Post-critical phase diagram of growing bilayers

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Growth-induced morphological instabilities are ubiquitously observed in biological systems across various length scales [1]. Indeed, instabilities in the form of wrinkling, folding and creasing are critical for certain biological functions. A common material arrangement consists of a thin stiff layer mounted on top of a thicker substrate, where either the thin outer layer, the thicker inner layer, or both can grow at specific rates. Depending on the level of differential growth between film and substrate ($G_\mathrm{f}/G_\mathrm{s}$) and the relative elastic moduli ($E_\mathrm{f}/E_\mathrm{s}$), different patterns such as sinusoidal wrinkling, perdiod doubling, period quadrupling, or creases can form. In this regard, an understanding of the morphological phase diagram of growing bilayers with regards to the moduli ratio and differential growth ratio would facilitate physical understanding and also contribute to the development of new diagnostics.

The existing work on growth in bilayers mainly focuses on the initial or intermediate post-critical regime [2], and often on the case of growth in the thin film only [3], *i.e.* $G_\mathrm{s} = 0$. The complete bifurcation landscape of bilayers in the deep post-critical regime remains an open question, especially how the post-critical phase diagram evolves as the ratio of elastic moduli ($E_\mathrm{f}/E_\mathrm{s}$) and differential growth ($G_\mathrm{f}/G_\mathrm{s}$) changes. Here, we derive wrinkling phase diagrams of growing bilayers with $E_\mathrm{f}/E_\mathrm{s} = 1.5 - 50$, which is in the range of most biologically observed bilayers for different film/substrate growth ratios $G_\mathrm{s}/G_\mathrm{f} = 0 - 10$. Our analysis is based on a hyperelastic, plane strain finite element model which is solved using a generalised path-following algorithm (numerical continuation) to explore the bifurcation structure beyond the first wrinkling instability and to trace critical boundaries. Phase diagrams are constructed in the parameter space of $E_\mathrm{f}/E_\mathrm{s}$-$G_\mathrm{s}/G_\mathrm{f}$-$g_\mathrm{p}$, where g_p is a growth parameter, and these diagrams break the design space into flat, sinusoidally wrinkled, period doubling/quadrupling, and creasing regimes. In addition, we uncover a relation between $E_\mathrm{f}/E_\mathrm{s}$ and $G_\mathrm{s}/G_\mathrm{f}$ that defines the boundary of supercritical to subcritical wrinkling (*i.e.* onset of Biot wrinkling).

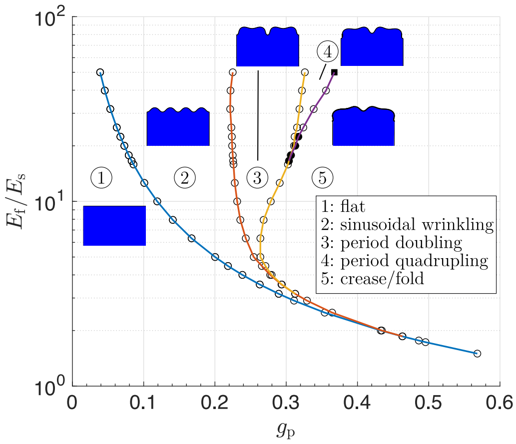

Fig. 1: Phase diagram of growing bilayers with isotropic growth $G_\mathrm{s}/G_\mathrm{f} = 1$. The wrinkling modes of the growing bilayer with $E_\mathrm{f}/E_\mathrm{s}$=50 are presented for illustration.

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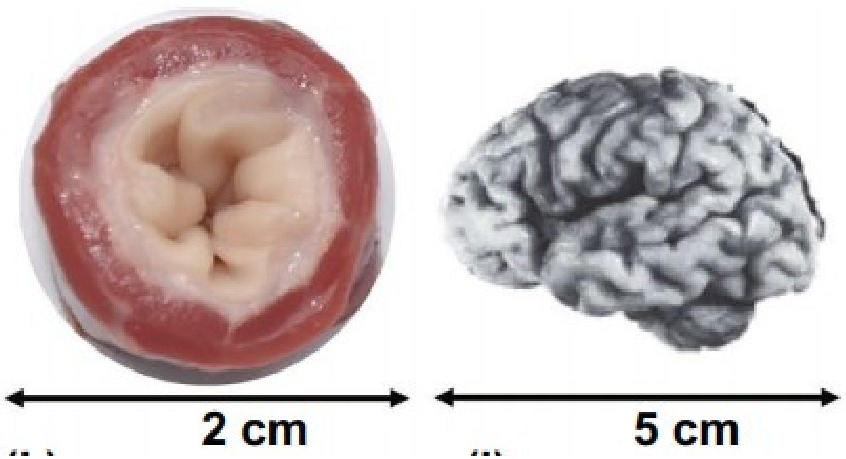
Background

- Growth-induced morphological instabilities of bilayers are critical for certain (mal)functionalities of biological systems. Understanding the mechanics helps to
  - develop new diagnostics
  - design bio-inspired engineering products, *e.g.* artificial digital skin



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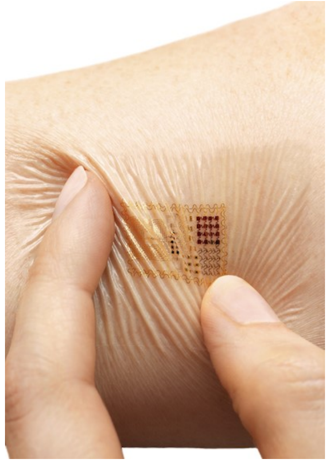

