_i(u) = (a_i + b_i\ u)\ \kappa(u), \qquad i = 1, 2, \tag{3.14}$$

where

$$\begin{aligned} a_i &= -\tilde{a} + \tilde{c} \cdot \mathrm{sign}(F_i), \\ b_i &= -\tilde{b} + \tilde{d} \cdot \mathrm{sign}(U_i), \end{aligned} \tag{3.15}$$

for $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \geq 0$, and where the factor $k(u)$ is a convenient smoothing function which is needed to verify the null flux condition (2.26), e.g.,

$$\kappa(u) = (1-u)\,(1+u)\,. \tag{3.16}$$

The two parameters $\tilde{a} \geq 0$ and $\tilde{b} \geq 0$ represent the base unconditioned levels, whereas $\tilde{d} \geq \tilde{b}$ and $\tilde{c} \geq \tilde{a}$ assign the variations on these levels due to the membership to i^{th}-party. That is, starting from a (loosely negative) judgment of the external global situation, a supporter of the dominant party accepts to reach a positive judgment of some affair which is negatively seen by the opposition. As well, if coefficient b_i of the personal status variable u is of the same sign of the average satisfaction U_i, then a centralizing cohesive drift is produced when U_i is negative, whereas a loosening and spreading action is expected on the adherents to a party whose spirit is positive.

Tendency Terms of Collective Nature. Mean variables play their main role here. Depending on the political power of the party, and on its spirit, a drift is produced on all its supporters, and along the same remarks seen above. This gives (see equation (2.24))

$$\varphi_{i,h}(u,w) = \left(c^{(U)}\delta_{ih}\frac{w}{N} + c^{(N)}\,(1-2\delta_{ih})\ \frac{u}{N}\right)\ \kappa(u), \qquad i,h \in \{1,2\}\,, \tag{3.17}$$

where $c^{(U)}, c^{(N)}$ are nonnegative reals, $\delta_{i,j}$ denotes the Kronecker symbol, and $\kappa(u)$ is given by equation (3.16). This yields the following global terms (see equations (2.16) and (2.18)) where F_i denotes for short $F_{i,h}$ since the families number is only $p = 2$:

$$K_i^{(m)}(t,u) := \left[c^{(U)}\ \frac{U_i(t)}{N} - c^{(N)}\ F_i(t)\ u\right]\ \kappa(u). \tag{3.18}$$

Consequently, the whole transport term appearing in equation (2.19) reads

$$K_i[\mathbf{f}](t,u) := \left[\left(a_i + c^{(U)}\ \frac{U_i(t)}{N}\right) + \left(b_i - c^{(N)}\ F_i(t)\right)u\right]\ \kappa(u), \qquad i = 1,2. \tag{3.19}$$

Global Effects on the Change-of-Party Procedure. The role played by the global variables in the above transport terms are, however, too restrictive. It is not possible or acceptable to underestimate the effects of the collective mass variables on the (albeit individual) procedure of choosing and changing party, which still is considered to happen on a strictly personal basis. A direct (linear) influence of the two variables F_i and U_j is here assumed on the changing side probabilities $p_{i,j}$, and following the same criteria used above. With the same notation as before, for $i, j \in \{1,2\}$, $i \neq j$, and with condition (2.13) to be verified by the terms $p_{i,j}$ for $i = j$, one has

$$\begin{aligned} \tilde{p}_{i,j} &= \tilde{p}_{i,j}^{(N)} \cdot \tilde{p}_{i,j}^{(U)}, \\ \tilde{p}_{i,j}^{(N)} &= \left[p_0^{(N)} + p^{(N)}\ F_i\right]_0^1, \\ \tilde{p}_{i,j}^{(U)} &= \left[p_0^{(U)} + p^{(U)}\ \left(1 - \frac{U_j}{N}\right)\right]_0^1, \end{aligned} \tag{3.20}$$

where $p^{(N)}$, $p^{(U)}$, $p_0^{(N)}$, $p_0^{(U)} \geq 0$, and where the symbol $[\cdot]_0^1$ is to mean

$$[x]_0^1 := \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases} \tag{3.21}$$

In the following section, specific simulations will be presented with particular attention to the time asymptotic behaviour of the system under exam.

4. ASYMPTOTIC BEHAVIOUR VIA COMPUTATION

Some models of social interactions and competition have been proposed in the preceding sections both under the theoretical point of view and applicative specialization. Numerical simulations are proposed in this section in order to visualize the behaviour of the system within the framework of the above description.

Simulations are developed in order to point out some peculiar aspects of the evolution with special attention to the asymptotic behaviour of the system. In fact, referring to population dynamics, it is important to understand whether the evolution of the population shows a trend toward an equilibrium solution and what are the controls on this behaviour. Further, one would like to understand if such a solution is asymptotically stable, or (relatively) structurally stable. Moreover, considering that we are dealing with a system of several families, it is useful to understand if the equilibrium solutions, if there are any, qualitatively differ one from the other. Specifically, the following qualitative and quantitative aspects can be investigated:

(i) existence of asymptotic, for $t \to \infty$, equilibrium solutions, or of periodic solutions;
(ii) qualitative shape of the equilibrium solutions depending on the control parameters and on initial data;
(iii) stability properties of the equilibrium solutions and of periodic solutions.

Of course, the above analysis can also be undertaken by analytic methods. Indeed, some mathematical results are already available in the literature. The analysis of existence of solutions of equation (2.12) and of their regularity properties have been studied in [26] by classical methods of semigroup theory.

