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BASIC APPLICATIONS OF THE q-DERIVATIVE FOR A GENERAL SUBFAMILY OF ANALYTIC FUNCTIONS SUBORDINATE TO k-JACOBSTHAL NUMBERS

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Abstract. This research paper deals with some radius problems, the basic geometric properties, general coefficient and inclusion relations that are established for functions in a general subfamily of analytic functions subordinate to k-Jacobsthal numbers.
Keywords: Fractional q-calculus operators, Starlike functions, subordination · k-Jacobsthal numbers, Generating functions

1. Background

The summation of analytic functions f in the open unit disc $\mathcal{D} = \{z : z \in \mathcal{C}, |z| < 1\}$ given with the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

has been shown by $\mathcal{A}$. Also, $\mathcal{S}$ is the subclass of $\mathcal{A}$ comprising of functions be univalent in $\mathcal{D}$.

The p functions which are analytic in $\mathcal{D}$ fulfilling $\mathrm{Re}(p(z)) > 0$ and $p(0) = 1$ construct the Carathéodory class which is represented by $\mathcal{P}^*$. Indeed, $p \in \mathcal{P}^*$ is represented as

$$p\left(z\right) = 1 + x_1 z + x_2 z^2 + x_3 z^3 + \cdots . \tag{1.2}$$

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with coefficients which satisfy $|x_n| \leq 2$ for $n \in \mathcal{N}$ [20].

If f_1 and f_2 are analytic in $\mathcal{D}$, then we call that f_1 is subordinate to f_2, showed by $f_1 \prec f_2$, for the Schwarz function

$$\varpi(z) = \sum_{n=1}^{\infty} \mathbf{c}_n z^n \quad \left(\varpi\left(0\right) = 0, \left|\varpi\left(z\right)\right| < 1\right), \tag{1.3}$$

analytic in $\mathcal{D}$ such that

$$f_1\left(z\right) = f_2\left(\varpi\left(z\right)\right) \qquad \left(z \in \mathcal{D}\right). \tag{1.4}$$

It is known that $|\mathbf{c}_n| < 1$ (see [5]) for $\varpi\left(z\right)$.

Among the first important papers which discuss topics from this area are Koebe [15], Alexander [1] and Bieberbach [2]. Koebe initiated in 1907 the study about univalent functions, while Bieberbach presented in 1916 would soon become a famous conjecture. Bieberbach [2] assumed that for $f \in \mathcal{S}$,

$$|a_n| \leq n \quad (n \geq 2). \tag{1.5}$$

He proved only for the case when $n = 2$. For many years this estimate has remained as a open problem for the mathematicians and has inspired the development of several remarkable techniques in the field. In 1985, Branges [3] demonstrated the Bieberbach's conjecture for all the coefficients n. After the proof of the conjecture, the searches of different subclasses of analytic and univalent functions have began to take shape, still remaining an interesting subject.

The q-calculus present various tools which have been studied in the fields of special functions and many other areas. While Jackson was one of the leading authors among mathematicians working on the q-calculus theory [9, 10], the use of q-calculus within the context of Geometric Function Theory was partially provided by Srivastava [29]. Afterwards, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were utilized for defining many subfamilies of univalent functions and their geometric properties in a compact disc were given [19, 21, 22]. In the most recent study by Srivastava [30], the concept of q-calculus were summarized again, a detailed literature was provided and finally, a new approach was given.

Now, we provide several elementary notations of the q-calculus due to [29] and [6]. In this paper, we assumed that $q \in (0,1)$ and the definitions of fractional q-calculus operators deal with the complex-valued function f.

Definition 1.1. [6, 29] The q-number $\left[\Upsilon\right]_q$ is expressed by

$$\left[\Upsilon\right]_q = \begin{cases} \frac{1-q^{\Upsilon}}{1-q}, & \Upsilon \in \mathcal{C} \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1}, & \Upsilon = n \in \mathcal{N} \end{cases}. \tag{1.6}$$

Definition 1.2. [6, 9, 29] The q-derivative of f, represented with $D_q f$, is defined in a given subset of $\mathcal{C}$ by

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \tag{1.7}$$

as long as $f'(0)$ exists.

