

# THE FUGLEDE CONJECTURE FOR CONVEX DOMAINS IS TRUE IN ALL DIMENSIONS

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ABSTRACT. A set  $\Omega \subset \mathbb{R}^d$  is said to be spectral if the space  $L^2(\Omega)$  has an orthogonal basis of exponential functions. A conjecture due to Fuglede (1974) stated that  $\Omega$  is a spectral set if and only if it can tile the space by translations. While this conjecture was disproved for general sets, it has long been known that for a *convex body*  $\Omega \subset \mathbb{R}^d$  the “tiling implies spectral” part of the conjecture is in fact true.

To the contrary, the “spectral implies tiling” direction of the conjecture for convex bodies was proved only in  $\mathbb{R}^2$ , and also in  $\mathbb{R}^3$  under the a priori assumption that  $\Omega$  is a convex polytope. In higher dimensions, this direction of the conjecture remained completely open (even in the case when  $\Omega$  is a polytope) and could not be treated using the previously developed techniques.

In this paper we fully settle Fuglede’s conjecture for convex bodies affirmatively in all dimensions, i.e. we prove that if a convex body  $\Omega \subset \mathbb{R}^d$  is a spectral set then  $\Omega$  is a convex polytope which can tile the space by translations. To prove this we introduce a new technique, involving a construction from crystallographic diffraction theory, which allows us to establish a geometric “weak tiling” condition necessary for a set  $\Omega \subset \mathbb{R}^d$  to be spectral.

## 1. INTRODUCTION

1.1. Let  $\Omega \subset \mathbb{R}^d$  be a bounded, measurable set of positive measure. We say that  $\Omega$  is *spectral* if there exists a countable set  $\Lambda \subset \mathbb{R}^d$  such that the system of exponential functions

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}, \quad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle}, \quad (1.1)$$

is orthogonal and complete in  $L^2(\Omega)$ , that is, the system constitutes an orthogonal basis in the space. Such a set  $\Lambda$  is called a *spectrum* for  $\Omega$ . The classical example of a spectral set is the unit cube  $\Omega = [-\frac{1}{2}, \frac{1}{2}]^d$ , for which the set  $\Lambda = \mathbb{Z}^d$  serves as a spectrum.

Which other sets  $\Omega$  can be spectral? The research on this problem has been influenced for many years by a famous paper [Fug74] due to Fuglede (1974), who suggested that there should be a concrete, geometric way to characterize the spectral sets. We say that  $\Omega$  *tiles the space by translations* if there exists a countable set  $\Lambda \subset \mathbb{R}^d$  such that the collection of sets  $\{\Omega + \lambda\}$ ,  $\lambda \in \Lambda$ , consisting of translated copies of  $\Omega$ , constitutes a partition of  $\mathbb{R}^d$  up to measure zero. In his paper, Fuglede stated the following conjecture: “A set  $\Omega \subset \mathbb{R}^d$  is spectral if and only if it can tile the space by translations”.

For example, Fuglede proved that a triangle and a disk in the plane are not spectral sets. He also proved that if  $\Omega$  can tile with respect to a *lattice* translation set  $\Lambda$  then the

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dual lattice  $\Lambda^*$  is a spectrum for  $\Omega$ , and conversely. Fuglede’s conjecture inspired extensive research over the years, and a number of interesting results establishing connections between spectrality and tiling had since been obtained.

The conjecture remained open for 30 years until a counterexample was discovered by Tao [Tao04], who constructed in dimensions 5 and higher an example of a spectral set which cannot tile by translations. Since then, counterexamples to both directions of the conjecture were found in dimensions  $d \geq 3$  (see [KM10, Section 4]). These examples are composed of finitely many unit cubes in special arithmetic arrangements. The conjecture is still open in dimensions  $d = 1$  and 2 in both directions.

On the other hand, it was believed that Fuglede’s conjecture should be true in all dimensions  $d$  if the set  $\Omega \subset \mathbb{R}^d$  is assumed to be a *convex body* (that is, a compact convex set with nonempty interior<sup>1</sup>). Indeed, all the known counterexamples to the conjecture are highly non-convex sets, being the union of a finite number of unit cubes centered at points of the integer lattice  $\mathbb{Z}^d$ . Moreover, it has long been known [Ven54, McM80] that a convex body which tiles by translations must be a polytope, and that it admits a face-to-face tiling by a lattice translation set  $\Lambda$  and therefore has a spectrum given by the dual lattice  $\Lambda^*$ . So this implies that for a convex body  $\Omega \subset \mathbb{R}^d$  the “tiling implies spectral” part of the conjecture is in fact true in any dimension  $d$ .

To the contrary, the “spectral implies tiling” direction of the conjecture for convex bodies was proved only in  $\mathbb{R}^2$  [IKT03], and also in  $\mathbb{R}^3$  under the a priori assumption that  $\Omega$  is a convex polytope [GL17]. In higher dimensions, this direction of the conjecture remained completely open (even in the case when  $\Omega$  is a polytope) and could not be treated using the previously developed techniques.

It is our goal in the present paper to establish that the result in fact holds in all dimensions and for general convex bodies. We will prove the following theorem:

**Theorem 1.1.** *Let  $\Omega$  be a convex body in  $\mathbb{R}^d$ . If  $\Omega$  is a spectral set, then  $\Omega$  must be a convex polytope, and it tiles the space face-to-face by translations along a lattice.*

This fully settles the Fuglede conjecture for convex bodies affirmatively: we obtain that a convex body in  $\mathbb{R}^d$  is a spectral set if and only if it can tile by translations.

1.2. There is a complete characterization due to Venkov [Ven54], that was rediscovered by McMullen [McM80, McM81], of the convex bodies which tile by translations:

*A convex body  $\Omega \subset \mathbb{R}^d$  can tile the space by translations if and only if it satisfies the following four conditions:*

- (i)  $\Omega$  is a convex polytope;
- (ii)  $\Omega$  is centrally symmetric;
- (iii) all the facets of  $\Omega$  are centrally symmetric;
- (iv) each belt of  $\Omega$  consists of either 4 or 6 facets.

*Moreover, a convex body  $\Omega$  satisfying the four conditions (i)–(iv) admits a face-to-face tiling by translations along a certain lattice.*

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<sup>1</sup>The compactness and nonempty interior assumptions can be made with no loss of generality, as any convex set of positive and finite measure coincides with a convex body up to a set of measure zero.

We recall that a *facet* of a convex polytope  $\Omega \subset \mathbb{R}^d$  is a  $(d-1)$ -dimensional face of  $\Omega$ . If  $\Omega$  has centrally symmetric facets, then a *belt* of  $\Omega$  is a system of facets obtained in the following way: Let  $G$  be a *subfacet* of  $\Omega$ , that is, a  $(d-2)$ -dimensional face. Then  $G$  lies in exactly two adjacent facets of  $\Omega$ , say  $F$  and  $F'$ . Since  $F'$  is centrally symmetric, there is another subfacet  $G'$  obtained by reflecting  $G$  through the center of  $F'$  (so in particular,  $G'$  is a translate of  $-G$ ). In turn,  $G'$  is the intersection of  $F'$  with another facet  $F''$ . Continuing in this way, we obtain a system of facets  $F, F', F'', \dots, F^{(m)} = F$ , called *the belt of  $\Omega$  generated by the subfacet  $G$* , such that the intersection  $F^{(i-1)} \cap F^{(i)}$  of any pair of consecutive facets in the system is a translate of either  $G$  or  $-G$ .

Fuglede's conjecture for convex bodies can thus be equivalently stated by saying that for a convex body  $\Omega \subset \mathbb{R}^d$  to be spectral, it is necessary and sufficient that the four conditions (i)–(iv) above hold.

In relation with the first condition (i), a result proved in [IKP99] states that if  $\Omega$  is a ball in  $\mathbb{R}^d$  ( $d \geq 2$ ) then  $\Omega$  is not a spectral set. In [IKT01] this result was extended to the class of convex bodies  $\Omega \subset \mathbb{R}^d$  that have a smooth boundary.

As for (ii), Kolountzakis [Kol00] proved that if a convex body  $\Omega \subset \mathbb{R}^d$  is spectral, then it must be centrally symmetric. If  $\Omega$  is assumed a priori to be a polytope, then another approach to this result was given in [KP02] (see also [GL17, Section 3]).

Recently, also the necessity of condition (iii) for spectrality was established. It was proved in [GL17, Section 4] that if a convex, centrally symmetric polytope  $\Omega \subset \mathbb{R}^d$  is a spectral set, then all the facets of  $\Omega$  must also be centrally symmetric. The proof is based on a development of the argument in [KP02].

The last condition (iv) was addressed so far only in dimensions  $d = 2$  and  $3$ . Iosevich, Katz and Tao proved in [IKT03] that if a convex polygon  $\Omega \subset \mathbb{R}^2$  is a spectral set, then it must be either a parallelogram or a (centrally symmetric) hexagon. In three dimensions, it was recently proved [GL16, GL17] that if a convex polytope  $\Omega \subset \mathbb{R}^3$  is spectral, then it can tile the space by translations (as a consequence, the condition (iv) must hold, although the proof does not establish it directly).

In this paper, we will show that the conditions (i) and (iv) are in fact necessary for the spectrality of a general convex body  $\Omega$  in every dimension, thus obtaining a proof of the full Fuglede conjecture for convex bodies. Our main results are as follows:

**Theorem 1.2.** *Let  $\Omega$  be a convex body in  $\mathbb{R}^d$ . If  $\Omega$  is a spectral set, then  $\Omega$  must be a convex polytope.*

**Theorem 1.3.** *Let  $\Omega \subset \mathbb{R}^d$  be a convex polytope, which is centrally symmetric and has centrally symmetric facets. If  $\Omega$  is a spectral set, then each belt of  $\Omega$  must consist of either 4 or 6 facets.*

Theorem 1.1 above follows as a consequence of these two theorems, combined with the results in [Kol00] (or [KP02]), [GL17, Section 4], and the Venkov-McMullen theorem.

1.3. In the above mentioned papers [IKP99], [IKT01], [KP02], [IKT03], [GL17] the approach relies on the asymptotic behavior of the Fourier transform of the indicator function of  $\Omega$ , and involves an analysis of its set of zeros. In the present paper we introduce a new approach to the problem, based on establishing a link between the notion of spectrality and a geometric notion which we refer to as “weak tiling”.

**Definition 1.4.** Let  $\Omega \subset \mathbb{R}^d$  be a bounded, measurable set. We say that another measurable, possibly unbounded, set  $\Sigma \subset \mathbb{R}^d$  admits a *weak tiling* by translates of  $\Omega$ , if there exists a positive, locally finite (Borel) measure  $\mu$  on  $\mathbb{R}^d$  such that  $\mathbb{1}_\Omega * \mu = \mathbb{1}_\Sigma$  a.e.

If the measure  $\mu$  is the sum of unit masses at the points of a locally finite set  $\Lambda$ , that is,  $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ , then the condition  $\mathbb{1}_\Omega * \mu = \mathbb{1}_\Sigma$  a.e. means that the collection  $\{\Omega + \lambda\}$ ,  $\lambda \in \Lambda$ , of translated copies of  $\Omega$ , constitutes a partition of  $\Sigma$  up to measure zero. In this case, we say that the weak tiling is a *proper tiling*.

For example, one can check that any bounded set  $\Omega$  of positive Lebesgue measure tiles the whole space  $\mathbb{R}^d$  weakly by translates with respect to the measure  $d\mu = m(A)^{-1} dx$  (where  $dx$  denotes the Lebesgue measure on  $\mathbb{R}^d$ ). This is in sharp contrast to the obvious fact that not every set  $\Omega$  can tile the space properly by translations.

We will prove the following theorem, which gives a necessary condition for spectrality in terms of weak tiling:

**Theorem 1.5.** *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ . If  $\Omega$  is spectral, then its complement  $\Omega^c = \mathbb{R}^d \setminus \Omega$  admits a weak tiling by translates of  $\Omega$ . That is, there exists a positive, locally finite measure  $\mu$  such that  $\mathbb{1}_\Omega * \mu = \mathbb{1}_{\Omega^c}$  a.e.*

Our proof of this result involves a construction due to Hof [Hof95], that is often used in mathematical crystallography in order to describe the diffraction pattern of an atomic structure (see also [BG13, Chapter 9]).

Notice that if the complement  $\Omega^c$  has a *proper* tiling by translates of  $\Omega$ , then it just means that  $\Omega$  can tile the space by translations. Theorem 1.5 thus establishes a weak form of the “spectral implies tiling” part of Fuglede’s conjecture, which is valid for all bounded, measurable sets  $\Omega \subset \mathbb{R}^d$ . We observe that the weak tiling conclusion cannot be strengthened to proper tiling without imposing extra assumptions on the set  $\Omega$ , since there exist examples of spectral sets which cannot tile by translations.

We will prove that if  $\Omega$  is a convex body in  $\mathbb{R}^d$ , and if it can tile its complement  $\Omega^c$  weakly by translations, then  $\Omega$  must in fact be a convex polytope (Theorem 4.1). We will also prove that if, in addition,  $\Omega$  is centrally symmetric and has centrally symmetric facets, then each belt of  $\Omega$  must have either 4 or 6 facets (Theorem 6.1). So in the latter case, it follows that  $\Omega$  can in fact tile its complement  $\Omega^c$  not only weakly, but even properly, by translations.

The potential applications of Theorem 1.5 are not limited to the class of convex bodies in  $\mathbb{R}^d$ . As an example, we will use this theorem to give a simple geometric condition which is necessary for the spectrality of a bounded, measurable set  $\Omega \subset \mathbb{R}^d$  (Theorem 3.5). Based on this condition we will prove that the boundary of a bounded, open spectral set must have Lebesgue measure zero (Theorem 3.6).

1.4. The rest of the paper is organized as follows.

In Section 2 we present some preliminary background. We fix notation that will be used in the paper and discuss basic results about measures and tempered distributions, spectral sets and weak tiling.

In Section 3 we prove that if a bounded, measurable set  $\Omega \subset \mathbb{R}^d$  is spectral, then its complement  $\Omega^c$  admits a weak tiling by translates of  $\Omega$  (Theorem 1.5). As an

application we show that a connected spectral domain cannot have any “holes”, and that the boundary of an open spectral domain must have Lebesgue measure zero.

In Section 4 we prove that if a convex body  $K \subset \mathbb{R}^d$  is a spectral set, then  $K$  must be a convex polytope (Theorem 1.2).

In the last two Sections 5 and 6 we establish that each belt of a spectral convex polytope  $K \subset \mathbb{R}^d$  must consist of either 4 or 6 facets (Theorem 1.3). The proof is based on an analysis of the measure  $\mu$  that provides a weak tiling of  $K^c$  by translates of  $K$ .

## 2. PRELIMINARIES

**2.1. Notation.** If  $A \subset \mathbb{R}^d$  then  $\text{int}(A)$  will denote the interior of  $A$ , and  $\text{bd}(A)$  the boundary of  $A$ . We use  $A^c$  to denote the complement  $\mathbb{R}^d \setminus A$  of the set  $A$ . If  $A, B \subset \mathbb{R}^d$  then  $A \Delta B$  is the symmetric difference of  $A$  and  $B$ . We denote by  $|A|$  the number of elements in  $A$ .