As it was done for the mathematical structure, simulations are presented in two successive steps: the first one is referred to the base model (2.12), the second one to the extended (2.19). The problem under exam is an initial value problem on a bipartite system, namely: the initial distributions $f_i(t = 0, u)$ being assigned for each of the two families $i = 1, 2$, the problem is solved by finding the consequent system dynamics ruled by equation (2.12) for the base model, and by equation (2.19) for the extended one, when the terms are specified as described in Section 3.

In all cases, the initial conditions have been assigned as follows. Let $\nu_i(u)$ denote the usual normal density with mean m_i^0 and variance σ_i^0

$$\nu_i(u) \;=\; \exp - \frac{\left(u - m_i^0\right)^2}{2\sigma_i^0}\,, \tag{4.1}$$

and let $\langle \nu_i \rangle(u)$ denote its normalized value on $[-1, +1]$, i.e.,

$$\langle \nu_i \rangle(u) := \frac{\nu_i(u)}{\int_{-1}^{+1} \nu_i(v),\, dv}. \tag{4.2}$$

The initial densities $f_i(t = 0, u)$, $i = 1, 2$, have been set according to

$$f_i(t = 0, u) =: f_i^0(u) \equiv N_i^0 \langle \nu_i \rangle(u), \tag{4.3}$$

where N_i^0 have been assigned such that $N_1^0 + N_2^0 = 10$. This procedure has been used to allow the initial conditions not only to have reasonable numerical values and sums N_i^0, but also to have possibly large border values: $f_i(t = 0, u = \pm 1)$ and conveniently flat distributions.

In all the simulations presented here, the above quantities have been valued as follows:

$$N_1^0 = 6.0, \quad N_2^0 = 4.0, \qquad m_1^0 = -0.4, \quad m_2^0 = +0.4, \qquad \sigma_1^0 = \sigma_2^0 = 0.36. \tag{4.4}$$

In Figure 2, the two initial value functions $f_1(t = 0, u)$ and $f_2(t = 0, u)$ are shown for comparison purposes.

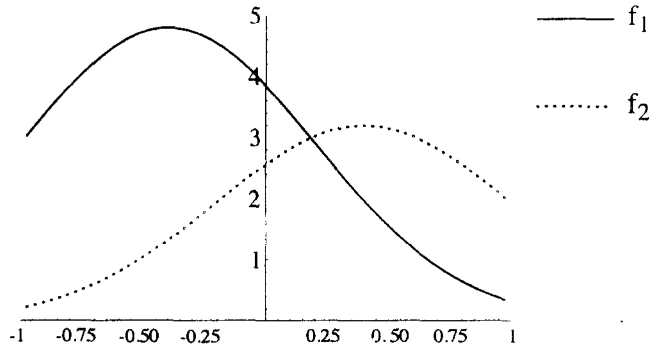

Figure 2.

REMARK 4.1. From the simulations performed, no particular dependence appeared of the asymptotic dynamics upon these initial conditions in the following sense. The values m_1^0, $m_2^0 \in [-1, +1]$ seem to be quite uneffective on the asymptotic behaviour. The role of the numbers N_1^0, $N_2^0 > 0$ appears to be restricted to the selection of which is the dominant (at $t = 0$) of the two parties. The role of the variances is to avoid exceedingly (and, hence, unrealistic) peaked initial conditions.

To conclude this general introduction, a few technical details may be sketched. A finite centered difference method has been used to solve the models equations. Call $\{u_{j+1/2}\}_{j=0,\dots,M}$ a set of M equally spaced nodes, and $\Delta u := u_{j+1/2} - u_{j+1/2}$. Call $\mathbf{f}_j^n$ the average value at time $t = n$ over the interval $[u_{j-1/2}, u_{j+1/2}]$ of the discretized version of $\mathbf{f}(t, u)$. Then, equations

$$\Delta u \, \frac{d}{dt} \, \mathbf{f}_j(t) + \mathbf{H}_j[\mathbf{f}](t) = \mathbf{g}_j[\mathbf{f}](t), \tag{4.5}$$

where

$$\mathbf{H}_j[\mathbf{f}](t) := \mathbf{K}^+_{j+1/2}(t)\mathbf{f}_j(t) + \mathbf{K}^-_{j+1/2}(t)\mathbf{f}_{j+1}(t) - \mathbf{K}^+_{j-1/2}(t)\mathbf{f}_{j-1}(t) - \mathbf{K}^-_{j-1/2}(t)\mathbf{f}_j(t), \tag{4.6}$$

$$\mathbf{g}_j[\mathbf{f}](t) := \left(\mathbf{G}_j[\mathbf{f}] - \mathbf{L}_j[\mathbf{f}] + \mathbf{B}_j[\mathbf{f}]\right)(t), \tag{4.7}$$

that rule the two-components vectors (in bold face as in equations (2.12) and (2.19)) relative to the two families under exam, yield the scheme

$$\mathbf{f}_j^{n+1} = \mathbf{f}_j^n + \Delta t \left(\mathbf{G}_j^n - \mathbf{L}_j^n + \mathbf{B}_j^n\right)[\mathbf{f}] - \frac{\Delta t}{\Delta u} \, \mathbf{H}_j^n[\mathbf{f}] \tag{4.8}$$

that has been used in the simulations. The results have also been tested, and negligible if any differences have been observed, with those obtained by means of the following more refined scheme. First, the equation

$$\mathbf{f}_j^* + \frac{\Delta t}{\Delta u} \, \theta \mathbf{H}_j^n[\mathbf{f}^*] = \mathbf{f}_j^n + \Delta t \left(\mathbf{G}_j - \mathbf{L}_j + \mathbf{B}_j\right)[\mathbf{f}^n] - \frac{\Delta t}{\Delta u} \, (1 - \theta)\mathbf{H}_j^n[\mathbf{f}^n], \tag{4.9}$$

for $\theta \in [0, +1]$, is used to compute

$$\mathbf{f}_j^* =: \Xi_j[\mathbf{f}^n], \tag{4.10}$$

then the advanced values are found by

$$\mathbf{f}_j^{n+1} = \frac{1}{2} \, \left(\mathbf{f}_j^n + \Xi_j \left[\Xi[\mathbf{f}^n]\right]\right). \tag{4.11}$$

4.1. The Base Model Simulations

Simulations of the model described by equation (2.12) are presented here; the terms therein contained being specified according to the discussion in Section 3.