We would like to point out that

$$\lim_{q\to 1^-} D_q f(z) = \lim_{q\to 1^-} \frac{f(z)-f(qz)}{(1-q)z} = f'(z). \tag{1.8}$$

Thus, with the help of (1.1) and (1.7), it can be deduced that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{1.9}$$

Now, we recall the Sălăgean q-differential operator D_q^λ by [12]

$$\begin{aligned} D_q^0 f(z) &= f(z), \\ D_q^1 f(z) &= z D_q f(z), \\ &\vdots \\ D_q^\lambda f(z) &= z D_q \left(D_q^{\lambda-1} f(z) \right). \end{aligned}$$

We note that

$$D_q^\lambda f(z) = z + \sum_{n=2}^{\infty} [n]_q^\lambda \, a_n z^n. \tag{1.10}$$

If we set $q \to 1^-$, we have the well known Sălăgean differential operator D^λ (see [24]).

The relation of subordination is used to establish many classes of functions encountered in the context of Geometric Functions Theory and to investigate some properties of these classes. Let us recall

$$S^*[\psi] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z), \ z \in \mathcal{D} \right\}, \tag{1.11}$$

where ψ is an analytic function in $\mathcal{D}$ with $\psi(0) = 1$. For $\psi(z) = \frac{1+z}{1-z}$, one can get the notable class $\mathcal{S}^*$ of starlike. Many researchers defined different classes of functions by using other functions instead of the ψ in (1.11). When the figure of the unit circle under these functions is calculated, very interesting results are obtained. In [23], Robertson showed that the figure of the unit circle is $\mathrm{Re}(w) > \gamma$

using $\psi(z) = \left(1 + (1-2\gamma)z\right)/(1-z)$. For this case, the set (1.11) becomes the class $\mathcal{S}^*(\gamma)$ of starlike functions of order γ. Janowski obtained that $\psi(\mathcal{D})$ is a disc in [11]. By taking $\psi(z) = (\frac{1+z}{1-z})^\beta$ $(0 < \beta \leq 1)$, the class of strongly starlike functions having order β is presented by Stankiewicz in [28] and demonstrated that the figure of unit circle under this class is an angle. With different choices of ψ, it is obtained that $\psi(\mathcal{D})$ is parabola in [7, 16], is ellipse and hyperbola in [13, 14]. If $\psi(z) = \sqrt{1+z}$ is chosen in order that $\sqrt{1} = 1$, then the right loop of the Lemniscate is obtained as the figure of $\psi(\mathcal{D})$. Obtained class is considered in [26, 27]. The $\psi(z)$ function discussed in the aforementioned studies is a convex univalent function. In [17], the authors gave some general results assuming ψ is univalent, $\psi(\mathcal{D})$ is symmetric about the real axis and starlike w.r.t. $\psi(0) = 1$ for (1.11). In cases where ψ is not univalent, the investigations are made for the class defined with (1.11) become quite difficult. Further, Masih and Kanas [18] introduced and studied the classes $S_L[\psi]$ and $K_L[\psi]$ $\left(\psi(z) = (1+\chi z)^2,\ 0 < \chi \leq 1/\sqrt{2}\right)$ related to a limaçon. Sharma et al. [25] studied the class $S_C[\psi]$ $\left(\psi(z) = 1 + 4z/3 + z^2/3\right)$ related to a cardioid. The classes $S_N[\psi]$ and $K_N[\psi]$ $\left(\psi(z) = 1 + z - z^3/3\right)$ associated with a nephroid domain were considered by Wani and Swaminathan [33].

In addition to these papers, the study of special functions, whose coefficients are composed of positive integer sequences such as Fibonacci, Pell, Lucas, Jacobsthal and their various generalizations, starts to gain attention in various fields of mathematics such as number theory, geometry, trigonometry, graph theory, linear algebra and combinatorics, and to link between these fields and analysis [28, 31, 32, 34].

Horadam [8] introduced the sequence of Jacobsthal numbers J_n by means of the recurrence relation

$$J_{n+1} = J_n + 2J_{n-1} \quad (n \geq 1, J_0 = 0, J_1 = 1).$$

A generalization of Jacobsthal numbers named k-Jacobsthal numbers is introduced in [4]. The sequence of k-Jacobsthal numbers, $J_{k,n}$, is represented recurrently by

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1} \quad (n \geq 1, J_{k,0} = 0, J_{k,1} = 1).$$

The n-th k-Jacobsthal number is given by

$$J_{k,n} = \frac{(k-\delta_k)^n - {\delta_k}^n}{\sqrt{k^2+8}}, \tag{1.12}$$

where $\delta_k = \frac{k-\sqrt{k^2+8}}{2}$. Notice that $\delta_k(k-\delta_k) = -2$ and $2\delta_k - k = -\sqrt{k^2+8}$. For unknown terminology and notations for k-Jacobsthal numbers, see [4, 8].

