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Local chromatic number and Sperner capacity

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Abstract

We introduce a directed analog of the local chromatic number defined by Erdős et al. [Discrete Math. 59 (1986) 21–34] and show that it provides an upper bound for the Sperner capacity of a directed graph. Applications and variants of this result are presented. In particular, we find a special orientation of an odd cycle and show that it achieves the maximum of Sperner capacity among the differently oriented versions of the cycle. We show that apart from this orientation, for all the others an odd cycle has the same Sperner capacity as a single edge graph. We also show that the (undirected) local chromatic number is bounded from below by the fractional chromatic number while for power graphs the two invariants have the same exponential asymptotics (under the co-normal product on which the definition of Sperner capacity is based). We strengthen our bound on Sperner capacity by introducing a fractional relaxation of our directed variant of the local chromatic number.

Keywords: Local chromatic number; Sperner capacity; Shannon capacity; Fractional chromatic number

1. Introduction

Colouring the vertices of a graph so that no adjacent vertices receive identical colours gives rise to many interesting problems and invariants, of which the book [17] gives an excellent survey. The best known among all these invariants is the chromatic number, the minimum number of colours needed for such proper colourings. An interesting variant was

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introduced by Erdős et al. [10] (cf. also [12]). They define the local chromatic number of a graph as follows.

Definition 1 (*Erdős et al. [10]*). The local chromatic number $\psi(G)$ of a graph G is the maximum number of different colours appearing in the closed neighbourhood of any vertex, minimized over all proper colourings of G. Formally,

$$\psi(G) := \min_{c\,:\,V(G)\to\mathbb{N}} \max_{v\in V(G)} |\{c(u) : u \in \Gamma_G(v)\}|,$$

where $\mathbb{N}$ is the set of natural numbers, $\Gamma_G(v)$, the closed neighborhood of the vertex $v \in V(G)$, is the set of those vertices of G that are either adjacent or equal to v and $c : V(G) \to \mathbb{N}$ runs over those functions that are proper colourings of G.

It is clear that $\psi(G)$ is always bounded from above by the chromatic number, $\chi(G)$. It is much less obvious that $\psi(G)$ can be strictly less than $\chi(G)$. Yet this is true; in fact, as proved in [10], there exist graphs with $\psi(G) = 3$ and $\chi(G)$ arbitrarily large.

Throughout this paper, we shall be interested in chromatic invariants as upper bounds for the Shannon capacity of undirected graphs and its natural generalization, Sperner capacity, for directed graphs. For the sake of unity in the treatment of undirected and directed graphs it is convenient and customary to treat Shannon capacity in terms that are complementary to Shannon's own, (cf. [24,20,14,18]). In this language Shannon capacity describes the asymptotic growth of the clique number in the co-normal powers of a graph. Shannon proved (although in different terms) that the Shannon capacity $c(G)$ of a graph is bounded from above by its fractional chromatic number.

We show that $\psi(G)$ is bounded from below by the fractional chromatic number of G. This proves, among other things, that $\psi(G)$ is always an upper bound for the Shannon capacity $c(G)$ of G, but it is not a very useful upper bound since it is always weaker than the fractional chromatic number itself. We make this seemingly useless remark only to stress that the situation is rather different in the case of directed graphs.

We introduce an analog of the local chromatic number for directed graphs and show that it is always an upper bound for the Sperner capacity of the digraph at hand. The proof is linear algebraic and generalizes an idea already used for bounding Sperner capacity in [6,1,11], cf. also [8]. To illustrate the usefulness of this bound we apply it to show, for example, that an oriented odd cycle with at least two vertices with outdegree and indegree 1 always has its Sperner capacity equal to that of the single-edge graph K_2. We also discuss fractional versions that further strengthen our bounds.

2. Local chromatic number for directed graphs

The definition of the directed version of $\psi(G)$ is straightforward.

Definition 2. The local chromatic number $\psi_d(G)$ of a digraph G is the maximum number of different colours appearing in the closed out-neighbourhood of any vertex, minimized over all proper colourings of G. Formally,

$$\psi_d(G) := \min_{c\,:\,V(G)\to\mathbb{N}} \max_{v\in V(G)} |\{c(w) : w \in \Gamma_G^+(v)\}|$$

where $\mathbb{N}$ is the set of natural numbers, $\Gamma_G^+(v)$, the closed out-neighborhood of the vertex $v \in V(G)$, is the set of those vertices $w \in V(G)$ that are either equal to v or else are endpoints of directed edges $(v, w) \in E(G)$, originated in v, and $c : V(G) \to \mathbb{N}$ runs over those functions that are proper colourings of G.

Our main goal is to prove that $\psi_d(G)$ is an upper bound for the Sperner capacity of digraph G.

3. Sperner capacity

Definition 3. For directed graphs $G = (V, E)$ and $H = (W, L)$, the co-normal (or disjunctive or OR) product $G \cdot H$ is defined to be the following directed graph:

$$V(G \cdot H) = V \times W$$

and

$$E(G \cdot H) = \{((v, w), (v', w')) : (v, v') \in E \quad \text{or} \quad (w, w') \in L\}.$$

The nth co-normal (or disjunctive or OR) power G^n of digraph G is defined as the n-fold co-normal product of G with itself, i.e., the vertex set of G^n is $V^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i \in V\}$, while its edge set is defined as

$$E(G^n) = \{(\boldsymbol{x}, \boldsymbol{y}) : \exists i \ (x_i, y_i) \in E(G)\}.$$

(An edge (a, b) always means an oriented edge in this paper as opposed to undirected edges denoted by $\{a, b\}$.)

Notice that G^n may contain edges in both directions between two vertices even if such a pair of edges is not present in G.

Definition 4. A subgraph of a digraph is called a symmetric clique if its edge set contains all ordered pairs of vertices belonging to the subgraph. (In other words, it is a clique with all its edges present in both directions.) For a directed graph G we denote the order (number of vertices) of its largest symmetric clique by $\omega_s(G)$.