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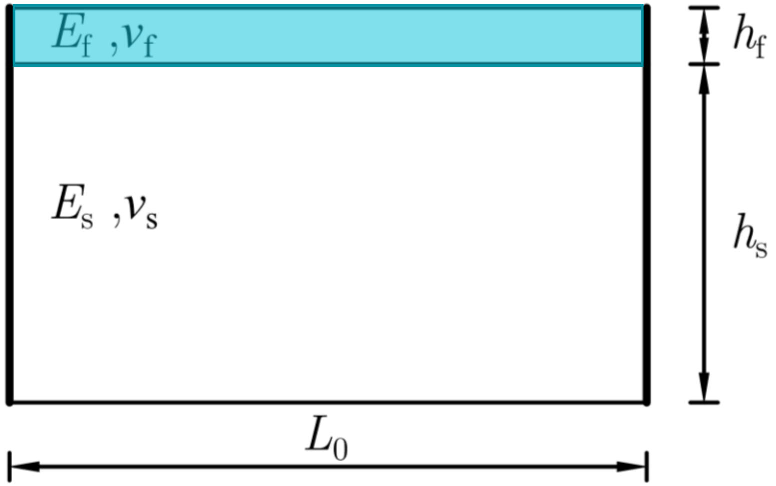
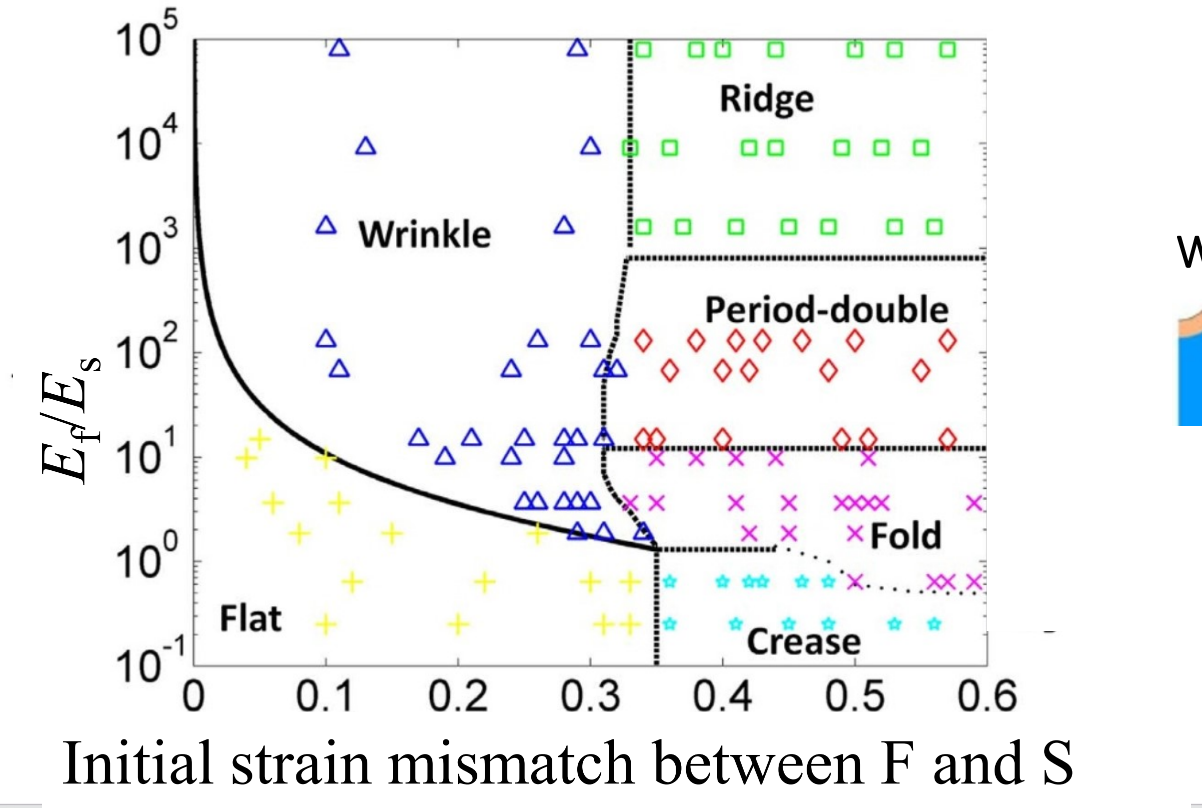
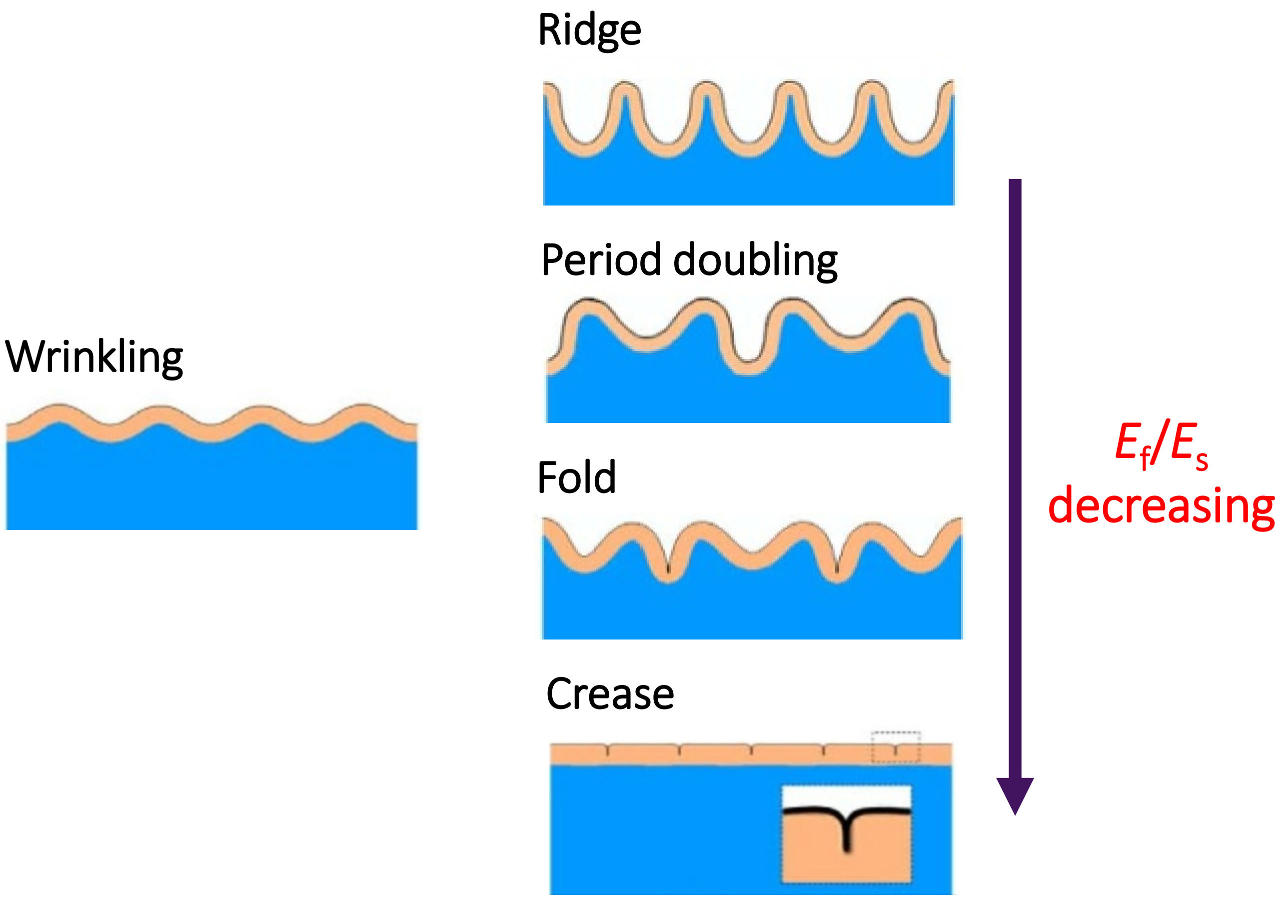


Morphological instabilities in bilayers and phase change diagram

- Using mechanical analog, it was identified that film/substrate stiffness ratio (E_f/E_s) and initial strain mismatch in film (F) and substrate (S) govern the morphological patterns[1].





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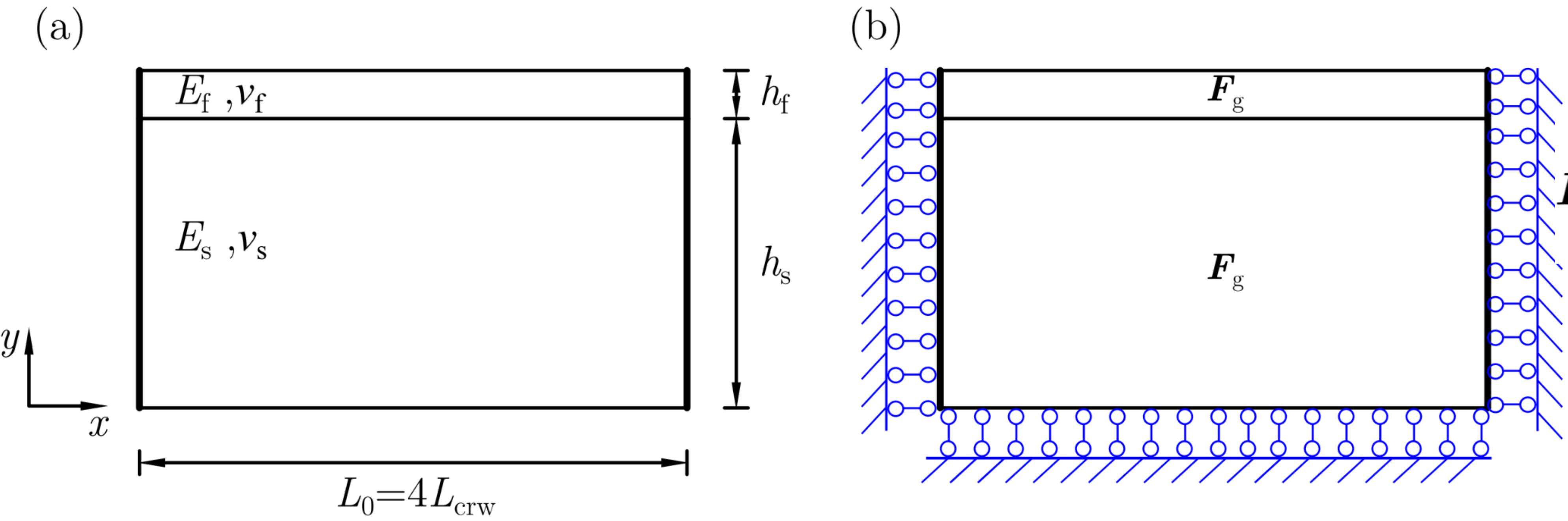
Gaps and problem definition

- Phase change diagram using strict morphoelasticity formulation, rather than mechanical analog.
- Limited understanding in the advanced post-critical behaviour of growing bilayers, *i.e.* what occurs beyond period doubling? Is it period *quadrupling* or *fold*?
- Limited understanding in the phase change in cases with $E_f/E_s<10$ due to highly nonlinearity, *i.e.* the transition from wrinkling to crease.
- A unified description of the critical strain due to growth and its analogy to the mechanical case.