2. The class $\mathcal{SJ}_k$

Definition 2.1. The function f belongs to $\mathcal{SJ}_k$ provided that

$$\frac{z\left[D_q^\lambda f(z)\right]'}{D_q^\lambda f(z)} \prec \tilde{j}_k(z), \qquad z \in \mathcal{D}, \tag{2.1}$$

where $f \in \mathcal{S}$, k is any positive real number and

$$\tilde{j}_k(z) = \frac{1+\delta_k^2 z^2}{1-k\delta_k z - 2\delta_k^2 z^2}, \qquad \delta_k = \frac{k-\sqrt{8+k^2}}{2}, \qquad z \in \mathcal{D}. \tag{2.2}$$

Lemma 2.1. *The figure of unit circle of $\tilde{j}_k(z)$ which is defined in (2.2) is the curve $\mathcal{C}_k$ with equation*

(2.3)

$$x = \frac{-1+\delta_k\, k\cos(t) + 2\,{\delta_k}^2 (\cos(t))^2 - {\delta_k}^2 + {\delta_k}^3 k\cos(t) + 2\,{\delta_k}^4}{-1+8\,{\delta_k}^2 (\cos(t))^2 + 2\,\delta_k\, k\cos(t) - 4\,{\delta_k}^3 k\cos(t) - 4\,{\delta_k}^2 - {\delta_k}^2 k^2 - 4\,{\delta_k}^4}$$

and

(2.4)

$$y = -\frac{\delta_k\ \sin(t)\left(6\,\delta_k\ \cos(t) - {\delta_k}^2 k + k\right)}{-1-4\,{\delta_k}^4 - 4\,{\delta_k}^3 k\cos(t) + \left(8\ (\cos(t))^2 - 4 - k^2\right){\delta_k}^2 + 2\,\delta_k\, k\cos(t)},$$

where t and k in $H = \left\{k \neq \pm\sqrt{-4+5\ \cos(t) \pm 3\ \sqrt{-1+(\cos(t))^2}}, t \in [0, 2\pi), k > 0\right\}$.

Proof. The proof can be obtained with some calculations by the help of Maple. For graphical demonstration see Figs. 2.1 and 2.2. $\square$

Example 2.1. The conchoid of Sluze is a family of plane curves given with the following well-known formula:

$$\alpha(x+\alpha)(x^2+y^2) = \beta^2 x^2. \tag{2.5}$$

It is useful here to use (2.3) and (2.4) to get the corresponding Cartesian equation of $\mathcal{C}_k$ for $k = 1$. Using basic trigonometric identities, we have

$$\frac{25x^2}{(4x-1)^2} + \frac{9x^2(x+1)^2}{y^2(4x-1)^2} = 1. \tag{2.6}$$

If we rewrite (2.6) in the following form

$$y^2 = -\frac{x^2(x+1)}{x-\frac{1}{9}} \tag{2.7}$$

which is obtained from (2.5) for $\alpha = -1/9$ and $\beta = \sqrt{10}/9$, see Fig. 2.3.

Remark 2.1. Observe that $\tilde{j}_k(0) = \tilde{j}_k(\frac{-k}{3\delta_k}) = 1$ and $\tilde{j}_k(1) = \tilde{j}_k(\frac{1+3\delta_k}{\delta_k(k\delta_k-3)}) = \frac{1+\delta_k^2}{1-k\delta_k-2\delta_k^2}$. Hence, $\tilde{j}_k$ is not univalent in $\mathcal{D}$.

Theorem 2.1. *The function $\tilde{j}_k$ is univalent in $\mathcal{D}_{R_k} = \{z : |z| < R_k\}$, where*

$$R_k = \frac{3-\sqrt{k^2+9}}{k\delta_k}, \qquad (k>0) \tag{2.8}$$

but it is not univalent in $\mathcal{D}_{R_k}$ for $r \geq R_k$.