Definition 5 (*Gargano et al. [14]*). The (non-logarithmic) Sperner capacity of a digraph G is defined as

$$\sigma(G) = \sup_n \sqrt[n]{\omega_s(G^n)}.$$

Remark. Denoting the number of vertices in a largest transitive clique of G by $tr(G)$, it is easy to show that $\sigma(G) = \sup_n \sqrt[n]{tr(G^n)}$ holds, cf. [14,22] and the references therein. (By a transitive clique we mean a clique where the edges are oriented transitively, i.e., consistently with some linear order of the vertices. It is allowed that some edges be present also in the reverse direction.) Since $tr(G^n) \geqslant [tr(G)]^n$ this remark implies that $tr(G) \leqslant \sigma(G)$ holds for any digraph G.

For an undirected graph G let us call the digraph we obtain from G by directing all its edges in both ways the symmetrically directed equivalent of G. In Shannon's own language the capacity (cf. [24]) of the complement of an (undirected) graph G can be defined as the Sperner capacity of its symmetrically directed equivalent. We denote this quantity by $c(G)$ and by slight abuse of the terminology we also refer to it as the Shannon capacity of G whenever it may not cause confusion.

Thus Sperner capacity is a generalization of Shannon capacity. It is a true generalization in the sense that there exist digraphs the Sperner capacity of which is different from the Shannon capacity ($c(G)$ value) of its underlying undirected graph. Denoting by G both an arbitrary digraph and its underlying undirected graph, it follows from the definitions that $\sigma(G) \leqslant c(G)$ always holds. The smallest example with strict inequality in the previous relation is a cyclically oriented triangle, cf. [8,6]. (See also [5] for an early and different attempt to generalize Shannon capacity to directed graphs.)

Shannon capacity is known to be a graph invariant that is difficult to determine (not only in the algorithmic but in any sense), and it is unknown for many relatively small and simple graphs, for example, for all odd cycles of length at least 7. This already shows that the more general invariant Sperner capacity cannot be easy to determine either. For a survey on graph invariants defined via powers, including Shannon and Sperner capacities, we refer the reader to [3]. There is an interesting and important connection between Sperner capacity and extremal set theory, introduced in [19] and fully explored in [15]. Several problems of this flavour are also discussed in [18].

4. Main result

Alon [1] proved that for any digraph G

$$\sigma(G) \leqslant \min\{\Delta_+(G), \Delta_-(G)\} + 1,$$

where $\Delta_+(G)$ is the maximum out-degree of the graph G and similarly $\Delta_-(G)$ is the maximum in-degree. The proof relies on a linear algebraic method similar to the one already used in [6] for a special case (cf. also [11] for a strengthening and cf. [2] for a general setup for this method in case of undirected graphs). We also use this method for proving the following stronger result:

Theorem 1.

$$\sigma(G) \leqslant \psi_d(G).$$

Proof. Consider a proper colouring $c : V(G) \to \mathbb{N}$ that achieves the value of $\psi_d(G)$. Let $N_c^+(v)$ denote the set of colours each of which appears as the colour of some vertex in the (open) out-neighbourhood of v in the colouring c.

For each vertex $\boldsymbol{a} = (a_1, \dots, a_n) \in V(G^n)$ we define a polynomial

$$P_{\boldsymbol{a},c}(x_1, \dots, x_n) := \prod_{i=1}^{n} \prod_{j \in N_c^+(a_i)} (x_i - j).$$

Let K be a symmetric clique in G^n. If $\boldsymbol{a} \in K$, $\boldsymbol{b} \in K$ then by definition $P_{\boldsymbol{a},c}(c(b_1), \dots, c(b_n)) = 0$ if $\boldsymbol{b} \neq \boldsymbol{a}$, while $P_{\boldsymbol{a},c}(c(a_1), \dots, c(a_n)) \neq 0$ by the properness of colouring c. This implies that the polynomials $\{P_{\boldsymbol{a},c}(x_1, \dots, x_n)\}_{\boldsymbol{a} \in K}$ are linearly independent over the reals. This can be shown in the usual way: substituting $c(\boldsymbol{b})$ into $\sum_{\boldsymbol{a} \in K} \lambda_{\boldsymbol{a}} P_{\boldsymbol{a},c}(\boldsymbol{x}) = 0$ we obtain $\lambda_{\boldsymbol{b}} = 0$ and this can be done for each $\boldsymbol{b} \in K$.

Since the degree of x_i in $P_{\boldsymbol{a},c}(\boldsymbol{x})$ is at most $\psi_d(G) - 1$, the dimension of the linear space generated by our polynomials is bounded from above by $[\psi_d(G)]^n$. By the previous paragraph, this is also an upper bound for $|K|$. Choosing K to be a symmetric clique of maximum size we obtain $\omega_s(G^n) \leqslant [\psi_d(G)]^n$ and thus the statement. □

Let G_{rev} denote the "reverse of G", i.e., the digraph we obtain from G by reversing the direction of all of its edges. Since obviously $\sigma(G) = \sigma(G_{\mathrm{rev}})$, Theorem 1 has the following trivial corollary:

Corollary 1.

$$\sigma(G) \leqslant \min\{\psi_d(G), \psi_d(G_{\mathrm{rev}})\}.$$

In Sections 7 and 8 we will strengthen Theorem 1 by introducing a fractional version of $\psi_d(G)$.

5. Application: odd cycles

We call an oriented cycle alternating if it has at most one vertex of outdegree 1. (In stating the following results we follow the convention that an oriented graph is a graph without oppositely directed edges between the same two points, while a general directed graph may contain such pairs of edges.) Clearly, in any oriented cycle the number of vertices of outdegree 2 equals the number of vertices of outdegree 0. Thus, in particular, an oriented odd cycle of length $2k+1$ is alternating if it has k points of outdegree zero, k points of outdegree 2 and only 1 point of outdegree 1. It takes an easy checking that up to isomorphism there is only one orientation of C_{2k+1} which is alternating.

Theorem 2. *Let G be an oriented odd cycle that is not alternating. Then*

$$\sigma(G) = 2.$$

Proof. Since any digraph with at least one edge has Sperner capacity at least 2 (see the Remark after Definition 5), it is enough to prove that 2 is also an upper bound.