If  $A \subset \mathbb{R}^d$ , then for each  $\tau \in \mathbb{R}^d$  we let  $A + \tau = \{a + \tau : a \in A\}$  denote the image of  $A$  under translation by the vector  $\tau$ . If  $s \in \mathbb{R}$ , then  $sA = \{sa : a \in A\}$  will denote the image of  $A$  under dilation with ratio  $s$ . If  $A, B$  are two subsets of  $\mathbb{R}^d$ , then  $A + B$  and  $A - B$  denote respectively their set of sums and set of differences.

We use  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  to denote the Euclidean scalar product and norm in  $\mathbb{R}^d$ .

By a *lattice* in  $\mathbb{R}^d$  we mean a set  $L$  which can be obtained as the image of  $\mathbb{Z}^d$  under an invertible linear transformation. The *dual lattice*  $L^*$  is the set of all vectors  $\lambda^* \in \mathbb{R}^d$  such that  $\langle \lambda, \lambda^* \rangle \in \mathbb{Z}$  for every  $\lambda \in L$ .

We denote by  $m(A)$  the Lebesgue measure of a set  $A \subset \mathbb{R}^d$ . We also use  $m_k(A)$  to denote the  $k$ -dimensional volume measure of  $A$  (so, in particular,  $m_d(A) = m(A)$ ).

If  $A \subset \mathbb{R}^d$  is a bounded, measurable set, then we define

$$\Delta(A) := \{x \in \mathbb{R}^d : m(A \cap (A + x)) > 0\}.$$

Then  $\Delta(A)$  is a bounded open set, and we have  $\Delta(A) = -\Delta(A)$  (which means that  $\Delta(A)$  is symmetric with respect to the origin). One can think of the set  $\Delta(A)$  as the measure-theoretic analog of the set of differences  $A - A$ . In particular, one can check that if  $A$  is an open set then  $\Delta(A) = A - A$ . In general we have  $\Delta(A) \subset A - A$ , but this inclusion can be strict.

The Fourier transform of a function  $f \in L^1(\mathbb{R}^d)$  is defined by

$$\widehat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle t, x \rangle} dx.$$

**2.2. Measures and distributions.** By a “measure” we will refer to a Borel (either positive, or complex) measure on  $\mathbb{R}^d$ . We use  $\text{supp}(\mu)$  to denote the closed support of a measure  $\mu$ . We denote by  $\delta_\lambda$  the Dirac measure consisting of a unit mass at the point  $\lambda$ . If  $\Lambda \subset \mathbb{R}^d$  is a countable set, then we define  $\delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ .

If  $\alpha$  is a tempered distribution on  $\mathbb{R}^d$ , and if  $\varphi$  is a Schwartz function on  $\mathbb{R}^d$ , then we use  $\langle \alpha, \varphi \rangle$  to denote the action of  $\alpha$  on  $\varphi$ . A tempered distribution  $\alpha$  is *positive* if we have  $\langle \alpha, \varphi \rangle \geq 0$  for any Schwartz function  $\varphi \geq 0$ . If a tempered distribution  $\alpha$  is positive, then  $\alpha$  is a positive measure. The Fourier transform  $\widehat{\alpha}$  of a tempered

distribution  $\alpha$  is defined by  $\langle \widehat{\alpha}, \varphi \rangle = \langle \alpha, \widehat{\varphi} \rangle$ . A tempered distribution  $\alpha$  is said to be *positive-definite* if  $\widehat{\alpha}$  is a positive distribution. See [BG13, Section 8.4], [Rud91].

If  $\mu$  is a measure on  $\mathbb{R}^d$ , then  $\mu$  is said to be *locally finite* if we have  $|\mu|(B) < \infty$  for every open ball  $B$ .

A measure  $\mu$  on  $\mathbb{R}^d$  is said to be *translation-bounded* if for every (or equivalently, for some) open ball  $B$  we have

$$\sup_{x \in \mathbb{R}^d} |\mu|(B + x) < \infty.$$

If a measure  $\mu$  on  $\mathbb{R}^d$  is translation-bounded, then it is a tempered distribution. If  $\mu$  is a translation-bounded measure on  $\mathbb{R}^d$ , and if  $\nu$  is a finite measure on  $\mathbb{R}^d$ , then the convolution  $\mu * \nu$  is a translation-bounded measure.

**Lemma 2.1.** *Let  $\nu$  be a finite measure on  $\mathbb{R}^d$ , and suppose that  $\mu$  is a translation-bounded measure on  $\mathbb{R}^d$  whose Fourier transform  $\widehat{\mu}$  is a locally finite measure. Then the Fourier transform of the convolution  $\mu * \nu$  is the measure  $\widehat{\mu} \cdot \widehat{\nu}$ .*

See e.g. [BG13, Section 8.6], [KL21, Section 2.5].

A sequence of measures  $\{\mu_n\}$  is said to be *uniformly translation-bounded* if for every (or equivalently, for some) open ball  $B$  one can find a constant  $C$  not depending on  $n$ , such that  $\sup_x |\mu_n|(B + x) \leq C$  for every  $n$ .

If  $\{\mu_n\}$  is a uniformly translation-bounded sequence of measures, then we say that  $\mu_n$  *converges vaguely* to a measure  $\mu$  if for every continuous, compactly supported function  $\varphi$  we have  $\int \varphi d\mu_n \rightarrow \int \varphi d\mu$ . In this case, the vague limit  $\mu$  must also be a translation-bounded measure. For a uniformly translation-bounded sequence of measures  $\{\mu_n\}$  to converge vaguely, it is necessary and sufficient that  $\{\mu_n\}$  converge in the space of tempered distributions. From any uniformly translation-bounded sequence of measures  $\{\mu_n\}$  one can extract a vaguely convergent subsequence  $\{\mu_{n_j}\}$ .

**Lemma 2.2.** *Let  $f \in L^1(\mathbb{R}^d)$ , and let  $\{\mu_n\}$  be a uniformly translation-bounded sequence of measures on  $\mathbb{R}^d$ , such that  $f * \mu_n = 1$  a.e. for every  $n$ . If  $\mu_n$  converges vaguely to a measure  $\mu$  then also  $f * \mu = 1$  a.e.*

*Proof.* Let  $\varphi$  be a continuous, compactly supported function on  $\mathbb{R}^d$ . Then the sequence of functions  $\mu_n * \varphi$  is uniformly bounded and converges pointwise to  $\mu * \varphi$ . Hence by the dominated convergence theorem we have  $f * (\mu_n * \varphi) \rightarrow f * (\mu * \varphi)$  pointwise. In turn, this implies that  $(f * \mu_n) * \varphi \rightarrow (f * \mu) * \varphi$  pointwise, since the convolution is associative (by Fubini's theorem). But  $f * \mu_n = 1$  a.e. for every  $n$ , so we conclude that  $(f * \mu) * \varphi = \int \varphi$ . Since this is true for an arbitrary  $\varphi$ , the assertion follows.  $\square$

**2.3. Spectra.** If  $\Omega$  is a bounded, measurable set in  $\mathbb{R}^d$  of positive measure, then by a *spectrum* for  $\Omega$  we mean a countable set  $\Lambda \subset \mathbb{R}^d$  such that the system of exponential functions  $E(\Lambda)$  defined by (1.1) is orthogonal and complete in the space  $L^2(\Omega)$ .

For any two points  $\lambda, \lambda'$  in  $\mathbb{R}^d$  we have  $\langle e_\lambda, e_{\lambda'} \rangle_{L^2(\Omega)} = \widehat{\mathbf{1}_\Omega}(\lambda' - \lambda)$ , where  $\widehat{\mathbf{1}_\Omega}$  is the Fourier transform of the indicator function  $\mathbf{1}_\Omega$  of the set  $\Omega$ . The orthogonality of the system  $E(\Lambda)$  in  $L^2(\Omega)$  is therefore equivalent to the condition

$$\Lambda - \Lambda \subset \mathcal{Z}(\widehat{\mathbf{1}_\Omega}) \cup \{0\}, \quad (2.1)$$

where  $\mathcal{Z}(\widehat{\mathbf{1}_\Omega}) := \{t \in \mathbb{R}^d : \widehat{\mathbf{1}_\Omega}(t) = 0\}$  is the set of zeros of the function  $\widehat{\mathbf{1}_\Omega}$ .

A set  $\Lambda \subset \mathbb{R}^d$  is said to be *uniformly discrete* if there is  $\delta > 0$  such that  $|\lambda' - \lambda| \geq \delta$  for any two distinct points  $\lambda, \lambda'$  in  $\Lambda$ . The condition (2.1) implies that every spectrum  $\Lambda$  of  $\Omega$  is a uniformly discrete set.

The set  $\Lambda$  is *relatively dense* if there is  $R > 0$  such that every ball of radius  $R$  contains at least one point from  $\Lambda$ . It is well-known that if  $\Lambda$  is a spectrum for  $\Omega$ , then  $\Lambda$  must be a relatively dense set (see e.g. [GL17, Section 2C]).

The following lemma gives a frequently used characterization of the spectra of  $\Omega$ .

**Lemma 2.3** (see [Kol04, Section 3.1]). *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ , and define the function  $f := m(\Omega)^{-2} |\widehat{\mathbf{1}}_\Omega|^2$ . Then a set  $\Lambda \subset \mathbb{R}^d$  is a spectrum for  $\Omega$  if and only if  $f * \delta_\Lambda = 1$  a.e.*

**2.4. Weak tiling.** If  $\Omega$  is a bounded, measurable set in  $\mathbb{R}^d$ , and if  $\Sigma$  is another, possibly unbounded, measurable set in  $\mathbb{R}^d$ , then we say that  $\Sigma$  admits a *weak tiling* by translates of  $\Omega$ , if there exists a positive, locally finite measure  $\mu$  on  $\mathbb{R}^d$  such that  $\mathbf{1}_\Omega * \mu = \mathbf{1}_\Sigma$  a.e.

The following lemma shows that the measure  $\mu$  in any weak tiling is not only locally finite, but in fact must be translation-bounded.

**Lemma 2.4.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded, measurable set of positive Lebesgue measure, and suppose that  $\mu$  is a positive, locally finite measure such that  $\mathbf{1}_\Omega * \mu \leq 1$  a.e. Then  $\mu$  is a translation-bounded measure.*

*Proof.* Fix  $r > 0$ , and let  $B_r$  be the open ball of radius  $r$  centered at the origin. It will be enough to show that there is a constant  $M > 0$  such that  $(\mathbf{1}_{B_r} * \mu)(x) \leq M$  for every  $x \in \mathbb{R}^d$ . Since  $\Omega$  is a bounded set, we can choose a sufficiently large number  $s > 0$  such that we have  $B_s + y \supset B_r$  for every  $y \in \Omega$ . It follows that  $\mathbf{1}_{B_s} * \mathbf{1}_\Omega \geq m(\Omega) \mathbf{1}_{B_r}$ . In turn, this implies that

$$m(\Omega) \mathbf{1}_{B_r} * \mu \leq (\mathbf{1}_{B_s} * \mathbf{1}_\Omega) * \mu = \mathbf{1}_{B_s} * (\mathbf{1}_\Omega * \mu) \leq \mathbf{1}_{B_s} * 1 = m(B_s).$$

We thus see that the constant  $M := m(B_s)/m(\Omega)$  satisfies  $\mathbf{1}_{B_r} * \mu \leq M$  as needed.  $\square$

The next lemma implies that if  $\Omega$  tiles  $\Sigma$  weakly by translations with respect to a measure  $\mu$ , then we must have  $m((\Omega + t) \cap \Sigma^c) = 0$  for every  $t \in \text{supp}(\mu)$ .

**Lemma 2.5.** *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ , and let  $\mu$  be a positive, locally finite measure on  $\mathbb{R}^d$ . Suppose that we have  $\mathbf{1}_\Omega * \mu = 0$  a.e. on the complement  $\Sigma^c$  of another measurable set  $\Sigma \subset \mathbb{R}^d$ . Then  $m((\Omega + t) \cap \Sigma^c) = 0$  for every  $t \in \text{supp}(\mu)$ .*

*Proof.* Let  $\varphi(t) := m((\Omega + t) \cap \Sigma^c)$ ,  $t \in \mathbb{R}^d$ . Suppose to the contrary that  $\varphi(t) > 0$  for some  $t \in \text{supp}(\mu)$ . Since  $\varphi$  is a continuous function, there is an open neighborhood  $U$  of  $t$  such that  $\varphi(t') > 0$  for all  $t' \in U$ . Since  $t \in \text{supp}(\mu)$  we must have  $\mu(U) > 0$ , and it follows that  $\int \varphi d\mu > 0$ . But on the other hand, using Fubini's theorem we have

$$\int \varphi d\mu = \int_{\Sigma^c} (\mathbf{1}_\Omega * \mu) dm = 0,$$

since we have assumed that  $\mathbf{1}_\Omega * \mu = 0$  a.e. on  $\Sigma^c$ . We thus obtain a contradiction.  $\square$

**Corollary 2.6.** *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ . Assume that the complement  $\Omega^c$  of  $\Omega$  admits a weak tiling by translates of  $\Omega$ , that is, there is a positive, locally finite measure  $\mu$  such that  $\mathbf{1}_\Omega * \mu = \mathbf{1}_{\Omega^c}$  a.e. Then  $\text{supp}(\mu) \subset \Delta(\Omega)^c$ .*

Indeed, this follows from Lemma 2.5 in the special case when  $\Sigma = \Omega^c$ .

## 3. SPECTRALITY AND WEAK TILING

In this section, our main goal is to prove Theorem 1.5, which states that if a bounded, measurable set  $\Omega \subset \mathbb{R}^d$  is spectral, then its complement  $\Omega^c$  admits a weak tiling by translates of  $\Omega$ . The proof of this result involves a construction due to Hof [Hof95], that is often used in mathematical crystallography in order to describe the diffraction pattern of an atomic structure.

We also present some simple applications of Theorem 1.5. As an example, we will use it to prove that if a bounded, open set  $\Omega \subset \mathbb{R}^d$  is spectral, then its boundary must have Lebesgue measure zero. Some other examples will also be discussed.

3.1. In [Hof95], Hof suggested a mathematical model for the diffraction experiment used in crystallography to analyze the atomic structure of a solid. In this model, the configuration of the atoms is represented by a uniformly discrete and relatively dense set  $\Lambda \subset \mathbb{R}^d$ . Hof used volume averaged convolution of the two measures  $\delta_\Lambda$  and  $\delta_{-\Lambda}$  in order to define the *autocorrelation measure*  $\gamma$  of the set  $\Lambda$ . He then argued that the diffraction by the atomic structure is described by the Fourier transform  $\widehat{\gamma}$  of the autocorrelation, which is a positive measure called the *diffraction measure* of the set  $\Lambda$  (for more details, see [BG13, Chapter 9]).