The first results obviously concern the easiest case, namely the evolution of the initial conditions given by equations (4.3) and (4.4), see Figure 2, when *no external actions* and *no party changes* are allowed but *only state transitions* due to internal personal interactions between individuals. In this case, the system shows stable asymptotic equilibrium distributions. Apart from the different values of N_i, which in this case are (obviously) separately constant, the shape of both the two asymptotic distributions is clearly even with respect to the state variable u and, hence, centered around the disinterested-ones value $u = 0$. This behaviour is shown in Figure 3 and it is exhibited by all three experiments produced by the different frequency functions discussed above, which in the following will be respectively denoted by η_d, η_0, η_c (see equations (3.9), (3.10), (3.12)) to depict short range, null, or global chances of interactions. Numerical values: $\tilde{e}_1 = \tilde{e}_2 = 0.2$, $\varepsilon = 0.2$, $d = 0.8$, $c = 10.0$.

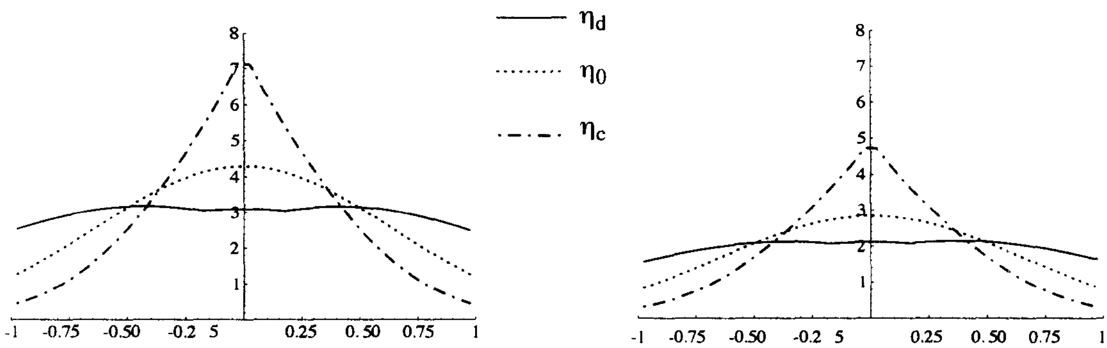

Figure 3.

It is of interest to point out from Figure 3 how the shapes of the undriven uninvolved asymptotic distributions depend on the frequency characters. When people of very different status are frequently allowed to interact, and be concerned in the politics game, as time goes by, they slowly reduce their density and merge to increase the central value of the population. On the contrary, if they are not involved and only restricted interactions happen on a class selective basis, then the border individuals remain numerous and even increase. The free unbiased interactions are clearly midway. The base model also shows that this behaviour does not depend on the party power. Quite heuristically, these base-model first results may be summarized as follows. In the absence of mass media, when no party changes are allowed but only satisfaction transitions, a stable centralized uninterested society develops, more (or less) spread out depending on how short (or long) the connection distances are. Moreover, the transient times that are needed to reach such stable asymptotic distributions directly depend on the interaction procedures, and increase with increasing the mixing among differently satisfied people.

The second set of results to be discussed concerns the model: *no external actions* although *with (unmodulated) party changes*. As mentioned above, this base model lacks fully-developed controls and it is able to describe only simple societies. In particular, family changes are assumed here to happen only on a probabilistic basis, with ground levels given by $p_0^{(N)}$ and $p_0^{(U)}$. In Figure 4, simulations are shown for small values of these ground probabilities, i.e., $p_0^{(N)} = 0.1$ and $p_0^{(U)} = 0.1$.

The results shown in Figure 5 refer to higher ground values: $p_0^{(N)} = 0.5$ and $p_0^{(U)} = 0.5$.

As it may be seen, they are almost identical to those of Figure 4, at least as long as only the asymptotic behaviours are concerned. The main differences reside in the length of the transient times needed to reach the stable asymptotics, which are much longer for the smaller ground levels

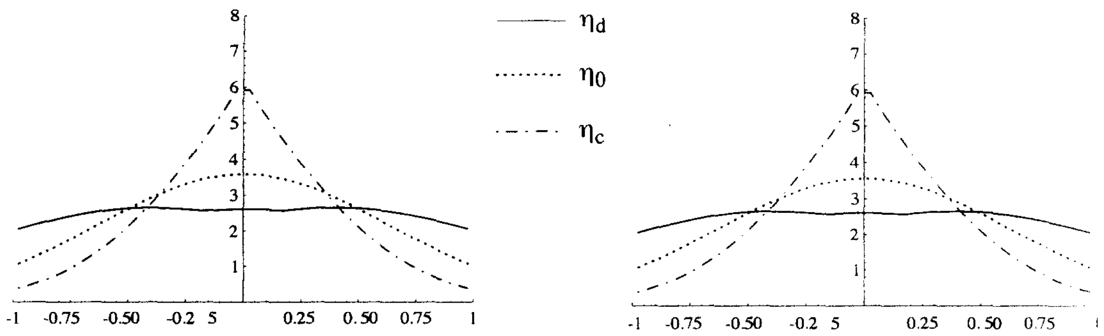

Figure 4.