Colour the vertices of G so that two points receive the same colour if and only if they have a common in-neighbour, i.e., a vertex sending an oriented edge to both of them. It is easy to check that this colouring is proper if and only if the odd cycle G is not alternating. In this case, our colouring also has the property that any vertex has only one colour in its out-neigbourhood proving $\psi_d(G) = 2$. Then the statement follows by Theorem 1. $\square$

Remark. It is easy to see that the following slightly stronger version of the previous theorem can be proven similarly: If G is a directed odd cycle not containing an alternating odd cycle, then $\sigma(G) = 2$.

The Sperner capacity of an alternating odd cycle can indeed be larger than 2. This is obvious for C_3, where the alternating orientation produces a transitive clique of size 3. A construction proving that the Sperner capacity of the alternating C_5 is at least $\sqrt{5}$ is given in [13], and this is further analyzed in [22]. The construction is clearly best possible by the celebrated result of Lovász [20] showing $c(C_5) = \sqrt{5}$.

In [22] the invariant $D(G) = \max \sigma(G)$ was defined where the maximization is over all orientations of G. It follows from the definitions that $D(G) \leqslant c(G)$, and it is asked in [22] whether one always has equality. No counterexample is known, while equality is trivial if $\chi(G) = \omega(G)$ (just orient a maximum size clique transitively) and it is proven for vertex-transitive self-complementary graphs in [22]. Denoting the alternatingly oriented C_{2k+1} by C_{2k+1}^{alt} Theorem 2 has the following immediate corollary:

Corollary 2.

$$D(C_{2k+1}) = \sigma(C_{2k+1}^{\mathrm{alt}})$$

holds for every positive integer k.

The discussion in this section becomes more relevant in the light of a recent result by Bohman and Holzman [7]. Until recently it was not known whether the Shannon capacity (in our complementary sense) of the odd cycle C_{2k+1}, i.e., $c(C_{2k+1})$ is larger than 2 for any value of $k > 2$. In [7] an affirmative answer to this question was given by an ingenious construction, showing that this is always the case, i.e., $c(C_{2k+1}) > 2$ for every positive integer k. This means that the bound provided by $\psi_d(G)$ goes beyond the obvious upper bound $c(G)$ of Sperner capacity in case of non-alternatingly oriented odd cycles, i.e., the following consequence of Theorem 2 can also be formulated:

Corollary 3. *If k is any positive integer and $C_{2k+1}^{\rightarrow}$ is a non-alternatingly oriented C_{2k+1}, then*

$$\sigma(C_{2k+1}^{\rightarrow}) < c(C_{2k+1}).$$

It is a natural idea to try to use the Bohman–Holzman construction for alternatingly oriented odd cycles and check whether the so-obtained sets of vertices inducing cliques in the appropriate power graphs will form transitive cliques in the oriented case. (If the answer were yes it would prove $D(C_{2k+1}) > 2$ for every k strengthening the result $c(C_{2k+1}) > 2$ of [7].) This idea turned out to work in the case of C_7, thus showing $D(C_7) > 2$. (To record this we list the 17 vertices of C_7^4 that form a transitive clique defined by their ordering on this list. The labels of the vertices of C_7 are the first 7 non-negative integers as in [7] and the unique point with outdegree and indegree 1 is the point labelled 5. Here we give the vertices simply as sequences. Thus the list is: 4444, 0520, 2030, 2051, 0605, 1205, 1320, 3006, 3012, 5106, 5112, 0561, 0613, 1213, 6130, 6151, 1361.) Strangely, however, the same construction did not work for C_9: after our unsuccessful attempts to prove a similar statement, Attila Sali wrote a computer program to check whether the clique of Bohman and Holzman in the 8th power of an alternating C_9 contains a transitive clique of the same size and the answer turned out to be negative. (Again, to record more than just this fact, we give six vertices of C_9^8 that form a directed cycle without inversely oriented edges in the clique of Bohman and Holzman whenever the path obtained after deleting vertex 5 of C_9 is oriented alternatingly. The existence of this cycle shows that the Bohman–Holzman clique does not contain a transitive clique whenever the only outdegree 1, indegree 1 point of the alternatingly oriented C_9 is 4 or 6 (or 5, but this case is less important), that is one of the neighbours of 5, the point the construction distinguishes. So the promised cycle is: 20302040, 12072040, 12140720, 40121207, 20401320, 07204012.) In spite of this, we believe that the Sperner capacity of alternating odd cycles will achieve the corresponding Shannon capacity value $c(C_{2k+1})$.

One more remark is in order. It is easy to check that the vertices of C_{2k+3}^{alt} can be mapped to those of C_{2k+1}^{alt} in an edge-preserving manner. This immediately implies that $\sigma(C_{2k+3}^{\mathrm{alt}}) \leqslant \sigma(C_{2k+1}^{\mathrm{alt}})$, i.e., if there were any odd cycle C_{2k+1} with $D(C_{2k+1}) = 2$, then the same must hold for all longer odd cycles as well.

6. The undirected case

Since identifying with any undirected graph G its symmetrically directed equivalent gives both $\sigma(G) = c(G)$ and $\psi_d(G) = \psi(G)$, it is immediate from Theorem 1 that $c(G) \leqslant \psi(G)$. We will show, however, that $\psi(G)$ is always bounded from below by the fractional chromatic number of G, which in turn is a well known upper bound for $c(G)$, cf. [24,20]. Thus, unlike in the directed case, the local chromatic number does not give us new information about Shannon capacity. Looking at it from another perspective, this relation tells us something about the behaviour of the local chromatic number. (For more on this other perspective, see the follow up paper [25].)

One of the main tools in the investigations of the local chromatic number in [10] is the recognition of the relevance of the universal graphs $U(m, k)$ defined as follows. (From now on we will use the notation $[m] = \{1, \ldots, m\}$).

Definition 6 (*Erdős et al. [10]*). Let the graph $U(m, k)$ for positive integers $k \leqslant m$ be defined as follows:

$$V(U(m, k)) := \{(x, A) : x \in [m], A \subseteq [m], |A| = k - 1, x \notin A\}$$

and

$$E(U(m, k)) := \{\{(x, A), (y, B)\} : x \in B, y \in A\}.$$

The relevance of these graphs is expressed by the following lemma. Recall that a homomorphism from some graph F to a graph G is an edge-preserving mapping of $V(F)$ to $V(G)$.