Here, we will apply this technique to a set  $\Lambda$  that constitutes a spectrum for a bounded, measurable set  $\Omega \subset \mathbb{R}^d$ . We will obtain the following result:

**Theorem 3.1.** *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ . If  $\Omega$  is a spectral set, then there exists a measure  $\gamma$  on  $\mathbb{R}^d$  with the following properties:*

- (a)  $\gamma$  is a positive, translation-bounded measure;
- (b) the support of  $\gamma$  is contained in the closed set  $\mathcal{Z}(\widehat{\mathbf{1}}_\Omega) \cup \{0\}$ ;
- (c)  $\gamma = \delta_0$  in some open neighborhood of the origin;
- (d)  $\widehat{\gamma}$  is also a positive, translation-bounded measure;
- (e)  $\widehat{\gamma} = m(\Omega) \delta_0$  in the open set  $\Delta(\Omega)$ .

A similar result can be proved in the more general context of locally compact abelian groups, but in this paper we work in the euclidean setting only.

We remark that a minor difference in our proof compared to the construction in [Hof95], is that we define the autocorrelation measure  $\gamma$  not as the volume averaged convolution of the two measures  $\delta_\Lambda$  and  $\delta_{-\Lambda}$ , but instead as a convolution averaged with respect to the number of points in  $\Lambda$ . It turns out that using this normalization allows in our context to establish more directly the desired properties of the measure  $\gamma$ .

*Proof of Theorem 3.1.* Suppose that  $\Omega$  is a spectral set, and let  $\Lambda$  be a spectrum for  $\Omega$ . Then  $\Lambda$  is a uniformly discrete and relatively dense set in  $\mathbb{R}^d$ . For each  $r > 0$  we denote  $\Lambda_r := \Lambda \cap B_r$ , where  $B_r$  is the open ball of radius  $r$  centered at the origin. Then there is  $r_0$  such that  $\Lambda_r$  is nonempty for every  $r \geq r_0$ . Consider a family of measures  $\{\gamma_r\}$ ,  $r \geq r_0$ , defined by  $\gamma_r := |\Lambda_r|^{-1} \delta_{\Lambda_r} * \delta_{-\Lambda_r}$ . Each  $\gamma_r$  is a positive, finite measure on  $\mathbb{R}^d$ , whose support is the finite set  $\Lambda_r - \Lambda_r$ . We have  $\widehat{\gamma}_r(t) = |\Lambda_r|^{-1} |\widehat{\delta}_{\Lambda_r}(t)|^2 \geq 0$ , hence  $\gamma_r$  is a positive-definite measure. Since  $\Lambda$  is uniformly discrete, the measures  $\gamma_r$  are uniformly translation-bounded, namely, for any open ball  $B$  we have  $\sup_x \gamma_r(B+x) \leq C(\Lambda, B)$ ,



where  $C(\Lambda, B)$  is a constant which does not depend on  $r$ . It follows that we may choose a sequence  $r_n \rightarrow \infty$  such that if we define  $\mu_n := \gamma_{r_n}$ , then the sequence  $\{\mu_n\}$  converges vaguely to some measure  $\gamma$  on  $\mathbb{R}^d$ .

The measure  $\gamma$  is translation-bounded, positive and positive-definite. It follows from the positive-definiteness of  $\gamma$  that its Fourier transform  $\widehat{\gamma}$  is a positive measure. Given any open ball  $B$ , let  $\varphi_B$  be a Schwartz function such that  $\varphi_B \geq \mathbf{1}_B$ . Then we have

$$\widehat{\gamma}(B+x) \leq \int \varphi_B(y-x) d\widehat{\gamma}(y) = \int e^{-2\pi i \langle x, t \rangle} \widehat{\varphi}_B(t) d\gamma(t),$$

and hence

$$\sup_{x \in \mathbb{R}^d} \widehat{\gamma}(B+x) \leq \int |\widehat{\varphi}_B(t)| d\gamma(t) < \infty$$

(the last integral is finite since  $\widehat{\varphi}_B$  has fast decay, while  $\gamma$  is translation-bounded). We conclude that  $\widehat{\gamma}$  is a translation-bounded measure.

Define now another sequence of measures  $\{\nu_n\}$  by  $\nu_n := |\Lambda_{r_n}|^{-1} \delta_\Lambda * \delta_{-\Lambda_{r_n}}$ . The measures  $\nu_n$  are uniformly translation-bounded, again due to the uniform discreteness of  $\Lambda$ . We claim that the sequence  $\{\nu_n\}$  converges vaguely to the same limit as the sequence  $\{\mu_n\}$ , namely, to the measure  $\gamma$ . To see this, it will be enough to check that the sequence of differences  $\{\nu_n - \mu_n\}$  converges vaguely to zero. Indeed, we have

$$\nu_n - \mu_n = |\Lambda_{r_n}|^{-1} \delta_{\Lambda \setminus \Lambda_{r_n}} * \delta_{-\Lambda_{r_n}}.$$

For any fixed  $s > 0$ , the total mass of the measure  $\nu_n - \mu_n$  in the open ball  $B_s$  is equal to  $|\Lambda_{r_n}|^{-1}$  times the number of pairs  $(\lambda, \lambda') \in \Lambda \times \Lambda$  such that  $|\lambda| < r_n$ ,  $|\lambda'| \geq r_n$  and  $|\lambda' - \lambda| < s$ . As  $n \rightarrow \infty$ , the number of such pairs is not greater than a constant multiple of  $r_n^{d-1}$  since  $\Lambda$  is a uniformly discrete set, while the number of elements in the set  $\Lambda_{r_n}$  is not less than a constant multiple of  $r_n^d$  since  $\Lambda$  is also relatively dense. Hence  $|\nu_n - \mu_n|(B_s)$  tends to zero as  $n \rightarrow \infty$ . Since this is true for any  $s > 0$ , it follows that the sequence  $\{\nu_n - \mu_n\}$  converges to zero, and so the sequence  $\nu_n$  converges to  $\gamma$ .

Let  $f := m(\Omega)^{-2} |\widehat{\mathbf{1}}_\Omega|^2$ . Since  $\Lambda$  is a spectrum for  $\Omega$ , by Lemma 2.3 we have  $f * \delta_\Lambda = 1$  a.e. As  $\nu_n$  is an average of translates of the measure  $\delta_\Lambda$ , we also have  $f * \nu_n = 1$  a.e. for every  $n$ . Since the measures  $\nu_n$  are uniformly translation-bounded and converge vaguely to  $\gamma$ , it follows from Lemma 2.2 that  $f * \gamma = 1$  a.e. as well. In turn, using Lemma 2.1 this implies that  $\widehat{f} \cdot \widehat{\gamma} = \delta_0$ . Since  $\widehat{f}(0) = \int f = m(\Omega)^{-1}$ , we deduce that  $\widehat{\gamma} = m(\Omega) \delta_0$  in the open set  $\{t : \widehat{f}(t) \neq 0\}$ . But the Fourier transform of  $f$  is the function  $\widehat{f} = m(\Omega)^{-2} \mathbf{1}_\Omega * \mathbf{1}_{-\Omega}$ , that is, we have  $\widehat{f}(t) = m(\Omega)^{-2} m(\Omega \cap (\Omega + t))$  for every  $t \in \mathbb{R}^d$ . Hence  $\widehat{f}(t) \neq 0$  if and only if  $t \in \Delta(\Omega)$ , and we conclude that  $\widehat{\gamma} = m(\Omega) \delta_0$  in the open set  $\Delta(\Omega)$ .

Since  $\Lambda$  is a uniformly discrete set, there exists  $\varepsilon > 0$  such that  $(\Lambda - \lambda) \cap B_\varepsilon = \{0\}$  for every  $\lambda \in \Lambda$ . In other words, for every  $\lambda \in \Lambda$  we have  $\delta_\Lambda * \delta_{-\lambda} = \delta_0$  in the ball  $B_\varepsilon$ . The measure  $\nu_n$  is an average of measures of the form  $\delta_\Lambda * \delta_{-\lambda}$  ( $\lambda \in \Lambda$ ), so we also have  $\nu_n = \delta_0$  in the ball  $B_\varepsilon$ . It follows that the same is true for the vague limit  $\gamma$  of the sequence  $\nu_n$ , that is, we have  $\gamma = \delta_0$  in the open ball  $B_\varepsilon$ .

The supports of all the measures  $\mu_n$  are contained in the set of differences  $\Lambda - \Lambda$ . Since  $\Lambda$  is a spectrum for  $\Omega$ , the set  $\Lambda - \Lambda$  is contained in the closed set  $\mathcal{Z}(\widehat{\mathbf{1}}_\Omega) \cup \{0\}$ . The measure  $\gamma$  is the vague limit of the sequence  $\mu_n$ , so it follows that the closed support of  $\gamma$  is also contained in the set  $\mathcal{Z}(\widehat{\mathbf{1}}_\Omega) \cup \{0\}$ . The theorem is thus proved.  $\square$

3.2. We can now use Theorem 3.1 to deduce Theorem 1.5.

*Proof of Theorem 1.5.* Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ , and assume that  $\Omega$  is spectral. We must show that  $\Omega^{\mathbb{C}}$  admits a weak tiling by translates of  $\Omega$ .

Indeed, let  $\gamma$  be the measure given by Theorem 3.1. We have  $\widehat{\mathbb{1}}_{-\Omega}(t) = \overline{\widehat{\mathbb{1}}_{\Omega}(t)}$ , and hence  $\text{supp}(\gamma)$  is contained in the set  $\mathcal{Z}(\widehat{\mathbb{1}}_{-\Omega}) \cup \{0\}$ . We also have  $\gamma = \delta_0$  in some open neighborhood of the origin. Since  $\widehat{\mathbb{1}}_{-\Omega}(0) = m(\Omega)$ , it follows that  $\gamma \cdot \widehat{\mathbb{1}}_{-\Omega} = m(\Omega) \delta_0$ .

The measure  $\gamma$  is translation-bounded, and its Fourier transform  $\widehat{\gamma}$  is a translation-bounded measure as well. Hence by Lemma 2.1, the Fourier transform of the measure  $\gamma \cdot \widehat{\mathbb{1}}_{-\Omega}$  is the function  $\widehat{\gamma} * \mathbb{1}_{\Omega}$ . We thus conclude that  $\widehat{\gamma} * \mathbb{1}_{\Omega} = m(\Omega)$  a.e.

The measure  $\widehat{\gamma}$  is positive, and satisfies  $\widehat{\gamma} = m(\Omega) \delta_0$  in the open set  $\Delta(\Omega)$ . We can therefore write  $\widehat{\gamma} = m(\Omega) (\delta_0 + \mu)$ , where  $\mu$  is a positive, translation-bounded (and hence locally finite) measure. The condition  $\widehat{\gamma} * \mathbb{1}_{\Omega} = m(\Omega)$  a.e. is then equivalent to  $\mathbb{1}_{\Omega} * \mu = \mathbb{1}_{\Omega^{\mathbb{C}}}$  a.e., and we obtain that  $\Omega^{\mathbb{C}}$  has a weak tiling by translates of  $\Omega$ .  $\square$

3.3. To illustrate the construction of the autocorrelation measure  $\gamma$  in the proof of Theorem 3.1, and the corresponding weak tiling of  $\Omega^{\mathbb{C}}$  obtained in Theorem 1.5, we now look at a few examples.

**Example 3.2.** Assume that  $\Omega \subset \mathbb{R}^d$  tiles the space with respect to a *lattice* translation set  $L$ . Let  $\Lambda$  be a spectrum of  $\Omega$  given by the dual lattice, that is,  $\Lambda = L^*$ . Then  $\Lambda - \lambda = \Lambda$  for every  $\lambda \in \Lambda$ , and hence the measures  $\nu_n := |\Lambda_{r_n}|^{-1} \delta_{\Lambda} * \delta_{-\Lambda_{r_n}}$ , whose vague limit is the autocorrelation measure  $\gamma$ , satisfy  $\nu_n = \delta_{\Lambda}$  for every  $n$ . We conclude that the autocorrelation measure  $\gamma$  is given by  $\gamma = \delta_{\Lambda} = \delta_{L^*}$ . In turn, by the Poisson summation formula we get  $\widehat{\gamma} = m(\Omega) \delta_L$ , and so the weak tiling measure  $\mu$  in Theorem 1.5 is given by  $\mu = \delta_{L \setminus \{0\}}$ . We thus see that the weak tiling of  $\Omega^{\mathbb{C}}$  is in this case a proper tiling.

**Example 3.3.** Let  $\Omega = [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$ . Then  $\Omega$  tiles the real line  $\mathbb{R}$  properly with respect to the translation set  $\Lambda = 2\mathbb{Z} \cup (2\mathbb{Z} + \frac{1}{2})$ . It is well-known and not difficult to verify that the same set  $\Lambda$  is also a spectrum for  $\Omega$ . The set  $\Lambda$  is periodic, but it is not a lattice. We calculate the autocorrelation measure  $\gamma$  as the vague limit of the measures  $\nu_n := |\Lambda_n|^{-1} \delta_{\Lambda} * \delta_{-\Lambda_n}$ , where  $\Lambda_n := \Lambda \cap [-2n, 2n)$ . Then all the measures  $\nu_n$ , as well as the vague limit  $\gamma$ , coincide with the measure  $\sum_{k \in \mathbb{Z}} \cos^2(\frac{\pi k}{4}) \delta_{k/2}$  (so that, in fact,  $\nu_n$  does not depend on  $n$ ). In turn, one can check using the Poisson summation formula that  $\widehat{\gamma} = \gamma$ . We thus obtain that the weak tiling measure  $\mu$  in Theorem 1.5 is a pure point measure (that is,  $\mu$  has no continuous part), but the weak tiling of  $\Omega^{\mathbb{C}}$  with respect to this measure  $\mu$  is not a proper tiling.

**Example 3.4.** Let  $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$  be the unit cube in  $\mathbb{R}^2$ , and let  $\alpha$  be an irrational real number. Define  $\Lambda$  to be the set of all points of the form  $(n, n^2\alpha + m)$  where  $n$  and  $m$  are integers. It is easy to see that  $\Omega$  tiles the plane  $\mathbb{R}^2$  with respect to the translation set  $\Lambda$ . It is known, see [JP99], that the translation sets for tilings by the unit cube  $\Omega$  coincide with the spectra of  $\Omega$ . Hence  $\Lambda$  is a spectrum for  $\Omega$ .

In this example, we will calculate the autocorrelation measure  $\gamma$  as the vague limit of the measures  $\nu_N := |\Lambda_N|^{-1} \delta_{\Lambda} * \delta_{-\Lambda_N}$ , where  $\Lambda_N := \Lambda \cap Q_N$  is the intersection of  $\Lambda$  with the cube  $Q_N := [-N, N]^2$ , that is, we will be averaging not with respect to balls, but instead with respect to cubes with sides parallel to the axes. Notice that the proof of Theorem 3.1 remains valid if the averages are taken with respect to cubes instead

of balls (in fact, Hof used cubes and not balls in [Hof95]), although the autocorrelation measure  $\gamma$  could, in principle, depend on the shape over which the average is taken.

