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AN ASYMPTOTIC FORMULA FOR THE DISPLACEMENT FIELD IN THE PRESENCE OF THIN ELASTIC INHOMOGENEITIES*

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Abstract. We consider a plane isotropic homogeneous elastic body with thin elastic inhomogeneities in the form of small neighborhoods of simple smooth curves. We derive a rigorous asymptotic expansion of the boundary displacement field as the thickness of the neighborhoods goes to zero.

Key words. Lamé system, thin inhomogeneities, asymptotic formulas

AMS subject classifications. 35C20, 35R05

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1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain representing the region occupied by an elastic material.

Let $\sigma_0 \subset \Omega$ be a simple smooth curve and define, for a positive small ϵ, the set

$$\omega_\epsilon = \{x \in \Omega \ : \ d(x, \sigma_0) < \epsilon\},$$

which represents an inclusion of small size made of a different elastic material.

Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the elastic tensor fields in $\Omega \setminus \overline{\omega}_\epsilon$ and ω_ϵ, respectively.

Given a traction field g on $\partial\Omega$, the displacement field u_ϵ, generated by this traction in the body containing the inclusion ω_ϵ, solves the following system of linearized elasticity:

$$\begin{cases} \operatorname{div}(\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon) = 0 & \text{in } \Omega, \\ (\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon) \cdot \nu = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbb{C}_\epsilon = \mathbb{C}_0 \chi_{\Omega \setminus \omega_\epsilon} + \mathbb{C}_1 \chi_{\omega_\epsilon}$, $\widehat{\nabla} u_\epsilon = \frac{1}{2}\left(\nabla u_\epsilon + (\nabla u_\epsilon)^T\right)$ is the symmetric deformation tensor and ν denotes the outward unit normal to $\partial\Omega$.

Let us also introduce the background displacement u_0, namely the solution of

$$\begin{cases} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} u_0) = 0 & \text{in } \Omega, \\ (\mathbb{C}_0 \widehat{\nabla} u_0) \cdot \nu = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The goal of this paper is to find an asymptotic expansion for $(u_\epsilon - u_0)_{|\partial\Omega}$ as $\epsilon \to 0$. An analogous expansion has been derived in [BFV] for the case of thin conductivity inclusions. These expansions represent a powerful tool to solve the inverse problem of identifying the inclusions, given boundary measurements (see, [ABF] and [ABF2] for the case of thin conductivity inclusions and [AK] for further references).

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In [AKNT] the authors derive an asymptotic expansion for the boundary displacement field $(u_\epsilon - u_0)_{|\partial\Omega}$ in the case of diametrically small elastic inclusions, namely inclusions of the form $z + \epsilon B$, where z is a point in Ω and B is a bounded domain containing the origin. The approach they use, based on the method of layer potentials (see [AK]), allows them to find a very accurate expansion. Unfortunately, this method does not seem to work in the case of thin elastic inclusions. Hence, in order to derive the expansion in our context, we apply similar arguments as in [BFV] for conductivity inclusions; more precisely, we use a variational approach and fine regularity estimates for solutions of elliptic systems with discontinuous coefficients obtained by Li and Nirenberg in [LN]. The main difficulty arising in the framework of linear elasticity, compared to the conductivity case, consists of finding an explicit representation formula for the tensor appearing in the first order term of the asymptotic expansion.

The plan of the paper is as follows: in section 2 we introduce some notation and state the main result. Section 3 is devoted to the derivation of some estimates and properties of the displacement field, and in section 4 we prove our main result.

2. The main result. Let us introduce some notation and assumptions that will be useful in what follows.

(a) We will assume that σ_0 is of class C^3 and that there exists some $K > 0$ such that

$$\begin{aligned} d(\sigma_0, \partial\Omega) &\geq K^{-1}, \\ \|\sigma_0\|_{C^3} &\leq K, \\ K^{-1} \leq \text{length}(\sigma_0) &\leq K. \end{aligned} \tag{3}$$

Moreover, we assume that for every $x \in \sigma_0$ there are two discs, B_1 and B_2, of radius K^{-1} such that

$$\overline{B}_1 \cap \overline{B}_2 = \overline{B}_1 \cap \sigma_0 = \overline{B}_2 \cap \sigma_0 = \{x\}.$$

The latter assumption guarantees that different parts of σ_0 do not get too close, so that ω_ϵ does not self-intersect for small ϵ.

(b) Ω and ω_ϵ are both homogeneous and isotropic, i.e., the elastic tensor fields $\mathbb{C}_0$ and $\mathbb{C}_1$ are of the following form:

$$(\mathbb{C}_m)_{ijlk} = \lambda_m \delta_{ij}\delta_{kl} + \mu_m(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}), \text{ for } i,j,k,l = 1,2, \ \ m = 0,1, \tag{4}$$

where (λ_0, μ_0) and (λ_1, μ_1) are the Lamè coefficients corresponding to $\Omega \setminus \overline{\omega}_\epsilon$ and ω_ϵ, respectively, and $(\lambda_0 - \lambda_1)^2 + (\mu_0 - \mu_1)^2 \neq 0$.

(c) There are two positive constants α_0 and β_0 such that

$$\min(\mu_0, \mu_1) \geq \alpha_0, \quad \min(2\lambda_0 + 2\mu_0, 2\lambda_1 + 2\mu_1) \geq \beta_0. \tag{5}$$

We note that the last conditions ensure that $\mathbb{C}_\epsilon$ is strongly convex in Ω, i.e., if we set $\xi_0 = \min(2\alpha_0, \beta_0)$, then

$$\mathbb{C}_\epsilon A \cdot A \geq \xi_0 |A|^2,$$

for any symmetric 2×2 matrix A, where $A \cdot B = \sum_{ij} a_{ij} b_{ij}$ and $|A|^2 = A \cdot A$.

