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EXPONENTIAL GROWTH OF SOLUTIONS FOR A VARIABLE-EXPONENT FOURTH-ORDER VISCOELASTIC EQUATION WITH NONLINEAR BOUNDARY FEEDBACK

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Abstract. In this paper we study a variable-exponent fourth-order viscoelastic equation of the form

$$|u_t|^{\rho(x)}u_{tt} + \Delta[(a + b|\Delta u|^{m(x)-2})\Delta u] - \int_0^t g(t-s)\Delta^2 u(s)ds = |u|^{p(x)-2}u,$$

in a bounded domain of R^n. Under suitable conditions on variable exponents and initial data, we prove that the solutions will grow up as an exponential function with positive initial energy level. Our result improves and extends many earlier results in the literature such as the one by Mahdi and Hakem (Ser. Math. Inform. 2020, https://doi.org/10.22190/FUMI2003647M).
Key words: Variable exponents, viscoelastic equation, weak solution.

1. Introduction

In this paper, we study the following variable-exponent fourth-order viscoelastic initial boundary value problem
(1.1)
$$|u_t|^{\rho(x)}u_{tt}+\Delta[(a+b|\Delta u|^{m(x)-2})\Delta u]-\int_0^t g(t-s)\Delta^2 u(s)ds = |u|^{p(x)-2}u,\ x \in \Omega, t > 0,$$

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$$(1.2)\qquad \begin{cases} u(x,t)=0, & x\in\Gamma_0, t>0 \\ a\Delta u(x,t)=\int_0^t g(t-s)\Delta u(s)ds-b|\Delta u|^{m(x)-2}\Delta u, & x\in\Gamma_1, t>0 \end{cases}$$

$$(1.3)\qquad u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\ x\in\Omega,$$

where $\Omega\subset R^n (n\geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega=\Gamma_0\cup\Gamma_1$ such that Γ_0 and Γ_1 are closed and disjointed. Here a, b are positive constants. Also $g(.), \rho(.)$, $m(.)$ and $p(.)$ are given real-valued functions that satisfy some conditions to be specified later.

Before going any further, it is worth pointing out some previous results. In bounded domains, there is an extensive literature on the existence, asymptotic behavior and nonexistence of solution for the plate equations. In the following part, the results are reviewed in two parts.

1.1. Case of constant exponents

It is known that the thin plate model leads to a differential equation of fourth-order. These kinds of equations have been extensively discussed by many authors. For example, Messaoudi [16] studied the following fourth-order equation

$$(1.4)\qquad u_{tt}+\Delta^2 u+a|u_t|^{m-2}u_t=b|u|^{p-2}u,$$

where $a,\ b>0$. He proved the existence of a local weak solution and showed that if $p>m$, this solution blows up in finite time with negative initial energy. Also, he proved that if $m\geq p$, the solution is global. Wu and Tsai [31] considered (1.4) and proved that the solution is global in time under some conditions without the relation between m and p. Moreover, similar to [16], they proved that if $p>m$, the local solution blows up in finite time. (see also [9, 27])

Shahrouzi [22] investigated the following fourth-order initial boundary value problem

$$u_{tt}+\Delta[(a_0+a|\Delta u|^{m-2})\Delta u]-b\Delta u_t=g(x,t,u,\Delta u)+|u|^{p-2}u,\quad x\in\Omega,\ t>0,$$

$$u(x,t)=0,\quad \Delta u(x,t)=c_0\partial_\nu u(x,t),\quad x\in\partial\Omega,\ t>0,$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\ x\in\Omega.$$

Under sufficient conditions on c_0 and b, when $p>m+1>3$, blow up of solutions was proved.

It is known that viscoelastic materials show natural damping properties, which is due to the special property of these substances in keeping memory of their past history. From mathematical point of view, these damping effects are modeled by integro-differential operators and for this reason, it is of special importance to study the viscoelastic initial boundary value problems. For more details about the viscoelastic problems and their applications, we refer to [1, 21].

In this regards, Rivera et al. [18] considered the plate model:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0, \tag{1.5}$$

and showed that if $g(.)$ is exponentially decayed, the energy of solution is exponentially decayed too. Tahamtani and Shahrouzi [30] studied the effect of a source term $|u|^p u$ on the plate model (1.5). They proved the existence of weak solutions by using the Faedo-Galerkin method. Also, blow-up of solutions was established with positive initial energy as well as non-positive initial energy.