It follows from the fact that  $\Lambda - \Lambda \subset \mathbb{Z} \times \mathbb{R}$  that all the measures  $\nu_N$ , and hence also the autocorrelation measure  $\gamma$ , are supported on  $\mathbb{Z} \times \mathbb{R}$ . The restriction of the measure  $\nu_N$  to a line of the form  $\{h\} \times \mathbb{R}$ , where  $h \in \mathbb{Z}$ , is given by

$$\frac{1}{2N} \sum_{-N \leq n < N} \sum_{k \in \mathbb{Z}} \delta_{(h, 2hn\alpha + h^2\alpha + k)}.$$

Since  $\alpha$  is irrational, it follows from Weyl's equidistribution theorem that  $\nu_N$  converges vaguely to the one-dimensional Lebesgue measure on each line  $\{h\} \times \mathbb{R}$  such that  $h \neq 0$ . On the other hand, on the line  $\{0\} \times \mathbb{R}$  all the measures  $\nu_N$  coincide with  $\delta_{\{0\} \times \mathbb{Z}}$ . We conclude that  $\gamma = \delta_0 \times \delta_{\mathbb{Z}} + \delta_{\mathbb{Z} \setminus \{0\}} \times m_1$ , where  $m_1$  denotes the one-dimensional Lebesgue measure. In turn, the diffraction measure  $\widehat{\gamma}$  is given by  $\widehat{\gamma} = \delta_{\mathbb{Z}} \times \delta_0 + m_1 \times \delta_{\mathbb{Z} \setminus \{0\}}$ . So in this example, we obtain that the weak tiling of  $\Omega^{\complement}$  given by Theorem 1.5 involves a measure  $\mu$  that has both a pure point part and a (singular) continuous part.

3.4. Our main application of Theorem 1.5 will be for the proof of Fuglede's conjecture for convex bodies in  $\mathbb{R}^d$ . This will be done in the following sections. But before that, we mention some simple applications of the theorem to other classes of domains.

The following result, for instance, excludes many sets from being spectral:

**Theorem 3.5.** *Let  $\Omega$  be a bounded, measurable set in  $\mathbb{R}^d$ . Suppose that there exists a measurable set  $S \subset \mathbb{R}^d$  with the following properties:*

- (i)  $m(S) > 0$ ;
- (ii)  $m(S \cap \Omega) = 0$ ;
- (iii) *for every  $x \in \mathbb{R}^d$ , if  $m((\Omega + x) \cap S) > 0$  then  $m((\Omega + x) \cap \Omega) > 0$ .*

*Then  $\Omega$  cannot be a spectral set.*

Informally speaking, the assumption in this theorem means that there exists a portion  $S$  of the complement  $\Omega^{\complement}$  of  $\Omega$ , such that no translated copy  $\Omega + x$  of  $\Omega$  can even partly cover  $S$  unless this translated copy also covers a part of  $\Omega$  as well. The theorem says that a set  $\Omega$  satisfying this assumption cannot be spectral.

*Proof of Theorem 3.5.* Suppose to the contrary that  $\Omega$  is spectral. Then by Theorem 1.5 the complement  $\Omega^{\complement}$  of  $\Omega$  admits a weak tiling by translates of  $\Omega$ , so there is a positive, locally finite measure  $\mu$  such that  $\mathbf{1}_{\Omega^{\complement}} * \mu = \mathbf{1}_{\Omega^{\complement}}$  a.e. Using condition (ii), we then have

$$m(S) = m(S \cap \Omega^{\complement}) = \int_S \mathbf{1}_{\Omega^{\complement}} dm = \int_S (\mathbf{1}_{\Omega} * \mu) dm = \int \varphi d\mu, \quad (3.1)$$

where  $\varphi(x) := m((\Omega + x) \cap S)$ . The measure  $\mu$  is supported on  $\Delta(\Omega)^{\complement}$  by Corollary 2.6, while the function  $\varphi$  vanishes on  $\Delta(\Omega)^{\complement}$  due to condition (iii). Hence the right hand side of (3.1) must be zero. It follows that  $m(S) = 0$ , a contradiction to condition (i).  $\square$

In Figure 3.1 we illustrate an example of a planar domain that can be shown to be non-spectral using Theorem 3.5. The essential feature of this domain is that it is connected but its complement is not connected, so that there is a ‘‘hole’’ inside the domain. We observe that if  $S$  is a set of positive measure contained in the hole, then

$S$  satisfies the conditions (i), (ii) and (iii) in Theorem 3.5, and therefore the domain cannot be spectral.

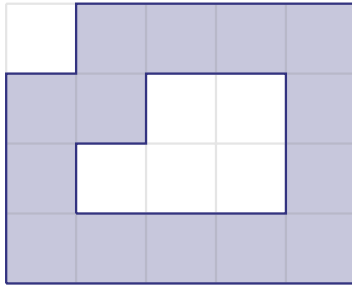


FIGURE 3.1. The illustration presents a planar domain that can be shown to be non-spectral using Theorem 3.5.

As another example, we can use Theorem 3.5 to obtain the following result:

**Theorem 3.6.** *Let  $\Omega$  be a bounded, open set in  $\mathbb{R}^d$ . If  $\Omega$  is spectral, then its boundary  $\text{bd}(\Omega)$  must be a set of Lebesgue measure zero.*

*Proof.* Suppose that  $\text{bd}(\Omega)$  has positive Lebesgue measure. We will show that the set  $S := \text{bd}(\Omega)$  satisfies the conditions (i), (ii) and (iii) in Theorem 3.5, and hence  $\Omega$  cannot be spectral. Indeed, condition (i) holds by assumption. Since  $\Omega$  is an open set, the two sets  $\Omega$  and  $S$  are disjoint, hence condition (ii) holds as well. To verify condition (iii), we let  $x \in \mathbb{R}^d$  be such that  $m((\Omega + x) \cap S) > 0$ . In particular, the set  $(\Omega + x) \cap S$  is nonempty. The set  $\Omega + x$  is open, while  $S$  is contained in the closure of  $\Omega$ , so it follows that  $(\Omega + x) \cap \Omega$  must be nonempty. But the set  $(\Omega + x) \cap \Omega$  is open, since it is the intersection of two open sets, so it can be nonempty only if  $m((\Omega + x) \cap \Omega) > 0$ . This confirms condition (iii), and thus the proof is concluded by Theorem 3.5.  $\square$

On the other hand, if  $\Omega$  is a convex body in  $\mathbb{R}^d$ , then there exists no set  $S$  satisfying the conditions (i), (ii) and (iii) in Theorem 3.5. So in order to study the spectrality problem for convex bodies, we must use the weak tiling condition  $\mathbb{1}_\Omega * \mu = \mathbb{1}_\Omega c$  a.e. in a stronger way. This will be done in the following sections.

#### 4. SPECTRAL CONVEX BODIES ARE POLYTOPES

In this section we prove Theorem 1.2, which states that if a convex body  $\Omega \subset \mathbb{R}^d$  is spectral, then  $\Omega$  must be a convex polytope. This shows that the first condition (i) in the Venkov-McMullen theorem is necessary not only for tiling by translations, but also for the spectrality of a convex body  $\Omega$ .

The fact that a convex body which tiles by translations must be a polytope, is a classical result that is due to Minkowski, see [McM80] or [Gru07, Section 32.2]. We will prove a stronger version of this result, which involves only a weak tiling assumption:

**Theorem 4.1.** *Let  $K$  be a convex body in  $\mathbb{R}^d$ . If the complement  $K^c$  of  $K$  has a weak tiling by translates of  $K$ , then  $K$  must be a convex polytope.*

Combining Theorem 4.1 with Theorem 1.5 yields that any spectral convex body in  $\mathbb{R}^d$  must be a convex polytope, and Theorem 1.2 thus follows.

The rest of the section is devoted to the proof of Theorem 4.1.

4.1. We start by recalling some basic facts about convex bodies in  $\mathbb{R}^d$ . For more details, the reader is referred to [Gru07, Sections 4.1, 5.1, 14.1].

Recall that a set  $K \subset \mathbb{R}^d$  is called a *convex body* if  $K$  is a compact, convex set with nonempty interior. In what follows, we assume  $K$  to be a convex body in  $\mathbb{R}^d$ .

A hyperplane  $H$  is called a *support hyperplane* of  $K$  at a point  $x \in \text{bd}(K)$ , if  $x \in H$  and  $K$  is contained in one of the two closed halfspaces bounded by  $H$ . In this case, we denote the halfspace containing  $K$  by  $H^-$ , and the other one by  $H^+$ .  $H^-$  is called a *support halfspace* of  $K$  at the point  $x$ . It can be represented in the form

$$H^- = \{z \in \mathbb{R}^d : \langle z, \xi \rangle \leq \langle x, \xi \rangle\},$$

where  $\xi$  is a vector in  $\mathbb{R}^d$  such that  $|\xi| = 1$ . We call  $\xi$  an *exterior normal unit vector* of  $K$ , or of  $H$ , at the point  $x$ .

For each  $x \in \text{bd}(K)$ , there exists at least one support hyperplane  $H$  of  $K$  at  $x$ . The support hyperplane at  $x$  need not, in general, be unique. If the support hyperplane is unique then  $x$  is called a *regular* boundary point of  $K$ , and otherwise it is called a *singular* boundary point.

For each vector  $\xi \in \mathbb{R}^d$ ,  $|\xi| = 1$ , there is a unique support hyperplane  $H(K, \xi)$  of  $K$  with exterior normal vector  $\xi$ . The set  $S(K, \xi) := K \cap H(K, \xi)$  is called the *support set* of  $K$  determined by  $\xi$ . The support set is a compact, convex subset of  $\text{bd}(K)$ .

A hyperplane  $H$  is said to *separate* two convex bodies  $K$  and  $K'$ , if  $K$  is contained in one of the two closed halfspaces bounded by  $H$ , while  $K'$  is contained in the other closed halfspace. Any two convex bodies  $K$  and  $K'$  with disjoint interiors have at least one separating hyperplane  $H$ .

A *convex polytope*  $P \subset \mathbb{R}^d$  is the convex hull of a finite number of points. Equivalently, a convex polytope is a bounded set  $P$  which can be represented as the intersection of finitely many closed halfspaces.

4.2.

**Lemma 4.2.** *Let  $K$  be a convex body in  $\mathbb{R}^d$ . If  $K$  is not a polytope, then there exists an infinite sequence  $x_n$  of regular boundary points of  $K$  such that the corresponding exterior normal unit vectors  $\xi_n$  are distinct, that is,  $\xi_n \neq \xi_m$  whenever  $n \neq m$ .*

*Proof.* We will rely on the fact that if  $K \subset \mathbb{R}^d$  is a convex body, then the set of regular boundary points of  $K$  constitutes a dense subset of  $\text{bd}(K)$  (see [Gru07, Section 5.1, Theorem 5.1 or Theorem 5.2]).

We construct the sequence  $x_n$  by induction. Let  $x_1$  be any regular boundary point of  $K$ , and  $\xi_1$  be the exterior normal unit vector at the point  $x_1$ . Now suppose that the points  $x_1, x_2, \dots, x_{n-1}$  have already been chosen, that they are regular boundary points of  $K$ , and their corresponding exterior normal unit vectors  $\xi_1, \xi_2, \dots, \xi_{n-1}$  are distinct.

Let

$$F_n := \bigcup_{j=1}^{n-1} S(K, \xi_j),$$

where  $S(K, \xi_j)$  is the support set of  $K$  determined by  $\xi_j$ . Then  $F_n$  is a closed subset of  $\text{bd}(K)$ . We claim that  $F_n$  must be a proper subset of  $\text{bd}(K)$ . Indeed, suppose to the

contrary that  $\text{bd}(K) = F_n$ . Then each point  $y \in \text{bd}(K)$  belongs to  $S(K, \xi_{j(y)})$  for some  $1 \leq j(y) \leq n-1$ . Hence the closed halfspace

$$H_y^- := \{z : \langle z, \xi_{j(y)} \rangle \leq \langle x_{j(y)}, \xi_{j(y)} \rangle\}$$

is a support halfspace of  $K$  at the point  $y$ . It is known (see [Gru07, Section 4.1, Corollary 4.1]) that if for each  $y \in \text{bd}(K)$ ,  $H_y^-$  is a support halfspace of  $K$  at the point  $y$ , then

$$K = \bigcap_{y \in \text{bd}(K)} H_y^-.$$

But the collection  $\{H_y^- : y \in \text{bd}(K)\}$  has at most  $n-1$  distinct members. It follows that  $K$  is the intersection of finitely many closed halfspaces, hence  $K$  is a convex polytope. Since we have assumed that  $K$  is not a polytope, we arrive at a contradiction. This establishes our claim that  $F_n$  must be a proper subset of  $\text{bd}(K)$ .

Since  $F_n$  is a closed set, it follows that its complement  $\text{bd}(K) \setminus F_n$  is a nonempty, relatively open subset of  $\text{bd}(K)$ . The set of regular boundary points of  $K$  is dense in  $\text{bd}(K)$ , hence we can choose a regular point  $x_n$  in  $\text{bd}(K) \setminus F_n$ . The exterior normal unit vector  $\xi_n$  at the point  $x_n$  is then distinct from all the vectors  $\xi_1, \xi_2, \dots, \xi_{n-1}$ . This completes the inductive construction, and thus concludes the proof of the lemma.  $\square$

### 4.3.

**Lemma 4.3.** *Let  $K$  be a convex body in  $\mathbb{R}^d$ , let  $x$  be a regular boundary point of  $K$ , and let  $\xi$  be the exterior normal unit vector at the point  $x$ . Suppose that  $U$  is an open neighborhood of the set  $-S(K, -\xi) + x$ , where  $S(K, -\xi)$  is the support set of  $K$  determined by the vector  $-\xi$ . Then there is an open neighborhood  $V$  of  $x$  such that for any  $t \in \Delta(K)^\circ$ , the set  $K + t$  cannot intersect  $V$  unless  $t \in U$ .*

*Proof.* Suppose to the contrary that the assertion is not true. Then there is a sequence  $t_j \in \Delta(K)^\circ \cap U^\circ$ , and for each  $j$  there is a point  $x_j \in K + t_j$ , such that  $x_j \rightarrow x$ . Since  $t_j$  must remain bounded as  $j \rightarrow \infty$ , we may assume, by passing to a subsequence if needed, that  $t_j \rightarrow t$ .

Since the set  $\Delta(K)^\circ$  is closed, the limit  $t$  of the sequence  $t_j$  belongs to  $\Delta(K)^\circ$ . Hence  $K$  and  $K + t$  are two convex bodies with disjoint interiors. It follows that there exists a hyperplane  $H$  separating  $K$  from  $K + t$ . The point  $x$  lies on both  $K$  and  $K + t$ , and therefore  $x$  must lie on  $H$ . It follows that  $x$  is a boundary point of both  $K$  and  $K + t$ , and  $H$  is a support hyperplane of both  $K$  and  $K + t$  at the point  $x$ . Since  $x$  is a regular boundary point of  $K$ ,  $H$  is the unique support hyperplane  $H(K, \xi)$  of  $K$  at the point  $x$ .

Now  $H(K, \xi)$  is a support hyperplane of  $K + t$  at the point  $x$ , with exterior normal vector  $-\xi$ . It follows that  $H(K, \xi) - t$  is a support hyperplane of  $K$  at the point  $x - t$ , with exterior normal vector  $-\xi$ . Since a support hyperplane of  $K$  with a given exterior normal vector is unique, we must have that  $H(K, \xi) - t = H(K, -\xi)$ , and hence the point  $x - t$  lies on the support set  $S(K, -\xi)$  determined by the vector  $-\xi$ . In other words, we have obtained that  $t \in -S(K, -\xi) + x$ . But since the latter set is contained in  $U$ , we conclude that  $t \in U$ . However this is not possible, since  $t$  is the limit of a sequence  $\{t_j\} \subset U^\circ$  and the set  $U$  is open. This contradiction completes the proof.  $\square$

4.4.