Model description

- An end-constrained planar bilayer system with isotropic growth pattern $\boldsymbol{F}_g$
- Growth-induced morphological patterns up to the crease or fold are studied, *i.e.* wrinkling patterns *without self-contact*.



$$\boldsymbol{F}_g = \begin{bmatrix} 1 + g_p & 0 \\ 0 & 1 + g_p \end{bmatrix}$$

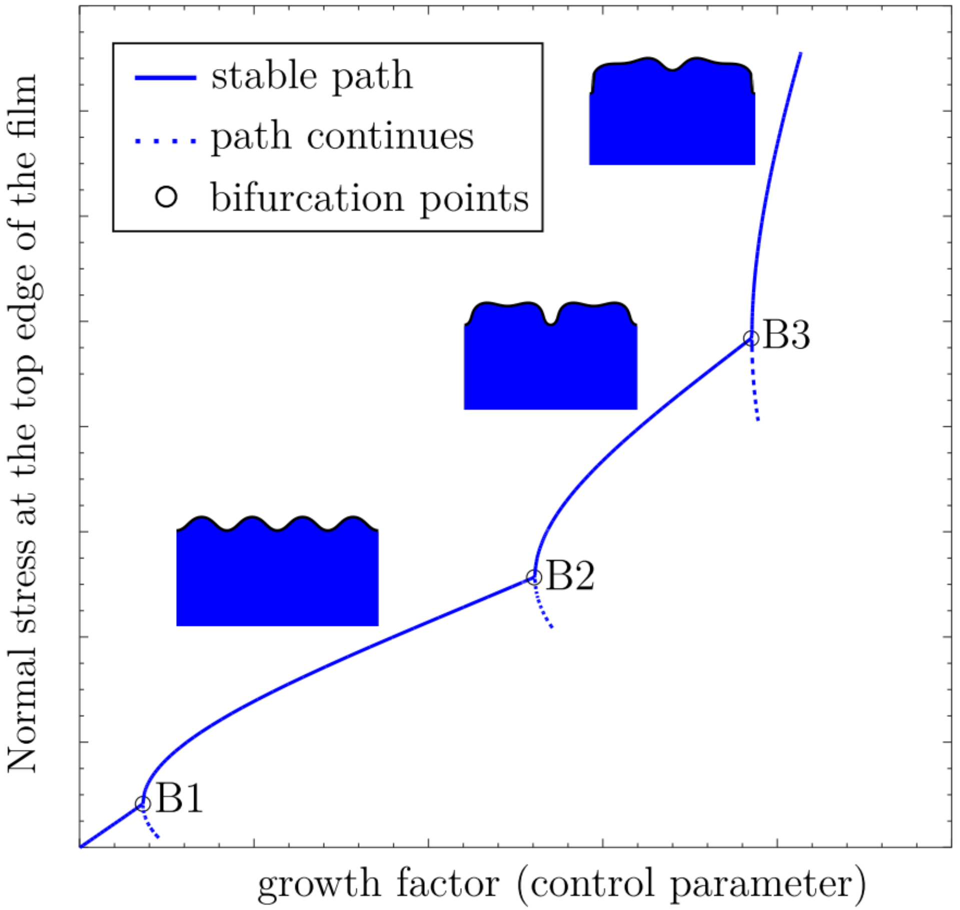
g_p: growth factor

Methods

- FE model
  - 2D morphoelastic *plane strain* element (multiplicative decomposition[1])

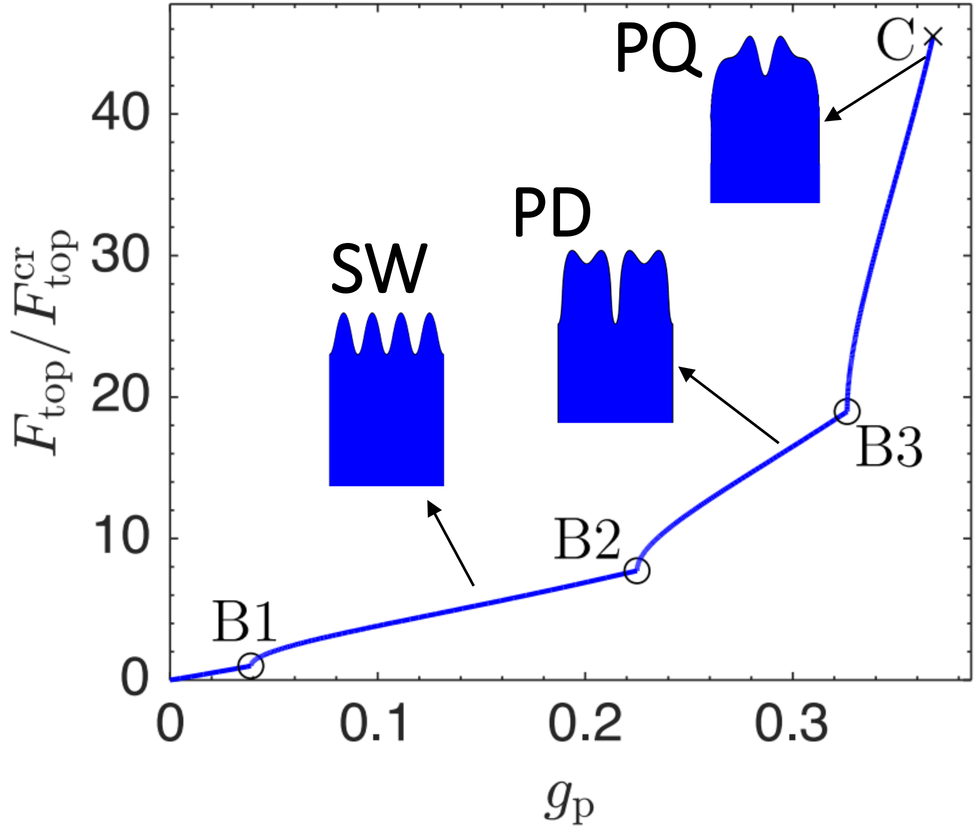
$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$$

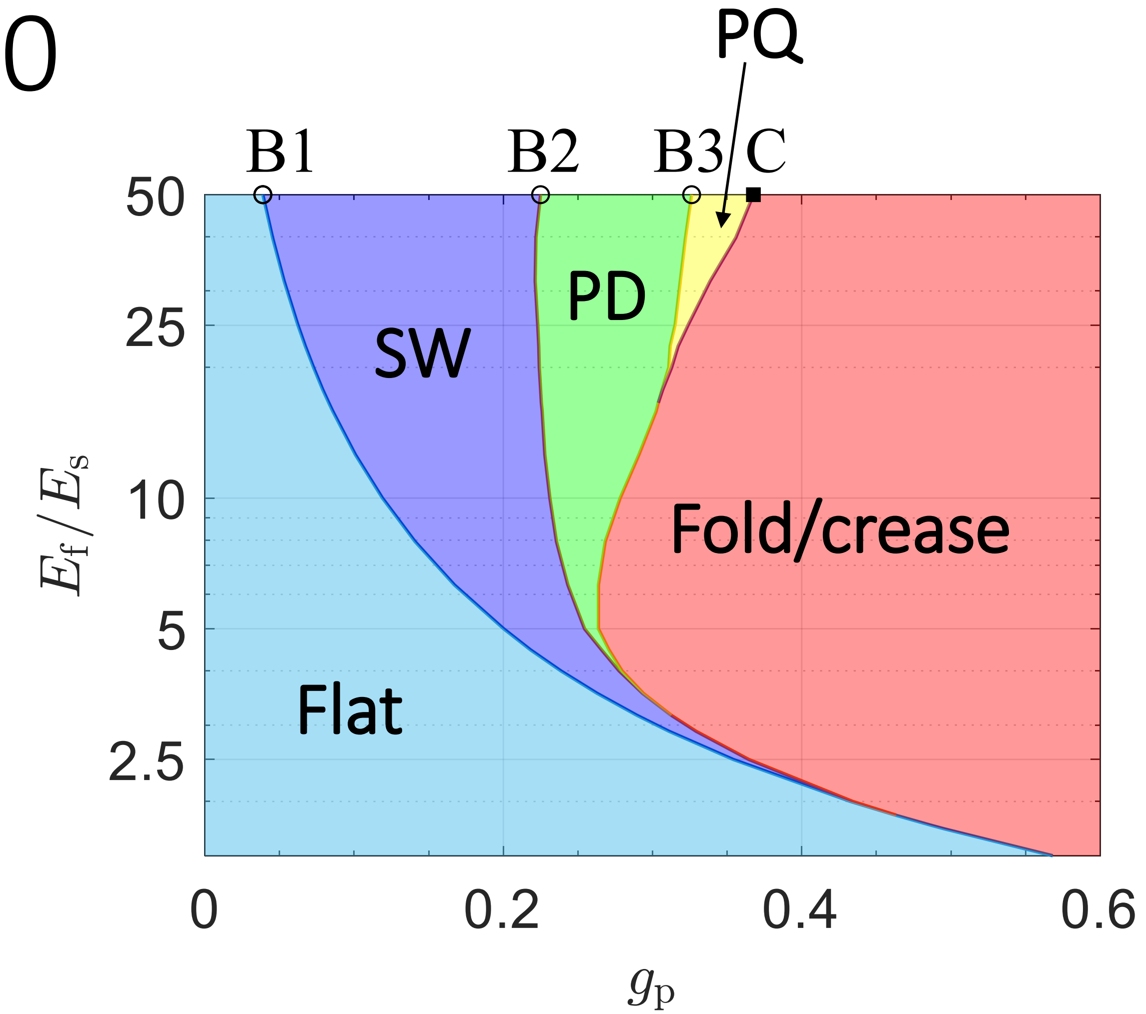
  - *Compressible* Neo-Hookean hyperelastic model with the Poisson's ratio being **0.46**.
- Generalized path-following solver[2]
  - Standard arc-length solver (Riks)
  - Pin-point critical points
  - Switch branch