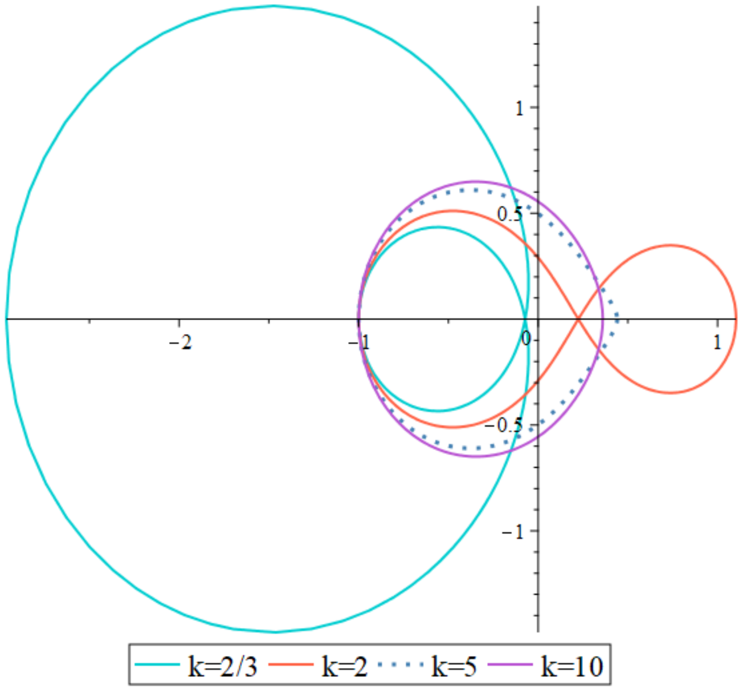

FIG. 2.1: The curve $\mathcal{C}_k$ for different values of k.

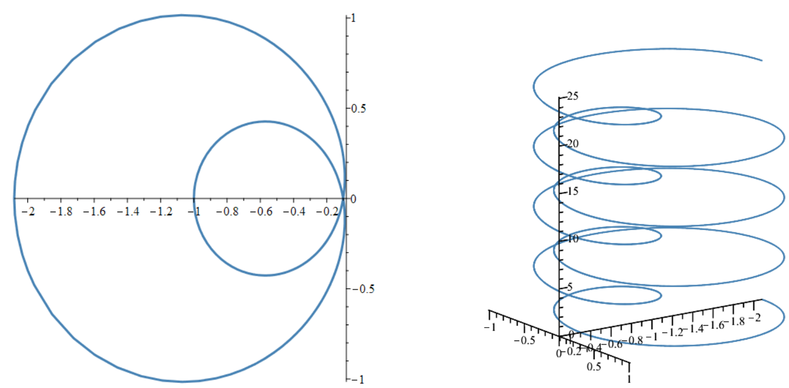

FIG. 2.2: The 2D curve and space curve $\mathcal{C}_k$ for $k = 0.5263157895$, respectively.

Proof. Assume that $\tilde{j}_k(z) = \tilde{j}_k(w)$ $(z, w \in \mathcal{D})$. This enables us to conclude that

$$\delta_k(z-w)\left[w - \frac{k + 3\delta_k z}{k\delta_k^2 z - 3\delta_k}\right] = 0. \tag{2.9}$$

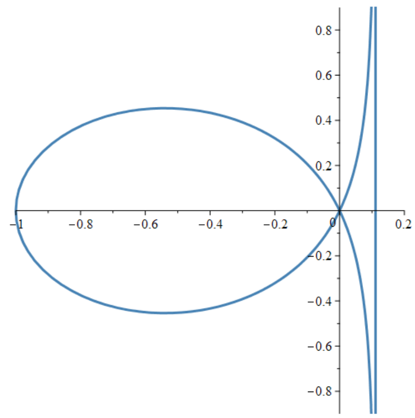

FIG. 2.3: The curve $\mathcal{C}_k$ for $k = 1$.

It can be obtained that the function

$$h_k(z) = \frac{k + 3\delta_k z}{k\delta_k^2 z - 3\delta_k} \tag{2.10}$$

transforms a circle $|z| = r < 3/(k\delta_k)$ into a circle centered at $m = \frac{3\,r^2 k\delta_k^3 + k + 3\delta_k}{r^2\delta_k^4 k^2 - 1}$ and of radius $\sigma = \frac{{\delta_k}^2 r^2 \left(3+\delta_k\, k^2 + 3\,{\delta_k}^2 k\right)^2}{\left(r^2 {\delta_k}^4 k^2 - 1\right)^2}$ with the diameter from $h_k(-r)$ to $h_k(r)$. Therefore, h_k transforms the circle $|z| = R_k$ into a circle with the diameter from $h_k(R_k) = R_k$ to the point $h_k(-R_k)$. Since $h_k'(x) = -\frac{9+k^2}{(-3+\ \delta_k\, xk)^2} < 0$ $(x \in \mathcal{R})$, $h_k(-R_k) > h_k(R_k) = R_k$ for each k. Hence, if $|w| \le R_k$ and $|z| \le R_k$, then