**Lemma 4.4.** *Let  $K \subset \mathbb{R}^d$  be a convex body whose complement  $K^c$  admits a weak tiling by translates of  $K$ , that is, there is a positive, locally finite measure  $\mu$  such that  $\mathbb{1}_K * \mu = \mathbb{1}_{K^c}$  a.e. Let  $x$  be a regular boundary point of  $K$ , and let  $\xi$  be the exterior normal unit vector at the point  $x$ . Then we have*

$$\mu(-S(K, -\xi) + x) \geq 1,$$

where  $S(K, -\xi)$  is the support set of  $K$  determined by the vector  $-\xi$ .

*Proof.* Suppose to the contrary that  $\mu(-S(K, -\xi) + x) < 1$ . Then there is an open neighborhood  $U$  of the set  $-S(K, -\xi) + x$  such that also  $\mu(U) < 1$ . By Lemma 4.3 there is an open neighborhood  $V$  of  $x$  such that for any  $t \in \Delta(K)^c$ , the set  $K + t$  cannot intersect  $V$  unless  $t \in U$ .

We now decompose the measure  $\mu$  into a sum  $\mu = \mu' + \mu''$ , where  $\mu' := \mu \cdot \mathbb{1}_U$  and  $\mu'' := \mu \cdot \mathbb{1}_{U^c}$ . The support of the measure  $\mu$  must be contained in  $\Delta(K)^c$  due to the weak tiling assumption (Corollary 2.6), and therefore  $\mu''$  is supported on the closed set  $\Delta(K)^c \cap U^c$ . Hence for any  $t \in \text{supp}(\mu'')$ , the set  $K + t$  does not intersect  $V$ , which implies that  $\mathbb{1}_K * \mu'' = 0$  a.e. in  $V$ . It follows that

$$\mathbb{1}_{V \cap K^c} = \mathbb{1}_V \cdot (\mathbb{1}_K * \mu) = \mathbb{1}_V \cdot (\mathbb{1}_K * \mu') + \mathbb{1}_V \cdot (\mathbb{1}_K * \mu'') = \mathbb{1}_V \cdot (\mathbb{1}_K * \mu') \quad \text{a.e.},$$

and therefore

$$\|\mathbb{1}_{V \cap K^c}\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathbb{1}_K * \mu'\|_{L^\infty(\mathbb{R}^d)} \leq \int d\mu' = \mu(U) < 1.$$

This implies that the set  $V \cap K^c$  must have Lebesgue measure zero. But this is not possible, since  $V$  is an open neighborhood of the point  $x \in \text{bd}(K)$ , and hence  $V \cap K^c$  has nonempty interior. We thus arrive at a contradiction, which concludes the proof.  $\square$

4.5.

**Lemma 4.5.** *Let  $K$  be a convex body in  $\mathbb{R}^d$ , and  $x, x'$  be two regular boundary points of  $K$ . Let  $\xi$  and  $\xi'$  be the exterior normal unit vectors at the points  $x$  and  $x'$  respectively, and let  $S(K, -\xi)$  and  $S(K, -\xi')$  be the support sets of  $K$  determined by  $-\xi$  and  $-\xi'$  respectively. If we have  $\xi \neq \xi'$ , then the two sets  $-S(K, -\xi) + x$  and  $-S(K, -\xi') + x'$  are disjoint.*

*Proof.* Suppose to the contrary that the two sets  $-S(K, -\xi) + x$  and  $-S(K, -\xi') + x'$  are not disjoint, so they have at least one point in common. Then there exist two points  $y \in S(K, -\xi)$  and  $y' \in S(K, -\xi')$  such that  $x - y = x' - y'$ .

The support hyperplanes of  $K$  with exterior normal vectors  $\xi$  and  $-\xi$  are respectively given by

$$H(K, \xi) = \{z : \langle z, \xi \rangle = \langle x, \xi \rangle\} \quad \text{and} \quad H(K, -\xi) = \{z : \langle z, \xi \rangle = \langle y, \xi \rangle\}. \quad (4.1)$$

Since  $K$  is contained in the closed slab bounded by these two hyperplanes, we have

$$K \subset \{z : \langle y, \xi \rangle \leq \langle z, \xi \rangle \leq \langle x, \xi \rangle\}. \quad (4.2)$$

The point  $x'$  is a regular boundary point of  $K$ , and so  $H(K, \xi')$  is the unique support hyperplane of  $K$  at the point  $x'$ . Since  $\xi \neq \xi'$ , it follows that  $x'$  does not lie on  $H(K, \xi)$ . Using (4.1) and (4.2) this implies that  $\langle x', \xi \rangle < \langle x, \xi \rangle$ .

Now denote  $h := x - y = x' - y'$ , then it follows that

$$\langle y', \xi \rangle = \langle x' - h, \xi \rangle < \langle x - h, \xi \rangle = \langle y, \xi \rangle.$$

But since  $y' \in K$  this contradicts (4.2), and thus the lemma is proved.  $\square$

4.6.

*Proof of Theorem 4.1.* Assume that  $K \subset \mathbb{R}^d$  is a convex body whose complement  $K^c$  has a weak tiling by translates of  $K$ . We will prove that  $K$  is necessarily a polytope.

Suppose to the contrary that  $K$  is not a polytope. Then by Lemma 4.2 there exists an infinite sequence  $x_n$  of regular boundary points of  $K$ , such that the corresponding exterior normal unit vectors  $\xi_n$  are mutually distinct.

The set  $K^c$  has a weak tiling by translates of  $K$ , so there is a positive, locally finite measure  $\mu$  such that  $\mathbb{1}_K * \mu = \mathbb{1}_{K^c}$  a.e. Let  $S(K, -\xi_n)$  be the support set of  $K$  determined by the vector  $-\xi_n$ , then by Lemma 4.4 we have  $\mu(-S(K, -\xi_n) + x_n) \geq 1$  for each  $n$ .

Since the exterior normal unit vectors  $\xi_n$  are distinct, we get from Lemma 4.5 that the sets  $-S(K, -\xi_n) + x_n$  are pairwise disjoint. On the other hand, all these sets are contained in  $K - K$ . Hence the set  $K - K$  contains an infinite sequence of pairwise disjoint subsets, such that the total mass of  $\mu$  in each one of these sets is at least 1. It follows that we must have  $\mu(K - K) = +\infty$ . But this is not possible, as  $K - K$  is a bounded set and  $\mu$  is a locally finite measure. We thus arrive at a contradiction, which concludes the proof of the theorem.  $\square$

## 5. SPECTRAL CONVEX POLYTOPES CAN TILE BY TRANSLATIONS, I

So far, we have established that a spectral convex body  $\Omega \subset \mathbb{R}^d$  must be a convex polytope (Theorem 1.2). We also know from the result in [Kol00] (or [KP02]) that  $\Omega$  must be centrally symmetric, and from the result in [GL17, Section 4] that all the facets of  $\Omega$  must be centrally symmetric as well. It now remains to prove that each belt of  $\Omega$  must consist of either 4 or 6 facets (Theorem 1.3). The proof will be given throughout the present section and the next one.

The key results of the present section are Lemma 5.8 and Lemma 5.9.

5.1. We begin by recalling some basic facts about convex polytopes in  $\mathbb{R}^d$ . For more details, we refer to [BG09, Section 1.A], [Gru07, Section 14.1] and [Sch14, Section 2.4].

A *convex polytope*  $A \subset \mathbb{R}^d$  is the convex hull of a finite number of points. Equivalently, a convex polytope is a bounded set  $A$  which can be represented as the intersection of finitely many closed halfspaces.

We denote by  $\text{aff}(A)$  the *affine hull* of a convex polytope  $A \subset \mathbb{R}^d$ , that is,  $\text{aff}(A)$  is the smallest affine subspace containing  $A$ . By the *relative interior* and *relative boundary* of  $A$  we refer respectively to the interior and boundary relative to  $\text{aff}(A)$ . The relative interior of  $A$  will be denoted by  $\text{relint}(A)$ .

A *face* of a convex polytope  $A$  is a support set of  $A$ , that is, the intersection of  $A$  with a support hyperplane of  $A$ . (There is also an alternative, equivalent definition, according to which a face of  $A$  is an *extreme subset* of  $A$ , that is, a convex subset  $F \subset A$  such that if  $x, y \in A$ ,  $(1 - \lambda)x + \lambda y \in F$ ,  $0 < \lambda < 1$ , then  $x, y \in F$ .)



If  $A \subset \mathbb{R}^d$  is a convex polytope with nonempty interior, then a  $(d-1)$ -dimensional face of  $A$  is called a *facet* of  $A$ , while a  $(d-2)$ -dimensional face is called a *subfacet* of  $A$ . If  $G$  is a subfacet of  $A$ , then there exist exactly two facets  $F_1$  and  $F_2$  of  $A$  which contain  $G$ , and we have  $G = F_1 \cap F_2$  (see [BG09, Proposition 1.12]).

A convex polytope  $A \subset \mathbb{R}^d$  is said to be *centrally symmetric* if the set  $-A$  is a translate of  $A$ . In this case, there is a unique point  $x \in \mathbb{R}^d$  such that  $-A + x = A - x$ . The point  $x$  is called the *center of symmetry* of  $A$ , and  $A$  is then said to be symmetric with respect to  $x$ .

If  $X, Y$  are two convex subsets of  $\mathbb{R}^d$ , then we use  $\text{conv}\{X, Y\}$  to denote the convex hull of the union  $X \cup Y$ .

A *prism* in  $\mathbb{R}^d$  is a convex polytope of the form  $\text{conv}\{F, F + \tau\}$ , where  $F$  is a  $(d-1)$ -dimensional convex polytope, and  $\tau$  is a vector such that  $F$  and  $F + \tau$  do not lie on the same hyperplane. The two sets  $F$  and  $F + \tau$  are called the *bases* of the prism.

A *slab* in  $\mathbb{R}^d$  is the closed region between two parallel hyperplanes, that is, a set of the form  $\{z : c_1 \leq \langle z, \xi \rangle \leq c_2\}$ , where  $\xi$  is a nonzero vector and  $c_1, c_2$  are constants.

5.2.

**Lemma 5.1.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope with nonempty interior. Then*

$$\text{int}(A) - \text{int}(A) = \text{int}(A) - A = \text{int}(A - A).$$

*Proof.* The fact that  $\text{int}(A)$  is a subset of  $A$  implies that  $\text{int}(A) - \text{int}(A) \subset \text{int}(A) - A$ . In turn,  $\text{int}(A) - A$  is the union of all sets of the form  $\text{int}(A) - x$  where  $x \in A$ , and all these sets are open. Hence  $\text{int}(A) - A$  is an open subset of  $A - A$ , and it follows that  $\text{int}(A) - A \subset \text{int}(A - A)$ . It remains to prove that  $\text{int}(A - A) \subset \text{int}(A) - \text{int}(A)$ . Indeed, let  $h \in \text{int}(A - A)$ . Then there is  $\varepsilon > 0$  such that  $h + \varepsilon h$  is in  $A - A$ . Let  $x, y \in A$  be such that  $h + \varepsilon h = y - x$ . Let  $a$  be an interior point of  $A$ , and define the points  $x' := \lambda x + (1 - \lambda)a$  and  $y' := \lambda y + (1 - \lambda)a$ , where  $\lambda := (1 + \varepsilon)^{-1}$ . Then  $x', y' \in \text{int}(A)$  and we have  $y' - x' = h$ , which shows that  $h \in \text{int}(A) - \text{int}(A)$ .  $\square$

**Lemma 5.2.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope with nonempty interior, and let  $F$  be a facet of  $A$ . Then*

$$\text{relint}(F - F) \subset \text{int}(A - A).$$

*Proof.* Let  $h \in \text{relint}(F - F)$ . Then there is an open set  $U$  such that  $h \in U \cap H \subset F - F$ , where  $H$  denotes the hyperplane through the origin parallel to  $F$ . It follows that if we choose  $\varepsilon > 0$  small enough then  $h + \varepsilon h$  is in  $F - F$ . Let  $x, y \in F$  be such that  $h + \varepsilon h = y - x$ . Let  $a$  be an interior point of  $A$ , and define the points  $x' = \lambda x + (1 - \lambda)a$  and  $y' = \lambda y + (1 - \lambda)a$ , where  $\lambda = (1 + \varepsilon)^{-1}$ . Then  $x', y' \in \text{int}(A)$  and we have  $y' - x' = h$ , which shows that  $h \in \text{int}(A) - \text{int}(A)$ . Using Lemma 5.1 we conclude that  $h \in \text{int}(A - A)$  and the lemma is proved.  $\square$

**Lemma 5.3.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope with nonempty interior. Let  $F$  be a facet of  $A$ , and let  $H^-$  be the support halfspace of  $A$  at the facet  $F$ . Suppose that  $E$  is a compact subset of  $\text{relint}(F)$ . Then there is an open neighborhood  $V$  of  $E$  such that the set  $V \cap H^-$  is contained in  $A$ .*

*Proof.* Let  $F_i, i = 1, 2, \dots, m$ , be all the facets of  $A$ , and suppose that  $F = F_1$ . For each  $i$ , let  $H_i$  be the hyperplane containing the facet  $F_i$ , and let  $H_i^-$  be the support halfspace

of  $A$  at the facet  $F_i$ . Then

$$A = H_1^- \cap H_2^- \cap \cdots \cap H_m^- \quad (5.1)$$

(see [Gru07, Theorem 14.2]). In particular, the set  $E$  is contained in all the halfspaces  $H_i^-$ . On the other hand no point of  $E$  can lie on any one of the hyperplanes  $H_i$ ,  $i \neq 1$ , since  $E$  is a subset of  $\text{relint}(F)$ . Hence there is an open neighborhood  $V$  of  $E$  such that  $V$  is contained in  $H_i^-$  for all  $i \neq 1$ . Using this together with (5.1) implies the claim.  $\square$

**Lemma 5.4.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope with nonempty interior, and let  $G$  be a subfacet of  $A$ . Let  $F_1$  and  $F_2$  be the two adjacent facets of  $A$  that meet at the subfacet  $G$ , and let  $H_1^-$  and  $H_2^-$  be the support halfspaces of  $A$  at the facets  $F_1$  and  $F_2$  respectively. Suppose that  $E$  is a compact subset of  $\text{relint}(G)$ . Then there is an open neighborhood  $V$  of  $E$  such that the set  $V \cap H_1^- \cap H_2^-$  is contained in  $A$ .*

*Proof.* We continue to use the same notations as in the proof of Lemma 5.3. The set  $E$  is again contained in all the halfspaces  $H_i^-$ . However no point of  $E$  can lie on any one of the hyperplanes  $H_i$ ,  $i \neq 1, 2$ , since  $E$  is a subset of  $\text{relint}(G)$ . Hence there is an open neighborhood  $V$  of  $E$  such that  $V$  is contained in  $H_i^-$  for all  $i \neq 1, 2$ . Using this with (5.1) we obtain the assertion of the lemma.  $\square$

5.3.

