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WASHINGTON UNIVERSITY IN ST. LOUIS

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Geometry and Dynamics of Rolling Systems

by

Bowei Zhao

A dissertation presented to
the Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

August 2022

St. Louis, Missouri

Table of Contents

List of Figures

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Bowei Zhao

Washington University

August 2022

Dedicated to the memory of Guido Weiss.

ABSTRACT OF THE DISSERTATION

Geometry and Dynamics of Rolling Systems

by

Zhao, Bowei

Doctor of Philosophy in Mathematics,

Washington University in St. Louis, 2022.

Professor Renato Feres, Chair

Billiard systems, broadly speaking, may be regarded as models of mechanical systems in which rigid parts interact through elastic impulsive collision forces. When it is desired or necessary to account for linear/angular momentum exchange in collisions involving a spherical body, a type of billiard system often referred to as no-slip has been used. Previous work indicated that no-slip billiards resemble non-holonomic systems, specifically, systems consisting of a ball rolling on surface. In prior research, such connections were only observed numerically and were restricted to very special surfaces. In this thesis, it is shown that no-slip billiard and rolling systems are directly related to each other under very general conditions. Our main result shows that no-slip billiards are truly the non-holonomic counterpart to standard billiard systems. In addition, to the best of our knowledge, we use a novel from of the rolling equations, showing that these systems are a one-parameter perturbation of the geodesic equation on a Riemannian manifold. This opens up a new area of investigation in the theory of geometric dynamical systems,

concerning what we call rolling flows. We introduced the main concepts related to the rolling flow but we leave further development for future research.

1. Introduction

1.1 Background

Typically in the literature of dynamical systems, the term "billiard system" is generally understood as a point particle on a planar domain that satisfies the specular reflection on the boundary of the domain. In this thesis, we want to understand billiard systems more generally, as discrete time dynamical systems involving elastic collisions. A natural question would be, suppose we change the particle to a little ball, what are the other possible behaviours in addition to the standard reflection rule?

In dimension 2, it has long been known that there are only two types of energy-preserving, time-reversible billiard systems. Figure 1.1 appeared in [1], used with permission from the authors, illustrates the difference between the trajectories of the two types of billiard systems in dimension 2.

The dashed trajectory in the picture is the standard specular reflection, although it is a ball in the picture instead of the point mass usually considered in standard billiards. The solid line is a trajectory that is similar to the bouncing behaviour of the popular toy first invented in the 1960s, the Wham-O Super Ball. The physicist Richard Garwin was interested in the unusual trajectory of the Wham-O Super ball, and his 1969 paper

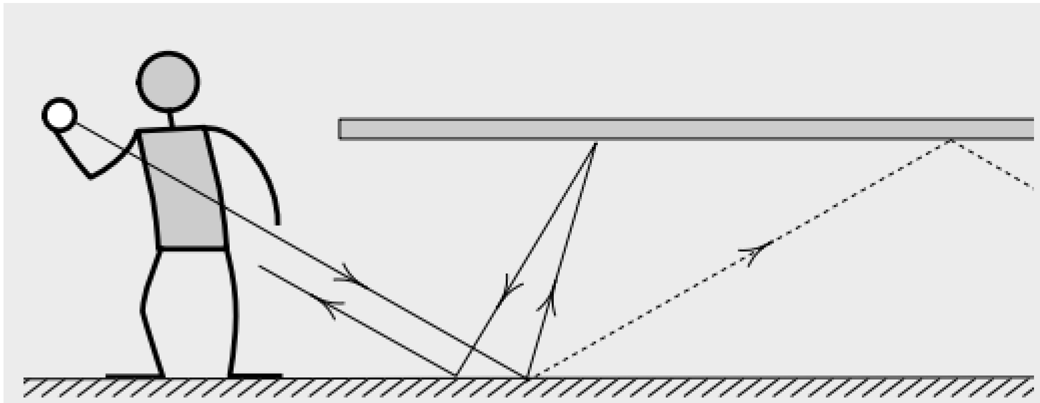

Figure 1.1. A ball thrown under the table.

Kinematics of an ultraelastic rough ball [2] was, to the best of our knowledge, the earliest research of what is called a no-slip billiard system. Figure 1.2 is processed from Wham-O commercial in the 1960s, showing some sample trajectories. As seen in the figure, the trajectories differs from those of specular reflection. In Garwin's model, instead of the specular reflection, the ball undergoes a kind of reflection where the ball experiences elastic conservative friction that upon impact, it causes some exchange in linear and angular momentum that will cause the ball to start rotating.

The difference between no-slip billiards and standard billiards is also shown in Figure 1.3 (Figure 11 in [3], used with permission from authors). The Bunimovich stadium and Sinai billiards, known to be chaotic billiards in the standard billiard case, exhibit some caustic in the no-slip case.

In the 1990s and early 2000s, there were a few papers both in physics and dynamical systems that studied the properties of the no-slip billiards, for example [4] [5] [6]. More systematic classification of rigid body collision on general dimensions begin in the 2010s by C. Cox and R. Feres in [7].

Figure 1.2. trajectories in the Wham-O Super Ball commercial.

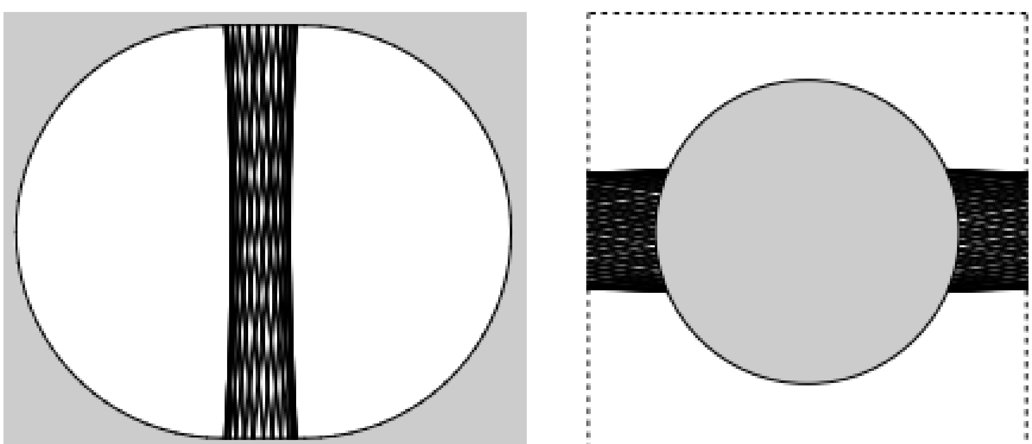

Figure 1.3. The Bunimovich stadium and Sinai billiards shows caustics in no-slip case

Another type of dynamical system that is studied in this thesis is the non-holonomic system. These are continuous time dynamical systems defined by differential equations.

They are not defined in terms of collisions between bodies. In particular, this thesis studies a typical type of non-holonomic systems, the rolling system. It is seemingly completely different from the no-slip billiards, but literature studying the properties of no-slip billiards, in particular a series of systematic studies on the dynamic system side by S. Cook, C. Cox, T. Chumley, R. Feres and H-K. Zhang, for example [1] [8] [3], have shown evidence that the two types of systems are actually connected in some ways.

In [1], the authors studied a type of no-slip billiard system which involved a ball bouncing in the inside of a cylinder. The setup of the system is shown in Figure 1.4:

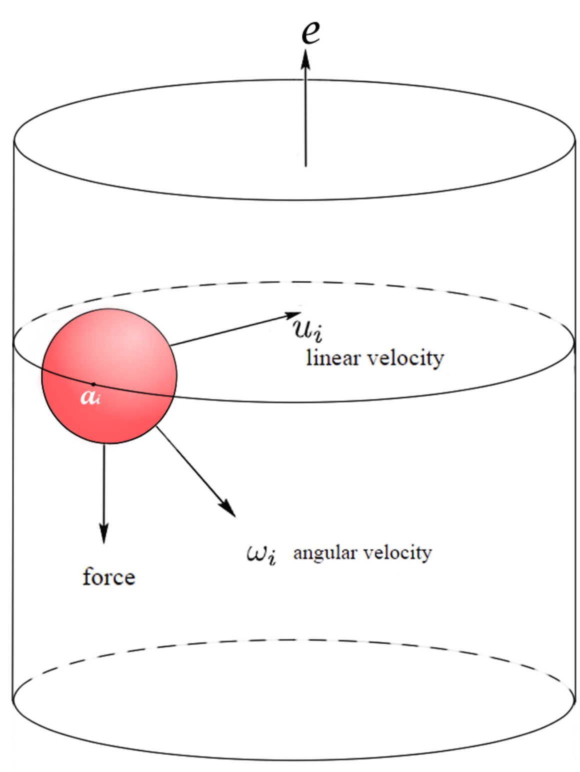

Figure 1.4. The ball is bouncing in the inside of the cylinder

A classical example of a non-holonomic mechanical system consists of a ball that rolls against the inner side of a vertical cylinder with enough speed so as not to lose contact with the surface. We imagine that the surface of the ball is ideally rough, or rubbery, so that a kind of conservative static friction causes it to roll without slipping. Previous studies of such non-holonomic mechanical systems have shown an indirect connection between rolling billiard systems and no-slip billiards. The next theorem (Chumley, Cook, Cox, Feres) was first proved in the 2016 paper *Rolling and no-slip bouncing in cylinders* [3], and to introduce the theorem it is necessary to define the transversal rolling impact. The authors showed that the no-slip bouncing in the cylinder follows a harmonic motion when a transversal rolling impact condition is satisfied in the first collision. Without giving a rigorous definition in this chapter, the transversal rolling impact condition says that if we look at the system from above, then at the moment of collision, the point of collision is stationary except for the velocity component perpendicular to the cylinder. So the condition requires that it does not have a velocity in the tangent direction. Figure 1.5 is originally from [1], used with permission from the authors, is one example of such similarities. The solid line shows the component of the motion along the axis of the cylinder as a function of time for the rolling ball, while the dots represent the height of the bouncing ball at the moments it is in contact with the inner surface of the cylinder.

Theorem (Chumley-Cook-Cox-Feres)**.** *Consider a no-slip billiard system in a circular cylinder in $\mathbb{R}^3$ with a particle subject to a constant force along the axis of the cylinder. If the first collision satisfies the transversal rolling impact condition and the first flight segment does not go through the axis of the cylinder, then the particle's trajectory is bounded. More specifically, the component of the motion along the axis is harmonic.*

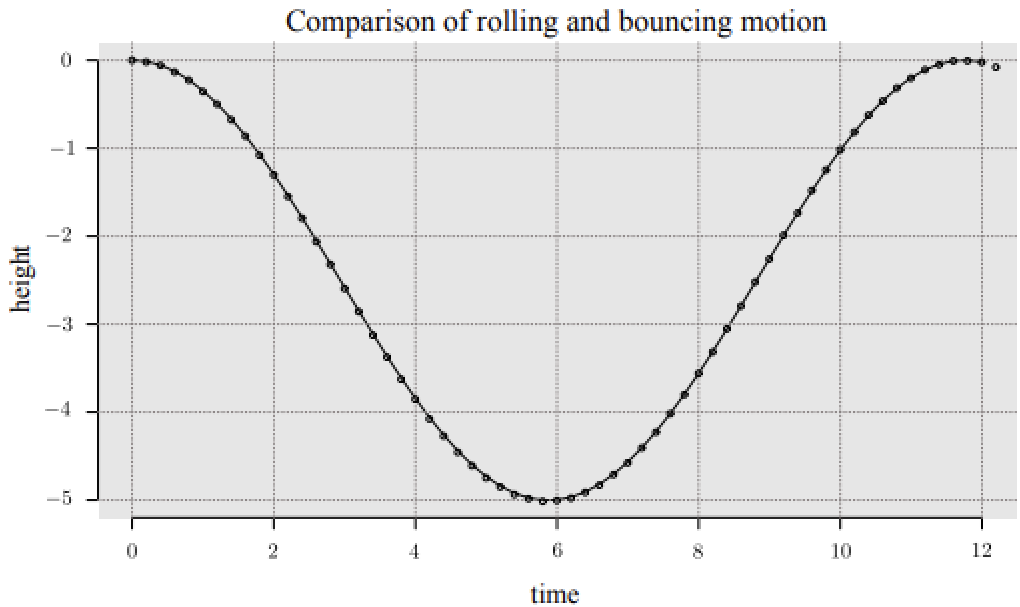

Figure 1.5. The bouncing trajectories are similar to rolling under very general conditions in the cylinder case.

Notice that in this theorem, the ball does not fall down from the cylinder, following a harmonic motion. The point for the theorem is that it is only true if the initial condition satisfies the transversal rolling impact condition, in other words, exhibits a rolling-like behaviour for the first collision. This is an indication that the no-slip billiard shares certain properties with non-holonomic systems. It shows that it behaves very similarly to a non-holonomic system, which is an indication of certain connections between the two types of systems.

As seen in the figure 1.5, the obvious way to relate the two types of systems is to make the jumping steps very small. For the cylinder it is easy to make the bouncing small, but for more complicated domains that are not convex, it would be hard to make collisions follow each other closely. The idea of taking the limit of the time between two

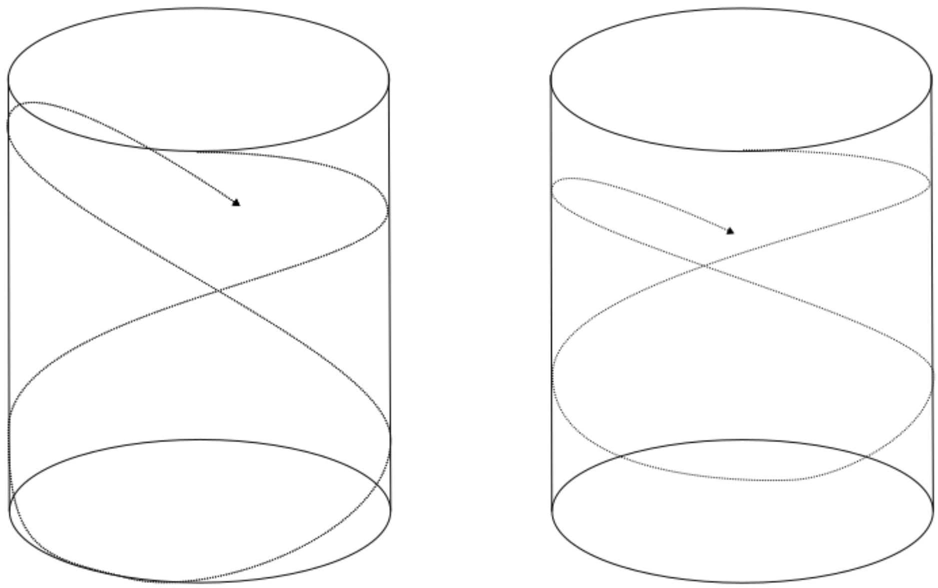

Figure 1.6. For the left cylinder the transversal rolling impact condition holds for the initial bounce, but on the right side a small deviation of this condition is introduced. Notice that the curve does not return to the initial height.

.

collisions become very small is only feasible in very special cases. In fact, the rolling limit of the billiard motion, even under most favourable conditions as in the cylinder, has not be analytically derived yet. One can get a sense of the subtlety of the limiting process by considering what happens when the transversal rolling impact condition on the first collision does not hold. Figure 1.6 which first appeared in [1], used with permission from the authors, gives a visual indication that obtaining a differential equation for the rolling motion heads to contend with the small scale roughness clearly seen in the figure.