[1] Goriely, A. (2017). The mathematics and mechanics of biological growth (Vol. 45). Springer.
[2] Groh, R.M.J., (2022) A morphoelastic stability framework for post-critical pattern formation in growing thin biomaterials, CMAME, 394,114839.

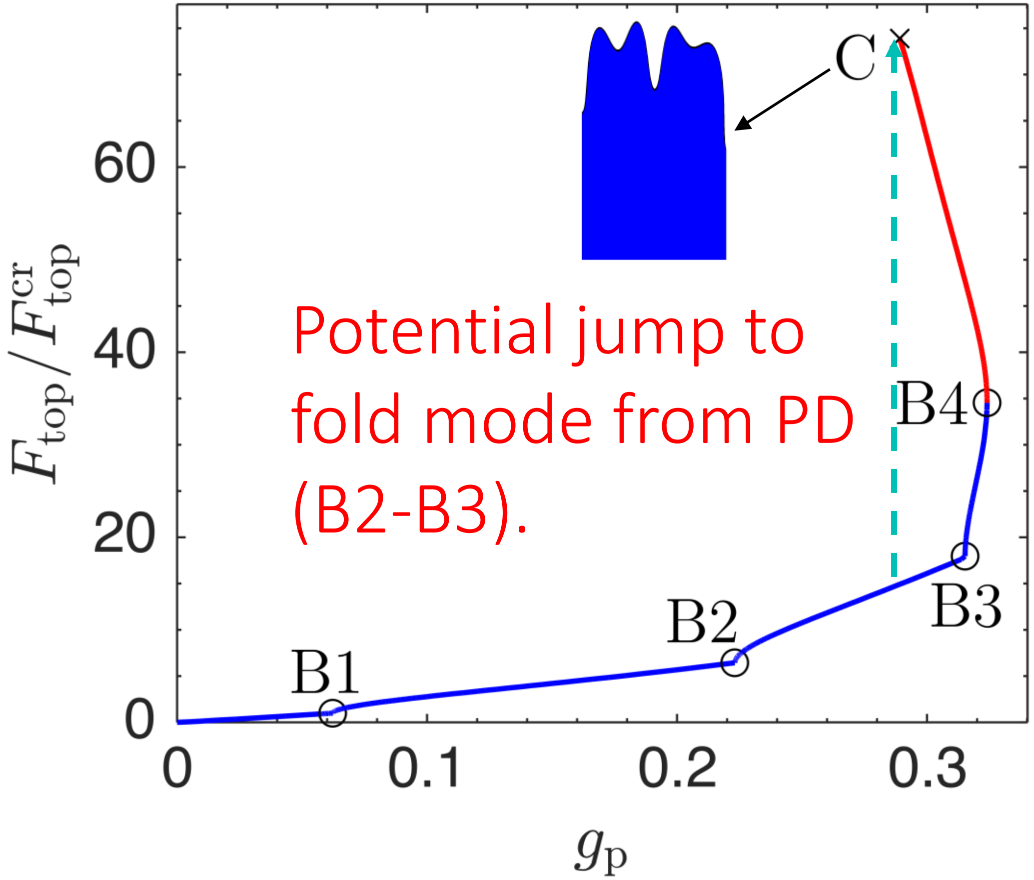
Isotropic growth, $E_f/E_s = 50$

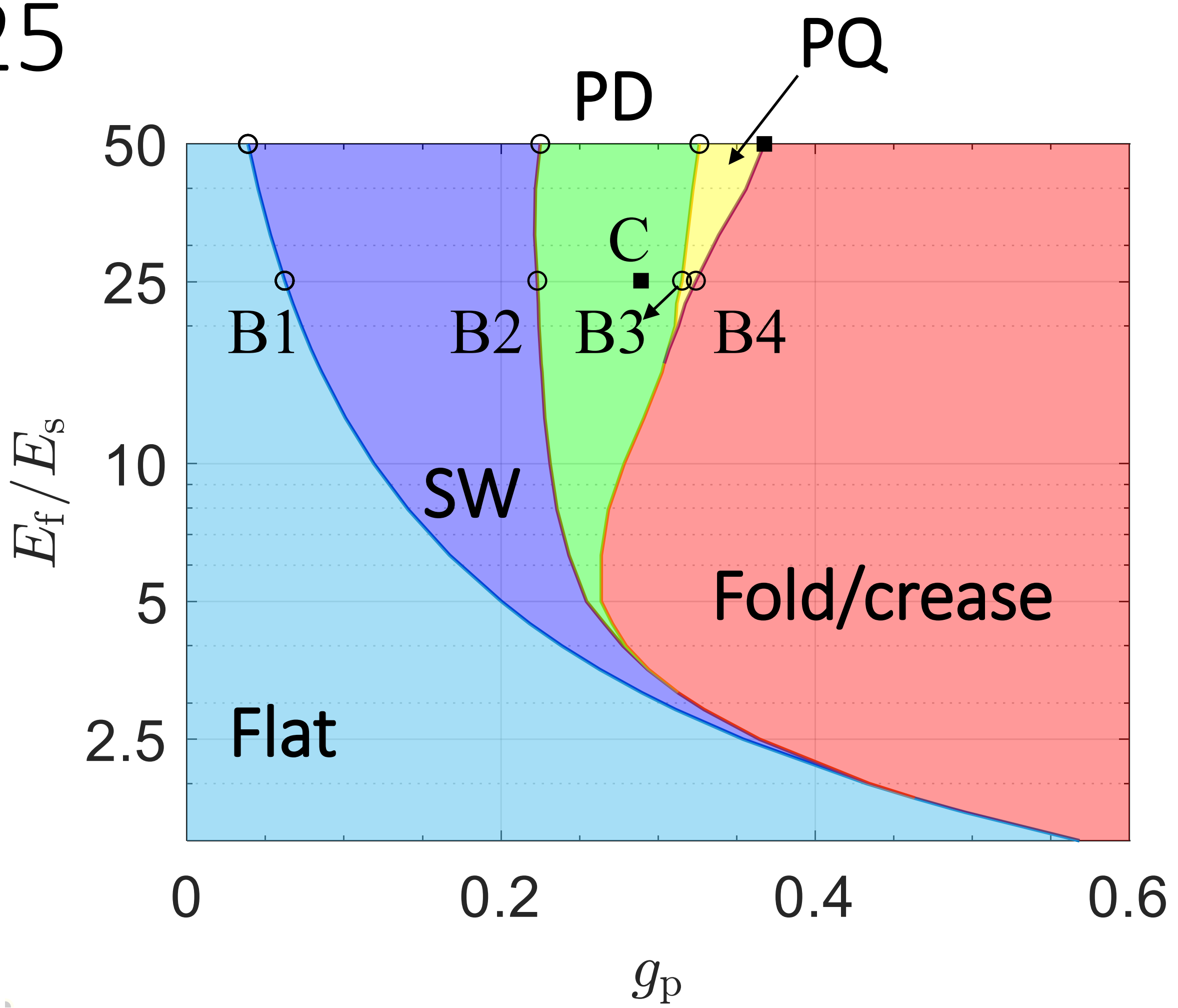




- Stable doubling in the periodicity from sinusoidal wrinkling (**SW**), period doubling (**PD**) up to *period quadrupling* (**PQ**) until fold occurs at C.