**Lemma 5.5.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope with nonempty interior, and let  $F$  be a facet of  $A$ . Suppose that  $\xi$  is the exterior normal unit vector of  $A$  at the facet  $F$ , and that  $H^- = \{z : \langle z, \xi \rangle \leq c\}$  is the support halfspace of  $A$  at the facet  $F$ . For each  $\delta > 0$  we let  $P_\delta = A \cap S_\delta$  be the intersection of  $A$  with the slab  $S_\delta = \{z : c - \delta \leq \langle z, \xi \rangle \leq c\}$ , and we let  $Q_\delta = \text{conv}\{F, F - \delta\xi\}$  be the prism with bases  $F$  and  $F - \delta\xi$ . Then we have  $m(P_\delta \Delta Q_\delta) = o(\delta)$  as  $\delta \rightarrow 0$  (see Figure 5.1).*

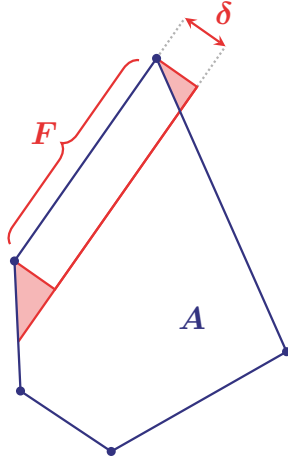


FIGURE 5.1. The shaded region in the illustration represents the symmetric difference  $P_\delta \Delta Q_\delta$  of the sets  $P_\delta$  and  $Q_\delta$  in Lemma 5.5.

*Proof.* Let  $H$  be the hyperplane containing the facet  $F$ . For each  $\varepsilon > 0$ , let  $F_{+\varepsilon}$  be the set of all points of  $H$  whose distance from  $F$  is at most  $\varepsilon$ , and  $F_{-\varepsilon}$  be the set of all points of  $F$  whose distance from the relative boundary of  $F$  is at least  $\varepsilon$ .

Let  $I_\delta = \{s\xi : 0 \leq s \leq \delta\}$  be the closed line segment connecting the origin to the point  $\delta\xi$ . Then there is  $\delta_0 = \delta_0(A, F, \varepsilon) > 0$  such that for every  $\delta < \delta_0$  we have

$$F_{-\varepsilon} - I_\delta \subset P_\delta \subset F_{+\varepsilon} - I_\delta.$$

(the left inclusion may be deduced from Lemma 5.3). Observe that  $Q_\delta = F - I_\delta$ , hence

$$P_\delta \Delta Q_\delta \subset (F_{+\varepsilon} \setminus F_{-\varepsilon}) - I_\delta.$$

This implies that

$$m_d(P_\delta \Delta Q_\delta) \leq \delta m_{d-1}(F_{+\varepsilon} \setminus F_{-\varepsilon}).$$

Since  $m_{d-1}(F_{+\varepsilon} \setminus F_{-\varepsilon})$  tends to zero as  $\varepsilon \rightarrow 0$ , the assertion follows.  $\square$

5.4.

**Lemma 5.6.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope with nonempty interior, and let  $G$  be a subfacet of  $A$ . Let  $F_1$  and  $F_2$  be the two adjacent facets of  $A$  that meet at the subfacet  $G$ , and let  $H_1^-$  and  $H_2^-$  be the support halfspaces of  $A$  at the facets  $F_1$  and  $F_2$  respectively. For each  $\delta > 0$  we let  $P_\delta$  be the set of all points of  $A$  whose distance from  $\text{aff}(G)$  is not greater than  $\delta$ , and we let  $Q_\delta = (G + S_\delta) \cap H_1^- \cap H_2^-$ , where  $S_\delta$  is a closed 2-dimensional ball of radius  $\delta$  centered at the origin and orthogonal to  $\text{aff}(G)$ . Then we have  $m(P_\delta \Delta Q_\delta) = o(\delta^2)$  as  $\delta \rightarrow 0$ .*

*Proof.* The proof is similar to that of Lemma 5.5. For each  $\varepsilon > 0$ , let  $G_{+\varepsilon}$  be the set of all points of  $\text{aff}(G)$  whose distance from  $G$  is at most  $\varepsilon$ , and  $G_{-\varepsilon}$  be the set of all points of  $G$  whose distance from the relative boundary of  $G$  is at least  $\varepsilon$ . If  $\delta_0 = \delta_0(A, G, \varepsilon) > 0$  is small enough, then for every  $\delta < \delta_0$  we have

$$(G_{-\varepsilon} + S_\delta) \cap H_1^- \cap H_2^- \subset P_\delta \subset (G_{+\varepsilon} + S_\delta) \cap H_1^- \cap H_2^-$$

(here, the left inclusion may be inferred from Lemma 5.4), and hence

$$P_\delta \Delta Q_\delta \subset ((G_{+\varepsilon} \setminus G_{-\varepsilon}) + S_\delta) \cap H_1^- \cap H_2^-.$$

This implies that

$$m_d(P_\delta \Delta Q_\delta) \leq m_{d-2}(G_{+\varepsilon} \setminus G_{-\varepsilon}) m_2(S_\delta).$$

But we have  $m_2(S_\delta) = \pi\delta^2$ , while  $m_{d-2}(G_{+\varepsilon} \setminus G_{-\varepsilon})$  tends to zero as  $\varepsilon \rightarrow 0$ .  $\square$

5.5.

**Lemma 5.7.** *Let  $A$  and  $B$  be two convex polytopes in  $\mathbb{R}^d$  with nonempty interiors. Let  $L$  be one of the facets of  $B$ , let  $\xi$  be the exterior normal unit vector of  $B$  at the facet  $L$ , and let  $E$  be a compact subset of  $\text{relint}(L)$ . Suppose that  $A$  has a facet  $F$  on which the exterior normal unit vector is  $-\xi$ , and let  $U$  be an open neighborhood of the set  $E - F$ . Then there is an open neighborhood  $V$  of  $E$  such that for any  $t$ , if  $A + t$  and  $B$  have disjoint interiors and if  $A + t$  intersects  $V$ , then  $t \in U$ .*

This can be proved in essentially the same way as Lemma 4.3 above.

5.6.

**Lemma 5.8.** *Let  $A$  and  $B$  be two convex polytopes in  $\mathbb{R}^d$  with nonempty, disjoint interiors, and suppose that  $A$  and  $B$  share a common facet  $F$ . Assume that  $\mu$  is a positive, locally finite measure such that  $\mathbb{1}_A * \mu \geq 1$  a.e. on  $A$ , while  $\mathbb{1}_A * \mu = 0$  a.e. on  $B$ . Let  $\mu'$  denote the restriction of the measure  $\mu$  to  $\text{relint}(F - F)$ . Then we have  $\mathbb{1}_F * \mu' \geq 1$  a.e. with respect to the  $(d - 1)$ -dimensional volume measure on the facet  $F$ .*

Notice that the convolution  $\mathbb{1}_F * \mu'$  vanishes outside of  $\text{aff}(F)$ , since  $F + t$  lies on  $\text{aff}(F)$  for every  $t \in \text{supp}(\mu')$ . An equivalent way to formulate the conclusion of the lemma is to say that if  $\sigma_F$  denotes the  $(d - 1)$ -dimensional volume measure restricted to the facet  $F$ , then we have  $(\sigma_F * \mu')(E) \geq \sigma_F(E)$  for every Borel set  $E$ .

*Proof of Lemma 5.8.* By applying a rotation and a translation we may assume that the facet  $F$  is contained in the hyperplane  $\{x : x_1 = 0\}$ . Hence  $F$  has the form  $F = \{0\} \times \Omega$ , where  $\Omega$  is a convex polytope in  $\mathbb{R}^{d-1}$  with nonempty interior. We may also suppose that  $A \subset \{x : x_1 \geq 0\}$  and  $B \subset \{x : x_1 \leq 0\}$ .

Let  $\eta > 0$ , and let  $\Sigma$  be a compact subset of  $\text{int}(\Omega)$ . Then the set  $E := \{0\} \times \Sigma$  is a compact subset of  $\text{relint}(F) = \{0\} \times \text{int}(\Omega)$ . Choose  $\varepsilon > 0$  such that the total mass of the measure  $\mu$  in the set  $((-\varepsilon, 0) \cup (0, \varepsilon)) \times \text{int}(\Omega - \Omega)$  is less than  $\eta$ , and define  $U := (-\varepsilon, \varepsilon) \times \text{int}(\Omega - \Omega)$ . Using Lemma 5.1 we see that  $U$  is an open neighborhood of the set  $E - F$ . By Lemma 5.7 there is an open neighborhood  $V$  of  $E$  such that for any  $t$ , if  $A + t$  and  $B$  have disjoint interiors and if  $A + t$  intersects  $V$ , then  $t \in U$ .

Consider the following three subsets of  $\mathbb{R}^d$ :

- (i)  $Y' := \{0\} \times \text{int}(\Omega - \Omega) = \text{relint}(F - F)$ ;
- (ii)  $Y'' := ((-\varepsilon, 0) \cup (0, \varepsilon)) \times \text{int}(\Omega - \Omega)$ ;
- (iii)  $Y''' := (Y' \cup Y'')^c = U^c$ .

Then the sets  $Y', Y'', Y'''$  are pairwise disjoint, and they cover the whole space. It follows that we may decompose the measure  $\mu$  into the sum  $\mu = \mu' + \mu'' + \mu'''$ , where the three measures  $\mu', \mu'', \mu'''$  are the restrictions of  $\mu$  to the sets  $Y', Y'', Y'''$  respectively.

The assumption that  $\mathbb{1}_A * \mu = 0$  a.e. on  $B$  implies that the sets  $A + t$  and  $B$  must have disjoint interiors for every  $t \in \text{supp}(\mu)$  (Lemma 2.5). Since the support of the measure  $\mu'''$  is contained in  $\text{supp}(\mu) \cap U^c$ , it follows that if  $t \in \text{supp}(\mu''')$  then  $A + t$  cannot intersect  $V$ . Hence we have  $\mathbb{1}_A * \mu''' = 0$  a.e. in  $V$ . We also have

$$\|\mathbb{1}_A * \mu''\|_{L^\infty(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} d\mu'' = \mu(Y'') < \eta.$$

Combining this with the assumption that  $\mathbb{1}_A * \mu \geq 1$  a.e. on  $A$ , this implies that

$$\mathbb{1}_A * \mu' = \mathbb{1}_A * \mu - \mathbb{1}_A * \mu'' - \mathbb{1}_A * \mu''' \geq 1 - \eta \quad \text{a.e. on } A \cap V. \quad (5.2)$$

For each  $\delta > 0$  we let  $P_\delta := A \cap S_\delta$  be the intersection of  $A$  with the slab  $S_\delta := [0, \delta] \times \mathbb{R}^{d-1}$ , and we also consider the prism  $Q_\delta := [0, \delta] \times \Omega$ . Then by Lemma 5.5, we can choose  $\delta$  small enough such that

$$m(P_\delta \Delta Q_\delta) \mu(F - F) < \delta \eta^2 m_{d-1}(\Sigma). \quad (5.3)$$

We can also assume, by choosing  $\delta$  small enough, that the set  $D_\delta := [0, \delta] \times \Sigma$  is contained in both  $V$  and  $A$  (the inclusion in  $A$  can be deduced from Lemma 5.3).

The support of the measure  $\mu'$  is contained in the hyperplane  $\{0\} \times \mathbb{R}^{d-1}$ . For each  $t \in \text{supp}(\mu')$  we therefore have  $(A+t) \cap S_\delta = P_\delta + t$ . This implies that  $\mathbb{1}_A * \mu' = \mathbb{1}_{P_\delta} * \mu'$  a.e. on the slab  $S_\delta$ . In particular, it follows from (5.2) that

$$\mathbb{1}_{P_\delta} * \mu' \geq 1 - \eta \quad \text{a.e. on } D_\delta. \quad (5.4)$$

Let  $D'_\delta$  be the set of all points  $x \in D_\delta$  such that  $(\mathbb{1}_{P_\delta \Delta Q_\delta} * \mu')(x) < \eta$ , then we have

$$\mathbb{1}_{Q_\delta} * \mu' \geq \mathbb{1}_{P_\delta} * \mu' - \mathbb{1}_{P_\delta \Delta Q_\delta} * \mu' \geq 1 - 2\eta \quad \text{a.e. on } D'_\delta,$$

which follows from (5.4). On the other hand, by (5.3) we have

$$\int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \Delta Q_\delta} * \mu') dm = m(P_\delta \Delta Q_\delta) \int_{\mathbb{R}^d} d\mu' < \delta \eta^2 m_{d-1}(\Sigma),$$

which in turn implies that

$$m(D_\delta \setminus D'_\delta) \leq \eta^{-1} \int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \Delta Q_\delta} * \mu') dm < \delta \eta m_{d-1}(\Sigma) = \eta m(D_\delta),$$

that is, we have  $m(D'_\delta) > (1 - \eta)m(D_\delta)$ . We conclude that

$$\mathbb{1}_{Q_\delta} * \mu' \geq 1 - 2\eta \quad \text{a.e. on } D'_\delta, \quad D'_\delta \subset D_\delta, \quad m(D'_\delta) > (1 - \eta)m(D_\delta). \quad (5.5)$$

Now recall that the support of the measure  $\mu'$  is contained in  $\{0\} \times \mathbb{R}^{d-1}$ . This implies that the value of  $(\mathbb{1}_{Q_\delta} * \mu')(x)$  does not depend, in the slab  $S_\delta$ , on the first coordinate  $x_1$  of the point  $x$ . Hence it follows from (5.5) that  $\mathbb{1}_F * \mu' \geq 1 - 2\eta$  a.e. with respect to the  $(d-1)$ -dimensional volume measure on some set of the form  $\{0\} \times \Sigma'$ , where  $\Sigma'$  is a subset of  $\Sigma$  (which depends on both  $\Sigma$  and  $\eta$ ) such that  $m_{d-1}(\Sigma') \geq (1 - \eta)m_{d-1}(\Sigma)$ .

However, as the number  $\eta$  was arbitrary, it follows that we must have  $\mathbb{1}_F * \mu' \geq 1$  a.e. with respect to the  $(d-1)$ -dimensional volume measure on  $\{0\} \times \Sigma$ . In turn,  $\Sigma$  was an arbitrary compact subset of  $\text{int}(\Omega)$ . Using the fact that  $\text{bd}(\Omega)$  is a set of measure zero in  $\mathbb{R}^{d-1}$ , this implies that  $\mathbb{1}_F * \mu' \geq 1$  a.e. with respect to the  $(d-1)$ -dimensional volume measure on  $\{0\} \times \Omega = F$ , which is what we had to prove.  $\square$

5.7.