(d) We shall prescribe a traction field $g \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$ satisfying the compatibility condition

$$\int_{\partial\Omega} g \cdot R = 0, \tag{6}$$

for every infinitesimal rigid displacement R, that is $R(x) = c + Wx$, where c is any constant vector in $\mathbb{R}^2$ and W any constant 2×2 skew matrix.

Under our assumptions there exist weak solutions u_ϵ and $u_0 \in H^1(\Omega, \mathbb{R}^2)$ to the problems (1) and (2), respectively (see, for example, [V] or [F]). Concerning uniqueness we recall that solutions of the above problems are uniquely determined up to infinitesimal rigid displacements. Hence, in order to uniquely identify such solutions, we assume that u_ϵ and u_0 satisfy the normalization conditions

$$\int_{\partial\Omega} u = 0, \quad \int_\Omega \nabla u - (\nabla u)^T = 0. \tag{7}$$

It is easy to see that if u_ϵ is solution of (1), then it solves the Lamé system

$$\begin{cases} \mu_0 \Delta u_\epsilon + (\lambda_0 + \mu_0)\nabla(\operatorname{div} u_\epsilon) & = 0 \quad \text{in} \quad \Omega \setminus \overline{\omega}_\epsilon, \\ \mu_1 \Delta u_\epsilon + (\lambda_1 + \mu_1)\nabla(\operatorname{div} u_\epsilon) & = 0 \quad \text{in} \quad \omega_\epsilon, \\ u_\epsilon^i & = u_\epsilon^e \quad \text{on} \quad \partial\omega_\epsilon, \\ \left(\mathbb{C}_0 \widehat{\nabla} u_\epsilon^i\right)\nu & = \left(\mathbb{C}_1 \widehat{\nabla} u_\epsilon^e\right)\nu \quad \text{on} \quad \partial\omega_\epsilon, \end{cases} \tag{8}$$

where ν is the outward unit normal to $\partial\omega_\epsilon$, and, for $x \in \partial\omega_\epsilon$,

$$u_\epsilon^e(x) = \lim_{\substack{y \to x \\ y \in \Omega\setminus\overline{\omega}_\epsilon}} u_\epsilon(y), \quad u_\epsilon^i(x) = \lim_{\substack{y \to x \\ y \in \omega_\epsilon}} u_\epsilon(y),$$

and

$$\widehat{\nabla} u_\epsilon^e(x) = \lim_{\substack{y \to x \\ y \in \Omega\setminus\overline{\omega}_\epsilon}} \widehat{\nabla} u_\epsilon(y), \quad \widehat{\nabla} u_\epsilon^i(x) = \lim_{\substack{y \to x \\ y \in \omega_\epsilon}} \widehat{\nabla} u_\epsilon(y).$$

For $y \in \Omega$, we will denote by $N(\cdot, y)$ the Neumann function related to Ω, i.e., the weak solution to the problem

$$\begin{cases} \operatorname{div}\left(\mathbb{C}_0 \widehat{\nabla} N(\cdot, y)\right) & = -\delta_y \, \mathrm{I}_d \quad \text{in} \quad \Omega, \\ \left(\mathbb{C}_0 \widehat{\nabla} N(\cdot, y)\right) \cdot \nu & = -\frac{1}{|\partial\Omega|} \, \mathrm{I}_d \quad \text{on} \quad \partial\Omega, \end{cases} \tag{9}$$

with the normalization conditions (7) and where I_d is the identity matrix in $\mathbb{R}^2$.

Note that $N(x, y)$ is regular for $x \neq y$ and, at $x = y$, has the same singularities of $\Gamma(x - y)$, where $\Gamma = (\Gamma_{ij})_{i,j=1}^2$ is the fundamental solution in the free space of the system

$$\operatorname{div}(\mathbb{C}_0 \widehat{\nabla} \cdot) = 0 \quad \text{in} \quad \mathbb{R}^2,$$

and is given by

$$\Gamma_{ij}(x) = \frac{A}{2\pi} \delta_{ij} \ln|x| - \frac{B}{2\pi} \frac{x_i x_j}{|x|^2},$$

where $A = \frac{1}{2}\left(\frac{1}{\mu_0} + \frac{1}{\lambda_0 + 2\mu_0}\right)$ and $B = \frac{1}{2}\left(\frac{1}{\mu_0} - \frac{1}{\lambda_0 + 2\mu_0}\right)$.

Let us fix an orthonormal system (n, τ) on σ_0 such that n is a unit normal vector field to the curve and τ is a unit tangent vector field. If σ_0 is a closed curve, then we will take n to point in the outward direction of the domain it encloses. Let κ denote the curvature of σ_0 and, for $a, b \in \mathbb{R}^2$, let $a \otimes b$ denote the tensor product $a \otimes b = a_i b_j$.

We are now ready to state our main result.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $\sigma_0 \subset\subset \Omega$ be a simple curve satisfying* (3). *Assume* (4), (5), *and* (6) *and let u_ϵ and u_0 be the solutions to* (1) *and* (2), *respectively, satisfying* (7). *For every $x \in \sigma_0$, there exists a fourth order symmetric tensor field $\mathcal{M}(x)$ such that, for $y \in \partial\Omega$ and $\epsilon \to 0$,*

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_{\sigma_0} \mathcal{M}(x)\widehat{\nabla} u_0(x) \cdot \widehat{\nabla} N(x,y)\, d\sigma_0(x) + o(\epsilon). \tag{10}$$

The term $o(\epsilon)$ is bounded by $C\epsilon^{1+\theta}\|g\|_{H^{-1/2}(\partial\Omega)}$, with $0 < \theta < 1$ and C depending only on θ, Ω, α_0, β_0, and K.