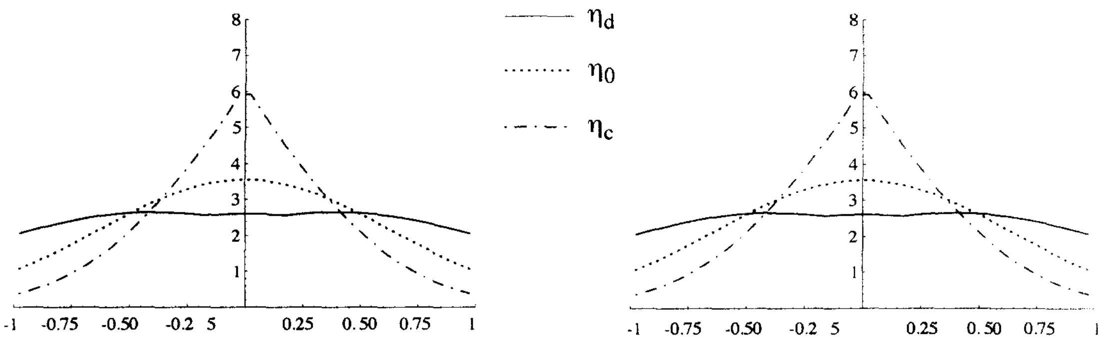

Figure 5.

($t > 100$ unit times, if an integration time of $\Delta t = 0.005$ is used) than they are for the bigger ground levels ($t > 15$ unit times).

Unlike the unmixed case of Figure 3, now the total number of adherent to each party is not a constant any more. On the contrary, the two parties converge to equally share the population without reference to their initial powers N_i^0, and the two asymptotic distributions become identical. Except for this fact, all the comments done for the preceding case hold in the present one as well: a simple society as the one described by this model converges to a completely centralized, stable, balanced configuration; the possibility of changing side acts in the sense of even enhancing this equilibrium. As already remarked, the higher the probability of changing side, the sooner the equilibrium is reached; yet, apart from this, the asymptotic behaviours reached by the two mentioned ground levels are the same.

4.2. The Extended Model Simulations

The discussion of the extended model results will be developed by first describing the effects of the global mean variables in a set of intermediate cases that precede the last, complete one.

A first intermediate picture is represented in Figure 6, where the *externally driven* and *internally free* case, i.e., when no interpersonal interaction terms are allowed. $\eta_{j,h} = 0$, is compared with the *externally driven* and *conservative* cases, i.e., when no party changes are considered: $\tilde{p}_{i,j} = 0$ for all $i \neq j$ in the three experiments $\eta = \eta_d,\ \eta_0,\ \eta_c$.

As it may be seen, interactions at a personal level only act in favour of mixing the population, so that the distribution curves result smoothed out. However, the details of their ultimate shapes depend on the control values, which have been chosen as follows:

$$\begin{aligned} \tilde{a} &= 0.5, & \tilde{c} &= 1.0, & c^{(U)} &= 1.0, \\ \tilde{b} &= 1.0, & \tilde{d} &= 2.0, & c^{(N)} &= 1.0. \end{aligned} \tag{4.12}$$

With this choice, the central and mean values of the personal trend of the people adhering to the dominant party ($F > 0$) are positive ($a = +0.5$), whereas they are negative for the weaker party

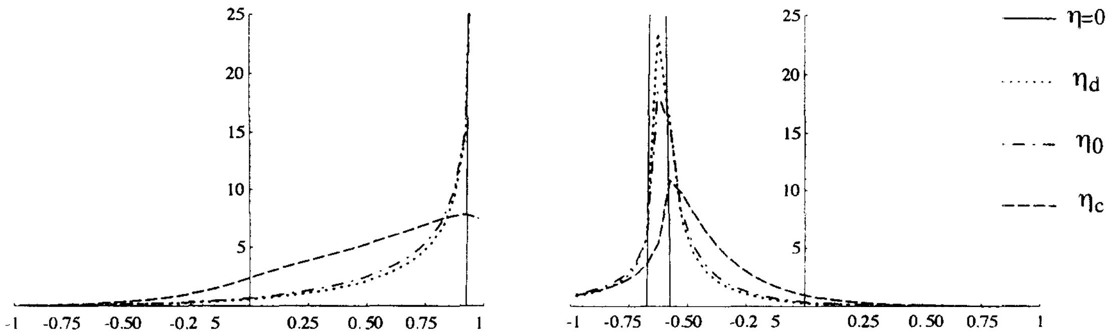

Figure 6.

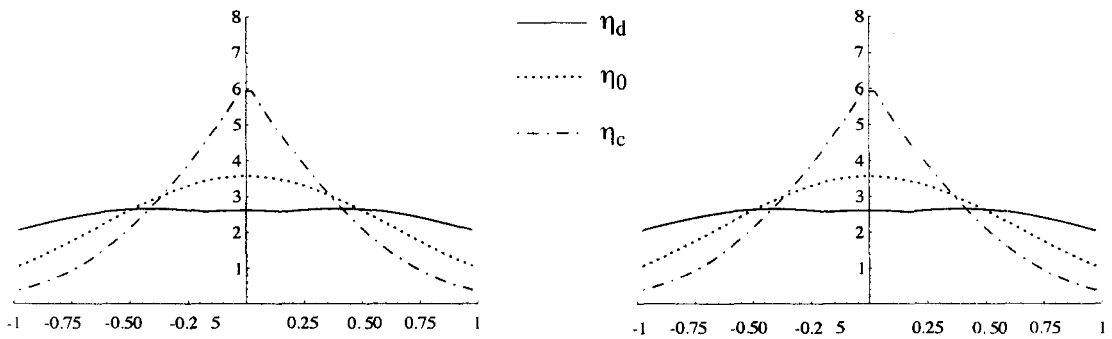

Figure 7.