$$w - \frac{k + 3\delta_k z}{k\delta_k^2 z - 3\delta_k} = 0 \tag{2.11}$$

for $w = z - R_k$ only. In this way, we conclude that (2.9) is not fulfilled when $|w| < R_k$ and $|z| < R_k$, which shows that $\tilde{j}_k(z)$ is univalent in the disc (2.8).

It follows that

$$\begin{aligned} \tilde{j}_k'(z) &= \frac{k\delta_k + 6\,\delta_k^2\, z - k\,\delta_k^3 z^2}{\left(1 - k\,\delta_k z - 2\,\delta_k^2 z^2\right)^2} \\ &= \frac{(z - R_k)\left(z - \frac{3+\sqrt{k^2+9}}{k\delta_k}\right)}{(1 - k\,\delta_k z - 2\delta_k^2 z^2)^2}. \end{aligned}$$

Since $\tilde{j}_k'(R_k) = 0$, $\tilde{j}_k$ is not univalent for $|z| \ge R_k$.

$\square$

Theorem 2.2. *Assume that $J_{k,n}$ is the sequence of the k-Jacobsthal numbers. If $\tilde{j}_k(z) = \frac{1+\delta_k^2 z^2}{1-k\delta_k z - 2\delta_k^2 z^2} = 1 + \sum\limits_{n=1}^{\infty} j_n z^n$, then $j_n = (J_{k,n-1} + J_{k,n+1})\delta_k^n$ for $n \ge 1$.*

Proof. Assume $\vartheta = \delta_k z$, $|\vartheta| < |\delta_k|$. Note that $\delta_k(k-\delta_k) = -2$ and $2\delta_k - k = -\sqrt{k^2+8}$. Then

$$\begin{aligned}
\tilde{j}_k(z) &= \tfrac{1+\delta_k^2 z^2}{1-k\delta_k z-2\delta_k^2 z^2} = \tfrac{1+\vartheta^2}{1-k\vartheta-2\vartheta^2} \\
&= \left(\vartheta+\tfrac{1}{\vartheta}\right)\tfrac{\vartheta}{1-k\vartheta-2\vartheta^2} \\
&= \left(\vartheta+\tfrac{1}{\vartheta}\right)\tfrac{1}{\sqrt{k^2+8}}\left(\tfrac{1}{1+\frac{2\vartheta}{\delta_k}} - \tfrac{1}{1+\frac{2\vartheta}{k-\delta_k}}\right) \\
&= \left(\vartheta+\tfrac{1}{\vartheta}\right)\tfrac{1}{\sqrt{k^2+8}}\sum_{n=1}^{\infty}(-1)^n\left[\left(\tfrac{2\vartheta}{\delta_k}\right)^n - \left(\tfrac{2\vartheta}{k-\delta_k}\right)^n\right] \\
&= \left(\vartheta+\tfrac{1}{\vartheta}\right)\sum_{n=1}^{\infty}\tfrac{(k-\delta_k)^n-{\delta_k}^n}{\sqrt{k^2+8}}\vartheta^n.
\end{aligned}$$

It follows from (1.12) that

$$\begin{aligned}
\tilde{j}_k(z) &= \left(\vartheta+\tfrac{1}{\vartheta}\right)\sum_{n=1}^{\infty} J_{k,n}\vartheta^n \\
&= 1+\sum_{n=1}^{\infty}(J_{k,n-1}+J_{k,n+1}){\delta_k}^n\vartheta^n
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.3. *Assume $f \in \mathcal{A}$ is in $\mathcal{SJ}_k$. Then*

$$|a_n| \le \frac{|\delta_k|}{[n]_q^\lambda}^{n-1} J_{k,n}, \tag{2.12}$$

where $J_{k,n}$ is the sequence of k-Jacobsthal number, $\delta_k = \frac{k-\sqrt{8+k^2}}{2}$.