Li and Gao [14] considered the following nonlinear Petrovsky type equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u,$$

and obtained the blow-up result with upper bounded initial energy. Furthermore, the blow up of solutions was proved for the linear damping case with non-positive initial energy.

Shahrouzi [23] studied the solution behavior of the nonlinear fourth-order equation of the form

$$u_{tt} + \Delta[(a + b|\Delta u|^{m-2})\Delta u] - \int_0^t g(t-s)\Delta^2 u(s)ds + cu_t = h(x, t, u, \Delta u) + |u|^{p-2}u,$$

and proved that if the initial data and parameters are taken in the appropriate domain, then solutions uniformly decay to zero with the same arbitrary rate as the memory kernel. Moreover, the blow up of solutions was proved under the conditions of positive initial energy and suitable domain of parameters.

In another study, Mustafa and Abusharkh [19] considered the following viscoelastic plate equation

$$u_{tt} + \Delta^2 u - \int_0^t \Delta[a(x)g(t-s)\Delta u(s)]ds + \theta(t)b(x)h(u_t) = 0,$$

in a bounded domain of R^n, where $a(.)$ and $b(.)$ are real functions such that $a(x) + b(x) \geq \beta > 0$ for all $x \in \Omega$. In addition, supposed that θ is a time-dependent coefficient of the frictional damping term. They obtained a general relation between the decay rate for the energy and the functions g, θ and h without imposing any growth assumption near the origin on h and strongly weakening the usual assumptions on g.

Al-Gharabli [2] investigated the stability of a viscoelastic plate equation of the form

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)ds = kuln|u|,$$

and proved explicit and general energy decay results when the relaxation function g satisfied $g'(t) \leq -\xi(t)G(g(t))$.

Yang [34] investigated the blow-up and lower bound of lifespan of solutions for the following equation
(1.6)
$$|u_t|^\rho u_{tt} - (1+b\|\nabla u\|_2^{2\gamma})\Delta u + \int_0^t g(t-s)\Delta u(s)ds + u_t|u_t|^{m-2} - \Delta u_t = |u|^{p-2}u.$$

By constructing a suitable auxiliary function to overcome the difficulty of gradient estimation and making use of differential inequality technique, he proved a finite time blow-up result when the initial data is at arbitrary energy level. Moreover, he derived a lower bound of the lifespan by constructing a control function with both nonlocal term and memory kernel. Song [28] considered equation (1.6) without strongly damping term and when ($b = 0$), he proved the nonexistence of solutions with positive initial energy and nonincreasing relaxation function. For more results related to the plate equations, we refer the reader to [3, 29].

1.2. Case of variable exponents

In recent years, much attention has been paid to the study of mathematical models of hyperbolic, parabolic and elliptic equations with variable exponents of nonlinearity. Equations with nonstandard growth conditions and variable-exponent nonlinearities occur in the mathematical modeling of various physical phenomena, e.g, the flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and image processing. For more details, see [7, 8] and references therein.

Messaoudi et al. [17] studied

$$u_{tt} - \Delta u + a|u_t|^{m(.)-2}u_t = |u|^{p(.)-2}u, \tag{1.7}$$

and proved the existence and blow-up of solutions under some conditions on variable exponents. Furthermore, Park and Kang [20] considered (1.7) in the presence of viscoelastic term and proved the blow-up result for the solutions with positive as well as non-positive initial energy when g is a nonincreasing positive function such that
$$\int_0^\infty g(s)ds < \frac{p_1(p_1-2)}{(p_1-1)^2}, \qquad 2 < p_1 \le p(x) \le p_2.$$

Shahrouzi [26] studied the behavior of solutions to the following initial-boundary value problem with variable-exponent nonlinearities

$$u_{tt} - \Delta u - div(|\nabla u|^{m(x)}\nabla u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(x,t,u,\nabla u) + \beta u_t$$
$$= |u|^{p(x)}u, \quad in \ \ \Omega \times (0,+\infty)$$
$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0 \\ \frac{\partial u}{\partial n}(x,t) = \int_0^t g(t-\tau)\frac{\partial u}{\partial n}(\tau)d\tau - |\nabla u|^{m(x)}\frac{\partial u}{\partial n} + \alpha u, & x \in \Gamma_1, t > 0 \end{cases}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad in \ \ \Omega.$$

Under appropriate conditions, he proved a general decay result associated with solution energy. Moreover, regarding arbitrary positive initial energy, blow up of solutions was proved. Recently, Shahrouzi in [25] investigated the blow-up result for a Lamé system of viscoelastic equation with variable-exponent nonlinearities and strong damping term.