Furthermore, on σ_0,

$$\mathcal{M}\widehat{\nabla} u_0 = a \operatorname{div} u_0\, I_d + b\widehat{\nabla} u_0 + c\left(\frac{\partial(u_0\cdot\tau)}{\partial\tau} + \kappa(u_0\cdot n)\right)\tau\otimes\tau + d\frac{\partial(u_0\cdot n)}{\partial n} n\otimes n,$$

where

$$a = (\lambda_1 - \lambda_0)\frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1}, \quad b = 2(\mu_1 - \mu_0)\frac{\mu_0}{\mu_1}, \tag{11}$$

$$c = 2(\mu_1 - \mu_0)\left[\left(\frac{2\lambda_1 + 2\mu_1 - \lambda_0}{\lambda_1 + 2\mu_1} - \frac{\mu_0}{\mu_1}\right)\right], \tag{12}$$

and

$$d = 2(\mu_1 - \mu_0)\frac{\mu_1\lambda_0 - \mu_0\lambda_1}{\mu_1(\lambda_1 + 2\mu_1)}. \tag{13}$$

The proof of the theorem is contained in section 4.

3. Energy and a priori estimates. In this section we will show that, for $\epsilon \to 0$,

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} = O(\epsilon^{1/2}). \tag{14}$$

In order to establish it we will need the following version of the Korn inequality.

LEMMA 3.1. *Let Ω be a Lipschitz connected open set in $\mathbb{R}^2$. Let $u \in H^1(\Omega, \mathbb{R}^2)$ and let $W_0 = \int_\Omega \frac{1}{2}\left(\nabla u - (\nabla u)^T\right)$.*

Then, there exists a constant C such that

$$\|\nabla u - W_0\|_{L^2(\Omega)} \le C\|\widehat{\nabla} u\|_{L^2(\Omega)}. \tag{15}$$

For the proof, see [T, section 3].

PROPOSITION 3.2. *Let u_ϵ and u_0 be solutions to* (1) *and* (2), *respectively. There exists a constant C depending on Ω, K, α_0, and β_0, such that*

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \le C\epsilon^{1/2}\|g\|_{H^{-1/2}(\partial\Omega)}. \tag{16}$$

Proof. Since $\int_{\partial\Omega}(u_\epsilon - u_0) = 0$, by the Poincaré inequality there exists a constant C, depending on Ω, such that

$$\int_\Omega |u_\epsilon - u_0|^2 \le C\int_\Omega |\nabla(u_\epsilon - u_0)|^2. \tag{17}$$

It thus suffices to estimate $\|\nabla(u_\epsilon - u_0)\|_{L^2(\Omega)}$.

By the strong convexity of $\mathbb{C}_\epsilon$ and the Korn inequality, in the form of Lemma 3.1, applied to $u_\epsilon - u_0$, and recalling that u_ϵ and u_0 satisfy (7), we get

$$
\begin{aligned}
\int_\Omega \mathbb{C}_\epsilon \widehat{\nabla}(u_\epsilon - u_0) \cdot \widehat{\nabla}(u_\epsilon - u_0) &\geq \xi_0 \int_\Omega |\widehat{\nabla}(u_\epsilon - u_0)|^2 \\
&\geq C \int_\Omega |\nabla(u_\epsilon - u_0)|^2, \qquad (18)
\end{aligned}
$$

where C depends on α_0, β_0, and Ω.

Now, observe that

$$
\int_\Omega \mathbb{C}_\epsilon \widehat{\nabla}(u_\epsilon - u_0) \cdot \widehat{\nabla}(u_\epsilon - u_0) = \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1) \widehat{\nabla} u_0 \cdot \widehat{\nabla}(u_\epsilon - u_0), \qquad (19)
$$

which follows by integration by parts and uses the fact that $(\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon) \cdot \nu = (\mathbb{C}_0 \widehat{\nabla} u_0) \cdot \nu$ on $\partial\Omega$. Indeed

$$
\begin{aligned}
&\int_\Omega \mathbb{C}_\epsilon \widehat{\nabla}(u_\epsilon - u_0) \cdot \widehat{\nabla}(u_\epsilon - u_0) - \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1) \widehat{\nabla} u_0 \cdot \widehat{\nabla}(u_\epsilon - u_0) \\
&= \int_{\Omega \setminus \omega_\epsilon} \mathbb{C}_0 \widehat{\nabla}(u_\epsilon - u_0) \cdot \widehat{\nabla}(u_\epsilon - u_0) + \int_{\omega_\epsilon} (\mathbb{C}_1 \widehat{\nabla} u_\epsilon - \mathbb{C}_0 \widehat{\nabla} u_0) \cdot \widehat{\nabla}(u_\epsilon - u_0) \\
&= \int_\Omega \mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon \cdot \widehat{\nabla}(u_\epsilon - u_0) - \int_\Omega \mathbb{C}_0 \widehat{\nabla} u_0 \cdot \widehat{\nabla}(u_\epsilon - u_0) \\
&= \int_{\partial\Omega} \left(\left(\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon \right) \cdot \nu - \left(\mathbb{C}_0 \widehat{\nabla} u_0 \right) \cdot \nu \right) \cdot (u_\epsilon - u_0) \, d\sigma = 0,
\end{aligned}
$$

hence (19) holds.