Lemma 1 (*Erdős et al. [10]*). *A graph G admits a proper colouring c with m colours and $\max_{v\in V(G)} |\{c(u) : u \in \Gamma_G(v)\}| \leqslant k$ if and only if there exists a homomorphism of G to $U(m,k)$. In particular, $\psi(G) \leqslant k$ if and only if there exists an m such that G admits a homomorphism to $U(m,k)$.*

We use these graphs to prove the relation between the fractional chromatic number and the local chromatic number.

Recall that the fractional chromatic number is $\chi^*(G) = \min \sum_{A\in S(G)} w(A)$ where $S(G)$ denotes the family of independent sets of graph G and the minimization is over all non-negative weightings $w : S(G) \to \mathbb{R}$ satisfying $\sum_{A\ni x} w(A) \geqslant 1$ for every $x \in V(G)$. It is straightforward from the definition that $\chi^*(G) \geqslant \omega(G)$ holds for any graph G. Another important fact we will use is that if G is vertex-transitive, then $\chi^*(G) = \frac{|V(G)|}{\alpha(G)}$. For a proof of this fact and for further information about the fractional chromatic number we refer to the books [23,16].

Theorem 3. *For any graph G*

$$\psi(G) \geqslant \chi^*(G).$$

The proof relies on the following simple observation:

Lemma 2. *For all $m \geqslant k \geqslant 2$ we have $\chi^*(U(m,k)) = k$.*

Proof. It is easy to check that $\chi^*(U(m,k)) \geqslant \omega(U(m,k)) = k$ thus we only have to prove that k is also an upper bound. It is straightforward from their definition that the graphs $U(m,k)$ are vertex-transitive. (Any permutation of $[m]$ gives an automorphism, and any vertex can be mapped to any other by such a permutation.) Consider those vertices (x, A) for which $x \leqslant a_i$ for all $a_i \in A$. These form an independent set S. Thinking about the vertices (x, A) as k-tuples with one distinguished element and the elements of S as those k-tuples whose distinguished element is the smallest one, we immediately get $\chi^*(U(m,k)) = \frac{|V(U(m,k))|}{\alpha(U(m,k))} \leqslant \frac{|V(U(m,k))|}{|S|} = k$ proving the statement. $\square$

Proof of Theorem 3. Let us have $\psi(G) = k$. This means that there is a homomorphism from G to $U(m,k)$ for some m (cf. Lemma 1). Since a homomorphism cannot decrease the fractional chromatic number, from Lemma 2 we obtain $\chi^*(G) \leqslant \chi^*(U(m,k)) = k = \psi(G)$. $\square$

In the rest of this section we formulate a consequence of Theorem 3 for the asymptotic behaviour of the local chromatic number with respect to the co-normal power of graphs.

It is a well-known theorem of McEliece and Posner [21] (cf. also Berge and Simonovits [4] and, for this particular formulation, [23]) that

$$\lim_{n\to\infty} \sqrt[n]{\chi(G^n)} = \chi^*(G).$$

It is equally well-known (cf., e.g., [23, Corollary 3.4.2]) that $\chi^*(G^n) = [\chi^*(G)]^n$. These two statements and Theorem 3 immediately imply the following:

Corollary 4.

$$\lim_{n\to\infty} \sqrt[n]{\psi(G^n)} = \chi^*(G).$$

Proof. By $\chi^*(G^n) \leqslant \psi(G^n) \leqslant \chi(G^n)$ we have $\chi^*(G) = \sqrt[n]{\chi^*(G^n)} = \lim_{n\to\infty} \sqrt[n]{\chi^*(G^n)} \leqslant \lim_{n\to\infty} \sqrt[n]{\psi(G^n)} \leqslant \lim_{n\to\infty} \sqrt[n]{\chi(G^n)} = \chi^*(G)$ where the last equality is by the McEliece–Posner theorem mentioned above. $\square$

Corollary 4 can be proved also in a different way using the techniques of [9]. This latter kind of proof can be generalized to show a similar statement for $\psi_d(G)$.

7. Fractional colourings

Now we define the fractional version of the local chromatic number. For $v \in V(G)$ let $\Gamma^+_G(v)$ denote, as before, the closed out-neighbourhood of v, i.e., the set containing v and its out-neighbours.

Definition 7. For a digraph G its (directed) fractional local chromatic number $\psi^*_d(G)$ is defined as follows:

$$\psi^*_d(G) := \min_{w} \max_{v\in V(G)} \sum_{\Gamma^+_G(v)\cap A\neq\emptyset} w(A),$$

where the minimization is over all fractional colourings w of G.

The fractional local chromatic number $\psi^*(G)$ of an undirected graph G is just $\psi^*_d(\hat{G})$ where $\hat{G}$ is the symmetrically directed equivalent of G.

An r-fold colouring of a graph G is a colouring of each of its vertices with r distinct colours with the property that the sets of colours assigned to adjacent vertices are disjoint. More formally, an r-fold colouring is a set-valued function $f : V(G) \to \binom{\mathbb{N}}{r}$ satisfying $f(u) \cap f(v) = \emptyset$ whenever $(u, v) \in E(G)$.

Definition 8. Let $\psi_d(G, r)$ denote the r-fold (directed) local chromatic number of digraph G defined as

$$\psi_d(G,r) := \min_{f} \max_{u\in V(G)} \left| \bigcup_{v\in\Gamma^+_G(u)} f(v) \right|,$$

where the minimization is over all r-fold colourings f of G.

The r-fold local chromatic number $\psi(G, r)$ of an undirected graph G is just $\psi_d(\hat{G}, r)$ where $\hat{G}$ is the symmetrically directed equivalent of G.

It is obvious that

$$\psi_d^*(G) = \inf_r \frac{\psi_d(G, r)}{r}$$

for every digraph G. This includes the equality

$$\psi^*(G) = \inf_r \frac{\psi(G, r)}{r}$$

for undirected graphs, too.