In another article, Mahdi and Hakem [15] established the weak existence theorem and found suitable assumptions on the initial data, $m(.)$ and $p(.)$ in which the solutions of the following equation blow up in a finite time:

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u.$$

In the plate model with variable exponent nonlinearities, Ferreira and Messaoudi [13], considered a nonlinear viscoelastic plate equation with a lower order perturbation of a $\overrightarrow{p}(x,t)-$Laplacian operator of the form

$$u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)} u + \int_0^t g(t-s)\Delta u(s)ds - \varepsilon \Delta u_t + f(u) = 0, \tag{1.8}$$

and proved a general decay result under suitable conditions on $g,\ f$ and the variable exponent of the $\overrightarrow{p}(x,t)-$ Laplacian operator. Next, Antontsev and Ferreira [4] studied (1.8) and proved the blow up of solutions under suitable conditions on $g,\ f$ and variable exponent with negative initial energy.

Recently, Antontsev et. al [5] considered the following nonlinear plate (or beam) Petrovsky equation

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u.$$

By using the Banach contraction mapping principle they obtained local weak solutions, under suitable assumptions on the variable exponents $p(.)$ and $q(.)$. Then, they proved that the solution is global if $p(.) \geq q(.)$ and blows up in a finite time if $p(.) < q(.)$. In this regards, also see [6, 24].

Motivated by the aforementioned works, we try to prove a exponential growth result for the problem (1.1)-(1.3). Subsequently, in Section 2, we recall some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$, the Sobolev space, $W^{1,p(\cdot)}(\Omega)$, which will be used for the main result. In Section 3, we prove the instability of solutions in infinity for appropriate initial data and suitable range of $\rho(.),\ m(.)$ and $p(.)$. The conclusions of the paper are presented in fourth section.

2. Preliminaries

In order to study problem (1.1)-(1.3), we need some additional conditions and theories about Lebesgue and Sobolev spaces with variable-exponents (for detailed, see

[10, 11, 12, 32, 33]). At first, we consider the following hypotheses for exponents and relaxation function g:
$(A1)$ $\rho(.)$, $m(.)$ and $p(.)$ are given continuous and measurable functions on $\overline{\Omega}$ such that

$$1 < \rho_1 \leq \rho(x) \leq \rho_2$$

$$2 < m_1 \leq m(x) \leq m_2$$

$$2 < p_1 \leq p(x) \leq p_2$$

with

$$\rho_1 := essinf_{x\in\overline{\Omega}}\rho(x),\ \rho_2 := esssup_{x\in\overline{\Omega}}\rho(x),$$

$$m_1 := essinf_{x\in\overline{\Omega}}m(x),\ m_2 := esssup_{x\in\overline{\Omega}}m(x),$$

$$p_1 := essinf_{x\in\overline{\Omega}}p(x),\ \ p_2 := esssup_{x\in\overline{\Omega}}p(x).$$

$(A2)$ $g : R^+ \to R^+$ is a C^1-nonincreasing function satisfying

$$g(0) > 0, \qquad a - \int_0^\infty g(s)ds > 0.$$

Let $p(x) \geq 1$ and measurable, we define

$$C_+(\overline{\Omega}) = \{h|h \in C(\overline{\Omega}),\ h(x) > 1\ for\ any\ x \in \overline{\Omega}\},$$

$$L^{p(x)}(\Omega) = \Big\{u|\ \text{u is a measurable real-valued function,}\ \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(x)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0\Big|\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\Big\}$$

Lemma 2.1. [10, 32] *Let Ω be a bounded domain in R^n*

(i) the space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$|\int_\Omega uvdx| \leq (\frac{1}{p_1} + \frac{1}{q_1})\|u\|_{p(x)}\|v\|_{q(x)}.$$

(ii) If $p, q \in C_+(\overline{\Omega})$, $q(x) \leq p(x)$ for any $x \in \overline{\Omega}$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous.