On the other hand, by the Hölder inequality,

$$
\begin{aligned}
&\int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1) \, \widehat{\nabla} u_0 \cdot \widehat{\nabla}(u_\epsilon - u_0) \, dx \\
&\leq \max\{2|\mu_0 - \mu_1|, |\lambda_0 - \lambda_1|\} \, \|\nabla u_0\|_{L^\infty(\omega_\epsilon)} |\omega_\epsilon|^{1/2} \|\nabla(u_\epsilon - u_0)\|_{L^2(\Omega)}. \qquad (20)
\end{aligned}
$$

In order to bound $\|\nabla u_0\|_{L^\infty(\omega_\epsilon)}$ note that for small ϵ, say $\epsilon < K/2$, the distance between ω_ϵ and $\partial\Omega$ is bounded from below by $K/2$. Hence, by standard interior regularity estimates for elliptic systems (see [C]),

$$
\|\nabla u_0\|_{L^\infty(\omega_\epsilon)} \leq C \|u_0\|_{H^1(\Omega)},
$$

where C depends on α_0, β_0, and K.

By the divergence theorem, the trace theorem (see [LM]), and the Poincaré inequality,

$$
\|u_0\|_{H^1(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)},
$$

where C depends only on Ω. Finally,

$$
\|\nabla u_0\|_{L^\infty(\omega_\epsilon)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}, \qquad (21)
$$

where C depends on Ω, α_0, β_0, and K.

So, by (18), (19), and (20) we obtain

$$\|\nabla(u_\epsilon - u_0)\|_{L^2(\Omega)} \le C|\omega_\epsilon|^{1/2}\|g\|_{H^{-1/2}(\partial\Omega)},$$

where $C = C(\Omega, \alpha_0, \beta_0, K)$. By assumption (3), we can estimate

$$|\omega_\epsilon| \le C\epsilon, \tag{22}$$

where C depends only on K. By putting together (21), (22), and the Poincarè inequality (17), we get (16). □

Besides the energy estimates (16), a key ingredient to establish the asymptotic expansion of Theorem 2.1 is a gradient estimate for elliptic systems modeling composite materials that has been established by Li and Nirenberg in [LN]. Here we state and use a simplified version of Proposition 5.1 in [LN].

Let D be the unit square $D = [-1, 1] \times [-1, 1]$, and let $f_0, \dots, f_{l+1} \in C^2([-1, 1])$ such that

$$-1 = f_0(x_1) < f_1(x_1) < \cdots < f_{l+1}(x_1) = 1 \quad \text{for} \quad x_1 \in [-1, 1].$$

Let

$$D_m = \{x = (x_1, x_2) \in D \,:\, f_{m-1}(x_1) < x_2 < f_m(x_1)\} \quad \text{for} \quad 1 \le m \le l + 1.$$

We suppose that the origin does not belong to the graphs of the functions f_j, and we denote by m_0 the index for which

$$f_{m_0}(0) < 0 < f_{m_0+1}(0).$$

Let us also set $\frac{1}{2}D = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$.

Let $\mathbb{C}$ be a bounded symmetric Lamé tensor defined in D and such that $\mathbb{C}$ is constant in each D_m with corresponding Lamé coefficients λ_m and μ_m. Then the following estimate holds.

Proposition 3.3. *Let $u \in H^1(D, \mathbb{R}^2)$ be a weak solution to*

$$\operatorname{div}\left(\mathbb{C}\widehat{\nabla} u\right) = 0 \quad in \quad D.$$

Then, for any $x \in \overline{D}_{m_0} \cap \frac{1}{2}D$,

$$|\nabla u(x) - \nabla u(0)| \le C\|u\|_{L^2(D)}|x|^\alpha, \tag{23}$$

where $\alpha \in (0, 1/4)$ and C depends on α, l, λ_m, μ_m, and $\|f_m\|_{C^2([-1,1])}$, for $m = 1, \dots, l + 1$.

For the proof of this result, see [LN, section 5].

4. Proof of Theorem 2.1. We divide the proof into several steps: in the first step we write $(u_\epsilon - u_0)_{|\partial\Omega}$ in terms of an integral over ω_ϵ of the product of $\widehat{\nabla} u_\epsilon^i$ and $\widehat{\nabla} N$.

In the second step, by using the estimate of Proposition 3.3, we reduce this integral to an integral over some part of $\partial\omega_\epsilon$. In the third part, we use the transmission conditions and introduce the tensor $\mathcal{M}$. The fourth part contains the conclusion of the proof.

First step.
We are going to show that, for $y \in \partial\Omega$,

$$(u_\epsilon - u_0)(y) = \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y). \tag{24}$$

In order to get (24) we recall that, for $y \in \Omega$, the function $N(\cdot, y)$ satisfies

$$\int_\Omega \mathbb{C}_0 \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla} v = -v(y) + \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v \quad \forall v \in H^1(\Omega).$$

By choosing $v = u_\epsilon - u_0$ and using the normalization (7) we get

$$\int_\Omega \mathbb{C}_0 \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla}(u_\epsilon - u_0) = -(u_\epsilon - u_0)(y). \tag{25}$$

Observe now that

$$\begin{aligned}\int_\Omega \mathbb{C}_0 \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla}(u_\epsilon - u_0) &= \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1)\,\widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla} u_\epsilon \\ &\quad + \int_\Omega \mathbb{C}_\epsilon \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla} u_\epsilon - \int_\Omega \mathbb{C}_0 \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla} u_0.\end{aligned} \tag{26}$$

Since u_ϵ and u_0 are solutions to (1) and (2), respectively, we have

$$\begin{aligned}\int_\Omega \mathbb{C}_0 \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla} u_0 &= \int_{\partial\Omega} g \cdot N(\cdot, y) \\ &= \int_\Omega \mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y),\end{aligned}$$

hence (26) becomes

$$\int_\Omega \mathbb{C}_0 \widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla}(u_\epsilon - u_0) = \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1)\,\widehat{\nabla} N(\cdot, y) \cdot \widehat{\nabla} u_\epsilon,$$

and, by inserting this last relation into (25,) we get (24) for $y \in \Omega$.