For a digraph G let $G[K_r]$ denote the graph obtained by substituting a symmetric clique of size r into each of its vertices. Formally this means

$$V(G[K_r]) = \{(v, i) : v \in V(G), i \in \{1, \dots, r\}\}$$

and

$$E(G[K_r]) = \{((u, i), (v, j)) : (u, v) \in E(G) \text{ or } u = v \text{ and } i \neq j\}.$$

It is easy to see that $\psi_d(G[K_r]) = \psi_d(G, r)$ for every digraph G and positive integer r. It is also not difficult to see that $\omega_s((G[K_r])^n) = r^n \omega_s(G^n)$ for every n. Indeed, any vertex of G^n can be substituted by r^n vertices of $(G[K_r])^n$ in the natural way and a symmetric clique K of G^n becomes a symmetric clique of size $r^n|K|$ in $(G[K_r])^n$ this way proving $\omega_s((G[K_r])^n) \geqslant r^n \omega_s(G^n)$. To see that equality holds let us denote by $\boldsymbol{a}(\boldsymbol{x})$ the unique vertex of G^n from which $\boldsymbol{x} \in (G[K_r])^n$ can be obtained by the previous substitution. (Thus the set $A_{\boldsymbol{x}} := \{\boldsymbol{y} : \boldsymbol{a}(\boldsymbol{y}) = \boldsymbol{a}(\boldsymbol{x})\}$ has r^n elements for every $\boldsymbol{x} \in V((G[K_r])^n)$.) The crucial observation is that if K is a symmetric clique in $(G[K_r])^n$ and $\boldsymbol{x} \in K$, then $K \cup A_{\boldsymbol{x}}$ is still a symmetric clique (it may be identical to K but may also be larger). Thus maximal symmetric cliques of $(G[K_r])^n$ can always be obtained as the union of some sets $A_{\boldsymbol{x}}$, which means that they can be obtained as "blown up" versions of symmetric cliques of G^n. This proves our claim that $\omega_s((G[K_r])^n) = r^n \omega_s(G^n)$. This equality implies $\sigma(G[K_r]) = r\sigma(G)$ for every digraph G and positive integer r.

The observations of the previous paragraph provide the following strengthening of Theorem 1.

Theorem 4. *For every digraph G*

$$\sigma(G) \leqslant \psi_d^*(G)$$

holds.

Proof. By Theorem 1 and the previous observations we have

$$\sigma(G) = \frac{\sigma(G[K_r])}{r} \leqslant \frac{\psi_d(G[K_r])}{r} = \frac{\psi_d(G, r)}{r}.$$

Since this holds for every r we can write

$$\sigma(G) \leqslant \inf_r \frac{\psi_d(G, r)}{r} = \psi_d^*(G). \qquad \square$$

We can formulate again the following trivial corollary:

Corollary 5.

$$\sigma(G) \leqslant \min\{\psi_d^*(G), \psi_d^*(G_{\text{rev}})\}.$$

To illustrate the usefulness of Theorem 4 we consider the complement of a 7-cycle with its only orientation in which all triangles are oriented cyclically. We denote this graph by D_7 (abbreviating double 7-cycle). None of the earlier bounds we know give a better upper bound for the Sperner capacity of D_7 than 3. Now we can improve on this.

Proposition 1.

$$\sqrt{5} \leqslant \sigma(D_7) \leqslant \frac{5}{2}.$$

Proof. The lower bound follows by observing that D_7 contains an alternating 5-cycle. The upper bound is a consequence of Theorem 4 since $\psi_d^*(D_7) = \frac{5}{2}$. We actually need here only $\psi_d^*(D_7) \leqslant \frac{5}{2}$ and this can be seen by giving weight $\frac{1}{2}$ to each 2-element stable set of D_7. □

This example can be further generalized as follows. Let D_{2k+1} denote the following oriented graph:

$$V(D_{2k+1}) = \{0, 1, \ldots, 2k\}$$

and

$$E(D_{2k+1}) = \{(u, v) : v \equiv u + j \ (\text{mod}\ 2k + 1), \quad j \in \{2, 3, \ldots, k\}\}.$$

Observe that this definition is consistent with the earlier definition of D_7 and that the underlying undirected graph of D_{2k+1} is the complement of the odd cycle C_{2k+1}. Now we can state the following:

Proposition 2.

$$\left\lceil \frac{k-1}{2} \right\rceil + 1 \leqslant \sigma(D_{2k+1}) \leqslant \frac{k}{2} + 1.$$

In particular, $\sigma(D_{2k+1}) = \frac{k}{2} + 1$ *if* k *is even.*

Proof. It is easy to verify for the transitive clique number that $tr(D_{2k+1}) = \lceil \frac{k-1}{2} \rceil + 1$ and this gives the lower bound. The upper bound is proven by assigning weight $\frac{1}{2}$ to every 2-element independent set of D_{2k+1} which clearly gives a fractional colouring. The weight thus assigned to any closed out-neighbourhood is $\frac{k}{2} + 1$ giving the upper bound by Theorem 4.

If k is even, the two bounds coincide. □

We remark that while the upper bound in Proposition 2 generalizes that of Proposition 1, the lower bound does not; it is weaker in case $k = 3$ than that of Proposition 1. Therefore we consider the oriented graph D_7 a particularly interesting instance of the problem.

As it was the case without fractionalization, Theorem 4 does not give us new information in the undirected case, i.e., about Shannon capacity. The reason for this is the following relation:

Theorem 5. *Let G be an undirected graph. Then*

$$\psi^*(G) = \chi^*(G).$$

To prove Theorem 5 we need the following generalization of the universal graphs $U(m, k)$:

Definition 9. We define the graph $U_r(m, k)$ for positive integers $2r \leqslant k \leqslant m$ as follows.

$$V(U_r(m, k)) := \{(X, A) : X, A \subseteq [m], X \cap A = \emptyset, |X| = r, |A| = k - r\}$$

and

$$E(U_r(m, k)) := \{\{(X, A), (Y, B)\} : X \subseteq B, Y \subseteq A\}.$$

Remark. Note that $U_1(m, k) = U(m, k)$, while $U_r(m, m) = K_{m:r}$, the Kneser graph with parameters m and r. Thus the graphs we just defined provide a common generalization of Kneser graphs and the universal graphs $U(m, k)$ of [10].