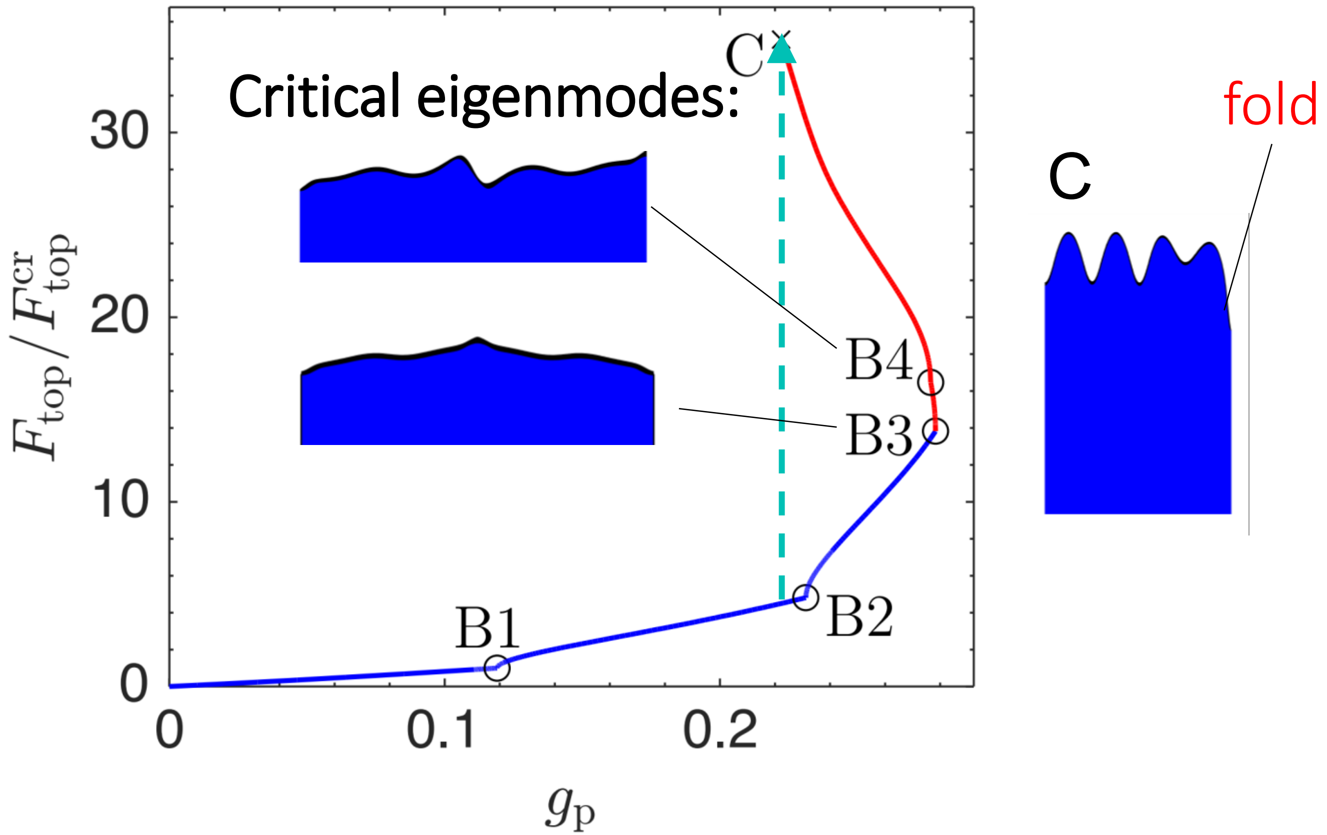
Isotropic growth, $E_f/E_s = 25$

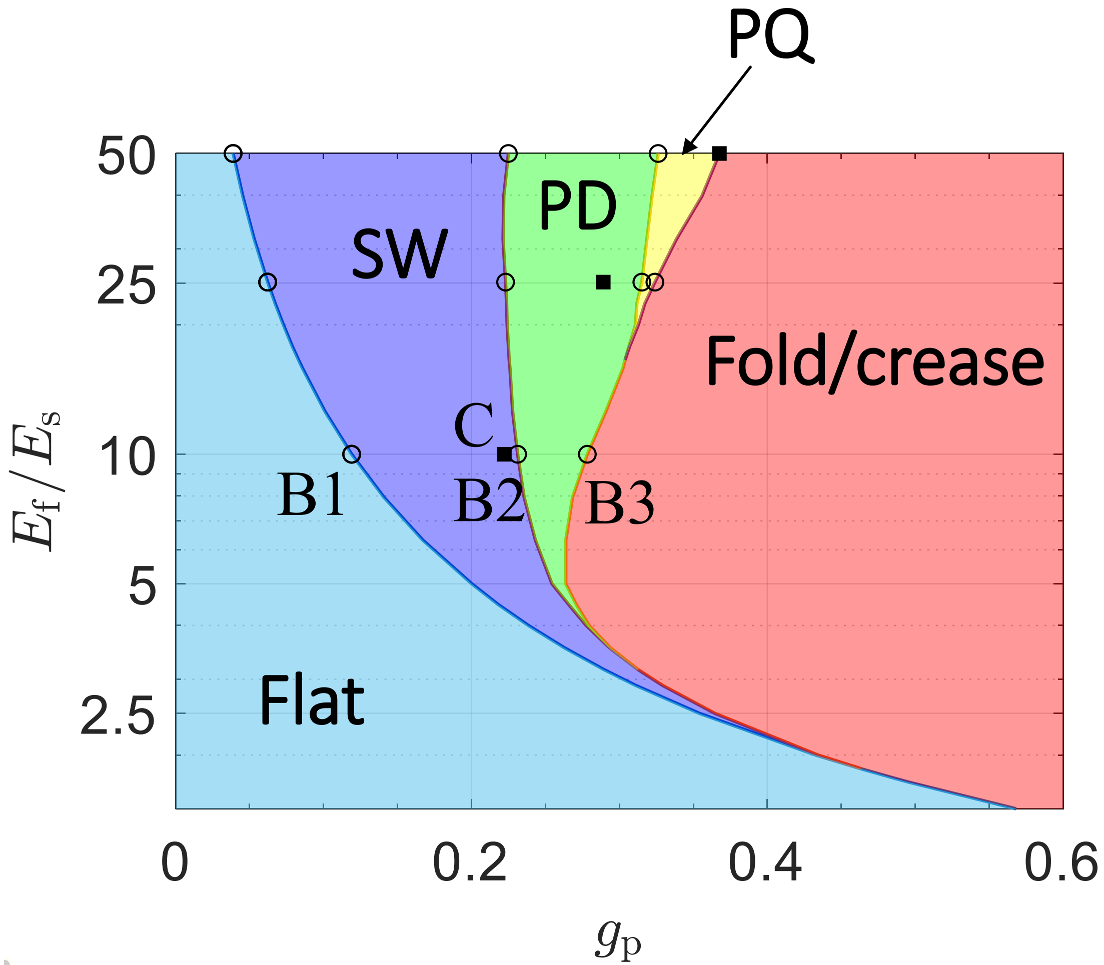




- Stable doubling in periodicity up to *period quadrupling* (B4), where bilayer loses stability and fold forms at C.