Finally, since $u_\epsilon - u_0$ is continuous up to $\partial\Omega$, we get (24) for any $y \in \partial\Omega$.

Second step.
Let β be a constant $0 < \beta < 1$, and set

$$\omega'_\epsilon = \left\{ x + \mu n(x) \,:\, x \in \sigma_0,\, d(x, \partial\sigma_0) > \epsilon^\beta,\, \mu \in (-\epsilon, \epsilon) \right\}.$$

Notice that if σ_0 is a closed simple curve, then $\omega'_\epsilon = \omega_\epsilon$.

Let us write

$$\begin{aligned}\int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y) &= \int_{\omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y) \\ &\quad + \int_{\omega_\epsilon \setminus \omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y).\end{aligned} \tag{27}$$

Concerning the last term in (27)

$$\begin{aligned}\left| \int_{\omega_\epsilon \setminus \omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y) \right| &\le \left| \int_{\omega_\epsilon \setminus \omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} (u_\epsilon - u_0) \cdot \widehat{\nabla} N(\cdot, y) \right| \\ &\quad + \left| \int_{\omega_\epsilon \setminus \omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla} u_0 \cdot \widehat{\nabla} N(\cdot, y) \right|.\end{aligned} \tag{28}$$

In order to bound the first term on the right-hand side (RHS) of (28) we use the energy estimate (16) and the fact that, for $y \in \partial\Omega$, $\|\nabla N(\cdot, y)\|_{L^\infty(\omega_\epsilon)}$ is bounded uniformly in ϵ. Moreover, since

$$|\omega_\epsilon \setminus \omega'_\epsilon| \le C\epsilon^{1+\beta},$$

we get

$$\left| \int_{\omega_\epsilon \setminus \omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\, \widehat{\nabla}(u_\epsilon - u_0) \cdot \widehat{\nabla} N(\cdot, y) \right| \le C\epsilon^{1+\beta/2} \|g\|_{H^{-1/2}(\partial\Omega)},$$

where $C = C(\Omega, K, \alpha_0, \beta_0)$.

In the last term on the RHS of (28) we use the regularity estimates for u_0 so that

$$\begin{aligned} \left| \int_{\omega_\epsilon \setminus \omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\, \widehat{\nabla} u_0 \cdot \widehat{\nabla} N(\cdot, y) \right| &\le C \|\nabla u_0\|_{L^\infty(\omega_\epsilon)} \|\nabla N(\cdot, y)\|_{L^\infty(\omega_\epsilon)} \cdot |\omega_\epsilon \setminus \omega'_\epsilon| \\ &\le C\epsilon^{1+\beta} \|g\|_{H^{-1/2}(\partial\Omega)}, \end{aligned}$$

where $C = C(K, \alpha_0, \beta_0)$ and so (27) becomes, for $\epsilon \to 0$,

$$\int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\, \widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y) = \int_{\omega'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\, \widehat{\nabla} u_\epsilon \cdot \widehat{\nabla} N(\cdot, y) + O(\epsilon^{1+\beta/2}). \tag{29}$$

Now let us denote by σ'_μ, for $\mu \in [-\epsilon, \epsilon]$, the curve

$$\sigma'_\mu = \left\{ x + \mu n(x) \,:\, x \in \sigma_0,\ d(x, \partial\sigma_0) > \epsilon^\beta \right\}.$$

For every point $x + \mu n(x) \in \omega'_\epsilon$ (for $\mu \in (-\epsilon, \epsilon)$), let us consider the point $x + \epsilon n(x) \in \sigma'_\epsilon$ and let us compare $\nabla u_\epsilon(x + \mu n(x))$ with $\nabla u^i_\epsilon(x + \epsilon n(x))$.

More precisely, we will establish that, for $\alpha \in (0, 1/4)$,

$$|\nabla u_\epsilon(x + \mu n(x)) - \nabla u^i_\epsilon(x + \epsilon n(x))| \le C\epsilon^{-\beta(2+\alpha)} \epsilon^\alpha \|g\|_{H^{-1/2}(\partial\Omega)}, \tag{30}$$

where $C = C(K, \alpha_0, \beta_0, \alpha)$.

Let ϵ be small enough to have

$$2\epsilon < \frac{\epsilon^\beta}{2\sqrt{2}}. \tag{31}$$

We note that the distance between $x + \mu n(x)$ and $x + \epsilon n(x)$ is smaller than 2ϵ and that, in a neighborhood of $x + \mu n(x)$ of radius ϵ^β, the boundary $\partial\omega_\epsilon$ is represented by graphs of smooth functions. In particular, if we set the origin to $x + \mu n(x)$, up to a rotation of the coordinate system (z_1, z_2), we have that there exist two functions g_1 and g_2 such that, for $-\frac{\epsilon^\beta}{\sqrt{2}} < z_1 < \frac{\epsilon^\beta}{\sqrt{2}}$,

$$\mathbb{C}_\epsilon(z_1, z_2) = \begin{cases} \mathbb{C}_0 & \text{if} \quad -\frac{\epsilon^\beta}{\sqrt{2}} < z_2 < g_1(z_1), \\ \mathbb{C}_1 & \text{if} \quad g_1(z_1) < z_2 < g_2(z_1), \\ \mathbb{C}_0 & \text{if} \quad g_2(z_1) < z_2 < \frac{\epsilon^\beta}{\sqrt{2}}. \end{cases}$$