The following lemma is the general version of Lemma 1 for multicolourings:

Lemma 3. *A graph G admits a proper r-fold colouring f with m colours in which the closed neighbourhood of every vertex contains at most k colours if and only if there exists a homomorphism from G to $U_r(m, k)$. In particular, $\psi(G, r) \leqslant k$ if and only if there exists an m alongside with a homomorphism from G to $U_r(m, k)$.*

Proof. The proof is more or less identical to that of Lemma 1 (cf. [10]). If the required colouring f exists then assign to each vertex v a pair of sets of colours (X, A) with $X = f(v)$ and $\bigcup_{\{u,v\}\in E(G)} f(u) \subseteq A$. If f has the required properties then this assignment is indeed a homomorphism to $U_r(m, k)$.

On the other hand, if the required homomorphism h exists then the r-fold colouring f defined by the X-part of $h(v) = (X, A)$ as $f(v)$ satisfies the requirements. □

The following lemma is a generalization of Lemma 2:

Lemma 4. *For all feasible parameters m, k, r*

$$\chi^*(U_r(m, k)) = \frac{k}{r}.$$

Proof. Think of the vertices of $U_r(m, k)$ as k-sets of the set $[m]$ with r elements of the k-set distinguished. The number of vertices is thus $\binom{m}{k}\binom{k}{r}$, while the number of those vertices in which the smallest element of the chosen k-set is among the distinguished ones is $\binom{m}{k}\binom{k-1}{r-1}$. Since the latter kind of vertices form an independent set in $U_r(m, k)$, we have $\alpha(U_r(m,k)) \geqslant \binom{m}{k}\binom{k-1}{r-1}$. The reverse inequality $\alpha(U_r(m,k)) \leqslant \binom{m}{k}\binom{k-1}{r-1}$ follows from the Erdős–Ko–Rado theorem: once the chosen k-set is fixed, we can have at most $\binom{k-1}{r-1}$ vertices (X_i, A_i) with the property that if $i \neq j$ then $X_i \cap X_j \neq \emptyset$. If $X_i \cup A_i = X_j \cup A_j$, then the latter is the very same condition as non-adjacency in $U_r(m,k)$. Thus we know $\alpha(U_r(m,k)) = \binom{m}{k}\binom{k-1}{r-1}$.

Since $U_r(m,k)$ is vertex-transitive (because any permutation of the elements of $[m]$ gives an automorphism), we have $\chi^*(U_r(m,k)) = \frac{|V(U_r(m,k))|}{\alpha(U_r(m,k))} = \frac{k}{r}$. □

Proof of Theorem 5. We know by Lemma 3 that $\psi(G,r) = k$ implies the existence, for some m, of a homomorphism from G to $U_r(m,k)$. Since a homomorphism cannot decrease the value of the fractional chromatic number, this implies $\chi^*(G) \leqslant \chi^*(U_r(m,k)) = \frac{k}{r} = \frac{\psi(G,r)}{r}$, where, in particular, the first equality holds by Lemma 4.

On the other hand, denoting by $\chi(G,r)$ the minimum number of colours needed for a proper r-fold colouring of G, $\inf_r \frac{\psi(G,r)}{r} \leqslant \chi^*(G)$ follows from $\inf_r \frac{\chi(G,r)}{r} = \chi^*(G)$ (cf. [16, Theorem 7.4.5]) and the obvious inequality $\psi(G,r) \leqslant \chi(G,r)$. □

We note that universal graphs can also be defined for the directed version of the local chromatic number. Denoting these graphs by $U_d(m,k)$ they have $V(U_d(m,k)) = V(U(m,k))$ while

$$E(U_d(m,k)) = \{((x,A),(y,B)) : y \in A\}.$$

To show the analog of Lemma 1 is straightforward. Comparing $U_d(m,k)$ to $U(m,k)$ one can see that the symmetrically directed edges of $U_d(m,k)$ are exactly the (undirected) edges present in $U(m,k)$. This means (but the same can be seen also directly) that $\omega_s(U_d(m,k)) = k$. On the other hand, naturally, $\psi_d(U_d(m,k)) = k$, thus for these graphs we have $\sigma(U_d(m,k)) = \omega_s(U_d(m,k))$ by Theorem 1 and the obvious inequality $\omega_s(G) \leqslant \sigma(G)$.

8. Fractional covers

A non-negative real-valued function $g : 2^{V(G)} \to \mathbb{R}$ is called a fractional cover of $V(G)$ if $\sum_{U \ni v} g(U) \geqslant 1$ holds for all $v \in V(G)$.

The most general upper bound on $\sigma(G)$ we prove in this paper is given by the following inequality that generalizes Theorem 4 along the lines of a result (Theorem 2) of [11].

Theorem 6. *For any digraph G we have*

$$\sigma(G) \leqslant \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G[U]),$$

where the minimization is over all fractional covers g of $V(G)$ and $G[U]$ denotes the digraph induced by G on $U \subseteq V(G)$.

By $\sigma(G) = \sigma(G_{\rm rev})$ we again have the following immediate corollary (cf. Corollary 1 of Theorem 1).

Corollary 6.

$$\sigma(G) \leqslant \min \left\{ \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G[U]), \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G_{\rm rev}[U]) \right\}.$$

The proof of Theorem 6 is almost identical to that of Theorem 2 of [11]. Yet, we give the details for the sake of completeness.

We need some lemmas. Following [2], we can speak about the *representation* of a (di)graph $G = (V, E)$ over a subspace $\mathcal{F}$ of polynomials in m variables over a field F. Such a representation is an assignment of a polynomial f_v in $\mathcal{F}$ and a vector $\boldsymbol{a}_v \in F^m$ to each vertex $v \in V$ such that the following two conditions hold:

(i) for each $v \in V$, $f_v(\boldsymbol{a}_v) \neq 0$,

and

(ii) if $(u, v) \in E(G)$ then $f_u(\boldsymbol{a}_v) = 0$.