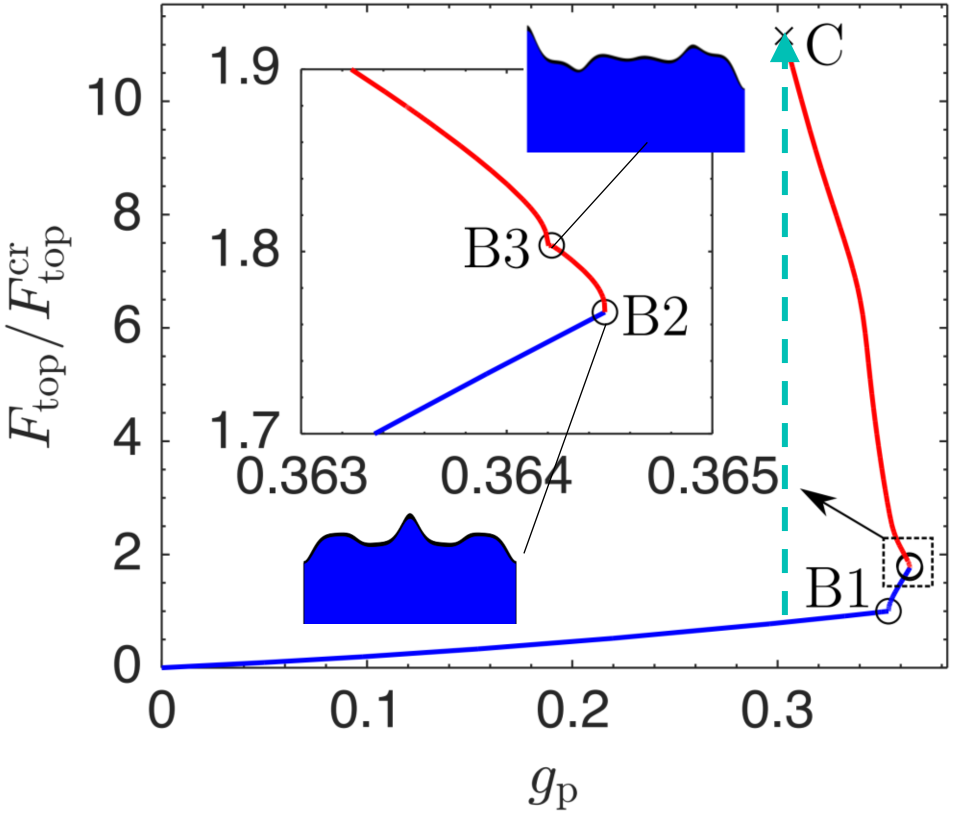
Isotropic growth, $E_f/E_s = 10$

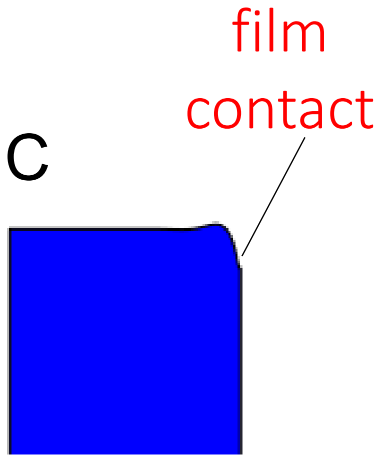


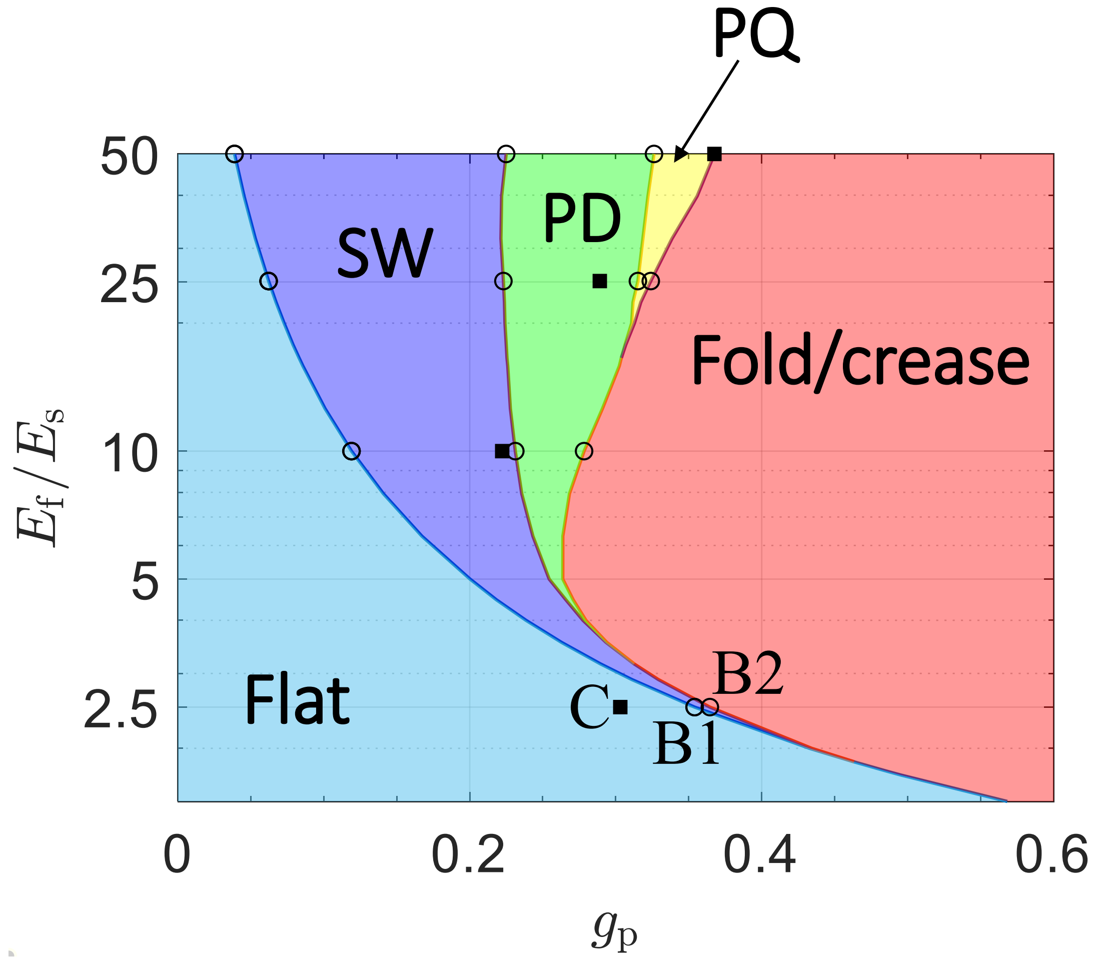


- Stable doubling in periodicity up to *period doubling* (B3), where bilayer loses stability and fold forms.