Borisov, Kilin and Mamaev in their 2010 paper [9] defined a new type of billiard system, which they call non-holonomic billiard, on a disc or strip by studying the rolling on a ellipsoid or a cylinder, which they flatten to the disc or strip. We observed that

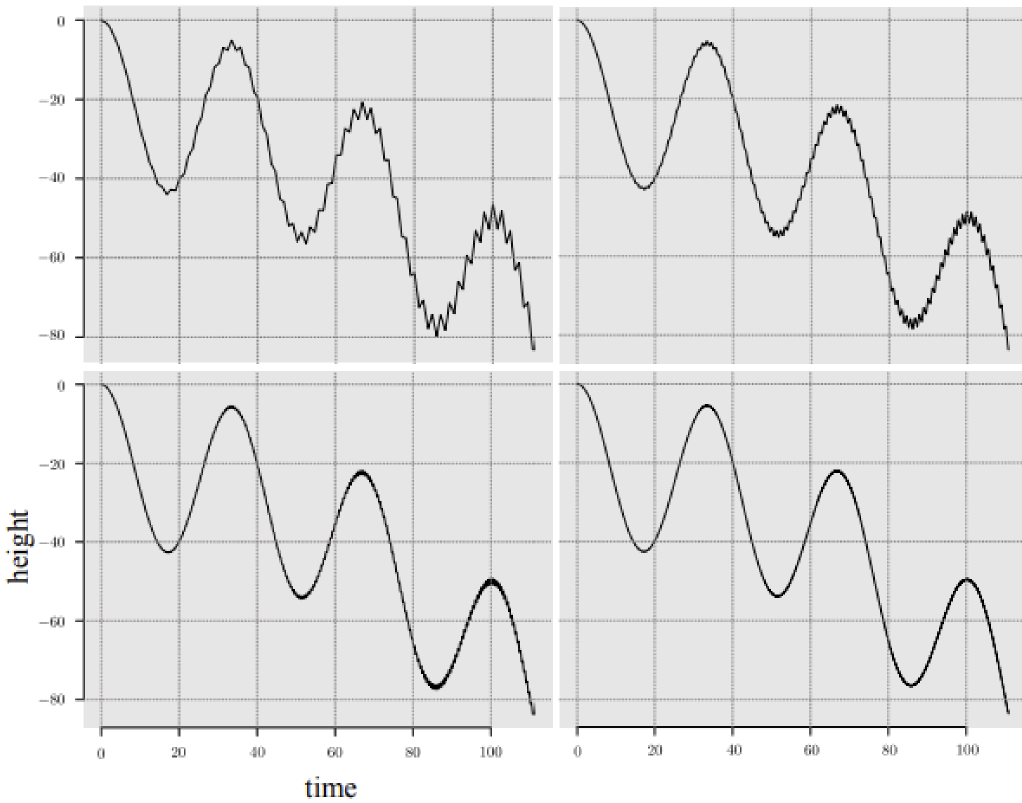

Figure 1.7. There will be some subtle issues when taking limit of the bouncing steps.

their non-holonomic billiard is in fact a very special case of the no-slip billiard described earlier. This led us to prove, by a different limiting procedure, that no-slip billiard can be obtained very generally by taking a projection of rolling systems, so the collision at the boundary in the no-slip billiard can be described as a "shadow" of the rolling system when the ball rolls over the edge of a flat plate in general dimensions. This holds for a rolling ball of finite radius when the plate is polygonal. In the general case, the conclusion holds in the limit as the radius approaches 0. This is the main result of the thesis. A precise statement of the following theorem will be given in Chapter 3.

Theorem. *Let P be a domain in $\mathbb{R}^k$ with piecewise smooth boundary, which we regard as a flat submanifold of $\mathbb{R}^{k+1}$. Consider of the motion of a ball having spherically symmetric mass distribution in $\mathbb{R}^{k+1}$ that rolls on P without slipping. Upon reaching the regular part of the boundary of P, the ball rolls around the edge and back into P either to the other side or back to the side from which it came. In the limit as the radius of the ball approaches* 0*, the resulting motion is that of a no-slip billiard system on P. When the moment of inertia of the ball is* 0*, one recovers ordinary billiard motion.*

A numerical example is illustrated in Figure 1.7. The ball is rolling on a disc in $\mathbb{R}^3$ with a small value of radius r. When a moment of inertia parameter η (to be defined in Chapter 2) is 0 , trajectories of the centre of the rolling ball (viewed from above) are indistinguishable from the trajectories for the ordinary billiard on a disc and exhibit the characteristic caustic circle. For a positive η, caustics split into two concentric circles, which looks exactly like the trajectories of a no slip billiard on the disc. This property in circular plates is easy to establish in the limit as the radius of the ball is sent to 0 , but the positive radius case is left for future works.

The main theorem shows that no-slip billiards arise from such rolling on submanifolds of Euclidean space under very general conditions. It will become apparent that these rolling systems (not only for domains in $\mathbb{R}^n$, but for fairly general submanifolds of Euclidean space) define one-parameter deformations of geodesic flows that depend on both the intrinsic and extrinsic submanifold geometry, the deformation parameter being the rolling ball's moment of inertia. The no-slip billiard system appears in the limit of the rolling motion as the radius of the ball approaches 0. We call these deformations of

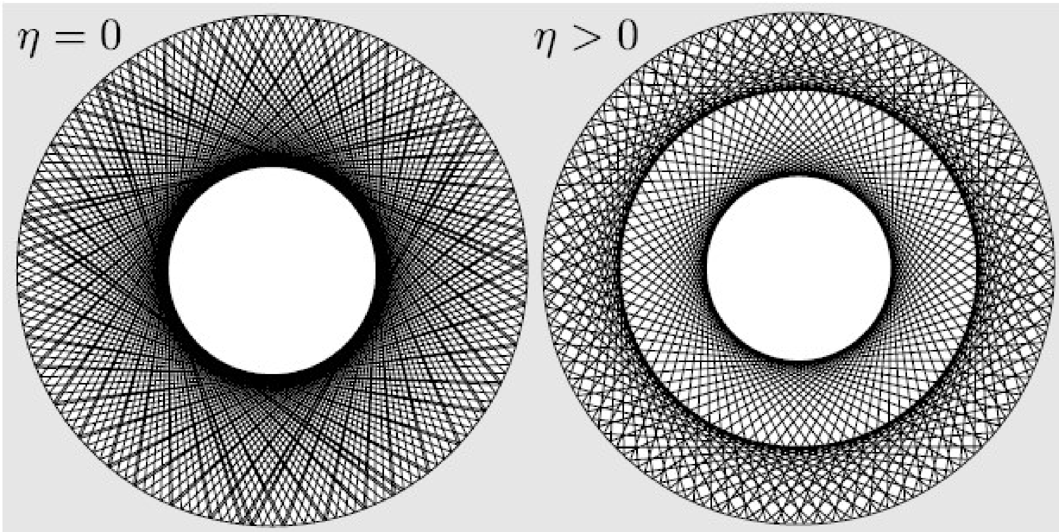

Figure 1.8. Caustics of rolling on a disc resembles those of no-slip billiards when viewed from above

geodesic flows rolling flows on pancake manifolds. (They should not be confused with the non-holonomic geodesic flows considered, for example, in [10].)

Rolling flows are a very natural class of dynamical systems that deserve independent study. They are akin to other, better known, systems of a differential geometric nature such as frame flows and magnetic geodesic flows. As a first step towards developing their dynamics and ergodic theory (for example, exploring stability properties along the lines of what is done in [1] for no-slip billiards in polygonal plates), our main result regarding the rolling flow is the following theorem:

Theorem. *In dimension 3, the canonical volume form is invariant under the rolling flow.*

A more detailed statement of this fact is given in Theorem 17, Chapter 4.

1.2 Structure of the thesis

The structure of the thesis will be centred around building up the main result in more detail. Chapter 2 reviews the derivation of the equations of a rolling ball. The behaviour of the rolling ball at the edge of P is not easy to describe explicitly when the boundary of P has non-zero curvature. For example, in the pancake hypersurface which is composed by two flat sheets connected by a curved part, it is possible the ball rolls part of the way on the curved part, and return to the side it was before collision. However, when the limit on the radius is taken is taken, the ball moves on P in a relatively simple way that can be described analytically, because it necessarily rolls to the other side in this case.

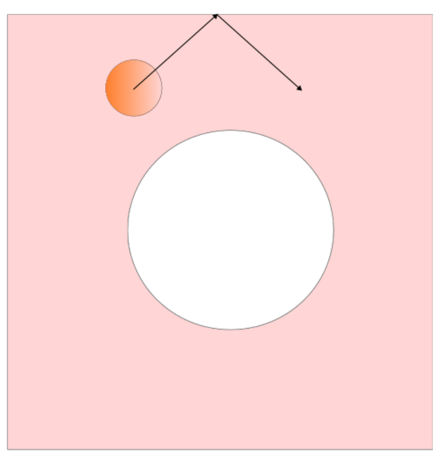

Figure 1.9. A simple Sinai billiard plate.

An elementary example is the rolling billiard counterpart of the Sinai billiard. The Sinai billiard plate is a submanifold of $\mathbb{T}^2 \times \mathbb{R}$, a 2-torus with a hole in the centre, in the shape of a disc. The rolling motion of the ball is no-slip, in other words it does not lose contact with P and rolls without slipping, or the point of contact at any given moment has zero velocity. Forces like gravity are neglected in the system, except forces of constraints. The pancake surface is the boundary of the regions in $\mathbb{T}^2 \times \mathbb{R}$ consisting of points whose distance from P is no greater than r. In the case where r is sufficiently small, the surface is differentially embedded. This example will be examined with more detail in Chapter 2.

Chapter 3 is where the main result will be stated and proved. Some background about no-slip billiards will be reviewed, including some properties of such systems that are fundamental for establishing their connections with general rolling systems. Then we do a review of previous work related to the connection between no-slip billiards and rolling billiards and describes how the rolling motion can be obtained by the bouncing in no-slip billiards when a certain condition is satisfied, a result in [1].

The main result of this thesis will be given in Chapter 3. It states that the no-slip billiard collision map arises from the limit as the radius of a ball rolling around the edge of the plate P goes to zero [11].

Chapter 4 gives a formal definition of the rolling flow, and an explicit equation of motion that connects to the geodesic flow.

The rolling flow is comparable with the geodesic flow, and in particular, when the function of moment of inertia $\eta = 0$, one gets an extension of geodesic flow. This shows clearly that the rolling flows are one-parameter deformations of geodesic flows. Then we establish the invariance of the canonical volume form in dimension 3.

Chapter 5 discusses some further direction to pursue, including a question that if canonical volume form is invariant under rolling flows.

1.3 Contributions

While [9] introduced a very interesting idea, no-slip billiards weren't mentioned in the paper. By considering rolling over ellipsoid and cylinders the authors defined a new type of billiard that they call non-holonomic billiards on discs and strips. This thesis is developed upon a similar idea, that no-slip billiard ought to be connected with rolling over the edge. We prove such a connection in very general conditions.

Another contribution of the thesis lies in the form of the rolling equations we arrive at by eliminating the non-holonomic constraint. We obtain an unconstrained motion on a Riemannian manifold M which flows over the hypersurface $\mathcal{N}$ of loci of centres of the rolling ball, that we can the bundle of tangential spins. In dimension 3 we have $M = \mathcal{N} \times \mathbb{R}$ and the equations of motion are:

$$\frac{\nabla u}{dt} = -\eta \mathfrak{s} J \mathbb{S}_x u, \quad \dot{\mathfrak{s}} = \eta \alpha(\mathbb{S}_x u, u)$$

Here $u \in T_x\mathcal{N}$ the centre velocity vector, $\mathfrak{s}$ the spin, α the area 2-form and J the standard complex structure on $\mathcal{N}$. Thus J_x, at each $x \in \mathcal{N}$, is the positive rotation by $\pi/2$ (relative to the choice of orientation set by ν). $\mathbb{S}_x : T_x\mathcal{N} \to T_x\mathcal{N}$ is the shape operator of $\mathcal{N}$ at x and ∇ is the surface's Levi-Civita connection. The parameter $\eta \in [0, 1)$ is a function of the ball's moment of inertia. When $\eta = 0$, the tangential spin $\int$ is constant and the centre follows a path with zero acceleration ($\frac{\nabla u}{dt} = 0$), so the motion of the centre becomes geodesic. This coupled system of equations, with coupling parameter η, makes it especially clear how the extrinsic geometry of η creates an acceleration that causes the centre of the rolling ball to deviate from geodesic motion in $\mathcal{N}$.

This is to our best of knowledge a new geodesic flow in the style of geodesic flows in the study of dynamical systems.

The sections of the main body of the thesis (Chapters 2,3,4) follows [11] closely. The contributions of the authors to the main result are divided as follows: Chris contributed to much of the computer work, some of which did not make it to the final version. In order to understand if that was the correct connection, it was important to look at examples, rolling over the edge of a straight line and a few other examples made it clear of the connection with the dynamic to the no-slip billiard. I helped with programming of simulation of the ball rolling on a polygonal billiard, which was the key to a clear understanding to rolling on polygonal billiard and the proper way to connect no-slip and rolling billiard. On the analytic side, I helped with some computations about the rolling systems, including materials about the rolling flow and stability result that didn't make into the paper. These will be further explored in future work. Writing the equation and so forth is a joint effort. Some of the work that has not been published in [11] are described in chapters 4 and 5.

2. The General Rolling System

In order to establish the main result of equivalence between rolling billiards and no-slip billiards, it is necessary to define these dynamical systems in detail. In this chapter give the definition of the rolling systems. Although rolling systems have been studied in prior literature as a typical non-holonomic system, we found it necessary to develop it in a self-contained way, since the available literature (mostly restricted to dimension 3) is not adequate for our goals. Section 2.2.2 is a summary of previous work.

2.1 Rolling billiards

We start from a k-dimensional manifold P and we consider P as a flat submanifold of $\mathbb{R}^{k+1}$. So it is still a domain in $\mathbb{R}^k \subset \mathbb{R}^{k+1}$. For a given radius of the ball $r > 0$, assume the boundary $\mathcal{N}$ of the set of points at distance less than or equal to r from P has a continuous and piecewise smooth unit normal vector field.

We describe the state of rolling motion by three parameters: $(x, u, \mathcal{S})$. Here $x \in \mathcal{N}$ is the centre of the ball, $u \in T_x\mathcal{N}$ is the centre velocity, and $\mathcal{S}$ is a skew-symmetric endomorphism of $T_x\mathcal{N}$. Let the vector space of such linear map be $\mathfrak{so}_x(\mathcal{N})$. So the set of the state of the rolling ball is made up of the vector bundle $\pi : \mathcal{M} = T\mathcal{N} \bigoplus \mathfrak{so}(\mathcal{N}) \to \mathcal{N}$.

The behaviour of the rolling ball at the edge of P is not easy to describe explicitly when the boundary of P has non-zero curvature. For example, in the pancake hypersurface which is composed by two flat sheets connected by a curved part, it is possible the ball rolls part of the way on the curved part, and return to the side it was before collision. However, when the limit is taken, the ball moves on P in a relatively simple way that can be described analytically, because it necessarily rolls to the other side in this case.

As a concrete example, consider the rolling system counterpart of the well-known Sinai billiard:

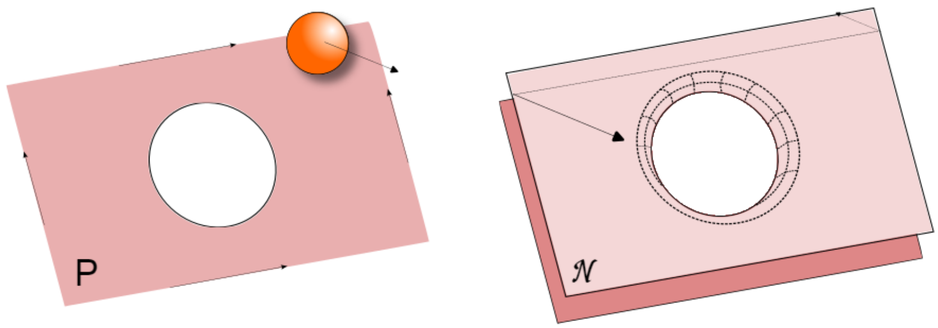

Figure 2.1. A pancake surface.

This pancake surface is called a *Sinai billiard* plate P. It is a submanifold of $\mathbb{T}^2 \times \mathbb{R}$, a 2-tours with a hole in the centre, in the shape of a disc. The ball follows the no-slip condition, so it does not lose contact with P and rolls without slipping, ie. the point of contact at any given moment has zero velocity. Forces like gravity are not considered in the system, except forces of constraints. The pancake surface is the boundary of the regions in $\mathbb{T}^2 \times \mathbb{R}$ consisting of points whose distance from P is no greater than r.