Notice that we adapted the description of a representation given in [2] to our terminology (where capacities are defined via cliques instead of stable sets) and to digraphs.

The following two lemmas are from [2]. Their proofs are essentially identical to those of Lemma 2.2 and Lemma 2.3 in [2] (after some trivial changes caused by the different language).

Lemma 5 (*Alon [2]*). *Let $G = (V, E)$ be a digraph and let $\mathcal{F}$ be a subspace of polynomials in m variables over a field F. If G has a representation over $\mathcal{F}$ then $\omega_s(G) \leqslant \dim(\mathcal{F})$.*

Lemma 6 (*Alon [2]*). *If G and H are two digraphs, G has a representation over $\mathcal{F}$ and H has a representation over $\mathcal{H}$, where $\mathcal{F}$ and $\mathcal{H}$ are spaces of polynomials over the same field F, then $\omega_s(G \cdot H) \leqslant \dim(\mathscr{F}) \cdot \dim(\mathcal{H})$.*

Remark. Lemmas 5 and 6 imply that if G and $\mathcal{F}$ are as in Lemma 5 then $\sigma(G) \leqslant \dim(\mathcal{F})$ (cf. [2, Theorem 2.4]). Notice that our Theorem 1 is a specialized version of this statement where the subspace $\mathcal{F}$ of polynomials is defined via a proper colouring of the vertices attaining the value of $\psi_d(G)$.

Our next lemma is analogous to Proposition 1 of [11].

Lemma 7. *Let $F_1, F_2, \ldots, F_n$ be digraphs. Then*

$$\omega_s(F_1 \cdot F_2 \cdot \cdots \cdot F_n) \leqslant \prod_{i=1}^{n} \psi_d^*(F_i).$$

Proof. First observe that the argument for $\omega_s((G[K_r])^n) = r^n \omega_s(G^n)$ that led us to state Theorem 4 generalizes to

$$\omega_s(F_1[K_r] \cdot F_2[K_r] \cdot \cdots \cdot F_n[K_r]) = r^n \omega_s(F_1 \cdot F_2 \cdot \cdots \cdot F_n).$$

(This is simply by realizing that in the argument mentioned above we have not used anywhere that in the n-fold product in question all the graphs were the same whereby we dealt with the nth power of a fixed graph.)

Take the representation (by subspaces of polynomials) given in the proof of Theorem 1 now for $F_1[K_r], F_2[K_r], \ldots, F_n[K_r]$, i.e., represent $F_i[K_r]$ for each i by the polynomials $\{P_{a,c_i}(x_i) := \prod_{j\in N^+_{c_i}(a)}(x_i - j)\}_{a\in V(F_i[K_r])}$, where c_i is a colouring of $V(F_i[K_r])$ that attains the value of $\psi_d(F_i[K_r])$. The dimension of this representation of $F_i[K_r]$ is bounded from above by $\psi_d(F_i[K_r])$. Now applying Lemma 6 we obtain

$$\omega_s(F_1[K_r]\cdot F_2[K_r]\cdot\cdots\cdot F_n[K_r])\leqslant\prod_{i=1}^n\psi_d(F_i[K_r])=\prod_{i=1}^n\psi_d(F_i,r).$$

Thus

$$\omega_s(F_1\cdot F_2\cdot\cdots\cdot F_n)=\frac{\omega_s(F_1[K_r]\cdot F_2[K_r]\cdot\cdots\cdot F_n[K_r])}{r^n}\leqslant\prod_{i=1}^n\frac{\psi_d(F_i,r)}{r}.$$

Since this last inequality is true for every positive integer r we can also write

$$\begin{aligned}\omega_s(F_1\cdot F_2\cdot\cdots\cdot F_n)&\leqslant\inf_r\prod_{i=1}^n\frac{\psi_d(F_i,r)}{r}=\liminf_r\prod_{i=1}^n\frac{\psi_d(F_i,r)}{r}\\&=\prod_{i=1}^n\liminf_r\frac{\psi_d(F_i,r)}{r}=\prod_{i=1}^n\psi_d^*(F_i).\qquad\Box\end{aligned}$$

Proof of Theorem 6. We call a function h assigning non-negative integer values to the elements of $2^{V(G)}$ an s-cover (s is a positive integer) of $V(G)$ if $\sum_{U\ni v}h(U)\geqslant s$ holds for all $v\in V(G)$.

It is clear that

$$\min_g\sum_{U\subseteq V(G)}g(U)\psi_d^*(G[U])=\inf_s\frac{1}{s}\min_h\sum_{U\subseteq V(G)}h(U)\psi_d^*(G[U]),$$

where the minimization on the left-hand side is over all fractional covers g while the minimization on the right-hand side is over all s-covers h.

Let us fix an s and let h be the s-cover achieving the minimum on the right-hand side. Let $\mathcal{U}$ be the multiset of those subsets of $V(G)$ that are assigned a positive value by h and let the multiplicity of $U\in V(G)$ in $\mathcal{U}$ be $h(U)$.

Fixing any natural number n denote by $\mathcal{U}^n$ the multiset of all n-fold Cartesian products of sets from $\mathcal{U}$. (The multiplicity of some $A=U_1\times U_2\times\cdots\times U_n\in\mathcal{U}^n$ is thus given by $h(U_1)\cdot h(U_2)\cdot\cdots\cdot h(U_n)$.)

We consider a maximum size symmetric clique K in G^n and observe that

$$s^n|K|\leqslant\sum_{\times_{i=1}^n U_i\in\mathcal{U}^n}\omega_s\left(G^n\left[\bigtimes_{i=1}^n U_i\right]\right).$$

Each summand in this last inequality satisfies by Lemma 7

$$\omega_s\left(G^n\left[\bigtimes_{i=1}^n U_i\right]\right) = \omega_s\left(\prod_{i=1}^n G[U_i]\right) \leqslant \prod_{i=1}^n \psi_d^*(G[U_i]).$$

Substituting this into the previous inequality we get

$$s^n|K| \leqslant \sum_{\times_{i=1}^n U_i \in \mathcal{U}^n} \prod_{i=1}^n \psi_d^*(G[U_i]) = \left[\sum_{U_i \in \mathcal{U}} \psi_d^*(G[U_i])\right]^n,$$

where the summations are meant with multiplicities.