Isotropic growth, E_f/E_s =2.5

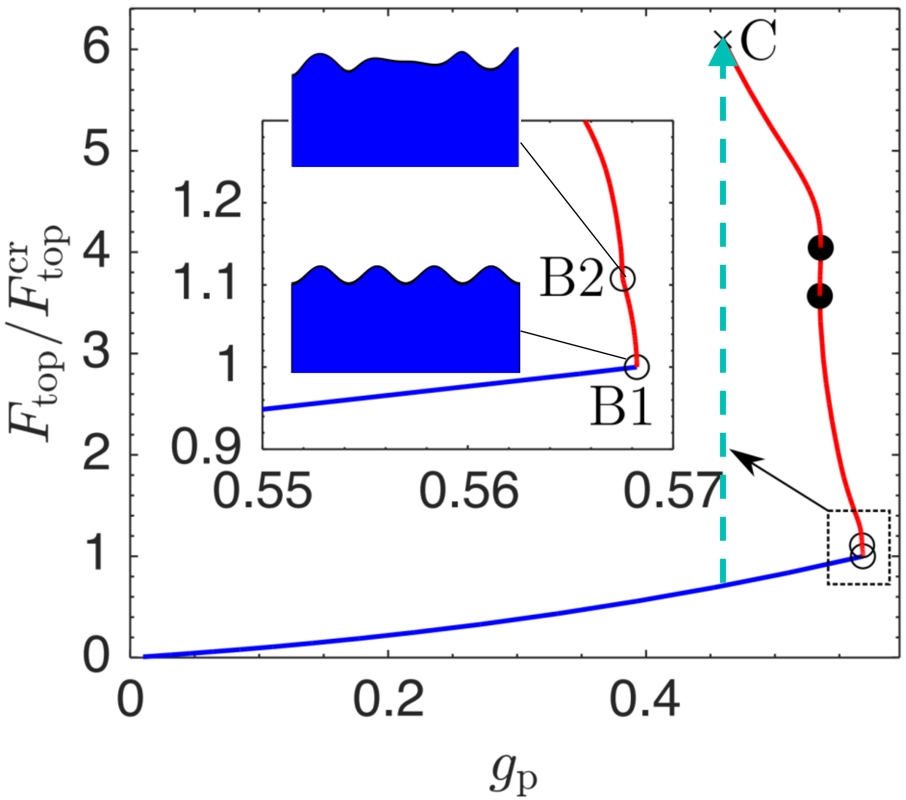


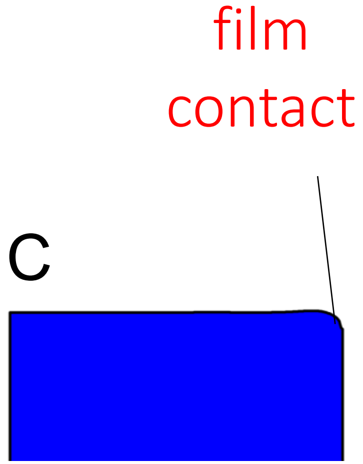


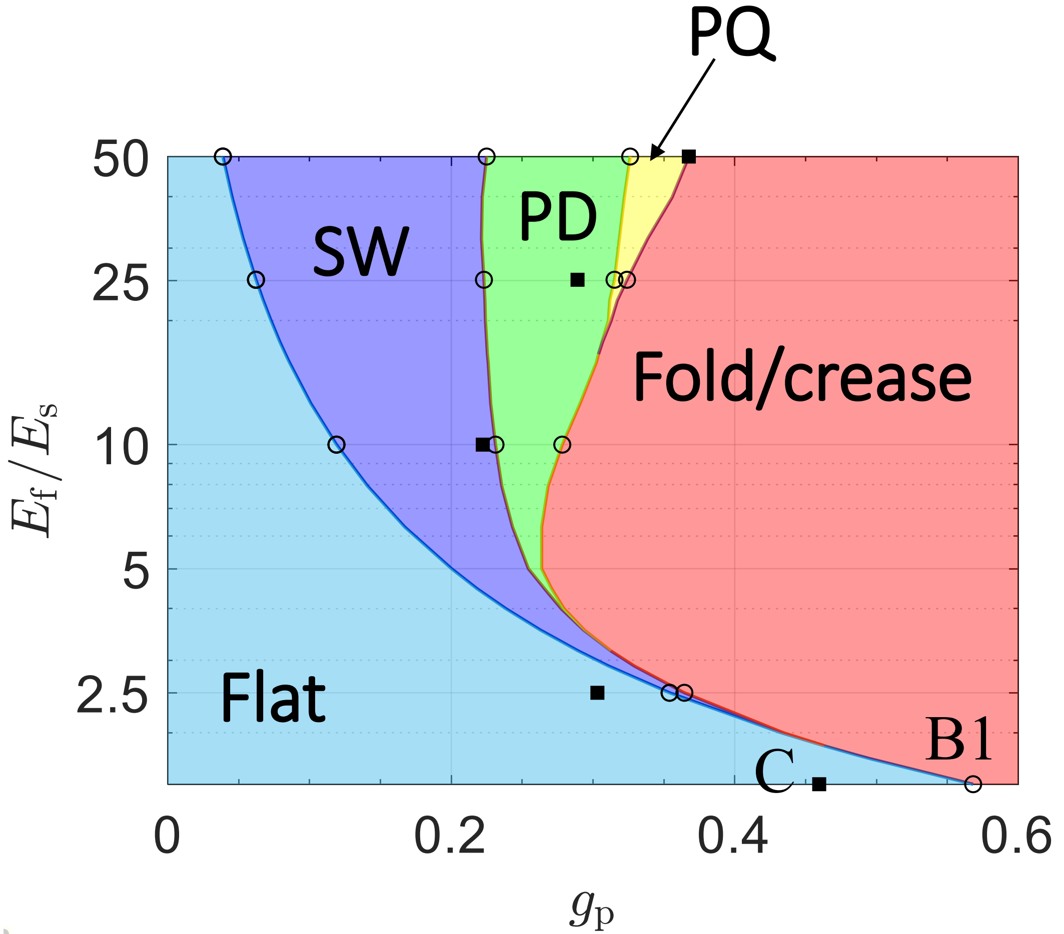


- *Stable sinusoidal wrinkling* up to B2, where bilayer loses stability and crease forms.