($a = -1.5$). As well, the linear trend term is positive ($b = +1.0$) if the average satisfaction of the party is positive, and it is negative otherwise ($b = -3.0$).

In Figure 6, the more cohesive, aggregate behaviour of the weaker party may be recognized, together with a spreading trend towards the highest satisfaction levels of the dominant one. As in the case depicted by Figure 3, here the total number of votes of each party is separately constant.

As a second intermediate picture, the case of a system with *no external actions* although with *full change-of-party procedure* is discussed, i.e., party changes happen not only on a probabilistic ground basis but also because of nonzero tuning values $p^{(N)}, p^{(U)}$ triggered by the global variables N and U as described in equation (3.20). Simulations have been done starting from the same ground levels used above, i.e., $p_0^{(N)} = p_0^{(U)} = 0.1$, $p_0^{(N)} = p_0^{(U)} = 0.5$, and for various tuning coefficients $p^{(N)} = p^{(U)}$. Apart from the length of the transient time, the system asymptotic behaviour is directly driven by the ratio of parameter $p^{(N)}$ with respect to the ground level $p_0^{(N)}$, and it only secondarily depends on the other two controls $p^{(U)}$ and $p_0^{(U)}$. Therefore, for convenience, these last have been set respectively equal to the former ones.

When $p^{(N)}$ is sufficiently small with respect to $p_0^{(N)}$ (depending on which one of the η-experiments is considered) the trend is comparable to the one depicted in Figure 4. On the contrary, for higher values of $p^{(N)}$, the power of the dominant party plays its role and the weaker party extinguishes. Figure 7 shows the asymptotic behaviour of both the undriven families in the case $p_0^{(N)} = p_0^{(U)} = 0.1$ and $p^{(N)} = p^{(U)} = 0.1$.

Figure 8 depicts the asymptotic behaviour of (only) the dominant party in the case $p_0^{(N)} = p_0^{(U)} = 0.1$, and now $p^{(N)} = p^{(U)} = 0.2$; the second graph being omitted because the population of the weaker party decreases more and more with the time, and becomes negligibly small after a transient time, which in these cases is appreciably long ($t > 200$ time units).

In total agreement with these two results are those obtained, although with much lower transient times ($t > 15$ time units), when the ground levels $p_0^{(N)} = p_0^{(U)}$ are raised to 0.5 and, respectively, the value 0.2 (or lower) and the value of 0.9 (or higher) are attributed to $p^{(N)} = p^{(U)}$.

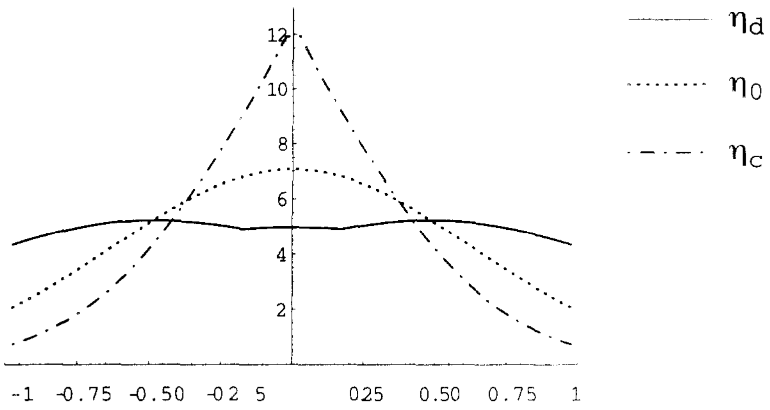

Figure 8.

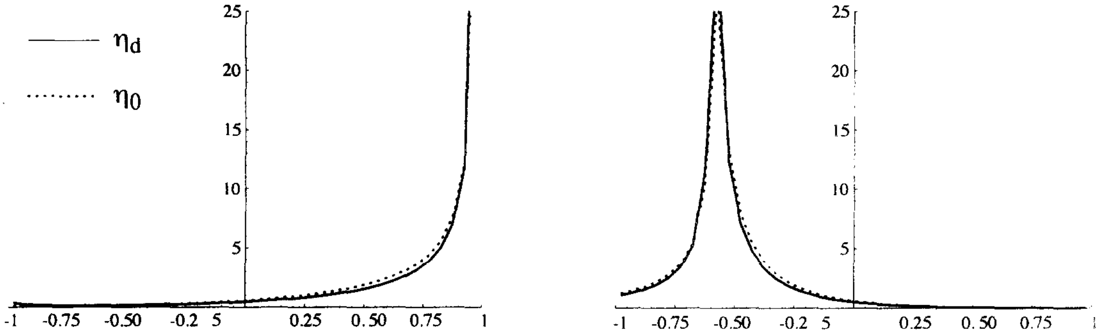

Figure 9.