Looking at this system, assuming mass distribution is rotationally symmetric, we look at the state of the system at a given moment of time. The trajectory given an initial value problem for Newton's differential equation is uniquely specified by a set of positions and velocities, called the state. Each state consists of the position of the centre of the ball (a point in $\mathcal{N}$) and three velocity components: two for the velocity of the centre of mass (centre velocity) and one for the angular velocity component (tangential spin) about the outward pointing unit normal vector ν to $\mathcal{N}$. The velocity components are subject to conservation of kinetic energy, and the tangential spin is a skew-symmetric linear map on $T_x\mathcal{N}$.

Now let $\mathfrak{n}$ be the inward pointing unit normal vector at a boundary point x. Let $U \in \mathfrak{so}(k+1)$ represent the angular velocity matrix of the rolling ball. Let $\Pi_x^P, x \in P$, be the orthogonal projection from $\mathbb{R}^{k+1}$ to $T_x\mathcal{N}$. Let Π_x be the orthogonal projection to the tangent space to ∂P at a boundary point x. γ is the moment of inertia (obtained from the matrix of second moments of the mass distribution, which must be a scalar matrix under the assumption that this distribution is rotationally symmetric). The parameter η mentioned earlier in Chapter 1 is $\eta = \frac{\gamma}{\sqrt{1+\gamma^2}}$. Both γ and η are independent of the radius r. We define

$$S = \Pi_x^P U \Pi_x^P (x \in P), \quad \mathcal{S} = r\eta S, \quad W = \mathcal{S}\mathfrak{n}(x)(x \in \partial P).$$

Finally, let $\bar{u} = \Pi_x u$. Now we have the following linear map for the change of velocity component:

Theorem 1. *In the limit as the radius of the rolling ball goes to 0, the velocity components of the ball immediately before and immediately after rounding the edge of the flat plate P at a boundary point x, are related by the linear map*

$$\Pi_x \mathcal{S} \Pi_x \mapsto \Pi_x \mathcal{S} \Pi_x, \quad \mathfrak{n}(x) \mapsto -\mathfrak{n}(x), \quad \begin{pmatrix} \bar{u} \\ W \end{pmatrix} \mapsto \begin{pmatrix} \cos(\pi\eta)I & \sin(\pi\eta)I \\ \sin(\pi\eta)I & -\cos(\pi\eta)I \end{pmatrix} \begin{pmatrix} \bar{u} \\ W \end{pmatrix}$$

The proof of this theorem will be given in chapter 3, when we obtain the main results.

2.2 Differential Geometry of Rolling

In this section, a review of equation of motion of a spherically symmetric mass distribution rolling without slipping on a submanifold of $\mathbb{R}^n$, possibly with boundaries and corners, will be given.

2.2.1 Constrained Rigid Motion

Let $SE(k) = SO(k) \ltimes \mathbb{R}^k$ denote the special Euclidean group of orientation preserving isometries of Euclidean space. Elements of $SE(k)$ can be written as $g = (a, A)$, where $a \in \mathbb{R}^k$ and $A \in SO(k)$. $SE(k)$ can be regarded as the configuration manifold of a rigid body $\mathcal{B} \subset \mathbb{R}^k$.

$\mathcal{B}$ here can be considered a ball of radius r, though a more general definition is a measurable set with mass distribution defined by a finite positive measure μ. A motion of $\mathcal{B}$ is a (smooth) path $g(t) = (a(t), A(t)) \in SE(k)$. We write $g = g(0), \xi = g'(0) = (a', A'), u_\xi = a'$, and $U_\xi = A'A^{-1} \in \mathfrak{so}(k)$ (the Lie algebra of the special orthogonal

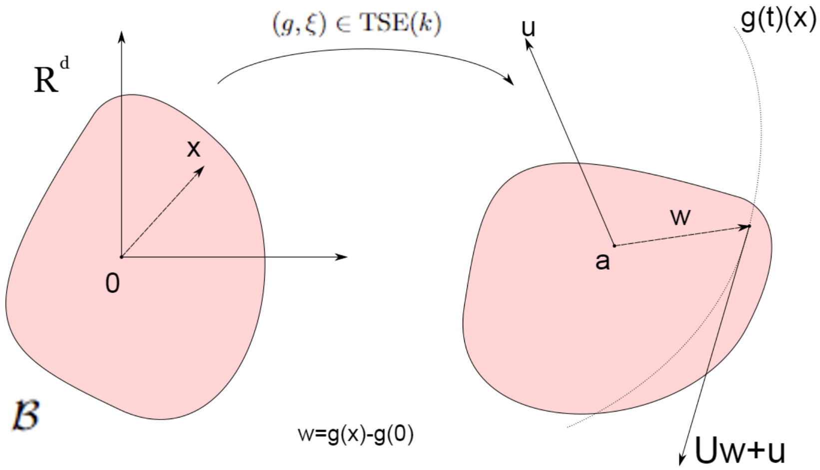

Figure 2.2. General Setup of the System.

group, consisting of m-by-m skew-symmetric real matrices). The velocity of material point $x \in \mathcal{B}$ at $t = 0$ is

$$V_x(g,\xi) = \frac{d}{dt}(A(t)x + a(t))\bigg|_{t=0} = A'x + a' = U_\xi Ax + u_\xi$$

We define U_ξ as the angular velocity matrix.

The kinetic energy of $\mathcal{B}$ can be written as a function of the state $\xi \in T_gSE(k)$ at configuration g as follows:

$$K_g(\xi) = \frac{1}{2}\int_{\mathcal{B}} |V_x(g,\xi)|^2 \, d\mu(x).$$

Upon integration it yields a quadratic form associated to the following symmetric bilinear form

$$\langle \xi, \zeta \rangle_g := m\left[u_\xi \cdot u_\zeta + \frac{1}{2}\,\mathrm{Tr}\left(\mathcal{L}\left(U_\xi\right)U_\zeta^\top\right)\right]$$

where $m = \mu(\mathcal{B})$ is the total mass of the body, $u_\xi \cdot u_\zeta$ is the ordinary dot product, and $\mathcal{L}(U)$ is a certain linear map on $\mathfrak{so}(m)$ that depends on the mass distribution μ, as defined in [12]. When $\mathcal{B}$ is a ball of radius r centered at the origin of $\mathbb{R}^k$ and μ is a rotationally symmetric mass distribution, $\mathcal{L}$ will be a scalar transformation. This result is Corollary 4 in [12], Section 4.2.

Here the resulting bilinear form induces the following Riemannian metric on $SE(k)$:

$$\langle \xi, \zeta \rangle_g := m \left[u_\xi \cdot u_\zeta + \frac{r^2 \gamma^2}{2} \operatorname{Tr}\left(U_\xi U_\zeta^\top \right) \right]$$

where γ is a moment of inertia parameter. Define $\eta = \frac{\gamma}{\sqrt{1+\gamma^2}} \in [0,1)$. When $\eta = \gamma = 0$, the mass is concentrated at the center of the ball. In this case body rotation does not contribute to the kinetic energy and the inner product becomes degenerate.

From now on in this section the motion of the ball is restricted to rolling over a submanifold P of $\mathbb{R}^k$ without slipping. The general setup can be found in the following figure. In this section we only give the necessary definitions to obtain the main result. A more systematic study of the no-slip rolling motion can be found in [3].

The locus of possible centres will be denoted $\mathcal{N} = \mathcal{N}(r)$. Thus $\mathcal{N}$ is the set of points in $\mathbb{R}^k$ at distance r from P. We assume that P is such that $\mathcal{N}$ is an embedded submanifold of $\mathbb{R}^k$ for sufficiently small r. Note that, when P has boundary, $\mathcal{N}$ may fail to be smoothly embedded even if P is smooth although the unit normal vector field $a \mapsto \nu(a)$ to $\mathcal{N}$ (pointing to the side of rolling) will typically be piecewise smooth and continuous. The no-slip constraint requires the velocity of the point on the ball in configuration g and in tangential contact with P to be zero.

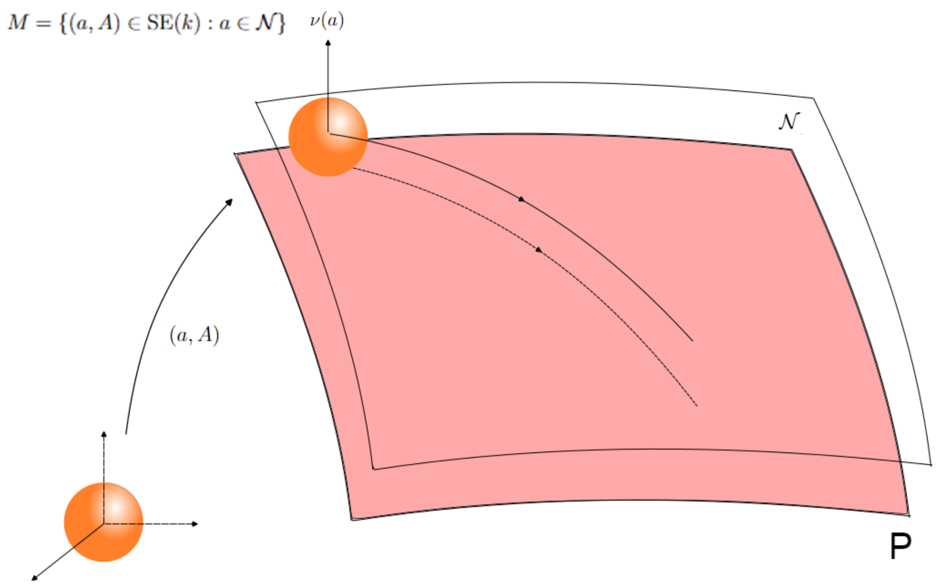

Figure 2.3. The rolling ball satisfies the nonholonomic constraint of being in tangential contact with the submanifold P. Therefore its centre lies on the hypersurface $\mathcal{N}$ at distance r from P, and the velocity at the point of contact is always 0

.

At a certain state (g, ξ), $g = (a, A)$, the velocity $V_x(g, \xi)$ where the contact point is $p = g(x)$ can be written as

$$V_x(g, \xi) = u_\xi - rU_\xi \nu(a)$$

Notice that $p = a - r\nu(a) = Ax + a$, which implies $Ax = -r\nu(a)$, combined with $V_x(g, \xi) = u_\xi + U_\xi Ax$ we can obtain the above equation.

In other words, $V_x(g, \xi)$ is the sum of the velocity u_ξ of the centre a of the ball in configuration (a, A) and the velocity of the contact point p relative to a, $-rU_\xi \nu(a)$.

Therefore the constraint equation can be written as

$$u_\xi = rU_\xi \nu(a).$$

This equation defines a vector subbundle which we call the rolling bundle $\mathfrak{R}$.

The formal definitions for the rolling bundle is as follows:

Definition 2 (The rolling bundle)**.** *The rolling bundle is the vector subbundle* $\mathfrak{R}$ *of* TM*, where* $M = \{g = (a, A) \in SE(k) : a \in \mathcal{N}\}$*, such that*

$$\mathfrak{R}_g = \{(u, UA) \in T_gM : u = rU\nu(a)\}.$$

Here the motion $t \mapsto g(t)$ satisfies that $g(t) \in M$ and $g'(t) \in \mathfrak{R}$ for all t.

There is also a similar no-slip bundle for no-slip billiard systems. It is one example of the similarities between the no-slip billiards and rolling systems, but since it is not related to the main results, it is not discussed in this these. For more detail one can refer to [1].

Definition 3. *On a vector space* V *with inner product* $\langle \cdot, \cdot \rangle$ *we define the cross-product* $(u \wedge v)w := \langle u, w\rangle v - \langle v, w\rangle u$ *for* $u, v, w \in V$*. Thus* $u \wedge v$ *is an element of* $\mathfrak{so}(V)$*.*

Notice that the wedge product here is different from the ordinary wedge product of forms. It is a generalisation of the ordinary cross product.

Proposition 4. *At* $g = (a, A) \in M$*, let* $\mathfrak{R}_g^\perp$ *denote the orthogonal complement of* $\mathfrak{R}_g$ *in* $T_gSE(k)$ *with respect to the kinetic energy Riemannian metric. Then*

$$\mathfrak{R}_g^\perp = \left\{ \left(\frac{1}{r\gamma^2} w \wedge \nu(a) A, w \right) : w \in \mathbb{R}^k \right\}$$

and $\mathfrak{R}_g^\perp \cap T_gM$ *has the same expression but with* $w \in T_a\mathcal{N}$*. It follows that* $\dim \mathfrak{R}_g^\perp = m$*,* $\dim \mathfrak{R}_g^\perp \cap T_gM = k - 1$ *and* $\dim \mathfrak{R}_g = \dim SE(k) - \dim \mathfrak{R}_g^\perp = \frac{k(k-1)}{2}$*.*

Proof. The main observation, from which the rest follows, is the orthogonality between $\mathfrak{R}_g$ and the subspace defined by the right-hand side of Eq. (5.2). This is shown by making use of the easily verified general identity

$$\frac{1}{2}\operatorname{Tr}\left((v\wedge w)U^\top\right) = (Uv)\cdot w.$$

Granted this identity and having in mind that $u = rU\nu(a)$ when $(u, UA) \in \mathfrak{R}_g$, we obtain

$$\begin{aligned}\left\langle (u,UA), \left(w, \frac{1}{r\gamma^2} w\wedge \nu(a)A\right)\right\rangle &= m\left\{u\cdot w + \frac{r^2\gamma^2}{2}\operatorname{Tr}\left(\frac{1}{r\gamma^2}w\wedge\nu(a)U^\top\right)\right\}\\ &= m\{rw\cdot U\nu(a) + r(Uw)\cdot\nu(a)\}\end{aligned}$$

but the last expression is 0 since U is skew-symmetric. □

2.2.2 Newton's Equation

To study the rolling billiard systems it is necessary to derive the Newton's equation for the rolling motion. The Riemannian metric given above on $SE(k)$ is a product metric that agrees with the Euclidean metric on the normal subgroup $\mathbb{R}^k$ and defines a bi-invariant Riemannian metric on $SO(k)$ for the rotationally symmetric mass distribution. The latter metric is $\langle X, Y\rangle = c\operatorname{Tr}\left(XY^\top\right)$ where c is a positive constant and $X, Y \in \mathfrak{so}(k)$. A basic property of the Levi-Civita connection on $SO(m)$ is that

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for left-invariant vector fields X, Y.

This is because for left-invariant X, Y,

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X Y + \nabla_Y X$$

The result then follows from the torsion-free condition $\nabla_X Y - \nabla_Y X = [X, Y]$.

Proposition 5. *Let $A(t)$ be a parametric curve in $SO(k)$ that is twice differentiable. Let $U(t) = \dot{A}(t)A(t)^{-1}$, then we have $\frac{\nabla \dot{A}}{dt} = \frac{\nabla \dot{A}}{dt} = \dot{U}A$.*

Proof. Let $E_1, E_2 ... E_n$ be orthonormal right-invariant vector fields on $SO(d)$, where $n = \frac{(k-1)k}{2}$. we have:

$$\begin{aligned}
\frac{\nabla \dot{A}}{dt} &= \sum_j \left\langle \dot{U}, E_j \right\rangle_I E_j(A) + \sum_j \langle U, E_j \rangle_I \nabla_{\dot{A}} E_j(A) \\
&= \dot{U}A + \sum_{j,k} \langle U, E_j \rangle_I \left\langle \dot{A}, E_k(A) \right\rangle_A \nabla_{E_k} E_j(A) \\
&= \dot{U}A + \frac{1}{2} \sum_{j,k} \langle U, E_j \rangle_I \langle U, E_k \rangle_I \left[E_k, E_j \right](A) \\
&= \dot{U}A + \frac{1}{2}[U, U] \\
&= \dot{U}A
\end{aligned}$$

Note that $\dot{A}(t) = \sum_j \langle U, E_j \rangle_I E_j(A(t))$ because $\left\langle \dot{A}, E_j \right\rangle_A = \langle U, E_j \rangle_I$.