By the a priori assumptions on σ_0 we know that

$$\|g_1\|_{C^2}, \quad \|g_2\|_{C^2} \le K.$$

Consider now the function

$$v(y) = u_\epsilon\left(\frac{\epsilon^\beta}{\sqrt{2}}y\right),$$

which is defined in $[-1,1]\times[-1,1]$. Notice that v solves

$$\operatorname{div}\left(\tilde{\mathbb{C}}\widehat{\nabla}v\right) = 0 \quad \text{in} \quad (-1,1)\times(-1,1),$$

where

$$\tilde{\mathbb{C}}(y_1,y_2) = \begin{cases} \mathbb{C}_0 & \text{if} \quad -1 < y_2 < f_1(y_1) = \sqrt{2}\epsilon^{-\beta}g_1(\frac{\epsilon^\beta}{\sqrt{2}}y_1), \\ \mathbb{C}_1 & \text{if} \quad f_1(y_1) < y_2 < f_2(y_1) = \sqrt{2}\epsilon^{-\beta}g_2(\frac{\epsilon^\beta}{\sqrt{2}}y_1), \\ \mathbb{C}_0 & \text{if} \quad f_2(y_1) < y_2 < 1. \end{cases}$$

Let us check that we can apply Proposition 3.3, with $l = 2$ and $m_0 = 1$. Since $|g_i(y_1)| < \frac{\epsilon^\beta}{\sqrt{2}}$ and the derivative of g_i is bounded by K for $i = 1,2$, then

$$\|g_i\|_{L^\infty} \le (K+1)\frac{\epsilon^\beta}{\sqrt{2}}, \quad i = 1,2.$$

From this last estimate, and from the C^2 bounds on g_1 and g_2, we get

$$\|f_i\|_{C^2([-1,1])} \le 2K+1 \quad \text{for} \quad i = 1,2.$$

Since we set the origin at the point $x + \mu n(x)$, we have that

$$|x + \epsilon n(x)| = |x + \mu n(x) + (\epsilon - \mu)n(x)| = |(\epsilon-\mu)n(x)| \le 2\epsilon.$$

Hence, if we set $\overline{y} = \sqrt{2}\epsilon^{-\beta}(x + \epsilon n(x))$ we have, by (31),

$$|\overline{y}| = \sqrt{2}\epsilon^{-\beta}|x + \epsilon n(x)| \le \sqrt{2}\epsilon^{-\beta}\cdot 2\epsilon \le \frac{1}{2},$$

and, $\overline{y} \in \overline{D}_{m_0} \cap \frac{1}{2}D$. By Proposition 3.3,

$$|\nabla v(\overline{y}) - \nabla v(0)| \le C\|v\|_{L^2(D)}|\overline{y}|^\alpha,$$

where C depends only on K, α_0, β_0, and $\alpha \in (0,1/4)$. If we read this estimate for the function u_ϵ we get

$$\begin{aligned} \left|\nabla u_\epsilon(x+\mu n(x)) - \nabla u_\epsilon^i(x+\epsilon n(x))\right| &\le C\|u_\epsilon\|_{L^2(\Omega)}\epsilon^{-\beta(2+\alpha)}|(\mu-\epsilon)n(x)|^\alpha \\ &\le C\|u_\epsilon\|_{L^2(\Omega)}\epsilon^{-\beta(2+\alpha)}\epsilon^\alpha. \end{aligned}$$

Since

$$\|u_\epsilon\|_{L^2(\Omega)} \le C\|\nabla u_\epsilon\|_{L^2(\Omega)} \le C\|g\|_{H^{-1/2}(\partial\Omega)}$$

we finally have (30).

Due to (30), we can approximate the values of ∇u_ϵ in ${\omega_\epsilon}'$ with the values on σ_ϵ'.

Let us denote by ${\sigma_0}' = \{x \in \sigma_0 \,:\, d(x,\partial\sigma_0) > \epsilon^\beta\}$. Due to the regularity assumption on σ_0,

$$d\sigma_x^\mu = (1+O(\epsilon))d\sigma_x^0,$$

where $d\sigma_x^\mu$ and $d\sigma_x^0$ denote the infinitesimal arclengths on σ'_μ and σ'_0, respectively.

Hence,

$$
\begin{aligned}
&\int_{\omega'_\epsilon} (\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u_\epsilon(x)\cdot\widehat{\nabla}N(x,y)\,dx\\
&=\int_{-\epsilon}^{\epsilon}\int_{\sigma'_\mu}(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u_\epsilon(x)\cdot\widehat{\nabla}N(x,y)\,d\sigma_x^\mu\,d\mu\\
&=\int_{-\epsilon}^{\epsilon}\int_{\sigma_0{}'}(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u_\epsilon(x+\mu n(x))\cdot\widehat{\nabla}N(x+\mu n(x),y)\,d\sigma_x^0\,d\mu+O(\epsilon^2)\\
&=\int_{-\epsilon}^{\epsilon}\int_{\sigma'_0}(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u_\epsilon(x+\epsilon n(x))\cdot\widehat{\nabla}N(x+\epsilon n(x),y)\,d\sigma_x^0\,d\mu\\
&\quad+O(\epsilon^{(1-\beta)(1+\alpha)})\\
&=2\epsilon\int_{\sigma'_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u^i_\epsilon\cdot\widehat{\nabla}N(\cdot,y)+O(\epsilon^{1+\alpha-\beta(2+\alpha)}),
\end{aligned}
\tag{32}
$$

for $\beta<\alpha(2+\alpha)^{-1}$.

Third step.

Let us now extend the fields n and τ from σ_0 to ω'_ϵ. For $x\in\sigma'_0$ we set n and τ equal to $n(x)$ and $\tau(x)$ all along the line segment $x+\mu n(x)$, for $\mu\in[-\epsilon,\epsilon]$.