The variable-exponent Lebesgue Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega)|\nabla u\ exists\ and\ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p'(x)}(\Omega)$, by the same way as the usual Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
If we define

$$p^*(x) = \begin{cases} \frac{Np(x)}{esssup_{x\in\overline{\Omega}}(N-p(x))}, & p^+ < N \\ \infty, & p^+ \geq N, \end{cases}$$

then we have

Lemma 2.2. [10, 32] *Let Ω be a bounded domain in R^n then for any measurable bounded exponent $p(x)$ we have*

(i) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable Banach spaces;

(ii) if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;

(iii) if $p(x)$ is uniformly continuous in Ω then there exists a constant $C_* > 0$, such that

$$\|u\|_{p(x)} \leq C_*\|\Delta u\|_{p(x)} \qquad \forall u \in W_0^{2,p(x)}(\Omega).$$

We know that the space $W_0^{1,p(x)}(\Omega)$ has an equivalent norm given by
$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$. Also, we recall the trace Sobolev embedding in Lebesgue space with a constant exponent

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1) \qquad for \qquad 2 \leq q < \frac{2(n-1)}{n-2}$$

where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

and the embedding inequality

$$\|u\|_{q,\Gamma_1} \leq B_q\|\nabla u\|_2,$$

where B_q is the optimal constant.
We recall the Young's inequality

$$XY \leq \theta X^{q(x)} + C(\theta, q(x))Y^{q'(x)}, \quad X, Y \geq 0, \quad \theta > 0, \quad \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \tag{2.1}$$

where $C(\theta, q(x)) = \frac{1}{q'(x)}(\theta q(x))^{-\frac{q'(x)}{q(x)}}$. In special case when $\theta = \frac{1}{q(x)}$, we have from (2.1)

$$XY \leq \frac{X^{q(x)}}{q(x)} + \frac{Y^{q'(x)}}{q'(x)}. \tag{2.2}$$

For completeness we state, without proof, the existence result for the problem (1.1)-(1.3):(see [13, 17])

Theorem 2.1. *(Local existence) Let* $(u_0, u_1) \in W_0^{2,m(.)}(\Omega) \times L^2(\Omega)$ *be given. Assume that* (A1) *and* (A2) *are satisfied; then the problem* (1.1)-(1.3) *has at least one weak solution such that*

$$u \in L^\infty\big((0,T), W_0^{2,m(.)}(\Omega)\big), \qquad u_t \in L^\infty\big((0,T), L^2(\Omega)\big) \cap L^{\rho(.)}(\Omega),$$

$$u_{tt} \in L^\infty\big((0,T), L^2(\Omega) \cap W_0^{-1,m'(.)}(\Omega)\big),$$

where $\frac{1}{m(x)} + \frac{1}{m'(x)} = 1$.

3. Exponential growth result

In this section we are going to prove instability of solutions in infinity for the problem (1.1)-(1.3). To prove the growth-up result for certain solutions with arbitrary positive initial energy, we assumed that:

(A3)

$$2 < m_1 \le m(x) \le m_2 \le \rho_2 + 2 \le p_1 \le p(x) \le p_2 < \infty,$$

$$\int_0^\infty g(s)ds \le \frac{a\rho_2(\rho_2+2)}{2(\rho_2^2 + 2\rho_2 + a)}.$$

The energy associated with problem (1.1)-(1.3) is given by

$$\begin{aligned} E(t) = &\int_\Omega \frac{1}{\rho(x)+2}|u_t|^{\rho(x)+2}dx + \frac{1}{2}(a - \int_0^t g(s)ds)\|\Delta u\|^2 + \frac{1}{2}(g * \Delta u)(t) \\ &+ b\int_\Omega \frac{1}{m(x)}|\Delta u|^{m(x)}dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx, \end{aligned} \tag{3.1}$$

where

$$(g * u)(t) = \int_0^t g(t-s)\int_\Omega |u(s) - u|^2 dxds.$$

Lemma 3.1. *(Monotonicity of energy) Let* u *be a local solution of* (1.1)-(1.3). *Then the energy functional along the solution satisfies*

$$E'(t) = \frac{1}{2}(g' * \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2 \le 0. \tag{3.2}$$

Proof. By multiplying equation (1.3) by u_t and integrating over Ω, using integration by parts, boundary conditions and hypotheses (A1) and (A2), we obtain (3.2) for any regular solution. This equality remains valid for weak solutions by a simple density argument. $\square$

Our main result in this section reads in the following theorem:

Theorem 3.1. *Suppose that the assumptions* (A1)-(A3) *hold. Moreover,* $E(0) > 0$ *is a given initial energy level. If we choose parameter* $a > 0$ *large enough and initial data* u_0, u_1 *satisfying*

$$\frac{1}{\rho_1 + 1}\int_\Omega u_0 u_1 |u_1|^{\rho_2} dx > \frac{\rho_2 + 2}{\rho_1 + 1} E(0), \tag{3.3}$$

then the solutions of (1.1)-(1.3) *grow up as an exponential function when time goes to infinity.*

To prove the above theorem, we assume that u is a global solution of the our problem. Let define

$$\psi(t) = \int_\Omega \frac{1}{\rho(x) + 1} u u_t |u_t|^{\rho(x)} dx. \tag{3.4}$$

Lemma 3.2. *Under the assumptions of Theorem* 3.1, $\psi(t)$ *satisfies, along the solution, the estimate*

$$|\psi(t)| \leq \frac{a\rho_2}{2\rho_1 + 2}\|\Delta u\|^2 + \frac{\rho_2 + 1}{(\rho_1 + 1)(\rho_2 + 2)}\int_\Omega |u_t|^{\rho(x)+2} dx. \tag{3.5}$$

Proof. By using (A1) and Young's inequality (2.2) with $X := |u|, Y := |u_t|^{\rho(x)+1}$ and $q(x) := \rho(x) + 2, q'(x) := \frac{\rho(x)+2}{\rho(x)+1}$, we get

$$\begin{aligned} |\psi(t)| \leq & \frac{1}{\rho_1 + 1}\left|\int_\Omega u|u_t|^{\rho(x)+1} dx\right| \\ \leq & \frac{1}{\rho_1 + 1}\int_\Omega \frac{1}{\rho(x) + 2}|u|^{\rho(x)+2} dx + \frac{1}{\rho_1 + 1}\int_\Omega \frac{\rho(x) + 1}{\rho(x) + 2}|u_t|^{\rho(x)+2} dx. \end{aligned} \tag{3.6}$$

On the other hand, let C_* be the best constant of embedding $H_0^2(\Omega) \hookrightarrow L^{\rho(\cdot)+2}(\Omega)$ (c.f. part (iii) of Lemma 2.2), then for sufficiently large a, we have

$$\begin{aligned} \int_\Omega |u|^{\rho(x)+2} dx \leq & \max\Big\{\Big(\int_\Omega |u|^{\rho(x)+2} dx\Big)^{\frac{\rho_1+2}{\rho(x)+2}}, \Big(\int_\Omega |u|^{\rho(x)+2} dx\Big)^{\frac{\rho_2+2}{\rho(x)+2}}\Big\} \\ \leq & \max\{C_*^{\rho_1+2}\|\Delta u\|^{\rho_1+2}, C_*^{\rho_2+2}\|\Delta u\|^{\rho_2+2}\} \\ \leq & \max\{C_*^{\rho_1+2}\|\Delta u\|^{\rho_1}, C_*^{\rho_2+2}\|\Delta u\|^{\rho_2}\}\|\Delta u\|^2 \\ \leq & \frac{a}{2}\rho_2(\rho_1 + 2)\|\Delta u\|^2. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we deduce

$$|\psi(t)| \leq \frac{a\rho_2}{2\rho_1 + 2}\|\Delta u\|^2 + \frac{\rho_2 + 1}{(\rho_1 + 1)(\rho_2 + 2)}\int_\Omega |u_t|^{\rho(x)+2} dx,$$

and proof of Lemma 3.2 is completed. $\square$

Lemma 3.3. *Under the assumptions of Theorem* 3.1, $\psi(t)$ *and* $E(t)$ *satisfy the following differential inequality*

$$\psi'(t) + (\rho_2 + 2)E(t) \geq (\rho_1 + 1)\psi(t) + \frac{\rho_2 + 2}{\rho_1 + 1}E'(t). \tag{3.8}$$

Proof. Multiplying equation (1.1) by u and integrating over Ω, we get

$$\int_\Omega |u|^{p(x)}dx = \int_\Omega u|u_t|^{\rho(x)}u_{tt}dx + \int_\Omega u\Delta[(a + b|\Delta u|^{m(x)-2})\Delta u]dx \\ - \int_0^t g(t-s)\int_\Omega u\Delta^2 u(s)dxds.$$