□

It follows from the proposition that if $g(t)$ is a twice differentiable path in $SE(k)$ and $\Pi_a : \mathbb{R}^k \to T_a\mathcal{N}$, then

$$\frac{\nabla \dot{g}}{dt} = (\dot{U}A, \Pi_a \ddot{u})$$

Let m again be the mass of the rolling body, then Newton's equation at the configuration $g = (a, A)$ can be written as

$$m\frac{\nabla \dot{g}}{dt} = N(g, \dot{g})$$

where $N(g, \dot{g}) \in \mathfrak{R}_g^{\perp}$ is the constraint force required in order to satisfy the condition $\dot{g} \in \mathfrak{R}_g$ with zero work done. An explicit expression is $N(g, \dot{g}) = \left(w, \frac{1}{r\gamma^2} w \wedge \nu(a)A\right)$ for some $w \in \mathbb{R}^k$.

We believe the following result is already know, but we derive it here because we cannot find an appropriate reference.

Proposition 6. *Suppose a ball with rotationally symmetric mass distribution and moment of intertia parameter* γ*, is rolling with the motion described by* $g(t) = (a(t), A(t))$*, subject to the nonholonomic constraint defined by the rolling distribution* $\mathfrak{R} \subset TM$*. Then*

$$\dot{U} = -\frac{r}{(1+\gamma^2)} (U\mathbb{S}_a U\nu(a)) \wedge \nu(a).$$

The linear velocity u *can be written as* $u = rU\nu(a)$*, where* U *is a solution of the above differential equation.*

Proof. Substituting the explicit expressions for N and $\nabla\dot{g}/dt$ into Newton's equation, and using the velocity constraint condition, we have the following system of differential equations:

$$\begin{cases} m\dot{U} &= \frac{1}{r\gamma^2} w \wedge \nu(a) \\ m\Pi_a \dot{u} &= w \\ u &= rU\nu(a) \end{cases}$$

Note that $\frac{d\nu(a(t))}{dt} = -\mathbb{S}_{a(t)}u$ where $\mathbb{S}_a$ is the shape operator of $\mathcal{N}$ at a, $\mathbb{S}_a v = -D_v\nu$, where D_u denotes the ordinary directional derivative of vectors in $\mathbb{R}^k$ and $v \in T_a\mathcal{N}$. Let ∇ denote the Levi-Civita connection on the hypersurface. Therefore differentiating the third equation gives:

$$\dot{u} = r\dot{U}\nu(a) - rU\mathbb{S}_a u$$

Note that $-(w \wedge \nu(a))\nu(a) = w - w \cdot \nu(a)\nu(a) = \Pi_a w$, this combined with the first equation gives

$$m\left(\dot{u} + rU\mathbb{S}_a u\right) = -\frac{1}{\gamma^2}\Pi_a w$$

It follows from the second equation that

$$w + mr\Pi_a U \mathbb{S}_a u = -\frac{1}{\gamma^2}\Pi_a w$$

Simplifying this equation, we can write w in terms of U :

$$w = -mr\frac{\gamma^2}{1+\gamma^2}\Pi_a U \mathbb{S}_a u$$

The equation in the statement of the proposition now follows after applying the system of three differential equations again. □

2.2.3 Rolling Motion in Terms of Tangential Spin and Velocity of Centre

In this section we give an alternative form of the rolling equation that will be more convenient in deriving the main result. Instead of using the full angular velocity matrix U, we write Newton's equation as a system involving the centre velocity u and a tensor on $\mathcal{N}$, which we call the tangential spin matrix.

Lemma 7. *Let ν be a unit vector in $\mathbb{R}^k$ and Π the orthogonal projection to the codimension 1 subspace perpendicular to ν. Then any $V \in \mathfrak{so}(k)$ can be written as*

$$V = \Pi V \Pi + \nu \wedge V\nu$$

This is an orthogonal decomposition with respect to the trace inner product (a.k.a Hilbert-Schmidt inner product).

Proof. Let $\Pi^{\perp}$ be the orthogonal projection to the line spanned by ν. Then a linear transformation V of $\mathbb{R}^k$ can be expressed as follows:

$$V = \Pi V \Pi + \Pi^{\perp} V \Pi + \Pi V \Pi^{\perp} + \Pi^{\perp} V \Pi^{\perp}$$

Since V is skew symmetric, the last term is 0.

It follows that

$$\Pi V \Pi^{\perp} w = (w \cdot \nu)\nu$$

and

$$\Pi^{\perp} V \Pi w = -(V\nu) \cdot w\nu$$

Now by the definition of the cross product $\wedge$, we have the decomposition in the lemma. □

Notice that it follows from the lemma that if $(u, U) \in \mathfrak{se}(k)$ is a state at configuration $g = (a, A)$ that satisfies the rolling constraint $u = rU\nu(a)$, then we have $U = \Pi_a U \Pi_a + \frac{1}{r}\nu(a) \wedge u$.

We can use the tangential part $S_a = \Pi_a U \Pi_a$ to describe the state. S_a is called the tangential angular velocity or the tangential spin. Therefore, under the no-slip constraint, the angular velocity matrix U satisfies

$$U = S_a + \nu(a) \wedge \frac{u}{r}$$

where S_a is the tangential spin at configuration $g = (a, A)$ and u is the velocity of the center point $a \in \mathcal{N}$.

Now we can rewrite the equation of motion in terms of S_a and u instead of U.

Lemma 8. *Let $a(t)$ be a differentiable curve in the hypersurface $\mathcal{N} \subset \mathbb{R}^k$, $u = \dot{a}$, and let $S(t) : T_{a(t)}\mathcal{N} \to T_{a(t)}\mathcal{N}$ be a differentiable field of symmetric linear maps along $a(t)$. Let ν denote a unit normal vector field on $\mathcal{N}$ and $\mathbb{S}$ the shape operator, $\mathbb{S}_a v = -D_v \nu$, where D_u denotes ordinary directional derivative of vectors in $\mathbb{R}^k$ and $v \in T_a\mathcal{N}$. Let ∇ denote the Levi-Civita connection on the hypersurface. Then*

$$\dot{S} = \frac{\nabla S}{dt} + \nu(a) \wedge S_a u$$

Proof. Let $E_1, \ldots, E_{k-1}$ be a local orthonormal frame of differentiable vector fields on $\mathcal{N}$: Then $\{E_i \wedge E_j : 1 \leq i < j \leq k-1\}$ is a basis of $\mathfrak{so}_a(\mathcal{N})$ and we may write

$$S = \sum_{i<j} s_{ji} E_i \wedge E_j.$$

Taking derivative where inner product $\langle \cdot, \cdot \rangle$ is the standard dot product,

$$\dot{E}_j = D_u E_j = \sum_{i=1}^{k-1} \langle E_i, D_u E_j \rangle E_i + \langle \nu(a), D_u E_j \rangle \nu(a) = \nabla_u E_j - \langle D_u \nu(a), E_j \rangle \nu(a)$$

Since $\mathbb{S}_a$ is symmetric, $-\langle D_u \nu(a), E_j \rangle = \langle \mathbb{S}_a u, E_j \rangle = \langle u, \mathbb{S}_a E_j \rangle$ and we obtain

$$D_u E_j = \nabla_u E_j + \langle u, \mathbb{S}_a E_j \rangle \nu(a)$$

It follows that

$$\begin{aligned} D_u (E_i \wedge E_j) &= (\nabla_u E_i + \langle u, \mathbb{S}_a E_i \rangle \nu(a)) \wedge E_j + E_i \wedge (\nabla_u E_j + \langle u, \mathbb{S}_a E_j \rangle \nu(a)) \\ &= \nabla_u (E_i \wedge E_j) + \nu(a) \wedge (\langle \mathbb{S}_a u, E_i \rangle E_j - \langle \mathbb{S}_a u, E_j \rangle E_i) \\ &= \nabla_u (E_i \wedge E_j) + \nu(a) \wedge [(E_i \wedge E_j) \mathbb{S}_a u] \end{aligned}$$

Finally,

$$\begin{aligned}
\dot{S} &= \sum_{i<j} \left[\dot{s}_{ji} E_i \wedge E_j + s_{ji} D_u \left(E_i \wedge E_j\right)\right] \\
&= \sum_{i<j} \left[\dot{s}_{ji} E_i \wedge E_j + s_{ji} \nabla_u \left(E_i \wedge E_j\right)\right] + \nu(a) \wedge \sum_{i<j} s_{ji} E_i \wedge E_j \mathbb{S}_a u \\
&= \frac{\nabla S}{dt} + \nu(a) \wedge S \mathbb{S}_a u
\end{aligned}$$

□

To simplify notations we write $\mathcal{S} = r\eta S$ as the tangential spin, where $\eta = r\sqrt{1+\gamma^2}$ is the moment of inertia parameter. Now for a given state, the kinetic energy will be a scalar multiple of the quantity $|u|^2 + \frac{1}{2}\operatorname{Tr}(\mathcal{S}\mathcal{S}^T)$.

Now we can write the equation of motion in terms of the spin $\mathcal{S}$ and the centre velocity u, without explicitly referring to the non-holonomic constraints.

Proposition 9. *The rolling motion with the hypersurface of the centres of the mass $\mathcal{N}$ under the no-slip constraint satisfies the system of equations*

$$\begin{aligned}
\frac{\nabla u}{dt} &= -\eta \mathcal{S} \mathbb{S}_a u \\
\frac{\nabla \mathcal{S}}{dt} &= \eta \left(\mathbb{S}_a u\right) \wedge u
\end{aligned}$$

where $u = \dot{a} \in T_a\mathcal{N}$ is the velocity of the center of the ball and $\mathcal{S}$ is the tangential spin. Here ∇ is the ordinary Levi-Civita connection of the hypersurface with the Riemannian metric induced by restriction of the dot product in $\mathbb{R}^k$. When the moment of inertia is zero ($\eta = 0$) the system reduces to geodesic motion on $\mathcal{N}$ with parallel tangential spin.

Proof. It follows from Lemma 7 that

$$(U\mathbb{S}_a U\nu(a)) \wedge \nu(a) = -\frac{1}{r}\nu(a) \wedge S\mathbb{S}_a u.$$

From this and Proposition 6 we have

$$\dot{U} = F(a) + \frac{1}{1+\gamma^2}\nu(a) \wedge S\mathbb{S}_a u.$$

Differentiating the constraint equation $u = rU\nu(a)$ in t, we have

$$\dot{u} = r\dot{U}\nu(a) - rU\mathbb{S}u.$$

We this we can rewrite the main equation of Proposition 6:

$$\frac{\nabla u}{dt} = rF(a)\nu(a) - r\eta^2 S\mathbb{S}_a u.$$

It follows from Lemma 7 and Lemma 8 that

$$\dot{U} = \frac{\nabla S}{dt} + \frac{1}{1+\gamma^2}\nu(a) \wedge S\mathbb{S}_a u + \frac{u}{r} \wedge \mathbb{S}_a u + \nu(a) \wedge F(a)\nu(a)$$

Combining this with the first equation in the proof we have

$$\frac{\nabla S}{dt} = -\frac{u}{r} \wedge \mathbb{S}_a u + F(a) - \nu \wedge F(a)\nu(a) = -\frac{u}{r} \wedge \mathbb{S}_a u + \Pi_a F(a)\Pi_a.$$

Now the equation in the statement of the proposition follows from defintion of $\mathcal{S}$. □

2.3 Elementary Examples

We consider here a few elementary examples for which the rolling equation can in principle be solved analytically. The case of rolling over $\mathbb{R}^k$ in $\mathbb{R}^n, k < n$, is of special interest. These naturally arise in the context of rolling on polyhedral shapes, where one needs to account for motion over faces of different dimensions. Thus let $P = \mathbb{R}^k$, regarded as a submanifold of $\mathbb{R}^n$. The locus of centres of the rolling ball or radius r is then $\mathcal{N} = \mathbb{R}^k \times S^{n-k-1}(r)$, where $S^\ell(r)$ is the sphere of radius r in $\mathbb{R}^{\ell+1}$ centered at the origin. At any given $x \in \mathcal{N}$ let $\Pi_x : T_x\mathcal{N} \to \mathbb{R}^k$ denote the orthogonal projection, where we identify $\mathbb{R}^k$ with its tangent space at any given point. We also write $\Pi_x^\perp = I - \Pi_x$. Now express the center of mass velocity u and tangential angular velocity operator $\mathcal{S}$ at any given point x as $u = u_0 + u_1$ and $\mathcal{S} = \mathcal{S}_{00} + \mathcal{S}_{01} + \mathcal{S}_{10} + \mathcal{S}_{11}$ where

$$u_0 = \Pi_x u, u_1 = \Pi_x^\perp u, \quad \mathcal{S}_{00} = \Pi_x \mathcal{S} \Pi_x, \mathcal{S}_{01} = \Pi_x \mathcal{S} \Pi_x^\perp, \mathcal{S}_{10} = \Pi_x^\perp \mathcal{S} \Pi_x, \mathcal{S}_{11} = \Pi_x^\perp \mathcal{S} \Pi_x^\perp$$

With these definitions, we can rewrite the rolling equations as the system of equations

$$\dot{u}_0 = \frac{\eta}{r}\mathcal{S}_{01}u_1, \dot{\mathcal{S}}_{00} = 0, \frac{\nabla u_1}{dt} = \frac{\eta}{r}\mathcal{S}_{11}u_1, \frac{\nabla \mathcal{S}_{11}}{dt} = 0, \frac{\nabla \mathcal{S}_{01}}{dt} = -\frac{\eta}{r}u_1^b \otimes u_0$$

Here u_1^b is the covector dual to u_1, so that $\left(u_1^b \otimes u_0\right) v = (u_1 \cdot v)\, u_0$.

Note that the tensors Π and $\Pi^\perp$ are parallel and the shape operator satisfies $\mathbb{S}_x = -\frac{1}{r}\Pi_x^\perp$.