Isotropic growth, E_f/E_s =1.5

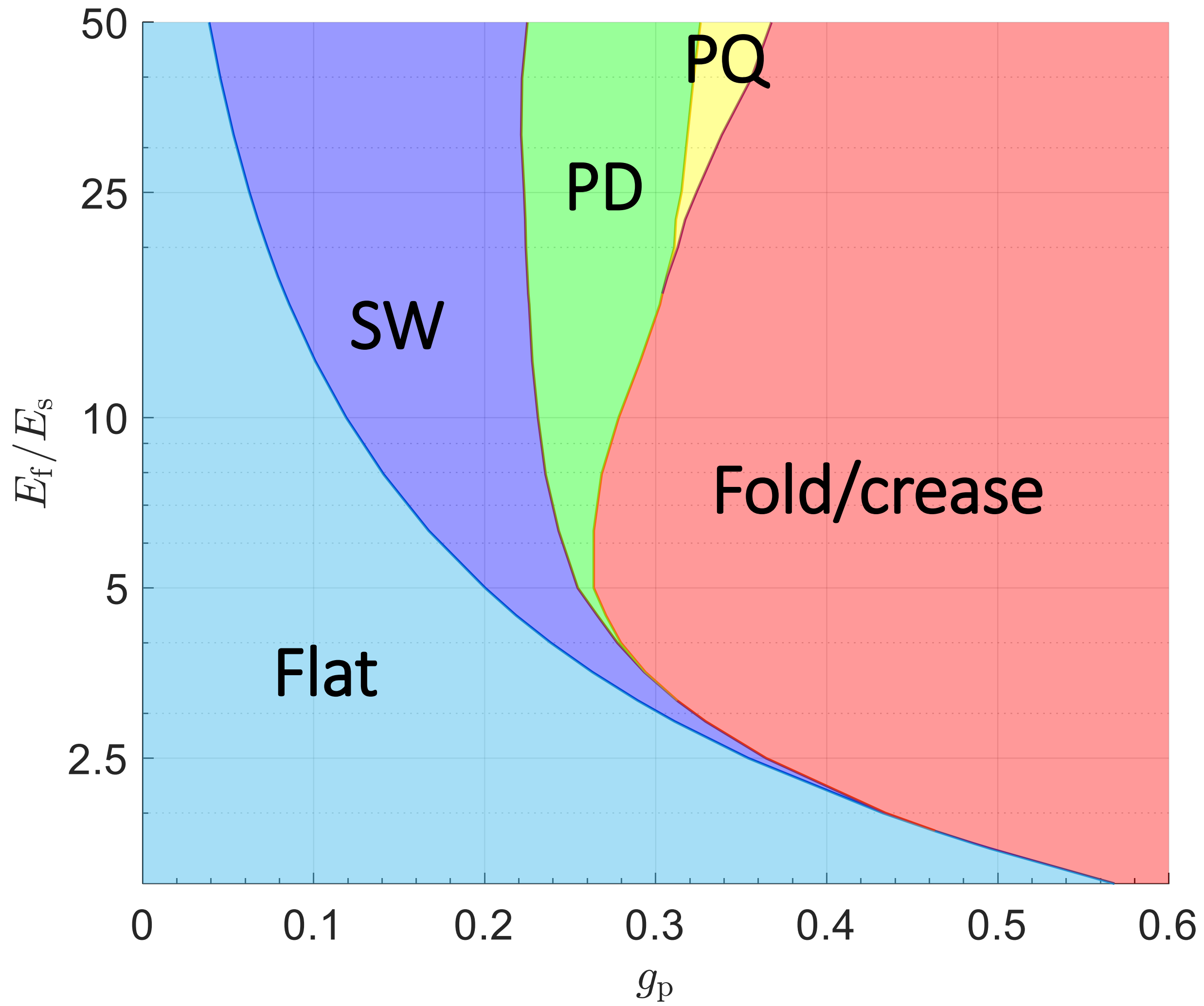






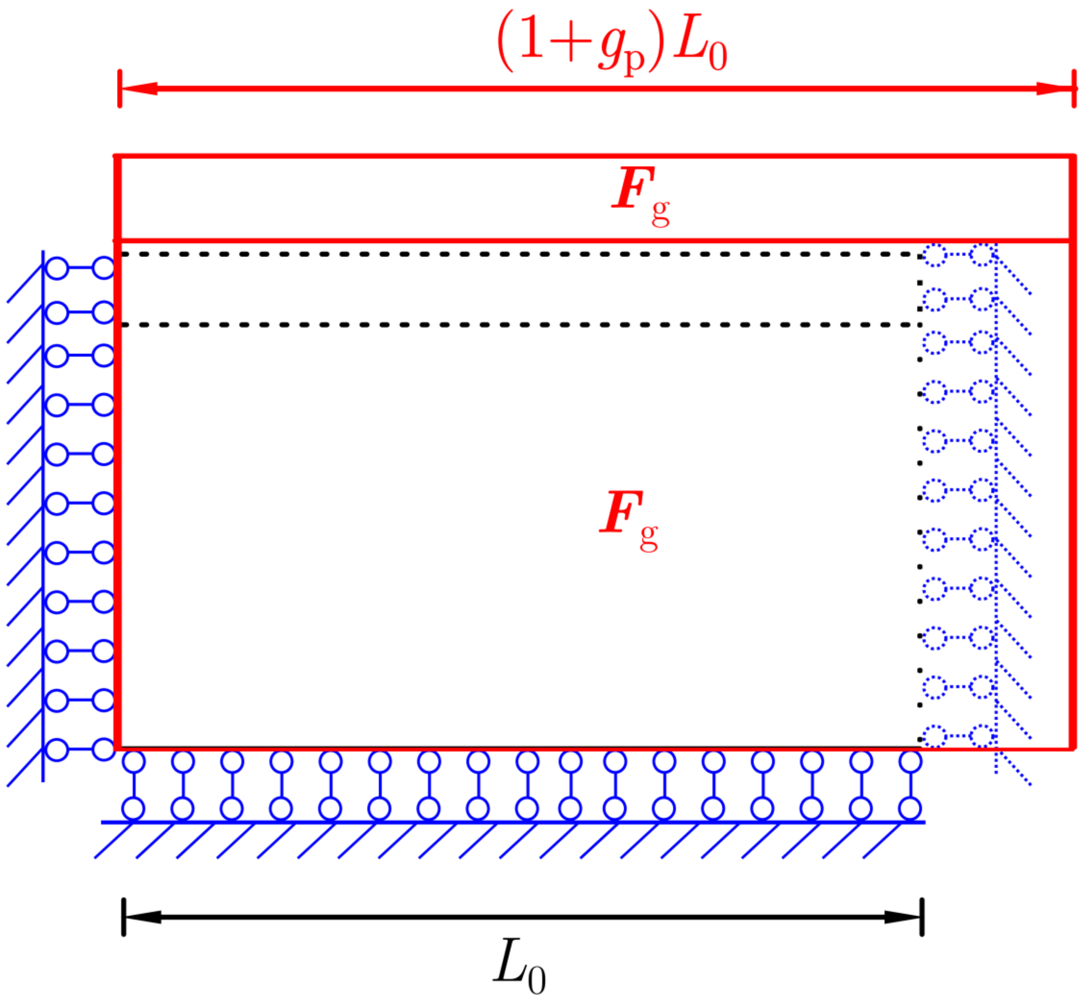
- No *stable* wrinkling pattern, where bilayer loses stability at the critical bifurcation point and crease forms immediately afterwards.

Phase change diagram for isotropic growth



- Growth factor at the critical state increases with decreasing E_f/E_s.
- Advanced stable wrinkling pattern, *i.e.* period quadrupling, exist in addition to period doubling.
- Wrinkling patterns are less developed with decreasing E_f/E_s.
- For E_f/E_s<1.73, no stable wrinkling pattern exist.

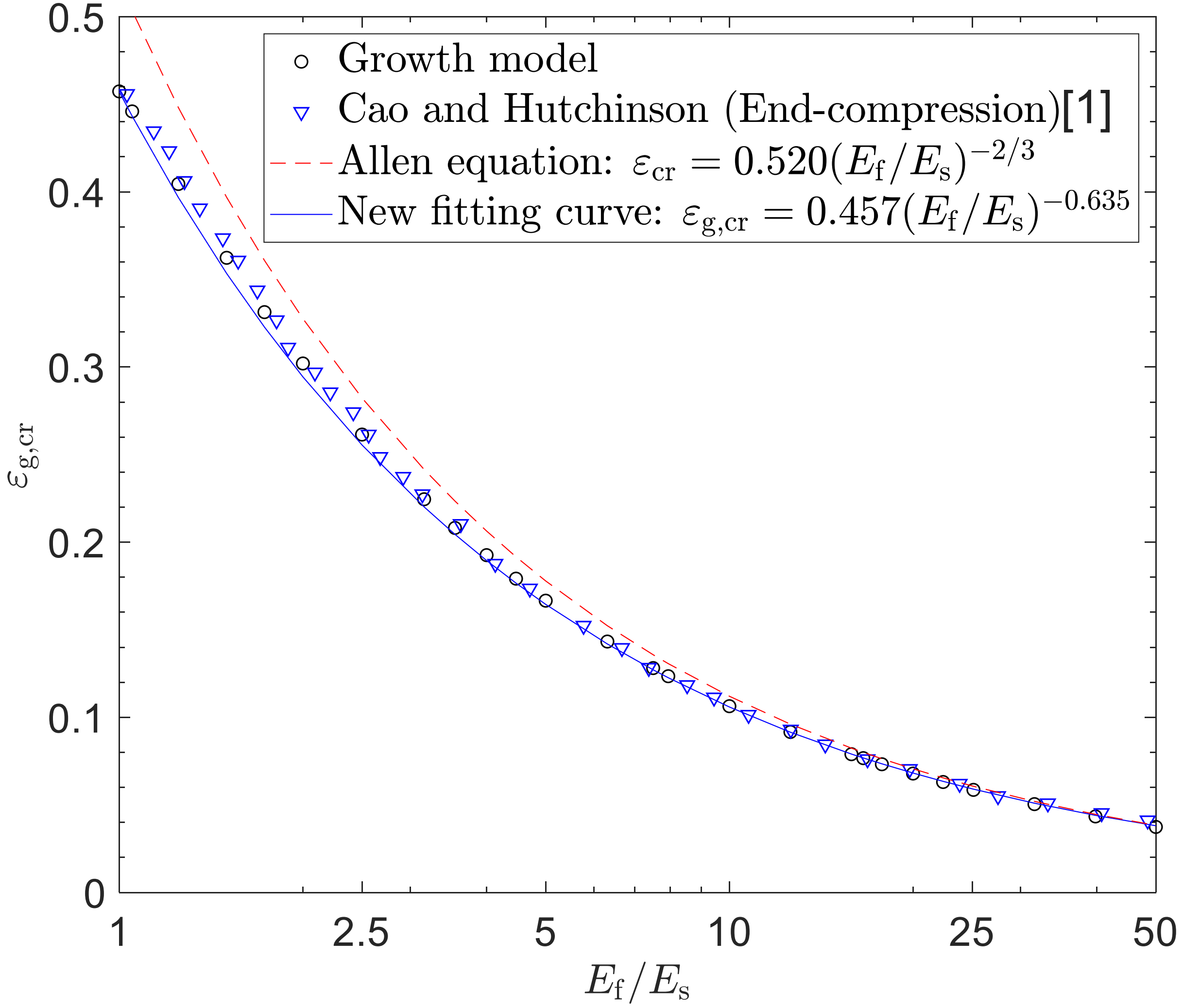
Unified behaviour of the isotropic growth and end-compression



- The nominal compressive **strain** of growing bilayer constrained at both ends with the isotropic growth pattern:

$$\varepsilon_g = \frac{g_p}{1+g_p}$$

Unified behaviour of the isotropic growth and end-compression



- The nominal **critical** growth strain—E_f/E_s shows identical relationship as bilayers under end-compression.
- New fitting function shows better correlation in the regime E_f/E_s=[1, 50] than the Allen equation (based on *very stiff* film—substrate system).
- Factor 0.457 is close to the Biot wrinkling strain (E_f/E_s=1)—0.456 (incompressibility assumption).

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Conclusion

- Generalised path-following technique[1] to identify the complete bifurcation diagram of growing bilayers and phase change diagram.
- With decreasing E_f/E_s, the response of growing bilayer becomes more governed by substrate, *i.e.* crease and fold mode is more favorable than further doubling in periodicity of wrinkling patterns.
- *Symmetry-breaking* bifurcations followed by unstable branches connect the stable wrinkling patterns and fold/crease mode for $E_f/E_s<50$.
- Under the nominal growth strain measurement, the critical strain—E_f/E_s relationship is identical to that of bilayers under uniformly end-compression.

[1] Groh, R.M.J., (2022) A morphoelastic stability framework for post-critical pattern formation in growing thin biomaterials, **CMAME**, 394,114839.

Thanks for your attention.

Q&A

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