Then, by using the boundary conditions, we easily obtain

$$\begin{aligned} \psi'(t) &= \int_\Omega \frac{1}{\rho(x)+1}|u_t|^{\rho(x)+2}dx + \int_\Omega u|u_t|^{\rho(x)}u_{tt}dx \\ &= \int_\Omega \frac{1}{\rho(x)+1}|u_t|^{\rho(x)+2}dx - a\|\Delta u\|^2 - b\int_\Omega |\Delta u|^{m(x)}dx \\ &\quad + \int_\Omega |u|^{p(x)}dx + \int_0^t g(t-s)\int_\Omega \Delta u\Delta u(s)dxds. \end{aligned} \tag{3.9}$$

For the last term on the right side of (3.9), we have

$$\begin{aligned} \int_0^t g(t-s)\int_\Omega \Delta u\Delta u(s)dxds &= \int_0^t g(t-s)\int_\Omega \Delta u(\Delta u(s) - \Delta u)dxds \\ &\quad + (\int_0^t g(s)ds)\|\Delta u\|^2, \end{aligned} \tag{3.10}$$

and by virtue of the Young's inequality (2.1) with $\theta = \frac{\rho_2+2}{2}$, $q(x) = q'(x) = 2$, we obtain

$$\begin{aligned} &|\int_\Omega \Delta u\int_0^t g(t-s)(\Delta u(s) - \Delta u)dsdx| \\ \leq&\frac{\rho_2+2}{2a}\int_\Omega \Big(\int_0^t g(t-s)|\Delta u(s) - \Delta u|ds\Big)^2 dx + \frac{a}{2(\rho_2+2)}\int_0^t g(s)ds\|\Delta u\|^2 \\ =&\frac{\rho_2+2}{2a}\int_\Omega \Big(\int_0^t \frac{g(t-s)}{\sqrt{g(t-s)}}\sqrt{g(t-s)}|\Delta u(s) - \Delta u|ds\Big)^2 dx \\ &+ \frac{a}{2(\rho_2+2)}\int_0^t g(s)ds\|\Delta u\|^2 \\ \leq&\frac{\rho_2+2}{2a}\Big(\int_0^t g(s)ds\Big)\int_\Omega\int_0^t g(t-s)|\Delta u(s) - \Delta u|^2 dsd \\ &+ \frac{a}{2(\rho_2+2)}\int_0^t g(s)ds\|\Delta u\|^2 \\ \leq&\frac{\rho_2+2}{2}(g*\Delta u)(t) + \frac{a}{2(\rho_2+2)}\int_0^t g(s)ds\|\Delta u\|^2, \end{aligned} \tag{3.11}$$

where condition (A2) and as a result $\int_0^t g(s)ds < \int_0^\infty g(s)ds < a$ has been used. Utilizing (3.11) into (3.10), we arrive at

$$\begin{aligned}\int_0^t g(t-s)\int_\Omega \Delta u\Delta u(s)dxds \geq & -\frac{\rho_2+2}{2}(g*\Delta u)(t)\\ & +(1-\frac{a}{2(\rho_2+2)})\int_0^t g(s)ds\|\Delta u\|^2.\end{aligned} \tag{3.12}$$

Therefore we get

$$\begin{aligned}\psi'(t) \geq & \int_\Omega \frac{1}{\rho(x)+1}|u_t|^{\rho(x)+2}dx - a\|\Delta u\|^2 - b\int_\Omega |\Delta u|^{m(x)}dx\\ & + \int_\Omega |u|^{p(x)}dx - \frac{\rho_2+2}{2}(g*\Delta u)(t)\\ & + (1-\frac{a}{2(\rho_2+2)})\int_0^t g(s)ds\|\Delta u\|^2.\end{aligned} \tag{3.13}$$