**Lemma 5.9.** *Let  $A$  be a convex polytope in  $\mathbb{R}^d$  with nonempty interior, and let  $G$  be a subfacet of  $A$ . Let  $F_1$  and  $F_2$  be the two adjacent facets of  $A$  that meet at the subfacet  $G$ , and let  $H_1^-$  and  $H_2^-$  be the support halfspaces of  $A$  at the facets  $F_1$  and  $F_2$  respectively. For each  $\delta > 0$  we let  $Q_\delta := (G + S_\delta) \cap H_1^- \cap H_2^-$ , where  $S_\delta$  is a closed 2-dimensional ball of radius  $\delta$  centered at the origin and orthogonal to  $\text{aff}(G)$ . Assume that  $\mu$  is a positive, finite measure supported on  $G - G$  and satisfying  $\mathbb{1}_G * \mu \geq 1$  a.e. with respect to the  $(d-2)$ -dimensional volume measure on  $G$ . Then for any  $\eta > 0$  we have*

$$m\{x \in Q_\delta : (\mathbb{1}_A * \mu)(x) < 1 - \eta\} = o(m(Q_\delta)), \quad \delta \rightarrow 0.$$

*Proof.* Let  $P_\delta = A \cap (\text{aff}(G) + S_\delta)$  be the set of all points of  $A$  whose distance from  $\text{aff}(G)$  is not greater than  $\delta$ . The assumption that  $\text{supp}(\mu) \subset G - G$  implies that we have  $(A+t) \cap (\text{aff}(G) + S_\delta) = P_\delta + t$  for every  $t \in \text{supp}(\mu)$ , and hence  $\mathbb{1}_A * \mu = \mathbb{1}_{P_\delta} * \mu$  a.e. on  $\text{aff}(G) + S_\delta$ . In particular,

$$\mathbb{1}_A * \mu = \mathbb{1}_{P_\delta} * \mu \quad \text{a.e. on } Q_\delta. \quad (5.6)$$

We have also assumed that  $\mathbb{1}_G * \mu \geq 1$  a.e. with respect to the  $(d-2)$ -dimensional volume measure on  $G$ . Using Fubini's theorem this implies that

$$\mathbb{1}_{Q_\delta} * \mu \geq 1 \quad \text{a.e. on } Q_\delta. \quad (5.7)$$

Denote by  $D_{\delta,\eta}$  the set of all points  $x \in Q_\delta$  such that  $(\mathbb{1}_{P_\delta \Delta Q_\delta} * \mu)(x) < \eta$ . Then

$$\mathbb{1}_A * \mu = \mathbb{1}_{P_\delta} * \mu \geq \mathbb{1}_{Q_\delta} * \mu - \mathbb{1}_{P_\delta \Delta Q_\delta} * \mu > 1 - \eta \quad \text{a.e. on } D_{\delta,\eta},$$

which follows from (5.6) and (5.7). Hence to prove the assertion of the lemma, it is enough to show that  $m(Q_\delta \setminus D_{\delta,\eta}) = o(m(Q_\delta))$  as  $\delta \rightarrow 0$ . We observe that  $m(Q_\delta) = m_{d-2}(G) \cdot \frac{1}{2}\theta\delta^2$ , where  $\theta$  is the dihedral angle of  $A$  at its subfacet  $G$ . Let  $\varepsilon > 0$ , then by Lemma 5.6 there is  $\delta_0 = \delta_0(A, G, \eta, \varepsilon) > 0$  such that for any  $\delta < \delta_0$  we have

$$m(P_\delta \Delta Q_\delta) \mu(G - G) < \varepsilon \eta m(Q_\delta).$$

It follows that

$$\int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \Delta Q_\delta} * \mu) dm = m(P_\delta \Delta Q_\delta) \int_{\mathbb{R}^d} d\mu < \varepsilon \eta m(Q_\delta),$$

which in turn implies

$$m(Q_\delta \setminus D_{\delta,\eta}) \leq \eta^{-1} \int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \Delta Q_\delta} * \mu) dm < \varepsilon m(Q_\delta).$$

This confirms that indeed we have  $m(Q_\delta \setminus D_{\delta,\eta}) = o(m(Q_\delta))$  as  $\delta \rightarrow 0$ .  $\square$

## 6. SPECTRAL CONVEX POLYTOPES CAN TILE BY TRANSLATIONS, II

In this section we prove the following theorem, which is the final result needed for the proof of Fuglede's conjecture for convex bodies:

**Theorem 6.1.** *Let  $A \subset \mathbb{R}^d$  be a convex polytope, which is centrally symmetric and has centrally symmetric facets. Assume that the complement  $A^c$  of  $A$  admits a weak tiling by translates of  $A$ , that is, there exists a positive, locally finite measure  $\mu$  such that  $\mathbb{1}_A * \mu = \mathbb{1}_{A^c}$  a.e. Then each belt of  $A$  consists of either 4 or 6 facets.*

Theorem 1.3 is then obtained as a consequence of Theorem 1.5 and Theorem 6.1.

The rest of the section is devoted to the proof of Theorem 6.1.

6.1. Let  $G$  be any one of the subfacets of  $A$ , and suppose that the belt of  $A$  generated by  $G$  has  $2m$  facets. Let  $F_0, F_1, F_2, \dots, F_{2m-1}, F_{2m} = F_0$  be an enumeration of the facets of the belt, such that  $F_{i-1}$  and  $F_i$  are adjacent facets for each  $1 \leq i \leq 2m$ . The intersection  $F_{i-1} \cap F_i$  of any pair of consecutive facets in the belt is then a translate of either  $G$  or  $-G$ . We shall suppose that  $G$  itself is given by  $G = F_1 \cap F_2$ .

Our goal is to show that under the assumptions in Theorem 6.1, the belt can have only 4 or 6 facets, that is, we must have  $m \leq 3$ .

The belt generated by  $G$  consists of  $m$  pairs of opposite facets  $\{F_i, F_{i+m}\}$  ( $0 \leq i \leq m-1$ ). We can assume, with no loss of generality, that  $A$  is symmetric with respect to the origin, that is,  $A = -A$ . Then for each facet  $F_i$  in the belt, its opposite facet is given by  $-F_i$ . Since the facets of  $A$  are centrally symmetric,  $-F_i$  is a translate of  $F_i$ , so there is a translation vector  $\tau_i$  which carries  $-F_i$  onto  $F_i$ , that is,  $F_i = -F_i + \tau_i$ .

6.2. Recall that we have assumed that the complement  $A^c$  of  $A$  admits a weak tiling by translates of  $A$ . This means that there exists a positive, locally finite measure  $\mu$  such that  $\mathbb{1}_A * \mu = \mathbb{1}_{A^c}$  a.e. For each  $0 \leq i \leq 2m$  we define

$$T_i := \text{relint}(F_i - F_i) + \tau_i, \quad \mu'_i := \mu \cdot \mathbb{1}_{T_i}, \quad \nu'_i := \mu'_i * \delta_{-\tau_i}. \quad (6.1)$$

We observe that  $\text{supp}(\mu'_i)$  is contained in the hyperplane passing through the point  $\tau_i$  and which is parallel to the facet  $F_i$ , while  $\text{supp}(\nu'_i)$  is contained in the hyperplane through the origin which is parallel to  $F_i$ .

We also notice that we have  $-F_i = F_i - \tau_i$  and hence  $T_i = 2 \text{relint}(F_i)$ . This implies that the sets  $T_i$  ( $0 \leq i \leq 2m - 1$ ) are pairwise disjoint, because any two distinct facets of  $A$  have disjoint relative interiors.

**Lemma 6.2.** *For each  $0 \leq i \leq 2m$  we have  $\mathbb{1}_{F_i} * \nu'_i \geq 1$  a.e. with respect to the  $(d - 1)$ -dimensional volume measure on the facet  $F_i$ .*

*Proof.* Let  $A' := A + \tau_i$  and  $B' := A$ . Then  $A'$  and  $B'$  are two convex polytopes in  $\mathbb{R}^d$  with nonempty, disjoint interiors, and they share  $F_i$  as a common facet. Let  $\rho := \mu * \delta_{-\tau_i}$ , then we have  $\mathbb{1}_{A'} * \rho = \mathbb{1}_{A^c}$  a.e., and in particular,  $\mathbb{1}_{A'} * \rho \geq 1$  a.e. on  $A'$ , and  $\mathbb{1}_{A'} * \rho = 0$  a.e. on  $B'$ . Since  $\nu'_i$  is the restriction of the measure  $\rho$  to  $\text{relint}(F_i - F_i)$ , we can apply Lemma 5.8 and conclude that  $\mathbb{1}_{F_i} * \nu'_i \geq 1$  a.e. with respect to the  $(d - 1)$ -dimensional volume measure on the facet  $F_i$ , as we had to show.  $\square$

6.3. Let  $C_i := \text{conv}\{F_i, F_i - \tau_i\}$  be the prism contained in  $A$  with bases  $F_i$  and  $F_i - \tau_i = -F_i$ . We also define the prism  $D_i := C_i + \tau_i = \text{conv}\{F_i, F_i + \tau_i\}$ .

**Lemma 6.3.** *For each  $0 \leq i \leq 2m$  we have  $\mathbb{1}_{C_i} * \mu'_i \geq 1$  a.e. on  $D_i$ .*

*Proof.* By Lemma 6.2 the facet  $F_i$  has a subset  $E_i$  of full  $(d - 1)$ -dimensional volume measure, such that we have  $(\mathbb{1}_{F_i} * \nu'_i)(y) \geq 1$  for every  $y \in E_i$ . By Fubini's theorem, the set of all points  $x$  of the form  $x = y + \lambda\tau_i$ ,  $y \in E_i$ ,  $0 \leq \lambda \leq 1$ , constitutes a subset  $D'_i$  of  $D_i$  of full  $d$ -dimensional volume measure. Since  $\text{supp}(\nu'_i)$  is contained in the hyperplane through the origin parallel to  $F_i$ , it follows that for any point  $x$  of the above form we have  $(\mathbb{1}_{D_i} * \nu'_i)(x) = (\mathbb{1}_{F_i} * \nu'_i)(y) \geq 1$ . We obtain that  $\mathbb{1}_{D_i} * \nu'_i \geq 1$  a.e. on  $D_i$ . But this is equivalent to the assertion of the lemma.  $\square$

In order to better understand the assertion of Lemma 6.3, observe that the prism  $D_i$  is contained (up to measure zero) in the complement  $A^c$  of  $A$ , and hence we trivially have the “weak covering” property  $\mathbb{1}_A * \mu \geq 1$  a.e. on  $D_i$ . However the point of the lemma is that only the part of  $A$  which lies in the prism  $C_i$ , and only the part of the measure  $\mu$  which lies in  $T_i$ , can in fact contribute to this covering (see Figure 6.1).

6.4. Let us recall that  $F_0, F_1, F_2, \dots, F_{2m-1}, F_{2m} = F_0$  is an enumeration of the facets of the belt generated by the subfacet  $G$  of the convex polytope  $A$ , and we have assumed that the subfacet  $G$  is given by  $G = F_1 \cap F_2$ .

For  $i \in \{1, 2\}$  (and only for these two values of  $i$ ) we denote

$$S_i := \text{relint}(G - G) + \tau_i, \quad \mu''_i := \mu'_i \cdot \mathbb{1}_{S_i}, \quad \nu''_i := \mu''_i * \delta_{-\tau_i}. \quad (6.2)$$

We notice that  $S_i$  is a subset of the set  $T_i$  defined in (6.1). This follows from Lemma 5.2 applied relative to the affine hull of the facet  $F_i$ . In particular, this shows that  $S_1$  and  $S_2$  are disjoint sets, because the sets  $T_1$  and  $T_2$  are disjoint.

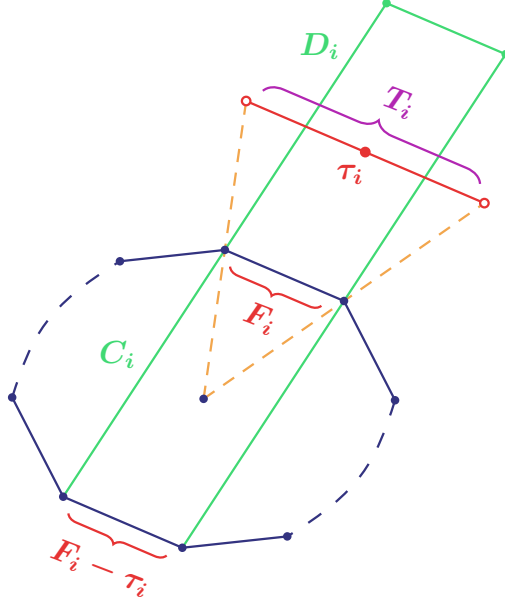


FIGURE 6.1. According to Lemma 6.3, the prism  $D_i$  is “weakly covered” by the translates of the prism  $C_i$  with respect to the measure  $\mu'_i$ .

It also follows that we have  $\mu''_i = \mu \cdot \mathbb{1}_{S_i}$ , that is, in the definition of the measure  $\mu''_i$  it does not matter whether we restrict  $\mu'_i$  or  $\mu$  to the set  $S_i$ .

**Lemma 6.4.** *Assume that the belt of  $A$  generated by the subfacet  $G$  has 6 or more facets. Then for each  $i \in \{1, 2\}$  we have  $\mathbb{1}_G * \nu''_i \geq 1$  a.e. with respect to the  $(d-2)$ -dimensional volume measure on the subfacet  $G$ .*

*Proof.* We suppose that  $i$  is an element of the set  $\{1, 2\}$  and we let  $j$  be the other element, so that  $(i, j) = (1, 2)$  or  $(2, 1)$ . The proof is divided into several steps.

*Step 1:* We first claim that if  $t \in \text{supp}(\nu'_i)$ , then  $F_i + t$  cannot intersect the interior of the prism  $D_j$ . For suppose that  $(F_i + t) \cap \text{int}(D_j)$  is nonempty. Let  $s := t + \tau_i$ , then  $s \in \text{supp}(\mu'_i)$  and we have  $(F_i - \tau_i + s) \cap \text{int}(D_j)$  is nonempty. Since  $F_i - \tau_i + s$  is a facet of the prism  $C_i + s$ , it follows that  $(C_i + s) \cap D_j$  has nonempty interior, and in particular we have  $m((C_i + s) \cap D_j) > 0$ . By Lemma 2.5 this implies that  $\mathbb{1}_{C_i} * \mu'_i$  cannot vanish a.e. on  $D_j$ , and hence there exist  $\eta > 0$  and a set  $E \subset D_j$ ,  $m(E) > 0$ , such that  $\mathbb{1}_{C_i} * \mu'_i \geq \eta$  a.e. on  $E$ . On the other hand we have  $\mathbb{1}_{C_j} * \mu'_j \geq \mathbb{1}_{D_j}$  a.e. by Lemma 6.3. Using the fact that  $C_i$  and  $C_j$  are both subsets of  $A$ , and that  $\mu'_i$  and  $\mu'_j$  are the restrictions of  $\mu$  to the two disjoint sets  $T_i$  and  $T_j$  respectively, we conclude that

$$\mathbb{1}_A * \mu \geq \mathbb{1}_{C_i} * \mu'_i + \mathbb{1}_{C_j} * \mu'_j \geq \eta \mathbb{1}_E + \mathbb{1}_{D_j} \geq (1 + \eta) \mathbb{1}_E \quad \text{a.e.}$$

However this contradicts the weak tiling assumption  $\mathbb{1}_A * \mu = \mathbb{1}_{A^c}$  a.e. This establishes our claim that  $F_i + t$  cannot intersect the interior of the prism  $D_j$ .