Proof. Suppose that $f \in \mathcal{SJ}_k$, $f(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$ with $a_0 = 0,\ a_1 = 1$. By using a relation of subordination, for a function ϖ, we have

$$\frac{z\left[D_q^\lambda f(z)\right]'}{D_q^\lambda f(z)} = \tilde{j}_k(\varpi(z)). \tag{2.13}$$

Next, we get

$$z\left[D_q^\lambda f(z)\right]' - D_q^\lambda f(z) = \delta_k^2\varpi^2(z)\left[2z\left[D_q^\lambda f(z)\right]' + D_q^\lambda f(z)\right] + \delta_k k\varpi(z)\, z\left[D_q^\lambda f(z)\right]' \tag{2.14}$$

then

$$\begin{aligned}(2.15)\qquad &\sum_{\ell=1}^{\infty}(\ell-1)\,[\ell]_q^{\lambda}\,a_\ell z^\ell+\sum_{\ell=n+1}^{\infty}C_\ell z^\ell=\delta_k^2\varpi^2\,(z)\sum_{\ell=1}^{n-2}(2\ell+1)\,[\ell]_q^{\lambda}\,a_\ell z^\ell\\ &+k\delta_k\varpi\,(z)\sum_{\ell=1}^{n-1}\ell\,[\ell]_q^{\lambda}\,a_\ell z^\ell.\end{aligned}$$

For $n\geq 2$, we deduce that

$$\begin{aligned}&\left|\sum_{\ell=1}^{\infty}(\ell-1)\,[\ell]_q^{\lambda}\,a_\ell z^\ell+\sum_{\ell=n+1}^{\infty}C_\ell z^\ell\right|^2\\ &=\left|\delta_k^2\varpi^2\,(z)\sum_{\ell=1}^{n-2}(2\ell+1)\,[\ell]_q^{\lambda}\,a_\ell z^\ell+k\delta_k\varpi\,(z)\sum_{\ell=1}^{n-1}\ell\,[\ell]_q^{\lambda}\,a_\ell z^\ell\right|^2\\ (2.16)\qquad &\leq\left|\delta_k^2\varpi\,(z)\sum_{\ell=1}^{n-1}(2\ell-1)\,[\ell-1]_q^{\lambda}\,a_{\ell-1}z^{\ell-1}+k\delta_k\sum_{\ell=1}^{n-1}\ell\,[\ell]_q^{\lambda}\,a_\ell z^\ell\right|^2\\ &\leq\sum_{\ell=1}^{n-1}\left|\delta_k^2(2\ell-1)\,[\ell-1]_q^{\lambda}\,a_{\ell-1}z^{\ell-1}+k\delta_k\ell\,[\ell]_q^{\lambda}\,a_\ell z^\ell\right|^2\\ &\leq\sum_{\ell=1}^{n-1}\left(\left|\delta_k^2(2\ell-1)\,[\ell-1]_q^{\lambda}\,a_{\ell-1}z^{\ell-1}\right|^2+\left|k\delta_k\ell\,[\ell]_q^{\lambda}\,a_\ell z^\ell\right|^2\right.\\ &\left.+2\left|k\delta_k^3\ell(2\ell-1)\,[\ell-1]_q^{\lambda}\,[\ell]_q^{\lambda}\,a_{\ell-1}a_\ell z^{2\ell-1}\right|\right).\end{aligned}$$

Since $z=re^{i\theta}$, as r approaches 1^-,

$$\begin{aligned}&\sum_{\ell=1}^{n}(\ell-1)^2\,[\ell]_q^{2\lambda}\,|a_\ell|^2+\sum_{\ell=n+1}^{\infty}|C_\ell|^2\\ \leq\quad&\delta_k^4\sum_{\ell=1}^{n-1}(2\ell-1)^2\,[\ell-1]_q^{2\lambda}\,|a_{\ell-1}|^2+k^2\delta_k^2\sum_{\ell=1}^{n-1}\ell^2\,[\ell]_q^{2\lambda}\,|a_\ell|^2\\ &+2k\,|\delta_k|^3\sum_{\ell=1}^{n-1}\ell\,(2\ell-1)\,[\ell-1]_q^{\lambda}\,[\ell]_q^{\lambda}\,|a_\ell|\,|a_{\ell-1}|\,.\end{aligned}$$