Since K is a maximum size symmetric clique of G^n and the multiplicity of U_i in $\mathcal{U}$ is $h(U_i)$, we obtained

$$\omega_s(G^n) \leqslant \frac{1}{s^n}\left[\sum_{U \subseteq V(G)} h(U)\psi_d^*(G[U])\right]^n.$$

This implies

$$\sigma(G) \leqslant \inf_s \ \frac{1}{s}\left(\sum_{U \subseteq V(G)} h(U)\psi_d^*(G[U])\right) = \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G[U]),$$

where the minimization is over all fractional covers g of $V(G)$, i.e., we arrived at the statement. □

To illustrate that the bound of Theorem 6 may indeed give an improvement over that of Theorem 4 (or, in fact, over that of Corollary 5) consider the following digraph G. Let $V(G) = \{1, 2, \dots, 2k+1, a, b\}$ and $E(G) = E(\vec{C}_{2k+1}) \cup \{(a,i), (i,b) : i \in \{1, \dots, 2k+1\}\}$, where $\vec{C}_{2k+1}$ is an arbitrary non-alternatingly oriented cycle on $2k+1$ vertices. It is easy to check that $\psi_d^*(G) = 3 + \frac{1}{k}$ (and also $\psi_d^*(G_{\text{rev}}) = 3 + \frac{1}{k}$), i.e., Theorem 4 gives $\sigma(G) \leqslant 3 + \frac{1}{k}$ only, while Theorem 6 gives $\sigma(G) \leqslant 3$. Indeed, using the fractional cover (which is also an integer cover) $g(V_1) = g(V_2) = 1$, where $V_1 = \{1, 2, \dots, 2k+1\}$, $V_2 = \{a, b\}$ (and $g(U) = 0$ for all other $U \subseteq V(G)$) we get $\sigma(G) \leqslant \psi_d^*(\vec{C}_{2k+1}) + \psi_d^*(\bar{K}_2) \leqslant \psi_d(\vec{C}_{2k+1}) + 1 = 3$. This bound is sharp since G contains transitive triangles.

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References

[1] N. Alon, On the capacity of digraphs, European J. Combin. 19 (1998) 1–5.

[2] N. Alon, The Shannon capacity of a union, Combinatorica 18 (1998) 301–310.

[3] N. Alon, Graph powers, in: B. Bollobás (Ed.), Contemporary Combinatorics, Bolyai Society Mathematical Studies, Springer, 2002, pp. 11–28.

[4] C. Berge, M. Simonovits, The coloring numbers of the direct product of two hypergraphs, in: Hypergraph Seminar (Proceedings of the First Working Seminar, Ohio State University, Columbus, Ohio, 1972; dedicated to Arnold Ross), Lecture Notes in Mathematics, vol. 411, Springer, Berlin, 1974, pp. 21–33.

[5] E. Bidamon, H. Meyniel, On the Shannon capacity of a directed graph, European J. Combin. 6 (1985) 289–290.

[6] A. Blokhuis, On the Sperner capacity of the cyclic triangle, J. Algebraic Combin. 2 (1993) 123–124.

[7] T. Bohman, R. Holzman, A nontrivial lower bound on the Shannon capacities of the complements of odd cycles, IEEE Trans. Inform. Theory 49 (2003) 721–722.

[8] R. Calderbank, P. Frankl, R.L. Graham, W. Li, L. Shepp, The Sperner capacity of the cyclic triangle for linear and non-linear codes, J. Algebraic Combin. 2 (1993) 31–48.

[9] I. Csiszár, J. Körner, Information theory: coding theorems for discrete memoryless systems, Academic Press, New York, 1982 and Akadémiai Kiadó, Budapest, 1981.

[10] P. Erdős, Z. Füredi, A. Hajnal, P. Komjáth, V. Rödl, Á. Seress, Coloring graphs with locally few colors, Discrete Math. 59 (1986) 21–34.

[11] E. Fachini, J. Körner, Colour number, capacity and perfectness of directed graphs, Graphs Combin. 16 (2000) 389–398.

[12] Z. Füredi, Local colourings of graphs (Gráfok lokális színezései), manuscript in Hungarian, September 2002.

[13] A. Galluccio, L. Gargano, J. Körner, G. Simonyi, Different capacities of a digraph, Graphs Combin. 10 (1994) 105–121.

[14] L. Gargano, J. Körner, U. Vaccaro, Sperner theorems on directed graphs and qualitative independence, J. Combin. Theory Ser. A 61 (1992) 173–192.

[15] L. Gargano, J. Körner, U. Vaccaro, Capacities: from information theory to extremal set theory, J. Combin. Theory Ser. A 68 (1994) 296–316.

[16] C. Godsil, G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics, vol. 207, Springer, New York, 2001.

[17] T.R. Jensen, B. Toft, Graph Colouring Problems, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, New York, 1995.

[18] J. Körner, A. Orlitsky, Zero-error information theory, IEEE Trans. Inform. Theory 44(6) (October 1998, commemorative issue) 2207–2229.

[19] J. Körner, G. Simonyi, A Sperner-type theorem and qualitative independence, J. Combin. Theory Ser. A 59 (1992) 90–103.

[20] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inform. Theory 25 (1979) 1–7.

[21] R.J. McEliece, E.C. Posner, Hide and seek, data storage, and entropy, Ann. Math. Statist. 42 (1971) 1706–1716.

[22] A. Sali, G. Simonyi, Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities, European J. Combin. 20 (1999) 93–99.

[23] E.R. Scheinerman, D.H. Ullman, Fractional Graph Theory, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, Chichester, 1997.

[24] C.E. Shannon, The zero-capacity of a noisy channel, IRE Trans. Inform. Theory 2 (1956) 8–19.

[25] G. Simonyi, G. Tardos, Local chromatic number, Ky Fan's theorem, and circular colorings, manuscript, arXiv:math.CO/0407075.