*Step 2:* Let  $H_i$  be the hyperplane containing the facet  $F_i$ , and define

$$B_i := H_i \cap D_j.$$

We claim that  $B_i$  is a  $(d-1)$ -dimensional convex polytope, that  $\text{relint}(B_i) \subset \text{int}(D_j)$ , and that  $G$  is a  $(d-2)$ -dimensional face of  $B_i$ .



First, it is clear that  $B_i$  is a convex polytope, being the intersection of a convex polytope and a hyperplane.

Next, recall that we have assumed the belt of  $A$  generated by the subfacet  $G$  to have 6 or more facets. This implies that the facet  $L_j := \text{conv}\{G, G - \tau_j\}$  of the prism  $C_j$  divides the dihedral angle of  $A$  at its subfacet  $G$  into two (strictly positive) angles  $\theta$  and  $\varphi$ , where  $\theta$  is the dihedral angle between  $L_j$  and  $F_i$ , and  $\varphi$  is the dihedral angle between  $L_j$  and  $F_j$ . Hence the hyperplane  $H_i$  divides the dihedral angle of the prism  $D_j$  at its subfacet  $G$  into two strictly positive angles  $\theta$  and  $\pi - \theta - \varphi$ . It follows that  $H_i$  must intersect the interior of the prism  $D_j$  and so  $B_i$  is a  $(d - 1)$ -dimensional convex polytope (see Figure 6.2).

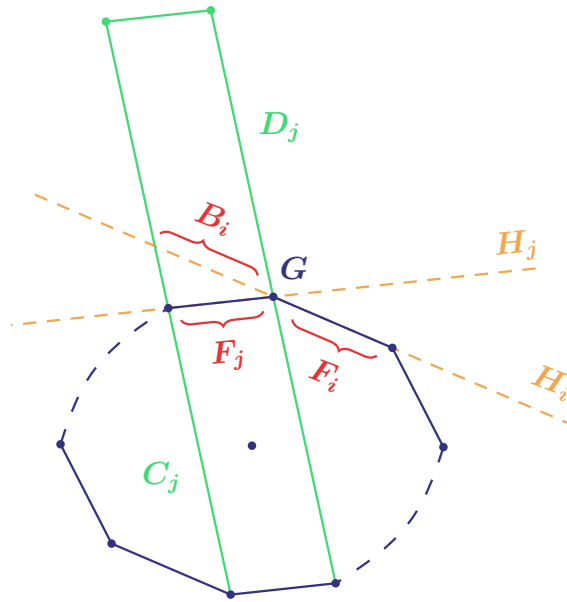


FIGURE 6.2. If the belt generated by the subfacet  $G$  has 6 or more facets, then the hyperplane  $H_i$  containing the facet  $F_i$  intersects the interior of the prism  $D_j$ , and  $B_i = H_i \cap D_j$  is a  $(d - 1)$ -dimensional convex polytope.

Let  $H_j$  be the hyperplane containing the facet  $F_j$ , and  $H_j^+$  be the closed halfspace bounded by  $H_j$  which contains the prism  $D_j$ . Then  $B_i$  is contained in  $H_j^+$  and we have  $B_i \cap H_j = G$ . This shows that  $H_j$  is a support hyperplane of  $B_i$  and  $G$  is a face (of dimension  $d - 2$ ) of  $B_i$ .

Finally, we show that  $\text{relint}(B_i) \subset \text{int}(D_j)$ . Indeed, let  $x \in \text{relint}(B_i)$ . Since  $H_i$  is the affine hull of  $B_i$ , this means that there is an open set  $V$  such that  $x \in V \cap H_i \subset B_i$ . In particular this implies that  $x \in D_j$ , so it is enough to prove that  $x$  cannot lie on  $\text{bd}(D_j)$ . Indeed, since  $x \in V \cap H_i \subset D_j$ , the point  $x$  can lie on  $\text{bd}(D_j)$  only if  $H_i$  is a support hyperplane of  $D_j$ . But this is not the case, since  $H_i$  intersects  $\text{int}(D_j)$ , so we must have  $x \in \text{int}(D_j)$ .

*Step 3:* It follows that if  $t \in \text{supp}(\nu'_i)$ , then  $F_i + t$  cannot intersect  $\text{relint}(B_i)$ . Indeed, as we have shown above,  $\text{relint}(B_i)$  is contained in the interior of the prism  $D_j$ , while  $F_i + t$  cannot intersect the interior of  $D_j$ . Hence  $F_i + t$  and  $\text{relint}(B_i)$  are disjoint sets for every  $t \in \text{supp}(\nu'_i)$ . From this we conclude that  $\mathbf{1}_{F_i} * \nu'_i = 0$  a.e. with respect to the  $(d - 1)$ -dimensional volume measure on  $B_i$ .

*Step 4:* The sets  $F_i$  and  $B_i$  are two  $(d-1)$ -dimensional convex polytopes contained in the same hyperplane  $H_i$ , they have disjoint relative interiors, and they share  $G$  as a common  $(d-2)$ -dimensional face. Recall that by Lemma 6.2 we have  $\mathbb{1}_{F_i} * \nu'_i \geq 1$  a.e. with respect to the  $(d-1)$ -dimensional volume measure on  $F_i$ , while we have just shown in Step 3 above that  $\mathbb{1}_{F_i} * \nu'_i = 0$  a.e. with respect to the  $(d-1)$ -dimensional volume measure on  $B_i$ . We may therefore apply Lemma 5.8 (relative to the hyperplane  $H_i$  containing  $F_i$  and  $B_i$ ) and conclude that the measure  $\nu''_i$  obtained by restricting the measure  $\nu'_i$  to  $\text{relint}(G - G)$ , satisfies  $\mathbb{1}_G * \nu''_i \geq 1$  a.e. with respect to the  $(d-2)$ -dimensional volume measure on  $G$ . So we obtain the assertion of the lemma.  $\square$

6.5. For  $i \in \{1, 2\}$ , we denote by  $N_i$  the hyperplane which contains the subfacet  $G$  and which is parallel to the facet  $F_0$  if  $i = 1$ , or parallel to the facet  $F_3$  if  $i = 2$ . Let  $N_i^-$  be the closed halfspace bounded by  $N_i$  that has exterior normal unit vector which is opposite to the exterior normal vector of  $A$  at the facet  $F_0$  (if  $i = 1$ ) or  $F_3$  (if  $i = 2$ ). It is not difficult to verify that  $N_i^-$  is the support halfspace of  $A + \tau_i$  at its facet  $F_0 - \tau_0 + \tau_1$  for  $i = 1$ , or  $F_3 - \tau_3 + \tau_2$  for  $i = 2$ .

**Lemma 6.5.** *Assume that the belt of  $A$  generated by the subfacet  $G$  has 8 or more facets. Then the set  $M := (A + \tau_1) \cap (A + \tau_2)$  is a convex polytope with nonempty interior,  $G$  is a subfacet of  $M$ , and  $N_1^-, N_2^-$  are the support halfspaces of  $M$  at its two facets which meet at the subfacet  $G$  (see Figure 6.3).*

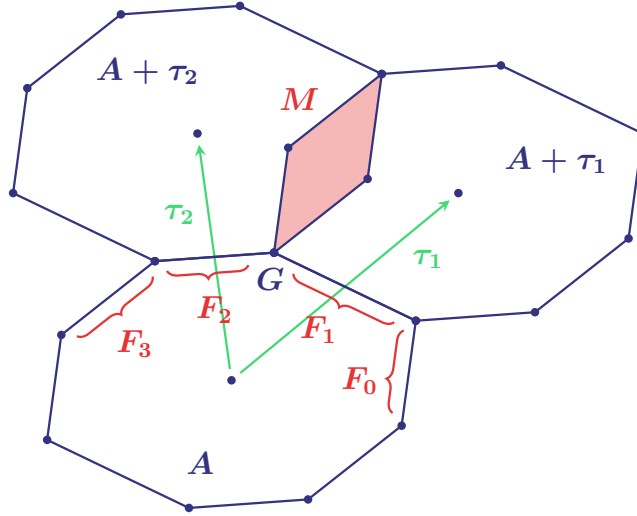


FIGURE 6.3. The shaded region in the illustration represents the convex polytope  $M = (A + \tau_1) \cap (A + \tau_2)$  in Lemma 6.5.

*Proof.* Let  $\alpha$ ,  $\beta$  and  $\gamma$  denote the dihedral angles of  $A$  at the subfacets  $G$ ,  $-G + \tau_1$  and  $-G + \tau_2$  respectively. If the belt of  $A$  generated by  $G$  has 8 or more facets, then we must have  $\alpha + \beta + \gamma > 2\pi$  (see Figure 6.4). On the other hand, each one of  $\alpha$ ,  $\beta$  and  $\gamma$  is strictly smaller than  $\pi$ . Hence the hyperplane  $N_1$  divides the dihedral angle of  $A + \tau_2$  at the subfacet  $G$  into two strictly positive angles  $2\pi - \alpha - \beta$  and  $\alpha + \beta + \gamma - 2\pi$ , while the hyperplane  $N_2$  divides the dihedral angle of  $A + \tau_1$  at  $G$  into two strictly positive angles  $2\pi - \alpha - \gamma$  and  $\alpha + \beta + \gamma - 2\pi$ . In particular the two hyperplanes  $N_1$  and  $N_2$  are not parallel, and thus  $N_1 \cap N_2 = \text{aff}(G)$ .

It is clear that  $M := (A + \tau_1) \cap (A + \tau_2)$  is a convex polytope, being the intersection of two convex polytopes. Since  $F_1$  is a facet of  $A + \tau_1$  and  $F_2$  is a facets of  $A + \tau_2$ , then  $G$  is a subfacet of both  $A + \tau_1$  and  $A + \tau_2$ . Let  $E$  be a closed  $(d - 2)$ -dimensional ball contained in  $\text{relint}(G)$ . For  $\delta > 0$  we denote  $D(E, \delta) := (E + S_\delta) \cap N_1^- \cap N_2^-$ , where  $S_\delta$  is a closed 2-dimensional ball of radius  $\delta$  centered at the origin and orthogonal to  $\text{aff}(G)$ . Using Lemma 5.4 we obtain that if  $\delta = \delta(A, G, E) > 0$  is small enough, then  $D(E, \delta)$  is contained both in  $A + \tau_1$  and in  $A + \tau_2$ , and hence  $D(E, \delta) \subset M$ . It follows that  $M$  has nonempty interior, and that  $N_1, N_2$  are support hyperplanes of  $M$  such that the corresponding support sets  $M \cap N_1$  and  $M \cap N_2$  are  $(d - 1)$ -dimensional, and hence these support sets are facets of  $M$ . We conclude that  $G$  is a subfacet of  $M$ , being the intersection of two adjacent facets  $M \cap N_1$  and  $M \cap N_2$  of  $M$ , and that  $N_1^-, N_2^-$  are the support halfspaces of  $M$  at its two facets which meet at the subfacet  $G$ .  $\square$

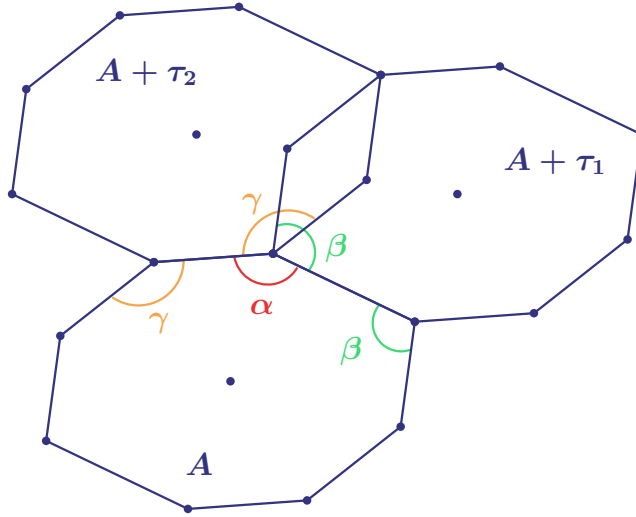


FIGURE 6.4. If the belt of  $A$  generated by the subfacet  $G$  has 8 or more facets, then the sum of the dihedral angles  $\alpha, \beta$  and  $\gamma$  at the subfacets  $G, -G + \tau_1$  and  $-G + \tau_2$  respectively is strictly greater than  $2\pi$ .

6.6. Now we can finish the proof of Theorem 6.1.

Indeed, suppose to the contrary that the belt of  $A$  generated by the subfacet  $G$  has 8 or more facets. By Lemma 6.5, the set  $M := (A + \tau_1) \cap (A + \tau_2)$  is a convex polytope with nonempty interior,  $G$  is a subfacet of  $M$ , and  $N_1^-, N_2^-$  are the support halfspaces of  $M$  at its two facets which meet at the subfacet  $G$ . For each  $i \in \{1, 2\}$ ,  $\nu_i''$  is a finite measure supported on  $G - G$ , and by Lemma 6.4 we have  $\mathbb{1}_G * \nu_i'' \geq 1$  a.e. with respect to the  $(d - 2)$ -dimensional volume measure on the subfacet  $G$ . Then we can apply Lemma 5.9 to the convex polytope  $M$ . It follows from the lemma that if we denote  $Q_\delta = (G + S_\delta) \cap N_1^- \cap N_2^-$ , where  $S_\delta$  is a closed 2-dimensional ball of radius  $\delta$  centered at the origin and orthogonal to  $\text{aff}(G)$ , then for any  $\eta > 0$  we have

$$m\{x \in Q_\delta : (\mathbb{1}_M * \nu_i'')(x) < 1 - \eta\} = o(m(Q_\delta)), \quad \delta \rightarrow 0, \quad i \in \{1, 2\}. \quad (6.3)$$

Fix any number  $0 < \eta < \frac{1}{2}$ , and let  $D(\delta, \eta)$  denote the set of all points  $x \in Q_\delta$  for which we have  $(\mathbb{1}_M * \nu_i'')(x) \geq 1 - \eta$  for both  $i = 1$  and  $i = 2$ . The set  $Q_\delta$  has positive

measure, and therefore it follows from (6.3) that if  $\delta$  is small enough then also  $D(\delta, \eta)$  has positive measure. On the other hand, we have

$$\mathbb{1}_A * \mu \geq \mathbb{1}_A * (\mu_1'' + \mu_2'') = \mathbb{1}_{A+\tau_1} * \nu_1'' + \mathbb{1}_{A+\tau_2} * \nu_2'' \geq \mathbb{1}_M * (\nu_1'' + \nu_2''),$$

where the first inequality is due to the fact that the measures  $\mu_1''$  and  $\mu_2''$  are obtained by restricting  $\mu$  respectively to the disjoint sets  $S_1$  and  $S_2$  defined in (6.2), while the second inequality holds since  $M$  is a subset of both  $A + \tau_1$  and  $A + \tau_2$ . This implies that we have  $\mathbb{1}_A * \mu \geq 2(1 - \eta) > 1$  a.e. on  $D(\delta, \eta)$ . However this contradicts the weak tiling assumption  $\mathbb{1}_A * \mu = \mathbb{1}_{A^c}$  a.e. We conclude that the belt of  $A$  generated by the subfacet  $G$  can have only 4 or 6 facets, and this completes the proof of Theorem 6.1.  $\square$

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