Example 1 (Codimension 1)**.** *In the codimension 1 case, n consists of two parallel planes of dimension k in $\mathbb{R}^{k+1}$ (a distance $2r$ apart) and $u_0 = u, \mathcal{S}_{00} = \mathcal{S}$. The rolling equations in Definition 14 reduce to $\dot{u} = 0$ and $\dot{\mathcal{S}} = 0$. Thus the ball rolls with constant center velocity and constant tangential angular rotation $\mathcal{S}$. In dimension $m = 3$, the latter means that the normal component of the angular velocity is constant.*

Example 2. *In the codimension 2 case we have $\mathcal{N} = \mathbb{R}^k \times S^1(r)$. This Riemannian manifold admits a parallel orthonormal frame of vector fields $E_1, \ldots, E_k, E$, where the E_i are tangent to $\mathbb{R}^k$ and E is tangent to the circle. We have $\mathcal{S}_{11} = 0$ since it is skew-symmetric and rank 1 . Therefore $\mu := u \cdot E = u_1 \cdot E$ is constant. This means that trajectories rotate around the circle factor at a constant rate μ. Let us define the quantity*

$w := \mathcal{S}_{01}E$. *Then* w *and* u_0 *are both vectors in* $\mathbb{R}^k$ *and are related by the system of linear equations*

$$\dot{u}_0 = \frac{\eta\mu}{r}w, \quad \dot{w} = -\frac{\eta\mu}{r}u_0.$$

Rolling trajectories project to ellipses in $\mathbb{R}^k$ *with the following parametric equation:*

$$x(t) = \cos(\omega t)\mathfrak{a} + \sin(\omega t)\mathfrak{b} + \mathfrak{c}$$

where $\omega = \eta\mu/r$ *and*

$$\mathfrak{a} = w(0)/\omega, \quad \mathfrak{b} = u_0(0)/\omega, \quad \mathfrak{c} = x(0) - (1/\omega)^2\dot{u}_0(0) = x(0) - (1/\omega)w(0)$$

The quantities $(\mathcal{S}E_i) \cdot E_j, 1 \leq i < j \leq k-1$ *are constants of motion.*

Example 3. *In the codimension 3 case,* $\mathcal{N} = \mathbb{R}^k \times S^2(r)$. *Points of* $\mathcal{N}$ *will be written* $x = (x_0, x_1)$ *where* x_0 *is the component in* $\mathbb{R}^k$ *and* x_1 *the component on the sphere. Let* $J_{x_1} : T_{x_1}S^2(r) \to T_{x_1}S^2(r)$ *denote positive rotation by* $\pi/2$ *(taking the outward pointing normal vector for the orientation of the sphere.) The tensor* J *is parallel. Then* $\mathcal{S}_{11} = \mathfrak{s}J$, $\frac{\nabla \mathcal{S}_{11}}{dt} = \dot{\mathfrak{s}}J$ *and the fourth among Equations (3) implies that* $\mathfrak{s}$ *is a constant of motion. The third equation then turns into a linear equation on the sphere:*

$$\frac{\nabla u_1}{dt} = \frac{\eta\mathfrak{s}}{r}Ju_1$$

An immediate consequence of this equation is that $|u_1|^2$ *is constant. Now define the quantity*

$$\mathscr{I} := \eta\mathfrak{s}x_1 + x_1 \times u_1,$$

which is a vector in $\mathbb{R}^3$ *(the orthogonal complement to* $\mathbb{R}^k$ *in* $\mathbb{R}^{k+3}$*.) Here* $\times$ *is the standard cross-product. Observe that*

$$\dot{\mathscr{I}} = \eta\mathfrak{s}u_1 + x_1 \times \frac{\nabla u_1}{dt} = \eta\mathfrak{s}\left(u_1 + \frac{x_1}{r} \times Ju_1\right) = 0$$

since u_1, Ju_1 *and* x_1/r *form a positive orthonormal basis of* $\mathbb{R}^3$. *Thus* $\mathscr{I}$ *is a constant of motion (only depending on the initial conditions). Also note that* $\mathscr{I}\cdot x_1 = \eta\mathfrak{s}r^2$ *is constant. This means that the projection to* $S^2(r)$ *of rolling trajectories are circles, traversed with uniform speed, given by the intersection of* $S^2(r)$ *and level sets of the function* $x_1 \mapsto \mathscr{I}\cdot x_1$, *which are cones. (See Figure 4.) Let* $w_1 := \mathcal{S}_{01}u_1$ *and* $w_2 := \mathcal{S}_{01}Ju_1$. *Then* $w_1, w_2, u_0 \in \mathbb{R}^k$ *are related by the linear system*

$$\dot{w}_1 = -\frac{\eta\,|u_1|^2}{r}u_0 + \frac{\eta\mathfrak{s}}{r}w_2, \dot{w}_2 = -\frac{\eta}{r}w_1, \quad \dot{u}_0 = \frac{\eta}{r}w_1$$

It is now a simple calculation to solve for w_1, w_2 *and* u_0, *as well as* $\mathcal{S}_{01}$. *The projection to* $\mathbb{R}^k$ *of rolling trajectories are ellipses.*

Example 4 (Rolling around a straight edge)**.** *We can use the analysis of Example 2 to obtain a billiard interpretation of the rolling around the edge of a half-space. Let* $P = \mathbb{R}^{k-1}\times[0,\infty)$ *be the half-space in* $\mathbb{R}^k$. *We view* P *as the submanifold of* $\mathbb{R}^{k+1}$ *consisting of points* $x = (x_1, \dots, x_{k+1})$ *such that* $x_k \geq 0$ *and* $x_{k+1} = 0$. *The manifold boundary of* P *is the subspace* $\mathbb{R}^{k-1}$ *corresponding to* $x_k = x_{k+1} = 0$. *Let* $\mathfrak{n} = (0, \dots, 0, 1, 0)$ *be the unit normal vector to* ∂P *pointing into* P. *Then* n *(defined for a radius* $r > 0$ *) is the product* $n = \mathbb{R}^{k-2}\times\mathscr{R}$ *where* $\mathscr{R}$ *is the piecewise smooth line in* $\mathbb{R}^2$ *depicted in the left-hand side of Figure 5. We wish to determine the map that gives the velocities of the rolling ball after rolling around the edge as a function of the velocities it had immediately before. Rolling around the edge itself is described in Example 2. Let* $E_1, \dots, E_{k-1}, E$ *be as in that example. We know that* $\mu = u\cdot E$ *is constant, so the time it takes the ball to roll around the edge (from the moment it leaves, say, the top sheet of* n *to the moment it enters the bottom one or vice versa) is* $T = \pi r/|\mu|$. *The tangential angular velocity matrix* $\mathcal{S}$

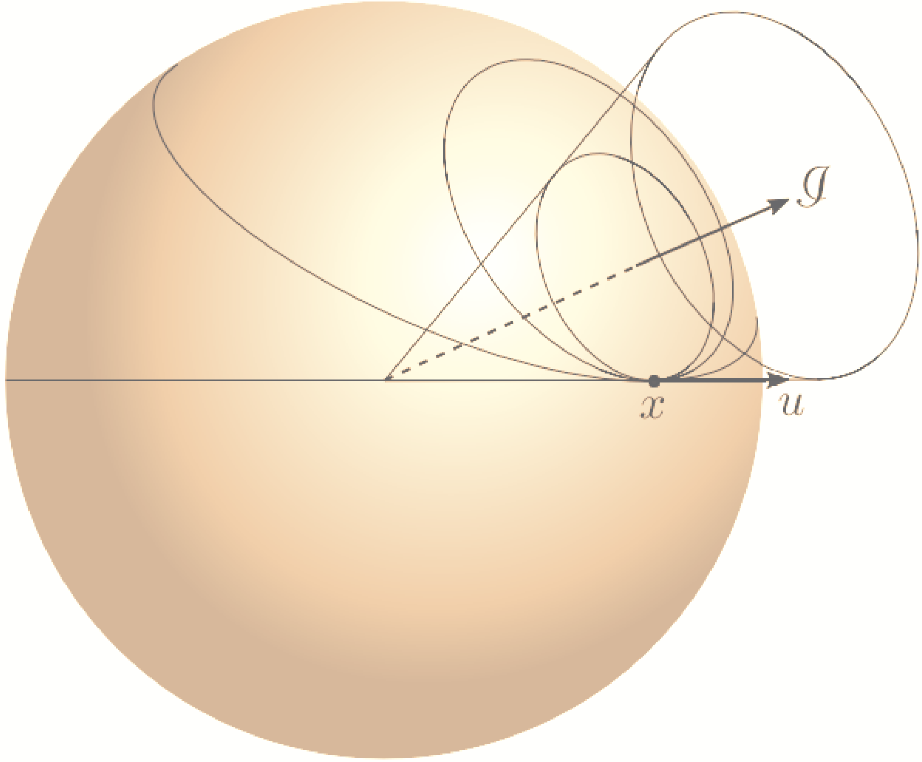

Figure 2.4. In Example 3, the rolling trajectories in $\mathbb{R}^3$ are the circles of intersection of the sphere with cones.

is fully specified by the components $w_i = (\mathcal{S}E)\cdot E_i, i = 1,\ldots,k-1$, and the constants $(\mathcal{S}E_i)\cdot E_j, 1 \le i < j \le k-1$. The quantities $u_0 = \sum_{i=1}^{k-1}(u\cdot E_i)\,E_i$ and $w = \sum_{i=1}^{k-1} w_i E_i$ satisfy, on the curved part of $\mathcal{N}$, the system of differential equations of Example 4 whose solution can be written as follows:

$$\begin{pmatrix} u_0^+ \\ \\ w^+ \end{pmatrix} = \exp\left\{ \frac{\eta\mu}{r} T \begin{pmatrix} 0 & I \\ \\ -I & 0 \end{pmatrix} \right\} \begin{pmatrix} u_0^- \\ \\ w^- \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\sigma\pi\eta)I & -\sin(\sigma\pi\eta)I \\ \\ \sin(\sigma\pi\eta)I & \cos(\sigma\pi\eta)I \end{pmatrix} \begin{pmatrix} u_0^- \\ \\ w^- \end{pmatrix}.$$

Here - and + indicate the velocities before and after rolling around the curved part of n *and* $\sigma \in \{+, -\}$ *is the sign of* μ*. This sign is positive when rolling begins at the top sheet of* $\mathcal{N}$ *and negative otherwise. For our later needs, we rewrite this relation as follows. Define* $W^{\pm} = \mathcal{S}$*, the sign indicates "before" and "after." Now we have*

$$\begin{pmatrix} u_0^+ \\ \\ W^+ \end{pmatrix} = \begin{pmatrix} \cos(\pi\eta)I & \sin(\pi\eta)I \\ \\ \sin(\pi\eta)I & -\cos(\pi\eta)I \end{pmatrix} \begin{pmatrix} u_0^- \\ \\ W^- \end{pmatrix}.$$

Notice that in Example 4, if one observes from above, the motion of a small ball that rolls around the edge of the half-space, the motion appears as a collision of a rotating disc (the flattened ball) with the boundary of P; the component of the velocity of the centre of the disc perpendicular to the boundary of P changes sign while the other components of this velocity and the components of the tangential angular velocity matrix $\mathcal{S}$ are exchanged according to the above equation. In particular, if $\eta = 0$, the disc undergoes ordinary billiard (specular) reflection and its direction of rotation is reversed. Also note that the linear map in the equation above does not depend on the radius r.

Example 5 (Rolling on a semi-infinite line in $\mathbb{R}^3$)**.** . *Example 4 and similar examples in higher codimension can be used as a foundation for rolling systems on polygonal or polyhedral convex shapes. For example, rolling on a convex polygonal plate in dimension*

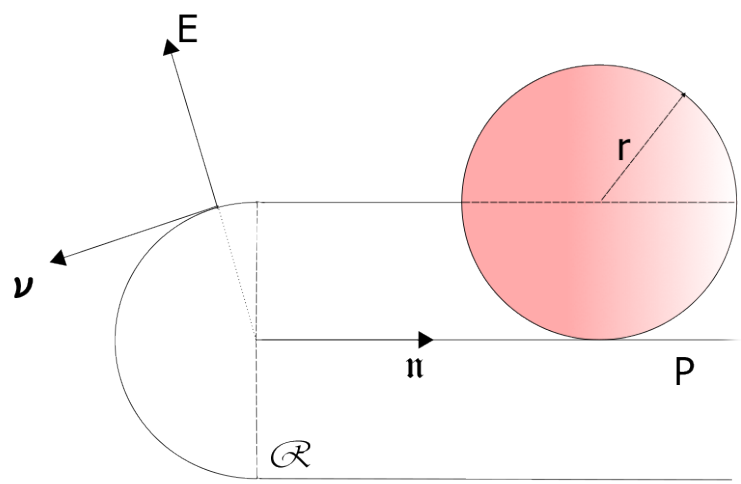

Figure 2.5. Example 4, transversal view.

3 *involves rolling on the surface interior, which is codimension* 1*, on the edges, which is codimension* 2*, and on the vertices, which is codimension* 3*. The rolling of a ball in dimension* 3 *over a semi-infinite straight line is one of the simplest examples of such a system. This is a combination of the codimensions* 2 *and* 3 *examples. Using notations from Example 2, the maximum displacement along the semi-infinite line as a function of the initial conditions and parameters* $\mathfrak{m}_d$ *is*

$$\mathfrak{m}_d = \frac{1}{\omega}\left[\sqrt{(u_0(0))^2 + (s(0))^2} - \mathfrak{s}(0)\right].$$

Recall that $\omega = \eta\mu/r$*, where* μ *is the constant velocity of rotation about the axis of the cylinder. Here the initial point is on the equator of the spherical cap and the initial velocity points into the cylindrical end. In particular, for a given set of initial conditions, this*

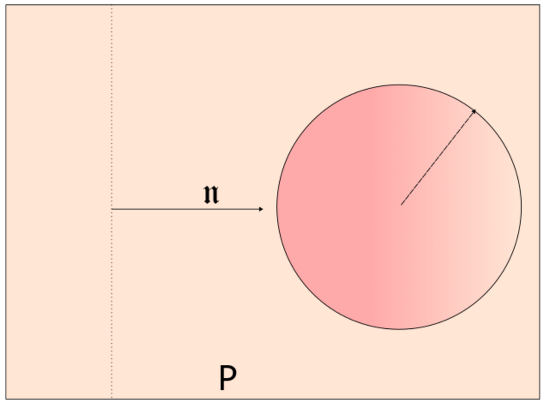

Figure 2.6. Example 4, viewed from above.

displacement is a linear function of η^{-1}. Naturally, as the moment of inertia parameter η approaches 0 and the trajectory of the centre of the ball approaches a geodesic path, this displacement approaches infinity. In each excursion from and back to the equator of the spherical cap, the initial and final values of $\mathfrak{s}$ are the same. In fact $\mathfrak{s}$ is constant on the spherical cap but not on the cylinder. u_0 changes sign, while the projection of u to the plane orthogonal to the axis of the cylinder simply rotates.

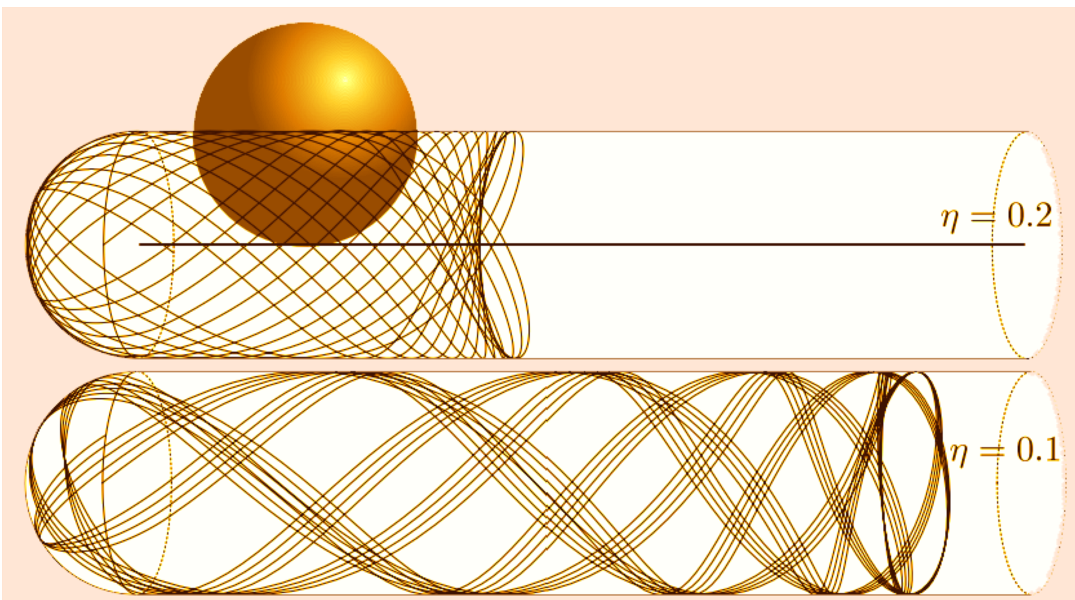

Figure 2.7. Example 5 illustrated in dimension 1. It can be described by combining the codimension 2 and codimension 3 examples.

3. Connection between rolling systems and no-slip billiards

Before establishing the connection of rolling systems and no-slip billiards, we would also have to characterise the no-slip collision map. These systems have been studied intensively in [3] and [7], and this chapter is a summary of the results in these papers. However, the notation used here is different from the paper to match with the main result in this thesis.

3.1 No-slip billiards

We begin by letting P be the closure of a domain in $\mathbb{R}^k$ with piecewise smooth boundary. To avoid adding more notation, we deviate slightly from the above description in Chapters 1 and 2 and assume from now on that P is the locus of centres of the ball rather than the full billiard table. The actual billiard table is then the union of P and the tubular neighbourhood of the boundary ∂P. Let $\mathfrak{n}(x)$ be the inward pointing unit normal vector defined at a point x on the smooth part of the boundary. Note that for a ball to be contained in P, its centre has to be at least r away from the boundary of P.