We will show, by using the transmission condition (8), that on σ'_ϵ,

$$
(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u^i_\epsilon=\mathcal{M}_\epsilon\widehat{\nabla}u^e_\epsilon, \tag{33}
$$

where

$$
\begin{aligned}
\mathcal{M}_\epsilon\widehat{\nabla}u^e_\epsilon&=a\,\mathrm{div}\,u^e_\epsilon\,\mathrm{I}_d+b\widehat{\nabla}u^e_\epsilon+c\left(\frac{\partial(u^e_\epsilon\cdot\tau)}{\partial\tau}+\kappa_\epsilon(u^e_\epsilon\cdot n)\right)\tau\otimes\tau\\
&\quad+d\frac{\partial(u^e_\epsilon\cdot n)}{\partial n}\,n\otimes n,
\end{aligned}
$$

with a, b, c, and d given by (11), (12), and (13), and κ_ϵ being the curvature of σ'_ϵ.

Let us express the transmission conditions (8) and $(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla} u^i_\epsilon$ in the n, τ coordinate system, namely

$$
\begin{aligned}
&\frac{\partial(u^i_\epsilon\cdot\tau)}{\partial\tau}+\kappa_\epsilon(u^i_\epsilon\cdot n)=\frac{\partial(u^e_\epsilon\cdot\tau)}{\partial\tau}+\kappa_\epsilon(u^e_\epsilon\cdot n),\\
&\frac{\partial(u^i_\epsilon\cdot n)}{\partial\tau}-\kappa_\epsilon(u^i_\epsilon\cdot \tau)=\frac{\partial(u^e_\epsilon\cdot n)}{\partial\tau}-\kappa_\epsilon(u^e_\epsilon\cdot \tau),\\
&\lambda_1\left(\frac{\partial(u^i_\epsilon\cdot\tau)}{\partial\tau}+\kappa_\epsilon(u^i_\epsilon\cdot n)+\frac{\partial(u^i_\epsilon\cdot n)}{\partial n}\right)+2\mu_1\frac{\partial(u^i_\epsilon\cdot n)}{\partial n}\\
&\qquad=\lambda_0\left(\frac{\partial(u^e_\epsilon\cdot\tau)}{\partial\tau}+\kappa_\epsilon(u^e_\epsilon\cdot n)+\frac{\partial(u^e_\epsilon\cdot n)}{\partial n}\right)+2\mu_0\frac{\partial(u^e_\epsilon\cdot n)}{\partial n},\\
&\mu_1\left(\frac{\partial(u^i_\epsilon\cdot\tau)}{\partial n}-\kappa_\epsilon(u^i_\epsilon\cdot \tau)+\frac{\partial(u^i_\epsilon\cdot n)}{\partial \tau}\right)=\mu_0\left(\frac{\partial(u^e_\epsilon\cdot\tau)}{\partial n}-\kappa_\epsilon(u^e_\epsilon\cdot \tau)+\frac{\partial(u^e_\epsilon\cdot n)}{\partial \tau}\right),
\end{aligned}
\tag{34}
$$

and

$$(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i=(\lambda_1-\lambda_0)\left[\frac{\partial(u_\epsilon^i\cdot\tau)}{\partial\tau}+\frac{\partial(u_\epsilon^i\cdot n)}{\partial n}+\kappa_\epsilon(u_\epsilon^i\cdot n)\right](n\otimes n+\tau\otimes\tau)$$
$$+\,2(\mu_1-\mu_0)\left[\left(\frac{\partial(u_\epsilon^i\cdot\tau)}{\partial\tau}+\kappa_\epsilon(u_\epsilon^i\cdot n)\right)\tau\otimes\tau+\frac{\partial(u_\epsilon^i\cdot n)}{\partial n}\,n\otimes n\right.$$
$$\left.+\,\frac{1}{2}\left(\frac{\partial(u_\epsilon^i\cdot\tau)}{\partial n}+\frac{\partial(u_\epsilon^i\cdot n)}{\partial\tau}-\kappa_\epsilon(u_\epsilon^i\cdot\tau)\right)(\tau\otimes n+n\otimes\tau)\right].\tag{35}$$

By solving the system (34) with respect to the derivatives of the components of u_ϵ^i, and inserting the result into (35), we derive (33).

Now, by inserting (33) into (32), we get

$$\int_{\sigma'_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i\cdot\widehat{\nabla}N(\cdot,y)=2\epsilon\int_{\sigma'_\epsilon}\mathcal{M}_\epsilon\widehat{\nabla}u_\epsilon^e\cdot\widehat{\nabla}N(\cdot,y)+O(\epsilon^{1+\alpha-\beta(2+\alpha)}).\tag{36}$$

Fourth step.
We will show that

$$\|\nabla u_\epsilon^e-\nabla u_0\|_{L^\infty(\sigma'_\epsilon)}\le C\epsilon^\gamma\|g\|_{H^{-1/2}(\partial\Omega)}\tag{37}$$

for some positive γ.

In order to prove the above inequality, we need the following theorem.

THEOREM 4.1 (mean value property). *Let Ψ be a biharmonic scalar, vector, or tensor field in a open bounded domain D. Then, for any ball $B_\rho(y)\subset\subset D$,*

$$\Psi(y)=\frac{1}{2\pi}\left[\frac{4}{\rho^2}\int_{B_\rho(y)}\Psi(x)\,dx-\frac{1}{\rho}\int_{\partial B_\rho(y)}\Psi(x)\,d\sigma_x\right].\tag{38}$$

For the proof of Theorem 4.1, see [N].

Since $\nabla u_\epsilon-\nabla u_0$ is biharmonic in $\Omega\setminus\overline{\omega}_\epsilon$ we might use the mean value property (38) for points in the set $\Omega_K\setminus\overline{\omega}_d$, where $\Omega_K=\left\{x\in\Omega:d(x,\partial\Omega)>\frac{1}{2K}\right\}$ and d is such that $2\epsilon<d$.