Hence,

$$\begin{aligned}(n-1)^2\,[n]_q^{2\lambda}\,|a_n|^2\quad&\leq\sum_{\ell=1}^{n-1}\delta_k^4\,(2\ell-1)^2\,[\ell-1]_q^{2\lambda}\,|a_{\ell-1}|^2\\ (2.17)\qquad &+\sum_{\ell=1}^{n-1}\left[k^2\delta_k^2\ell^2-(\ell-1)^2\right][\ell]_q^{2\lambda}\,|a_\ell|^2\\ &+\sum_{\ell=1}^{n-1}2k\,|\delta_k|^3\,\ell\,(2\ell-1)\,[\ell-1]_q^{\lambda}\,[\ell]_q^{\lambda}\,|a_\ell|\,|a_{\ell-1}|\,.\end{aligned}$$

The inequality (2.12) is provided for $n = 1$. Then, suppose that the estimation (2.12) satisfies $l \leq n$ for all $l \in \mathcal{N}$. Thus from (2.12) and (2.17), we get

$$\begin{aligned}
& n^2 [n+1]_q^{2\lambda} |a_{n+1}|^2 \\
& \leq \sum_{\ell=1}^{n} \delta_k^4 (2\ell-1)^2 [\ell-1]_q^{2\lambda} |a_{\ell-1}|^2 + \sum_{\ell=1}^{n} \left[k^2\delta_k^2\ell^2 - (\ell-1)^2\right] [\ell]_q^{2\lambda} |a_\ell|^2 \\
& + \sum_{\ell=1}^{n} 2k |\delta_k|^3 \ell (2\ell-1) [\ell-1]_q^{\lambda} [\ell]_q^{\lambda} |a_\ell| |a_{\ell-1}| \\
& \leq \sum_{\ell=1}^{n} \delta_k^4 (2\ell-1)^2 [\ell-1]_q^{2\lambda} \left(\frac{|\delta_k|^{\ell-2} J_{k,\ell-1}}{[\ell-1]_q^{\lambda}}\right)^2 \\
& + \sum_{\ell=1}^{n} \left[k^2\delta_k^2\ell^2 - (\ell-1)^2\right] [\ell]_q^{2\lambda} \left(\frac{|\delta_k|^{\ell-1} J_{k,\ell}}{[\ell]_q^{\lambda}}\right)^2 \\
& + \sum_{\ell=1}^{n} 2k |\delta_k|^3 \ell (2\ell-1) [\ell-1]_q^{\lambda} [\ell]_q^{\lambda} \left(\frac{|\delta_k|^{\ell-2} J_{k,\ell-1}}{[\ell-1]_q^{\lambda}}\right) \left(\frac{|\delta_k|^{\ell-1} J_{k,\ell}}{[\ell]_q^{\lambda}}\right) \\
& = \sum_{\ell=1}^{n} \delta_k^{2\ell} (2\ell-1)^2 J_{k,\ell-1}^2 + \sum_{\ell=1}^{n} \left[k^2\delta_k^2\ell^2 - (\ell-1)^2\right] |\delta_k|^{2\ell-2} J_{k,\ell}^2 \\
& + \sum_{\ell=1}^{n} 2k\delta_k^{2\ell} \ell (2\ell-1) J_{k,\ell-1} J_{k,\ell} \\
& = \sum_{\ell=1}^{n} |\delta_k|^{2\ell} \left[(2\ell-1) J_{k,\ell-1} + k\ell J_{k,\ell}\right]^2 - \sum_{\ell=1}^{n} \left[(\ell-1) |\delta_k|^{\ell-1} J_{k,\ell}\right]^2 \\
& \leq \sum_{\ell=1}^{n} |\delta_k|^{2\ell} \ell^2 J_{k,\ell+1}^2 - \sum_{\ell=1}^{n} (\ell-1)^2 |\delta_k|^{2\ell-2} J_{k,\ell}^2 \\
& = n^2 |\delta_k|^{2n} J_{k,n+1}^2
\end{aligned} \tag{2.18}$$

This shows us that the inequality (2.12) holds for all $n \in \mathcal{N}$. Theorem 2.3 is proved. □

3. Concluding remarks

In the current research, utilizing the relation of subordination, we have established a new family of q-starlike functions by using k-Jacobsthal numbers. We have investigated radius problems, which are necessary to establish the geometry of the image domain. We have derived general coefficient related to k-Jacobsthal numbers. Some basic properties are also obtained.

Finally, many other problems like coefficient bounds, Fekete-Szegö and Hankel determinant inequalities can be determined for the class $\mathcal{SJ}_k$ as a future work. Also, the approach presented here has been extended to establish new subfamilies of univalent functions with the other special number sequences.

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