The configuration manifold M of a ball moving without constraints in the interior of P is the subset $\{(x, R) \in SE(k) : x \in P\}$ of the Euclidean group $SE(k)$. Here x denotes the translation part and R is the rotation. The boundary of M, ∂M consists of the elements $q = (x, R)$ where x lies on the boundary of P. A subbundle $\mathfrak{R}$ of the tangent bundle to the boundary of M can be defined by the linear condition that a vector $v \in \mathfrak{R}_q$ describes a state in which the point on the ball's surface in contact with the boundary of P has zero velocity. Notice that the corners are singular points, since in billiard systems the trajectories are not prolonged at corners, so the tangent bundle is defined outside of corners. We denote tangent vectors to M by $(u, S) \in T_xP \times \mathfrak{so}(k)$, where u is the centre velocity and $S = \dot{R}R^{-1} \in \mathfrak{so}(k)$ is the angular velocity matrix in the Lie algebra of the rotation group $SO(k)$.

To define the collision map it is first necessary to give a Riemannian metric on M:

Let $\xi = (u_\xi, S_\xi), \zeta = (u_\zeta, S_\zeta)$ be tangent to M at (x, R), m be the mass of the ball and γ be the moment of inertia parameter, then

$$\langle \xi, \zeta \rangle = m \left\{ \frac{(r\gamma)^2}{2} \operatorname{Tr}\left(S_\xi S_\zeta^\top\right) + u_\xi \cdot u_\zeta \right\}$$

is the Riemannian metric on M.

Let $\eta = \frac{\gamma}{\sqrt{1+\gamma^2}}$, and let $\cos\beta = \frac{1-\gamma^2}{1+\gamma^2}$ and $\sin\beta = \frac{2\gamma}{1+\gamma^2}$. Now the collision map at a boundary point $q = (x, R)$ of M is a linear map $C_q : T_qM \to T_qM$ that sends vectors pointing out of M to vectors pointing into it and satisfies the following requirements:

1. Conservation of energy, ie., C_q is an orthogonal linear map; 2. Conservation of linear and angular momentum of the unconstrained motion. 3. Time reversibility. C_q is in fact a linear involution; 4. Impulse forces at collision are applied only at the single point of contact.

The explicit expression of the collision map is now $C_q(u, S) =$

$\Big(\cos\beta u - \frac{\sin\beta}{\gamma}(u \cdot \mathfrak{n}(x))\mathfrak{n}(x) + \sin\beta\gamma r S\mathfrak{n}(x), S + \frac{\sin\beta}{\gamma r}\mathfrak{n}(x) \wedge [u - rS\mathfrak{n}(x)]\Big)$.

This result is proposition 15 of [3]. To simplify the notations we'll denote $\mathcal{S} = r\gamma S$ and $W = \mathcal{S}\mathfrak{n}(x)$. Now the Riemannian metric can be written as $\langle \xi, \zeta \rangle = m\{\frac{1}{2}Tr(\mathcal{S}_\xi \mathcal{S}_\zeta^T) + u_\xi \cdot u_\zeta\}$.

It follows from Lemma 7 that if $(u, U) \in \mathfrak{se}(m)$ defines a state at configuration $g = (a, A)$ that satisfies the rolling constraint $u = rU\nu(a)$, then

$$U = \Pi_a U \Pi_a + \frac{1}{r}\nu(a) \wedge u.$$

Rather than using U to describe the state (from which we obtain u using the constraint equation), we use its tangential part

$$S_a := \Pi_a U \Pi_a$$

and u, from which the other components of U can be derived. We will refer to S_a as the body's tangential angular velocity or tangential spin.

Let Π_x be the orthogonal projection from $\mathbb{R}^k$ to the tangent space of the boundary of P at $x \in \partial P$. It follows from the theorem that elements of $\mathfrak{so}(k)$ may be written as

$$\mathcal{S} = \Pi_x \mathcal{S} \Pi_x + \mathfrak{n}(x) \wedge \mathcal{S}\mathfrak{n}(x)$$

Then the effect of C_q is to map

$$\Pi_x \mathcal{S} \Pi_x \mapsto \Pi_x \mathcal{S} \Pi_x, \quad \mathfrak{n}(x) \mapsto -\mathfrak{n}(x), \quad \begin{pmatrix} \bar{u} \\ W \end{pmatrix} \mapsto \begin{pmatrix} \cos\beta I & \sin\beta I \\ \sin\beta I & -\cos\beta I \end{pmatrix} \begin{pmatrix} \bar{u} \\ W \end{pmatrix},$$

where $\bar{u} = \Pi_x u$. This is the map in Proposition 3 of [1] in general dimensions. The linear map in general dimensions was given in [12], Section 2.4, but not in matrix form. Note that both W and $\bar{u}$ are tangent to the boundary of P at x. This orthogonal transformation of $\bar{u}$ and W is the characteristic exchange of linear and angular velocities of no-slip collisions. Notice that although the equation here is two dimensional, it can be generalised to higher dimensions by multiplying an identity matrix to the coefficients.

We can now define no-slip billiards as the system whose orbits in the interior of P consist of straight line segments with constant u and constant $\mathcal{S}$, and at the boundary undergoes a change of velocities according to the above collision map C_q. When the mass distribution of the ball is entirely concentrated at the centre, $\gamma = 0$ and the collision map reduces to a transformation that decouples linear and angular velocities: the centre velocity u transforms according to the standard billiard reflection, and the components of the angular velocity contained in W switches sign while the other components remain the same.

As mentioned earlier, many of the concepts about no-slip billiards have a rolling billiard counterpart. When the context is not clear, we use superscript notations C^b and C^r, γ^b and γ^r, W^b and W^r to distinguish, where r stands for "rolling" and b stands for "billiard".

3.2 No-slip billiards as a limit of rolling systems

Before we state our main result, let us give a formal definition of the non-holonomic collision map.

Definition 10 (No-slip collision map)**.** *Let P be an $(k-1)$-dimensional flat plate in $\mathbb{R}^k$ with smooth manifold boundary P_0. At any $x \in P_0$ we define the vector space $V_x = T_xP_0 \oplus T_xP_0 \oplus \mathbb{R}\mathfrak{n}(x)$ consisting of vectors $\left(W, u_0, u^\perp\right)$, where $u = u_0 + u^\perp \in T_xP$ and*

$\mathfrak{n}(x)$ is the inward pointing unit normal vector to P_0. Then the non-holonomic collision map $\mathcal{C}_x : V_x \to V_x$ is defined as

$$\mathcal{C}_x \left(W, u_0, u^{\perp}\right) = \left(\sin(\pi\eta)u_0 - \cos(\pi\eta)W, \cos(\pi\eta)u_0 + \sin(\pi\eta)W, -u^{\perp}\right)$$

Note that, when $\eta = 0, \mathcal{C}_x$ reflects u specularly and reverses the sign of W.

The following theorem, highlighting the relation between no-slip billiards and general rolling systems, is the main result of this thesis.

Theorem 11. *Let P be an $(k-1)$-dimensional flat plate in $\mathbb{R}^k$ with smooth boundary P_0 whose principal curvatures (as a hypersurface in P) are uniformly bounded. In the limit when the radius of the ball approaches 0 , solutions of the rolling ball equation have the following description: On $P \backslash P_0$ the point-mass moves with constant velocities u and $\mathcal{S}$; upon reaching a boundary point $x \in P_0$, the vector $(u, W) \in T_xP \oplus T_xP_0$, where $W = \mathcal{S}\mathfrak{n}(x)$, undergoes a reflection according to the no-slip collision map $\mathcal{C}_x$ of the previous definition. It should be noted that the moment of inertia parameter η, which appears in the rolling equation and in the billiard map $\mathcal{C}_x$, is independent of the radius.*

Proof. To begin, let us assume that the radius r of the ball is sufficiently small so that the map $\pi : \mathcal{N}(r) = \mathcal{N} \to P$ that associates to each $p \in \mathcal{N}(r)$ the closest point in P is well defined. This is possible due to the assumption that the principal curvatures of P_0 are bounded. The hypersurface $\mathcal{N}$ is piecewise smooth and consists of the union of two parallel copies of P, lying $2r$ apart from each other, and half the boundary of the tube of radius r centred around P_0. We call the two copies of P the two sheets of $\mathcal{N}$ and the half-tube the curved part of $\mathcal{N}$.

Let $x, u, \mathcal{S}$ be initial conditions for the rolling equation, where $x \in \mathcal{N}$ is a point on the interface where the curved part of $\mathcal{N}$ meets the flat sheets. Note that this interface is the union of two diffeomorphic copies of P_0. Here $u \cdot \mathfrak{n}(x) < 0$, so the centre of mass velocity u points towards the curved part of $\mathcal{N}$; $\mathcal{S}$ is the tangential angular velocity tensor. Set $W := \mathcal{S}\mathfrak{n}(x)$. Let $u(t) = \dot{x}(t)$ and $\mathcal{S}(t)$ satisfy the rolling equations

$$\frac{\nabla u}{dt} = -\eta \mathcal{S} \mathbb{S}_x u, \quad \frac{\nabla \mathcal{S}}{dt} = \eta \left(\mathbb{S}_x u\right) \wedge u$$

with the given initial conditions. We follow the solution from time 0 till the moment (if it happens) when the ball reaches the interface submanifold again.

The shape operator $\mathbb{S}$ naturally becomes singular as r approaches 0 . In fact, on the intersection of the curved part of $\mathcal{N}$ with the 2-plane perpendicular to $T_{\pi(x)}P_0$ Notice that the principal curvature is $-1/r$. We call this intersection the meridian of $\mathcal{N}$ at x. This produces a discontinuity of velocities at the limit. It is also to be expected that the duration of the rolling on the curved part of $\mathcal{N}$ approaches 0 in the limit. With these issues in mind, we transform the original equations of motion by making a time change and applying an appropriate homothety. The resulting system will be of the kind considered in Example 4 (the rolling of a finite radius ball on a straight edge).

Here are some of the details. Let c^2 be the square norm of $(u, \mathcal{S})$, a quantity proportional to the energy of the initial condition. Introduce a new time given by $\tau = \frac{c}{r}t$ and define the homothety $h : x \in \mathbb{R}^k \to x/r \in \mathbb{R}^k$. Let $\bar{\mathcal{N}}$ be the image of $\mathcal{N}(r)$ under h, appropriately translated so the projection $\pi(x)$ of the initial point lies at the origin. Note

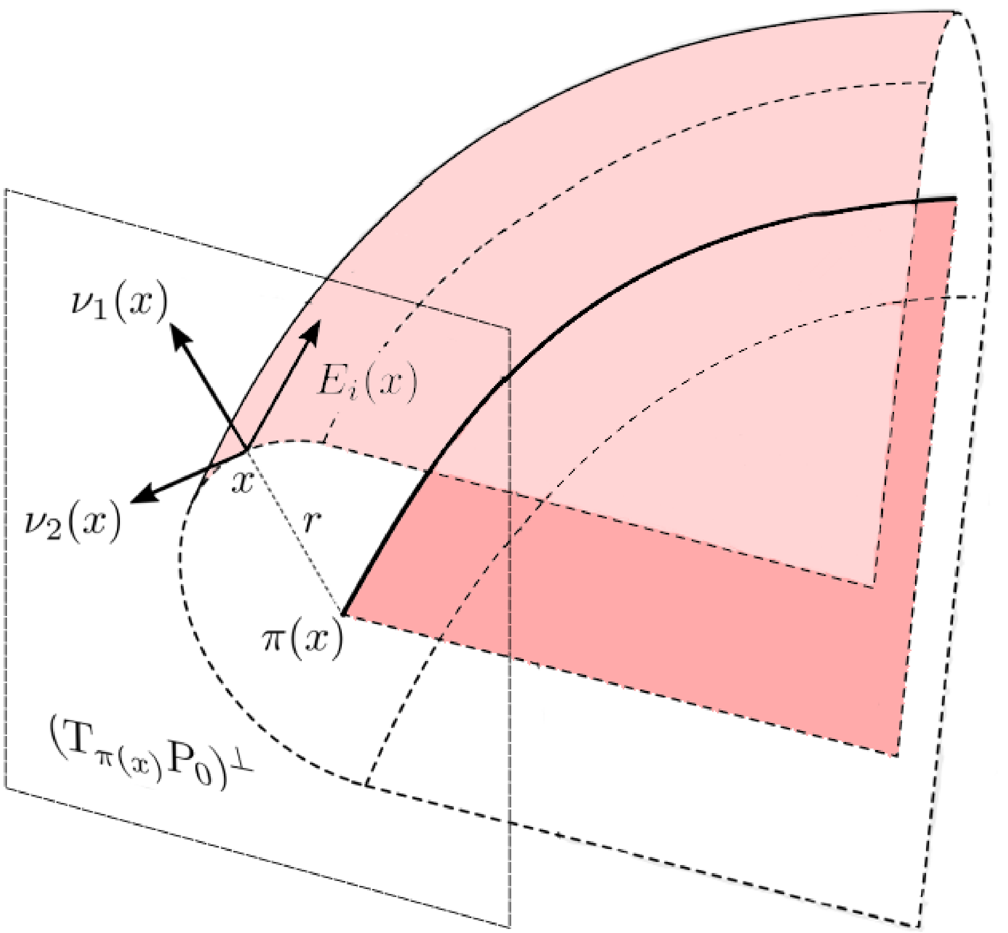

Figure 3.1. This figure illustrates the hypersurface $\mathcal{N} = \mathcal{N}(r)$ associated to the manifold P.

that, as r approaches $0, \bar{\mathcal{N}}$ looks increasingly like the straight edge situation of Example 4. Now define

$$\bar{x}(\tau) = h(x(ct/r)), \quad \overline{\mathcal{S}}(\tau) = h\left(\frac{r}{c}\mathcal{S}(ct/r)\right).$$

For any given value of r, the rolling equations turn into

$$\frac{\nabla \bar{u}}{d\tau} = -\eta \overline{\mathcal{S}} \mathbb{S}_{\bar{x}} \bar{u}, \quad \frac{\nabla \overline{\mathcal{S}}}{d\tau} = \eta \left(\mathbb{S}_{\bar{x}} \bar{u}\right) \wedge \bar{u}$$

where the new shape operator $\mathbb{S}$ at $\bar{x}(\tau)$ equals $r\mathbb{S}$ at $x(ct/r)$. The norm of the new initial velocities $(\bar{u}, \overline{\mathcal{S}})$ is 1 for all r. The principal curvature on the meridian circles become -1

for all r, and the other principal curvatures approach 0 . In the limit, this shape operator becomes $-E^b \otimes E$ where E is a unit vector field tangent to the meridian circle and E^b is its dual vector relative to the dot-product.

The meridian circles are geodesics so $\nabla_E E = 0$, and E has constant norm, so $E \cdot \nabla_v E = 0$ for any tangent vector v. Writing $\bar{u}^\perp$ for the component of $\bar{u}$ perpendicular to E, we obtain $\bar{u} \cdot \nabla_{\bar{u}} E = \bar{u}^\perp \cdot \nabla_{\bar{u}^\perp} E$. Then, using the equations of motion,

$$\frac{d}{d\tau}\bar{u} \cdot E = \frac{\nabla \bar{u}}{d\tau} \cdot E + \bar{u} \cdot \nabla_{\bar{u}} E = -\eta E \cdot (\overline{\mathcal{S}}\overline{\mathbb{S}}\bar{u}) + \bar{u}^\perp \cdot \nabla_{\bar{u}^\perp} E.$$

As r approaches $0, \overline{\mathbb{S}}\bar{u}$ converges to a vector parallel to E; since $\overline{\mathcal{S}}$ is skew-symmetric, the term $E \cdot (\overline{\mathcal{S}}\overline{\mathbb{S}}\bar{u})$ approaches 0 . Notice that E is normal to the isometric copies of the rescaled P_0, so the quantity $\nabla_{\bar{u}^\perp} E$ is the negative of the shape operator of this submanifold. Thus the term $\bar{u}^\perp \cdot \nabla_{\bar{u}^\perp} E$ also approaches 0 due to the assumption that the principal curvatures of P_0 are bounded. The conclusion is that, in the limit, $\bar{u} \cdot E = \mu$ is a constant of motion and, in any fixed neighbourhood of the initial (rescaled) point, the hypesurface $\bar{n}$ approaches that of the rolling around a straight edge example. By introducing an orthonormal frame $E_1, \ldots, E_{k-2}$ of parallel vector fields tangent to the rescaled (and straightened) P_0, and using $\overline{\mathbb{S}}E_i = 0$, we obtain the system of equations

$$\frac{d}{dt}\left(E_i \cdot \bar{u}\right) = \eta\mu \left(E_i \cdot \overline{\mathcal{S}}E\right), \quad \frac{d}{dt}\left(E_i \cdot \overline{\mathcal{S}}E\right) = -\eta\mu E_i \cdot \bar{u}$$

But these are precisely the equations of Example 4 . By reversing the rescaling on velocities we obtain from the conclusion of that example the collision map C we are after.