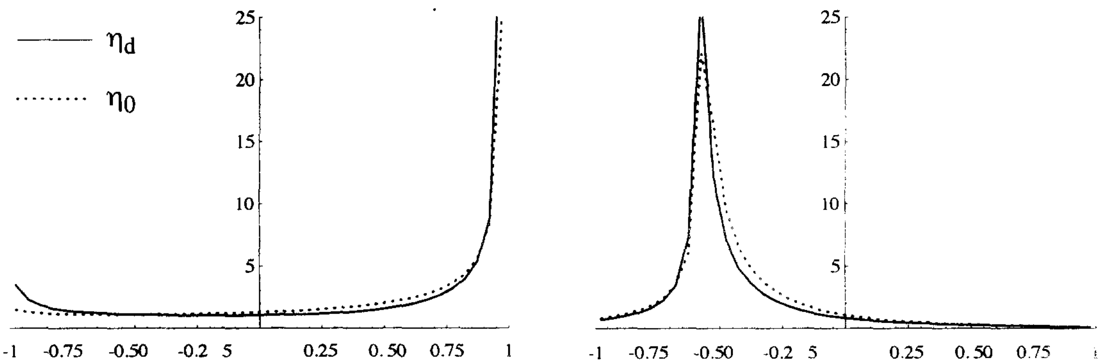

Figure 10.

A third intermediate picture refers to an *externally driven* case although *with (unmodulated) party changes*, i.e., the case which stems from Figure 6 when the change of party is allowed only on a simple probabilistic basis: nonzero $p_0^{(N)} = p_0^{(U)}$ yet $p^{(N)} = p^{(U)} = 0$. Figure 9 depicts the behaviour given by $p_0^{(N)} = p_0^{(U)} = 0.1$ for times greater than 400 time units, whereas Figure 10 represents the case $p_0^{(N)} = p_0^{(U)} = 0.5$, for times greater than 15 time units.

Surprisingly enough, in both these cases, the behaviour shown by the third experiment, η_d is drastically different from those followed by the former two experiments and that is depicted in Figures 9 and 10. In fact, already from Figure 6, it is apparent that the mixing effect of the nonnull long range interaction frequencies plays a fundamental role on the dynamics. The distribution function of the dominant party is no longer peaked at the higher border, $u = 1$. On the contrary, its border value is much lower than those reached in the other two experiments. This same

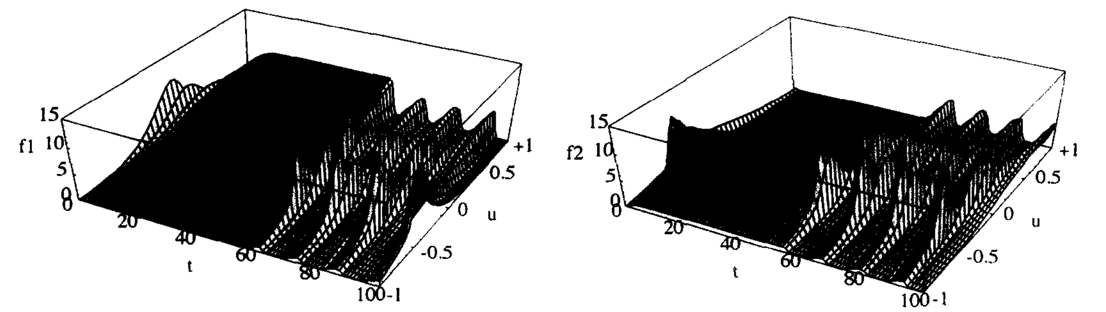

Figure 11.

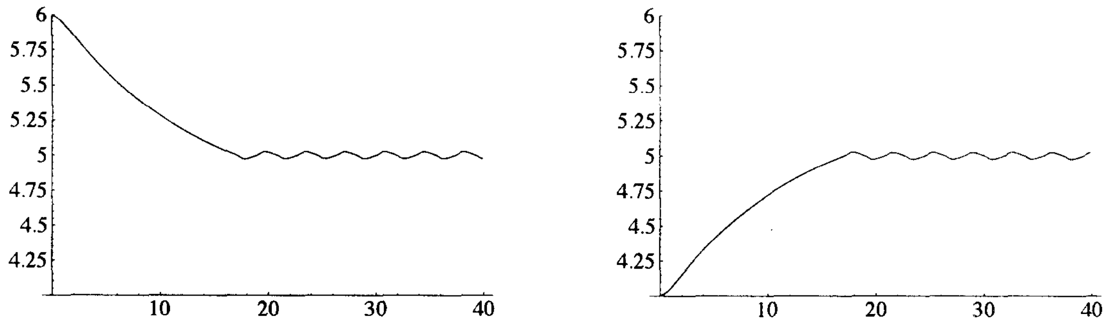

Figure 12.

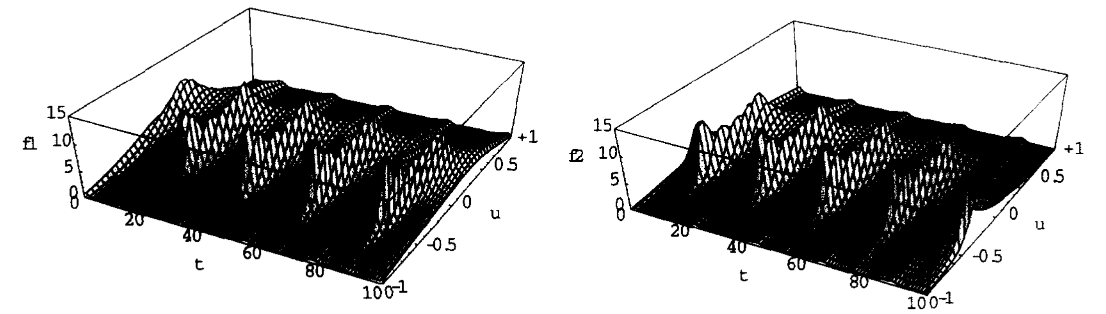

Figure 13.