Taking into account (3.1), we deduce

$$\begin{aligned}\psi'(t) \geq & -(\rho_2+2)E(t) + (\rho_2+2)\int_\Omega \frac{1}{\rho(x)+2}|u_t|^{\rho(x)+2}dx\\ & + \int_\Omega \frac{1}{\rho(x)+1}|u_t|^{\rho(x)+2}dx + \frac{1}{2}[\rho_2 a - (\rho_2+\frac{a}{\rho_2+2})\int_0^t g(s)ds]\|\Delta u\|^2\\ & + b(\rho_2+2)\int_\Omega \frac{1}{m(x)}|\Delta u|^{m(x)}dx - b\int_\Omega |\Delta u|^{m(x)}dx\\ & + \int_\Omega |u|^{p(x)}dx - (\rho_2+2)\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx,\end{aligned}$$

by the assumption (A1), last inequality becomes

$$\begin{aligned}\psi'(t) \geq & -(\rho_2+2)E(t) + (1+\frac{1}{\rho_2+1})\int_\Omega |u_t|^{\rho(x)+2}dx\\ & + (1-\frac{\rho_2+2}{p_1})\int_\Omega |u|^{p(x)}dx + \frac{1}{2}[\rho_2 a - (\rho_2+\frac{a}{\rho_2+2})\int_0^t g(s)ds]\|\Delta u\|^2\\ & + b(\frac{\rho_2+2}{m_2}-1)\int_\Omega |\Delta u|^{m(x)}dx,\end{aligned}$$

and by using (A3), we obtain

$$\begin{aligned}\psi'(t) \geq & -(\rho_2+2)E(t) + (1+\frac{1}{\rho_2+1})\int_\Omega |u_t|^{\rho(x)+2}dx\\ & + \frac{1}{2}[\rho_2 a - (\rho_2+\frac{a}{\rho_2+2})\int_0^t g(s)ds]\|\Delta u\|^2.\end{aligned} \tag{3.14}$$

Utilizing (3.4) into (3.14), it is easy to see that

$$\begin{aligned}\psi'(t) \geq & (\rho_1+1)(\psi(t)-\frac{\rho_2+2}{\rho_1+1}E(t))+(1+\frac{1}{\rho_2+1})\int_\Omega |u_t|^{\rho(x)+2}dx \\ & +\frac{1}{2}[\rho_2 a-(\rho_2+\frac{a}{\rho_2+2})\int_0^t g(s)ds]\|\Delta u\|^2-(\rho_1+1)\psi(t).\end{aligned} \tag{3.15}$$

Thanks to the conclusion of Lemma 3.2, we get from (3.15)

$$\begin{aligned}\psi'(t) \geq & (\rho_1+1)(\psi(t)-\frac{\rho_2+2}{\rho_1+1}E(t))+(1+\frac{1}{\rho_2+1}-\frac{\rho_2+1}{\rho_2+2})\int_\Omega |u_t|^{\rho(x)+2}dx \\ & +\frac{1}{2}[\frac{\rho_2 a}{2}-(\rho_2+\frac{a}{\rho_2+2})\int_0^t g(s)ds]\|\Delta u\|^2.\end{aligned}$$

Now, by using (A3) and (3.2), we have

$$\begin{aligned}\psi'(t) \geq & (\rho_1+1)(\psi(t)-\frac{\rho_2+2}{\rho_1+1}E(t)) \\ \geq & (\rho_1+1)(\psi(t)-\frac{\rho_2+2}{\rho_1+1}E(t))+\frac{\rho_2+2}{\rho_1+1}E'(t).\end{aligned}$$

and proof of Lemma 3.3 is completed. □

Proof of Theorem 3.1. Let define

$$H(t)=\psi(t)-\frac{\rho_2+2}{\rho_1+1}E(t),$$

by using the inequality (3.8), we get

$$H'(t) \geq (\rho_1+1)H(t). \tag{3.16}$$

Then, by (3.3), it holds that $H(0)=\psi(0)-\frac{\rho_2+2}{\rho_1+1}E(0)>0$. Therefore, by (3.16), we conclude that

$$H(t) \geq e^{(\rho_1+1)t}H(0), \quad \forall t \geq 0.$$

This completes the proof of Theorem 3.1.

4. Conclusion

In recent years, due to the very wide applications of partial differential equations with non-standard growth conditions, there many works concerning the wave equations with variable-exponent nonlinearities have been published. However, to the best of our knowledge, there were no exponential growth results for the viscoelastic plate equation involving variable-exponent nonlinearities and boundary feedback. In this work, we used the modified energy method to establish the exponential growth of solutions for the boundary value problem (1.1)-(1.3). Indeed, we proved that for the appropriate range of the variable exponents $\rho(.)$, $m(.)$ and $p(.)$ the global solutions are exponential growth (without damping term), while in the previous studies (with damping term) the authors proved that there exists a finite time such that the solutions blow up (see [4, 15, 17, 20, 24]).

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