□

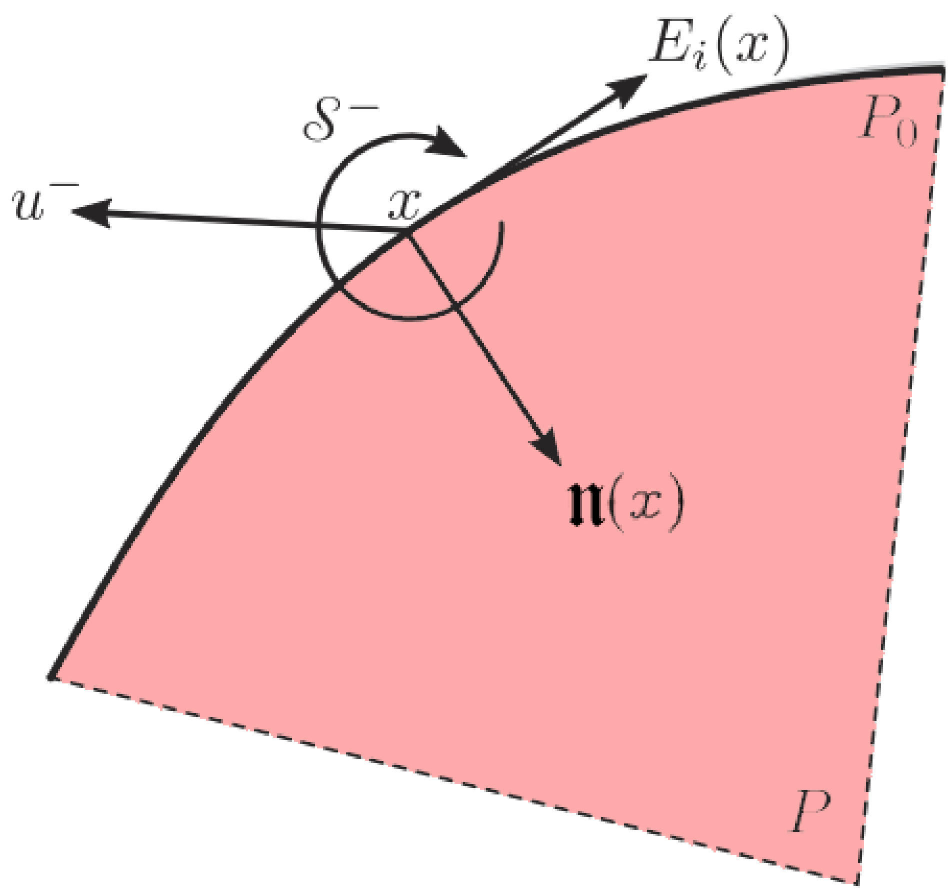

Figure 3.2. When $\lim r \to 0$, one obtains a nonholonomic billiard system with collision map $\mathcal{C} : (u^-, W^-) \mapsto (u^+, W^+)$, using the notations in Example 4

4. The Rolling Flow

4.1 Definition of the Rolling flow

Recall that the state of the rolling system can be specified by a triple $(x, u, \mathcal{S})$ where $x \in n$ is the centre of the ball, $u \in T_x\mathcal{N}$ is the centre velocity, and $\mathcal{S}$ is a skew symmetric endomorphism of $T_x\mathcal{N}$ that we call the tangential spin. We denote the vector space of such linear maps as $\mathfrak{so}_x(\mathcal{N})$. The latter is a fibre of the vector bundle of skew-symmetric maps, which we denote by $\mathfrak{so}(\mathcal{N})$. Thus the set of states of the rolling ball comprises the vector bundle $\pi : \mathcal{M} = T\mathcal{N} \oplus \mathfrak{so}(\mathcal{N}) \to \mathcal{N}$. Given the rolling constraint, the other components of the angular velocity matrix not in $\mathfrak{so}(\mathcal{N})$ can be recovered from u and $\mathcal{S}$. This vector bundle is given the Riemannian metric derived from the kinetic energy. If $e_i = (u_i, \mathcal{S}_i)$, for $i = 1, 2$, lie in the fiber above $x \in \mathcal{N}$, then the following is true up to a scalar coefficient:

$$\langle e_1, e_2 \rangle = u_1 \cdot u_2 + \frac{1}{2} \operatorname{Tr}\left(\mathcal{S}_1 \mathcal{S}_2^\top\right).$$

To define the rolling flow it is helpful to first define the Newton's equation on $\mathcal{M}$.

Theorem 12. *Let $f : \mathcal{M} \to \mathcal{M}$ be the bundle map defined as $f(e) = -\eta(\mathcal{S}\mathbb{S}_x u, u \wedge \mathbb{S}_x u)$. Here $e = (u, \mathcal{S})$ and $\mathbb{S}$ is the shape operator on $\mathcal{N}$. Define ∇ the connection on the vector* *bundle $\mathcal{M}$ induced from the Levi-Civita connection on $\mathcal{N}$. The rolling equation can be written as $\frac{\nabla e}{dt} = f(e)$. The components can be described by the system of differential equations:*

$$\begin{cases} \dot{x} & = u \\ \frac{\nabla u}{dt} & = -\eta \mathcal{S} \mathbb{S}_x u \\ \frac{\nabla \mathcal{S}}{dt} & = \eta (\mathbb{S}_x u) \wedge u \end{cases}$$

The rolling flow will be a flow on $\mathcal{M}$. Before giving a formal definition, it is worthwhile to review the traditional geodesic flows. For geodesic flows, $\mathcal{M}$ would correspond to the tangent bundle of a Riemannian manifold $\mathcal{N}$. In our case, besides the velocity $u = \dot{x} \in T_x\mathcal{N}$, we also have the tangential spin velocity $\mathcal{S}$. The moment of inertia parameter η describe the connection between this two components. When $\eta = 0$, the motion on $T\mathcal{N}$ is the geodesic flow, independent of the tangential spin, while $\mathcal{S}$ is transported along geodesics by parallel translation. This is similar to the previously studied orthogonal frame flows, except that in this case instead of the orthonormal frame, it is a tensor that is related to the state of spinning of a frame relative to itself that is parallel transported.

If follows from definition of force term f in Newton's equation that $\langle e, f(e) \rangle = 0$. Also by energy conservation, solution curves $e(t)$ have constant energy: $\mathscr{E}(e(t)) = \mathscr{E}(e(0))$, where

$$\mathscr{E}(e) = \frac{1}{2}\|e\|^2 = \frac{1}{2}\left(|u|^2 + \frac{1}{2}\operatorname{Tr}\left(\mathcal{S}\mathcal{S}^\top\right)\right).$$

The connection induces a splitting $T\mathcal{M} = E^V \oplus E^H$ as a direct sum into vertical and horizontal subbundles, and a connection map $K_e : T_e\mathcal{M} \to T_x\mathcal{M} \to \mathcal{M}_x$ where $\mathcal{M}+x, x = \pi(e)$ is the vector fibre of $\mathcal{M}$ at $x \in \mathcal{N}$.

The rolling flow is a flow defined on $\mathcal{M}$ as follows:

Definition 13. *Let Z be the horizontal vector field on $\mathcal{M}$ such that $d\pi_e Z(e) = e$ for all $e \in \mathcal{M}$; and let V be the vertical vector field such that $K_e V(e) = \frac{1}{\eta} f(e)$. Then define a vector field $X = Z + \eta V$. The rolling flow is the flow on $\mathcal{M}$ generated by the vector field X.*

We can see from the definition that when $\eta = 0$, the rolling flow is the same as the geodesic flow on $T\mathcal{N}$, which is extended by adding parallel transported tangential spin. The motion on $T\mathcal{N}$ is geodesic flow, independent of the tangential spin, and $\mathcal{S}$ is transported along geodesics by parallel translation. In the geodesic flow case, $\mathcal{M}$ would be the tangent bundle of a Riemannian manifold $\mathcal{N}$. It can also be seen that $e(t)$ is an integral curve of V if and only if, regarded as a vector field along $x(t) = \pi(e(t))$, it satisfies Newton's equation. (See chapter 4 for more details).

Note that $\mathcal{M}$ may be regarded as the tangent bundle of a manifold M. Let $\pi : \mathcal{M} \to \mathcal{N}$ be the fibre bundle of linear isometries on tangent spaces of $\mathcal{N}$. By definition, the fibre M_x over $x \in \mathcal{N}$ consists of all the orientation preserving linear maps $A : T_x\mathcal{N} \to T_x\mathcal{N}$ such that $\langle Au, Av\rangle_x = \langle u, v\rangle_x$ for all $x \in \mathcal{N}$ and all $u, v \in T_x\mathcal{N}$. In other words, this is a bundle of groups and the fibre over x is the special orthogonal group on $T_x\mathcal{N}$. We denote this group by $SO_x(\mathcal{N})$ with its Lie algebra $\mathfrak{so}_x(\mathcal{N})$.

The rolling flow can be regarded as a dynamical system on the energy level sets of the tangent bundle of M. Here π is the base-point map $\pi : TM \to M$. TM can be identified with the vector bundle $\pi : \mathcal{M} = T\mathcal{N} \oplus \mathfrak{so}(\mathcal{N}) \to \mathcal{N}$ as follows. Let $e \in TM$ have base-point $A \in M$, where $A \in SO_x(\mathcal{N})$. Let $A(t) \in M$ be a parametrized differentiable curve representing e in the sense that $A(0) = A$ and $A'(0) = e$. We can then map e to $(u, \mathcal{S})$ where

$$u = \frac{d}{dt}\pi(A(t))\bigg|_{t=0} \in T_x\mathcal{N}, \quad \mathcal{S} = \frac{\nabla A(t)}{dt}|_{t=0}A^{-1} \in \mathfrak{so}_x(\mathcal{N})$$

where ∇ is the covariant derivative obtained through the imbedding $M \subset T^*\mathcal{N} \otimes T\mathcal{N}$.

The following proposition shows that flow lines of the rolling flow project to the trajectories of the rolling motion.

Proposition 14. *A differentiable curve $e(t) \in M$ is an integral curve of X if and only if, regarded as a vector field along $c(t) = \pi(e(t)) = (x(t), A(t)) \in M$, it satisfies*

$$\frac{\nabla e}{dt} = f(e), \quad (u, \mathcal{S}) = e$$

where $u = \dot{x}$ and $\mathcal{S} = \frac{\nabla A}{dt}A^{-1}$. Recall that we identify $\dot{A}$ at $A \in M$ with $\mathcal{S} = \frac{\nabla A}{dt}A^{-1}$ under the identification $TM \cong T\mathcal{N} \oplus \mathfrak{so}(\mathcal{N})$.

Proof. Let $e(t)$ be a flow line of X and define $(x(t), A(t)) = \pi(e(t))$. Then the equation $e' = X$ implies

$$(u, \mathcal{S}) \cong (\dot{x}, \dot{A}) = d\pi_e\dot{e} = d\pi_e X(e) = d\pi_e Z = e$$

since F is vertical; and

$$\frac{\nabla e}{dt} = K_e e'(0) = K_e X(e) = f(e)$$

Conversely,

$$K_e\dot{e} = \frac{\nabla e}{dt} = f(e) = K_e X$$

and

$$d\pi_e\dot{e} = \frac{d}{dt}\pi(e(t)) = (\dot{x}, \dot{A}) = e = \pi_e X(e).$$

Therefore $\dot{e} = X(e)$. □

Thus the flow lines of X in M project under π to solution curves in M of the rolling equation.

4.2 Volume Invariance

In this section, we want to understand if the canonical volume form is invariant under the rolling flow. we begin by defining the kinetic energy function $\mathscr{E} : \mathcal{M} \to \mathbb{R}$ by:

$$\mathscr{E}(e) = \mathscr{E}(u, \mathcal{S}) = \frac{1}{2}\left(\|u\|^2 + \frac{1}{2}\operatorname{Tr}\left[\mathcal{S}\mathcal{S}^\top\right]\right)$$

.

Let $\pi : M \to M$ be the base-point map and ∇ the Riemannian connection induced from the Levi-Civita connection on $\mathcal{N}$. Recall the connection map, $K_e : T_e\mathcal{M} \to T_{\pi(e)}\mathcal{M}$, where $e \in \mathcal{M}$: If $\xi \in T_e\mathcal{M}$ is represented by a differentiable curve $e(t)$, so that $e(0) = e$ and $e'(0) = \xi$, then

$$K_e\xi := \frac{\nabla e}{dt}(0)$$

The kernel $E_e^H \subset T_e m$ of K_e is the horizontal subspace. The vertical subspace $E_e^V \subset T_e\mathcal{M}$ is the kernel of $d\pi_e : T_e\mathcal{M} \to T_{\pi(e)}M$. If $v \in T_{\pi(e)}M$, then $t \mapsto e + tv$ is a curve in $\mathcal{M}$ contained in the fiber above $\pi(e)$. We write $\tilde{v}_e \in E_e^V$ for the vertical vector represented by this curve. Note that $\tilde{v}_e$ is characterized by the equations $K_e\tilde{v}_e = v, d\pi_e\tilde{v}_e = 0$. Similarly, define $\bar{v}_e \in E_e^H$ by the equations $K_e\bar{v}_e = 0, d\pi_e\bar{v}_e = v$. We call these vectors the vertical and horizontal lifts of v, respectively. If X is a vector field on M, we obtain the vertical, $\widetilde{X}$, and horizontal, $\bar{X}$, lifts of X. These are vector fields on $\mathcal{M}$.

We can now write $T\mathcal{M} = E^V \oplus E^H$, a direct sum of vector bundles. The Sasaki metric on $\mathcal{M}$ is the Riemannian metric given in terms of the Riemannian metric on M by

$$\langle \xi_1, \xi_2 \rangle_e = \langle d\pi_e \xi_1, d\pi_e \xi_2 \rangle_{\pi(e)} + \langle K_e \xi_1, K_e \xi_2 \rangle_{\pi(e)}$$

Define the contact form as the 1-form θ on $\mathcal{M}$ given by

$$\theta_e(\xi) = \langle e, d\pi_e \xi \rangle_{\pi(e)}$$

Given a local orthonormal frame of vector fields E_0, E_1, E_2 on $\mathcal{N}$, notice that $\bar{E}_0$, $\bar{E}_1$, $\bar{E}_2$ are tangent to N, the energy level set.

Let M be a n-dimensional Riemannian manifold, then the canonical volume form on M is the n-form

$$\Omega = d\theta \wedge \cdots \wedge d\theta$$

.

Now we want to answer the following question:

Question 1. *Is the volume form Ω is invariant under the rolling flow, and is the corresponding volume form on the constant energy hypersurface, $\Omega^{\mathcal{E}}$, invariant under the rolling flow restricted to the constant energy hypersurface?*

The answer to Question 1 is positive in dimension 3. To understand this, we first need to prove some properties for the bracket relations of the horizontal and vertical lifts.

Proposition 15. *Let M be a Riemannian manifold and $\mathcal{M} = TM$. Given vector fields X, Y on M and $e \in \mathcal{M}$, the following bracket relations hold for the vertical and horizontal* *lifts of* X *and* Y *: 1.* $[\widetilde{X}, \widetilde{Y}] = 0$*; 2.* $[\bar{X}, \widetilde{Y}] = \widetilde{\nabla_X Y}$*; 3.* $[\bar{X}, \bar{Y}]_e = [\overline{X, Y}]_e + (R(\overline{X, Y})e)_e$*. where* R *is the curvature tensor on* M*.*

Proof. The first identity is immediate due to the property that the vertical lifts ($\tilde{X}$ and $\tilde{Y}$) project to the zero vector field and the projection of the Lie bracket of projectable vector fields is the Lie bracket of the projections from M to $\mathcal{N}$. The identities 1 and 2 correspond to identities (ii) and (i) of Lemma 2.112, page 105 in [13].