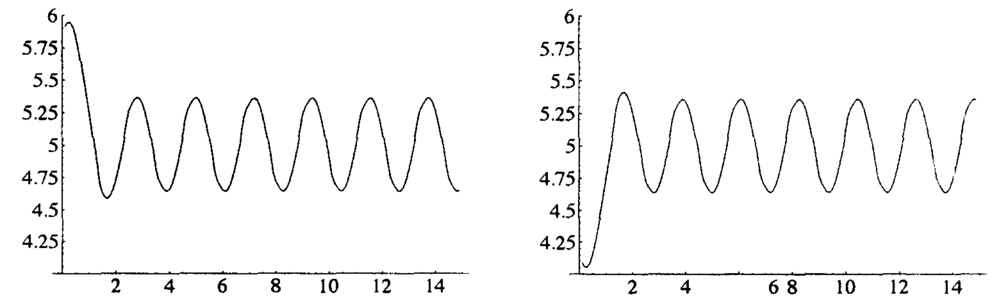

Figure 14.

peculiarity is responsible for the unexpected and completely different behaviour of Experiment 3; a difference which already appears for $p^{(N)} = p^{(U)} = 0$ provided nonnull values are attributed to $p_0^{(N)}$ and $p_0^{(U)}$. Indeed, after a transient time which depends on the ground levels $p_0^{(N)}$ and $p_0^{(U)}$, the system initiates and stably maintains an autonomous oscillatory behaviour, as may be seen in Figures 11 and 12. The first one depicts the 3D plots of the distribution functions of the two parties versus time t and state variable u, and the second one shows the total number of the individuals adhering to each of the two parties versus time, in the case $p_0^{(N)} = p_0^{(U)} = 0.1$ and $p^{(N)} = p^{(U)} = 0$.

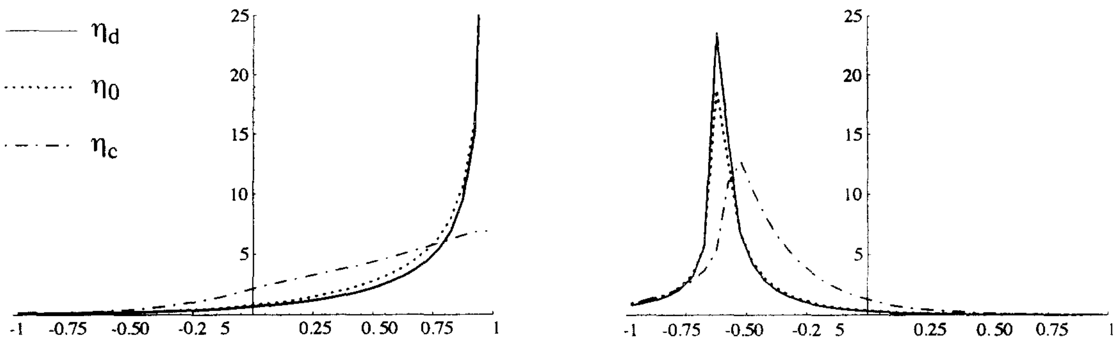

Figure 15.

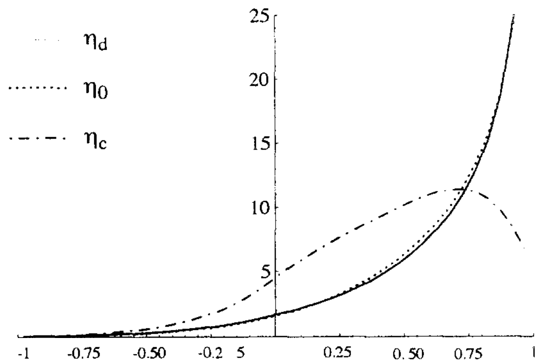

Figure 16.

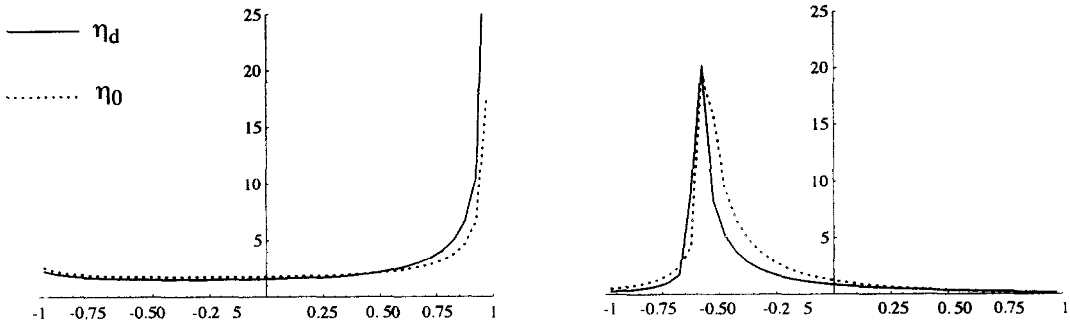

Figure 17.