Observe that, by (38), for every $y\in\Omega_K\setminus\overline{\omega}_d$ and for $0<\lambda\le\frac{d}{2}$,

$$\nabla(u_\epsilon-u_0)(y)=\frac{1}{2\pi}\left[\frac{4}{\lambda^2}\int_{B_\lambda(y)}\nabla(u_\epsilon-u_0)-\frac{1}{\lambda}\int_{\partial B_\lambda}\nabla(u_\epsilon-u_0)\right].\tag{39}$$

By using the divergence theorem we can rewrite (39) as follows:

$$\nabla(u_\epsilon-u_0)(y)=\frac{1}{2\pi}\left[\frac{4}{\lambda^2}\int_{\partial B_\lambda(y)}(u_\epsilon-u_0)\otimes\tilde{\nu}\,d\sigma-\frac{1}{\lambda}\int_{\partial B_\lambda}\nabla(u_\epsilon-u_0)\right],$$

where $\tilde{\nu}$ is the outward normal vector to ∂B_λ. If we multiply the last relation by λ^3 and integrate from 0 to $\rho=\frac{d}{2}$ we get

$$\nabla(u_\epsilon-u_0)(y)=\frac{12}{\pi}\left[\frac{4}{d^4}\int_{B_{\frac{d}{2}}(y)}(u_\epsilon-u_0)\otimes\underline{r}\,dx-\frac{1}{d^4}\int_{B_{\frac{d}{2}}(y)}r^2\nabla(u_\epsilon-u_0)\,dx\right],\tag{40}$$

where $\underline{r}(x) = x - y$, $r = |\underline{r}|$.

From (40) and (16) we have that

$$\|\nabla(u_\epsilon - u_0)\|_{L^\infty(\Omega_K \setminus \omega_d)} \le C d^{-2} \epsilon^{\frac{1}{2}}, \tag{41}$$

where $C = C(\Omega, K, \alpha_0, \beta_0)$. Let $x = z + \epsilon n(z)$ be any point on σ'_ϵ and let x_d be the point $x_d = z + dn(z)$. Since the entire line segment from x to x_d lies in $\Omega \setminus \overline{\omega}_\epsilon$ and has distance greater than $\frac{\epsilon^\beta}{2}$ from $\partial\sigma_0$, by Proposition 3.3 and arguing similarly as we did to prove (30), we have

$$|\nabla u^e_\epsilon(x) - \nabla u_\epsilon(x_d)| \le C\epsilon^{-(2+\alpha)\beta} d^\alpha \|g\|_{H^{-1/2}(\partial\Omega)}, \tag{42}$$

where $C = C(K, \alpha_0, \beta_0)$.

By combining (41) and (42) we have that for any $x \in \sigma'_\epsilon$

$$\begin{aligned} |\nabla u^e_\epsilon(x) - \nabla u_0(x)| &\le |\nabla u^e_\epsilon(x) - \nabla u_\epsilon(x_d)| \\ &\quad + |\nabla u_\epsilon(x_d) - \nabla u_0(x_d)| + |\nabla u_0(x_d) - \nabla u_0(x)| \\ &\le C\left(d^\alpha \epsilon^{-(2+\alpha)\beta} + d^{-2}\epsilon^{\frac{1}{2}} + d\right) \|g\|_{H^{-1/2}(\partial\Omega)}. \end{aligned}$$

By choosing

$$d = \epsilon^{\frac{1}{2(\alpha+2)} + \beta}$$

we have (37) with $\gamma = \frac{\alpha}{2(\alpha+2)} - 2\beta$. Notice that $\gamma > 0$ if we choose $\beta < \frac{\alpha}{4(\alpha+2)}$.

By using (37), we have

$$2\epsilon \int_{\sigma'_\epsilon} \mathcal{M}_\epsilon \widehat{\nabla} u^e_\epsilon \cdot \widehat{\nabla} N(\cdot, y) = 2\epsilon \int_{\sigma'_\epsilon} \mathcal{M}_\epsilon \widehat{\nabla} u_0 \cdot \widehat{\nabla} N(\cdot, y) + O(\epsilon^{1+\gamma}).$$

Now, we recall that $d\sigma^\epsilon_x = (1 + O(\epsilon)) d\sigma_0$ and observe that, by assumption (3), $\mathcal{M}_\epsilon = (1 + O(\epsilon))\mathcal{M}$. Hence

$$2\epsilon \int_{\sigma'_\epsilon} \mathcal{M}_\epsilon \widehat{\nabla} u^e_\epsilon \cdot \widehat{\nabla} N(\cdot, y) = 2\epsilon \int_{\sigma_0} \mathcal{M} \widehat{\nabla} u_0 \cdot \widehat{\nabla} N(\cdot, y) + O(\epsilon^{1+\gamma}). \tag{43}$$

Finally, if we compare the remainders in the expansion in formulas (29), (32), (36), and (43), we have that (10) holds, if we choose $\alpha \in (0, 1/4)$ and $\beta \in (0,1)$ such that $\beta < \frac{\alpha}{4(\alpha+2)}$. □

Remark 4.2. The asymptotic expansion also holds in the case where $\sigma_0 = \cup_{i=1}^M \sigma_i$ and $\sigma_1, \ldots, \sigma_M$ are disjoint and far from each other. In that case

$$(u_\epsilon - u_0)(y) = 2\epsilon \sum_{i=1}^M \int_{\sigma_i} \mathcal{M}_i \widehat{\nabla} u_0 \cdot \widehat{\nabla} N(\cdot, y)\, d\sigma_i + o(\epsilon),$$

where $\mathcal{M}_i$ is the restriction to σ_i of the tensor $\mathcal{M}$.

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