The third identity can also be found in [13] by combining Definition 2.43 on page 67 and Proposition 2.66 on page 82. □

Proposition 16. *Let* M *be a Riemannian manifold and* X, Y *be vector fields on* M*. Suppose* E_i*,* $i = 1, ..n$ *is a local orthonormal frame of* $\mathcal{M} = TM$*. Let* θ *be the 1-form on* $\mathcal{M}$ *defined earlier in the section.* Z *and* W *are vector fields on* $\mathcal{M}$ *defined similar to Definition 13, where for* $e \in \mathcal{M}$*,* Z *is defined by* $d\pi_e Z(e) = e$*,* $K_e Z(e) = 0$ *and* W *by* $d\pi_e W(e) = 0$ *and* $K_e W(e) = e$*, then we have the following bracket relations: 1.* $\bar{Y}\theta(\bar{X}) = \theta(\overline{\nabla_Y X})$*. 2.* $[\bar{Y}, W] = 0$*. 3.* $[W, Z] = Z$*.*

Proof. For the first relation, first note that $\bar{Y}_e\theta(\bar{X}) = \langle e, \nabla_Y X\rangle_{\pi(e)}$.

This is because $\bar{Y}_e = \frac{d}{dt}|_{t=0}\Phi_t^Y(\pi(e))e$, where Φ_t^Y is the parallel translation along a path $\gamma(t)$. This is a direct application of Proposition 2.57 in [13].

Therefore, $\bar{Y}_e\theta(\bar{X}) = \frac{d}{dt}|_{t=0}\langle \Phi_t^Y(\pi(e))e, X(\gamma(t))\rangle = \langle \frac{\nabla e}{dt}(0), X_{\pi(e)}\rangle + \langle e, \nabla_Y X\rangle$.

Now $\bar{Y}\theta(\bar{X}) = \theta(\overline{\nabla_Y X})$ since $\frac{\nabla e}{dt}(0) = 0$.

Then we can expand the expression as follows:

$$
\begin{aligned}
[\bar{Y}, W] &= \sum_{i=1}^{n} [\bar{Y}, \theta(\bar{E}_i)\bar{E}_i] \\
&= \sum_{i=1}^{n} [(\bar{Y}\theta(\bar{E}_i))\bar{E}_i + \theta(\bar{E}_i)[\bar{Y}, \bar{E}_i]] \\
&= \sum_{i=1}^{n} (\theta(\overline{\nabla_Y E_i})\bar{E}_i + \theta(\bar{E}_i)\widetilde{\nabla_Y E_i})
\end{aligned}
$$

Here we used properties in Proposition 15. Now for each e, we can extend it to a local vector field E, so

$$
\begin{aligned}
&\langle E, \nabla_Y E_i \rangle E_i + \langle E, E_i \rangle \nabla_Y E_i \\
&= \nabla_Y(\langle E, E_i \rangle E_i) - \langle \nabla_Y E, E_i \rangle E_i - \langle E, E_i \rangle \nabla_Y E_i + \langle E, E_i \rangle \nabla_Y E_i
\end{aligned}
$$

The last two terms cancel out, and the first two term will also cancel out when the sum is taken over all i. This proves the second relation.

For the third relation, Let $a_i(e) = \langle \bar{E}_i, e \rangle$. Then we have:

$$
\begin{aligned}
[W, Z] &= \sum (W a_i)\bar{E}_i + \sum a_i [W, \bar{E}_i] \\
&= \sum a_i \bar{E}_i \\
&= Z
\end{aligned}
$$

Notice that $[W, \bar{E}_i] = 0$ since $(W a_i)(e) = \frac{d}{dt}|_{t=0} a_i(e + te) = a_i(e)$.

$\square$

Now we are ready to prove the main result of this section in dimension 3.

Theorem 17. *Suppose M is a 3-dimensional Riemannian manifold. Then the symplectic form Ω on the tangent bundle of M is invariant under the rolling flow. Moreover, the restriction of the contact form θ to level sets of the energy function is also invariant under the rolling flow.*

Proof. In dimension 3, M is the product $\mathcal{N} \times \mathbb{R}$.

Like before, let $\mathcal{M} = TM$ equipped with the Sasaki metric, and let E_0, E_1, E_2 be a local orthonormal frame on $\mathcal{M}$, where E_0 is tangent to $\mathbb{R}$. and E_1 and E_2 are eigenvectors of the shape operator $\mathbb{S}$, and their corresponding eigenvalues are κ_1 and κ_2. Notice that the eigenvalue for E_0 is zero. $\bar{E}_0, \bar{E}_1, \bar{E}_2$, $\widetilde{E}_0, \widetilde{E}_1, \widetilde{E}_2$ be the vertical and horizontal lifts as defined earlier.

Let $a_i(e) = \langle \bar{E}_i, e\rangle$. These are one-form on M that becomes functions on $\mathcal{M}$: $\mathcal{M} \mapsto \mathbb{R}$. Since V is a vertical lift of f by definition,

$$\begin{aligned} f(q, a_0E_0 + a_1E_1 + a_2E_2) &= -a_0[a_1\kappa_1E_2 - a_2\kappa_2E_1] + a_1a_2(\kappa_1 - \kappa_2)E_0 \\ &= a_1a_2(\kappa_1 - \kappa_2)E_0 + a_0a_2\kappa_2E_1 - a_0a_1\kappa_1E_2 \end{aligned}$$

Then we can write $V = f_0\widetilde{E}_0 + f_1\widetilde{E}_1 + f_2\widetilde{E}_2$.

Here $f_0 = a_1a_2(\kappa_1 - \kappa_2)$, $f_1 = a_0a_2\kappa_2$, $f_2 = -a_0a_1$.

Recall that Z is the vector field defined by $d\pi_e Z(e) = e$ and $K_e Z(e) = 0$, let's define the vector field W by $d\pi_e W(e) = 0$ and $K_e W(e) = e$, then $W \perp N$. Notice that Z and W are vector fields that are only on $\mathcal{M}$. We need to check if the Lie derivatives of θ applied to Z,W, and horizontal and vertical lifts of general vector fields on M are zero.

For a general vector field Y on M, we have $(\mathcal{L}_X\theta)(\widetilde{Y}) = X\theta(\widetilde{Y}) - \theta(\mathcal{L}_X\widetilde{Y})$.

Note that $\theta(\widetilde{Y}) = \langle Z, \widetilde{Y}\rangle = 0$, and $\mathcal{L}_X\widetilde{Y} = [Z + \eta V, \widetilde{Y}] = \widetilde{\nabla_e Y} - \widetilde{Y} + \eta[V, \widetilde{Y}] = 0$. Therefore $(\mathcal{L}_X\theta)(\widetilde{Y}) = 0$.

On the other hand, $(\mathcal{L}_X\theta)(Z) = X\langle Z, Z\rangle - \langle Z, [X, Z]\rangle = 0 + \langle Z, [Z, V]\rangle = 0$, since $[Z, V] = \sum(Zf_j)\widetilde{E}_j + \sum f_j[Z, \widetilde{E}_j]$.

Using the properties in Proposition 15 and 16, we have

$$
\begin{aligned}
(\mathcal{L}_X\theta)(\bar{Y}) &= X\langle Z, \bar{Y}\rangle - \langle Z, [X, \bar{Y}]\rangle \\
&= X\langle Z, \bar{Y}\rangle - \langle Z, [Z, \bar{Y}]\rangle - \eta\langle Z, [V, \bar{Y}]\rangle \\
&= X\langle Z, \bar{Y}\rangle - \langle Z, \bar{\nabla}_e Y\rangle - \eta\langle Z, [\sum f_j\widetilde{E}_j, \bar{Y}]\rangle \\
&= Z\langle e, Y\rangle - \langle e, \nabla_u Y\rangle \\
&= \frac{d}{dt}|_{t=0}\langle \Phi_t(e), Y(\pi \circ \Phi_t(e)\rangle - \langle e, \nabla_e Y\rangle \\
&= \langle e, \nabla_e Y\rangle - \langle e, \nabla_e Y\rangle \\
&= 0
\end{aligned}
$$

The only issue is $(\mathcal{L}_X\theta)(W)$. Note that

$$
\begin{aligned}
(\mathcal{L}_X\theta)(W) &= X\langle Z, W\rangle - \langle Z, [X, W]\rangle \\
&= \langle Z, ([Z, W] + \eta[V, W])\rangle \\
&= -\|Z\|^2
\end{aligned}
$$

This is because the first component is Z by Proposition 15 and the second component is vertical. Therefore this expression is constant on the level set.

Recall $W \perp N$, where N is the level set of the energy function as defined in Section 4.1, so the Lie derivative of θ is zero when restricted to the level sets of the energy function. Therefore, although θ is not invariant under the rolling flow, it is invariant under the flow on energy level sets. In particular, the symplectic form and volume form are also invariant on energy level sets.

In fact, both the symplectic form and the volume form are invariant under the rolling flow on $\mathcal{M}$. This can be seen as follows: if we take the Lie derivative of the symplectic form for any two vector fields ξ, ζ, we can use the Cartan's formula:

$$(\mathcal{L}_X\theta)(\xi,\zeta) = d(\mathcal{L}_X\theta)(\xi,\zeta)$$
$$= \xi(\mathcal{L}_X\theta)(\zeta) - \zeta(\mathcal{L}_X\theta)(\xi) - (\mathcal{L}_x\theta)([\xi,\zeta])$$

If both of ξ, ζ is tangent to N, then the expression is zero. If not, then without loss of generality we can replace ξ with W, the perpendicular vector field, and ζ tangent to N. Then $(\mathcal{L}_X d\theta)(W,\zeta) = 0$ by Cartan's formula:

$$(\mathcal{L}_X d\theta)(W,\zeta) = d(\mathcal{L}_X\theta)(W,\zeta) = W(\mathcal{L}_{\mathcal{X}}\theta)(\zeta) - \zeta(\mathcal{L}_{\mathcal{X}}\theta)(W) - (\mathcal{L}_{\mathcal{X}}\theta)([W,\zeta]).$$

The first term is zero since ζ is tangent to N, and we have checked this case earlier. The second term is $-\zeta\|Z\|^2$. Since $\|Z\|^2$ is constant on level set, and ζ is tangent to level set, this term is also zero. The third term is also zero following Proposition 15, since $[W,Z] = Z$ is tangent to N, so this again is equivalent to case that we showed earlier. The third term is zero because $[W,Z] = Z$ by Proposition 16 so it is again tangent to N.

□

5. Future Directions and Open Problems

Once we have defined the rolling flow, there are some natural questions that follow. For example, one can compare the rolling flows with similar properties of no-slip billiards and geodesic flows, and there are many problems that needs to be solved from both the dynamical systems side and ergodic theory side. From a dynamical direction, the very first natural thing to do would be to pursue results that are similar to the results about dynamics of polygonal no-slip billiards in [1]. We expect that many of those results would hold for rolling flows in the polygonal case, although things are move involved for the positive radius case. For example, thereoms 1 and 16 in [1]. The results about curved sides are more challenging, but still good hope we can produce results for the positive radius case. Another question is understanding the breakdown of ergodicity for a sufficiently large moment of inertia parameter η for deformations of hyperbolic geodesic flows, or the effect of the same parameter on Lyapunov exponents and the metric entropy. A first step in taking the subject along this direction is to investigate whether the canonical (Liouville) volume form defined on the system's phase space is invariant under the rolling flow. This is true in dimension 3 as shown in Theorem 25, but the general dimension case is work in progress. To understand this problem we also want to study more examples for some non-polygonal plates, such as the stadium plate in figure 5.1.

At the present stage there are questions we'd like to ask about the rolling systems that are of differential geometry nature or of dynamical systems nature. Recall our definition

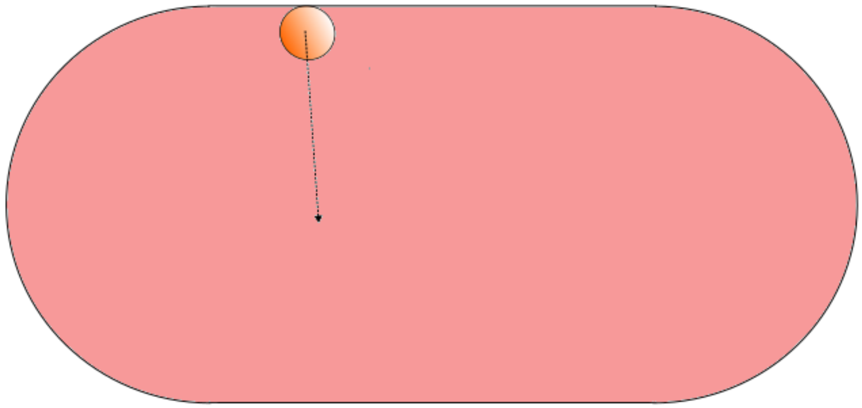

Figure 5.1. The Bunimovich Stadium plate. For more information, see [14] and figure 5 in [8]

.

of the configuration manifold $M = \{(x, A) \in T\mathcal{N} \otimes T^*\mathcal{N} : A \in SO(T_x\mathcal{N})\}$. Note that $\langle Au, Av\rangle_x = \langle u, v\rangle_x \; \forall u, v \in T_x\mathcal{N}$. We have the connection on $(\mathcal{N}, \langle\cdot,\cdot\rangle^\circ)$, ∇° induced by the Levi-Civita connection. The big problem is to investigate the Riemannian geometry of M. We want to have a concrete description of Levi-Civita connection ∇ on $(M, \langle\cdot,\cdot\rangle)$. It is also important to study the curvature properties of M, given $(\mathcal{N}, \langle\cdot,\cdot\rangle^\circ)$, as well as dynamical properties of geodesic flows of M compared with geodesic flows of $\mathcal{N}$.

Going back to the topic of rolling flows, more broadly one can ask about the dynamics of rolling flows along the lines of the theory for geodesic flows and frame flows [15], which is a generalisation of the geodesic flows. A key question to understand in order to develop an ergodic theory for rolling flows is to understand the issue of volume invariance, invariant measures for the flows in particular whether there are nice, smooth measures related to the canonical volume form. This requires understanding better the Riemannian geometry of the Riemannian manifold M that we define the rolling flow for. Manifold M is the manifold of rotations over $\mathcal{N}$, which still needs to be studied in detail.

This leads to the natural question that has been stated as Question 1 in Chapter 4: is the canonical volume form, which is invariant under geodesic flow of M, also invariant under the rolling flow?.

If the answer to Question 1 is negative, then this leads to the question if there are smooth invariant measures of the form $\rho\Omega$, where ρ is the smooth density and the Ω is the canonical volume form. Is the rolling flow time reversible, like the geodesic flow? If it is time reversible, then what does it imply for the question of volume invariance would be the next step to pursue.

In trying to answer the question of invariance of Ω, we realise that it is necessary to investigate the relationship between the Levi-Civita connection ∇ of $(M, \langle\cdot,\cdot\rangle)$ and the connection ∇° on $\mathcal{M}$ comes from Levi-Civita connection ∇° of $(\mathcal{N}, \langle\cdot,\cdot\rangle^\circ)$. Notice that ∇° gives a connection on $TM \cong \pi^*\mathcal{M}$ where ∇ is a connection on TM. They are both metric connections; $\nabla_X Y - \nabla^\circ_X Y = \tau(X,Y) \in TM$. If $\tau = 0$, then we have a proof of invariance of Ω under rolling flow.

However if $\tau \neq 0$, we need to have a better understanding of the Lie derivative with respect to V, which is the generator of the rolling flow, $\mathscr{L}_V\omega$, where ω is the symplectic form on TM, $\Omega = \omega \wedge ... \wedge \omega$. If $\mathscr{L}_V\Omega \neq 0$, then we need to find $h : TM \to \mathbb{R}$ such that $\mathscr{L}_V\Omega = h\Omega$. In other words is there $\rho > 0$ such that $\mathscr{L}_V(\rho\Omega) = 0$?

These questions opens up many different cases, and will help us have a more systematic dynamic and ergodic theory for the nonholonomic systems.

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