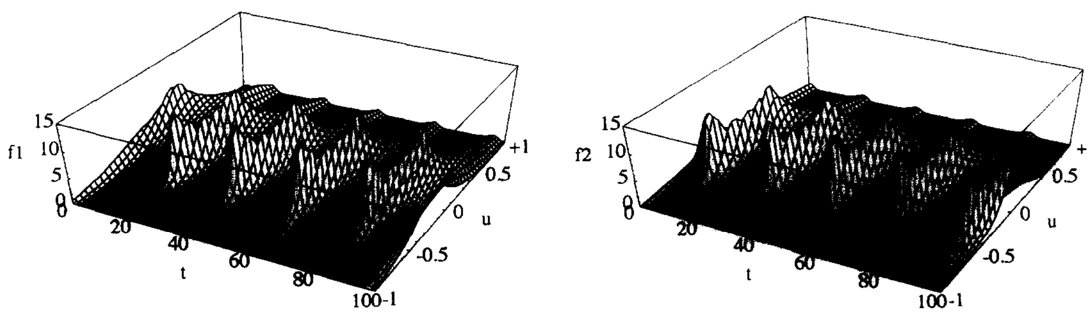

Figure 18.

Quite similar behaviour is obtained, and analogous plots shown in Figures 13 and 14, in the case $p_0^{(N)} = p_0^{(U)} = 0.5$ and $p^{(N)} = p^{(U)} = 0$, the main differences with the previous case residing in the length of the transient time.

Finally, the complete picture of the *externally driven* with *full change-of-party procedure* may now be presented.

When the modulation due to the nonzero values of the controls $p^{(N)}$ and $p^{(U)}$ is added to the system, the various behaviours shown in the case $p^{(N)} = 0$ are confirmed as long as coefficients $p^{(N)}$ are sufficiently small with respect to the ground levels $p_0^{(N)}$. On the other hand, when the modulating factors are greater than the ground levels, then results similar to those depicted in Figure 6 are obtained. In particular, Figure 15 shows that equality between the modulation and the ground level does not initiate the oscillatory status if the ground level is small: $p_0^{(N)} = p_0^{(U)} = p^{(N)} = p^{(U)} = 0.1$, and a picture like Figure 6 is reached after a transient time of 200 time units.

As well, Figure 16 shows that the same qualitative pictures are obtained (and for only the dominant family) when $p_0^{(N)} = p_0^{(U)} = 0.1$ and $p^{(N)} = p^{(U)} = 0.5$; the loser party again becoming negligibly small.

On the contrary, small ratios or even equality between the modulating and ground levels are sufficient to sustain oscillations if the ground level is big: $p_0^{(N)} = p_0^{(U)} = p^{(N)} = p^{(U)} = 0.5$, as is shown in Figures 17 and 18.

5. CRITICAL ANALYSIS AND PERSPECTIVES

Some models of social interaction and competition have been proposed in this paper. All the models considered are such that the total number of individuals in the population is constant with time. On the contrary, the individual membership to the families which partition the population may change, and the individual state be modified, by mass effects and pair interactions. Belonging to a certain party modifies the microscopic way of interacting. The individual state is identified by a variable related to the social satisfaction. The description of the system is acquired by the statistical distributions over such a state.

Together with pairwise interactions, global actions are also considered, which take into account the effects of the external world over the system on both a collective and an individual basis. Changes of parties happen as individual events, but are influenced by the membership to a certain party via the mass variables connected with it. The models define the time evolution of the statistical distributions by means of a suitable set of nonlinear integro-differential equations.

We do not naively claim that the class of mathematical models proposed in this paper can effectively describe the complex variety of the phenomena related to a physical system such as the one here considered. On the other hand, the various experiments proposed in Section 4 show that several interesting behaviours can effectively be described by the model.

Consequently, a critical analysis will conclude this paper, which is addressed to indicate research perspectives, some of them already the object of active speculation. In detail, we have the following.

(i) If the mathematical structure of the evolution model is modified, then it should be enlarged to include the description of additional phenomena. For instance, one may consider models with time structure and/or memory terms.

(ii) A zeroth-family of the *abstention-party*, subject to rules totally different from those of the other parties, should be considered to complete a proper validation of the various terms.

(iii) A set of state variables should be introduced to allow direct interactions among the various parties and to justify a detailed party-dynamics and competition.

(iv) A suitable complexity analysis should be developed in order to properly understand the limits of pair interactions. For instance, the same individual can interact with several other people. As a matter of fact, individuals are not particles with short range interactions. Indeed, long range interactions may involve more than two subjects and, hence, considerably increase the complexity of the mathematical problem.

(v) The base model is able to represent only very simple societies, where only individual connections play a role, and consequently, the final picture is sufficiently poor. When mass interactions are added, yet a complete and uncritic support is maintained by each individual to his own party, a set of unbalanced distributions is obtained (see Figure 6) which rewards the dominant party supporters and only them. When the change-of-party procedure is dominated by the power, then the loser party necessarily extinguishes (see Figure 8). On the other hand, a live and balanced dynamics may be obtained the more the individuals of different status are allowed to interact and be involved. Ground levels are necessary to trigger the dynamics and avoid overly unbalanced distributions. Transient times are reduced by conveniently high mixing procedures.

(vi) The mathematical structure can further be developed in order to deal with a variable number of individuals. This means including influx and out flux conditions and possibly related interactions.

(vii) The general mathematical model may include the description of the dynamics in the physical space. This means that individuals may move in space and interactions modify their motion. For instance, stochastic interaction models, such as those in [27,28], can be considered. Of course, the space structure may not be relevant for a class of systems such as the one dealt with in this paper, but it may be of importance for relatively less complex societies as those in the paper by Jager and Segel [29].

The above indications should be regarded as samples of several conceivable developments. In all cases, realistic applications of models generally originate interesting and challenging related mathematical problems, e.g., the well posedness of the evolution problem, or the existence and stability of equilibrium or stationary solutions. Some of them are already available in the pertinent literature, an account is given in [1, Chapter 3]. On the other hand, various problems are still open, for instance, qualitative analysis towards asymptotic behaviour and stability properties, and are currently under investigation.

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