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Semi-active damping by variational control algorithms

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ABSTRACT: Semi-active damping control of oscillating devices and structures is a challenging field that conjugates a relative actuators simplicity together with the chance of studying new algorithms and system architectures. This paper proposes an overview of the investigation recently developed by the authors in the field of semi-active damping using the variational optimal control. The method produces a general class of controllers that can be applied to linear as well as nonlinear suspension systems and vibrating structures. Their performances are systematically investigated and compared with the sky-hook benchmark.

KEY WORDS: Variational, Control, Suspension, Vibration, Algorithm, Semi-Active, Damping

1 INTRODUCTION

Control and actuation techniques are the basis for many innovations in vehicle dynamics, especially in vibration attenuation. Actual industrial production is populated by two families of actuation technology: active and semi-active. While the first generally guarantees the finest performances, it is accompanied by several drawbacks related to heavy and large actuators, high power motors, heavy energy consumption and as a direct consequence, by expensive devices. The class of semi-active controls represents indeed a very valuable alternative and effective choice. These devices are in fact characterized by reasonable costs, light modification of the original passive mechanical design, limited dimensions and weight. Moreover, the combination of lightness and poor energy absorption, makes the semi-active technology tuned with market requirement of eco-compatibility and “green” standards.

The active technology provides the designer with a complete freedom in following the performance requirements of vibration reduction with impressive results. On the other hand, the semi-active philosophy, because of the restraint imposed to the actuators to be energetically passive, would benefit, in a remarkable way, of suitable selections of the control strategy. In particular, the equations of the mechanical system under control is in this case characterized by the presence of time-varying mechanical damping and/or stiffness and in some cases of time-varying inertial properties. These modifications are indeed produced using a very low amount of energy, compared to the average energy of the vibrating masses.

Automotive production in the last decade made a large use of semi-active vibration technology [1,2], because of the available techniques, for example, to simply modify the dissipation characteristics of the suspension systems. Therefore, one can find a wide scientific and technical literature on the subject together with many control algorithms [3-5]. Hydraulic, pneumatic and electrical solutions were patented along the last twenty years to control the mechanical damping and/or stiffness in a semi-active way, leading to many industrial products. Among them even purely mechanically controlled systems developed by Boge (the Nivomat system) are present. However, the advantage of an electrical control of damping at a reasonable low cost, discloses new possibilities for vibration attenuation and suitably designed control strategies. Among them, certainly the original magneto-rheological (MR) [6] technology developed originally by Lord Corporation [7] presents many advantages in terms of simplicity with electronic interfaces. One of the most successful control algorithm in this field is the famous sky-hook [8].

This paper is focused on dedicated control algorithms to be used in this industrial context, and finalized to produce a new controller to optimize the suspension performances. Some of these results, are part of a university project in progress at the Department of Mechanical and Aerospace Engineering of Sapienza, and have been recently patented as a part of an integrated semi-active technology for vibration suppression [9].

The present paper is finalized to the synthesis of a new algorithm for the optimal control of semi-active dampers in a prototype suspension system. The results are obtained using a variational approach to the optimal control, together with a special form for the objective function. The theory is developed in general but, interestingly, for particular cases of engineering interest, a closed form dependency of the damping on the actual measured states of the suspension system is obtained, the basis for a direct synthesis of a new controller. The presented control strategy is an original engineering application of variational control as part of a powerful mathematical field, the theory of control of partial differential equations [10-13]. In fact, although the theory of variational control is an established tool in the context of control theory [14-17], its systematic application to optimal semi-active damping has not yet explored in the technical literature.

The organization of the paper is as follows. In section 2, the theory is synthetically presented, illustrating its general mathematical basis, the prototype equations of the controlled system, the master equation for the controller and the clipping technique to adapt the method to the semi-active requirement. Section 3 shows indeed numerical simulations to test the effectiveness of the found control laws.

2 A VARIATIONAL APPROACH TO DAMPING CONTROL

Usually the variational approach is used in the control theory for finding an optimal control finalized to minimize or maximize some objective function in integral form.

Before describing how a variational approach can be suitably used in our context, some preliminary concepts are introduced. In control theory the mathematical description of the system is expressed as a set of n coupled first-order ordinary differential equations, known as the state equations, in which the time derivative of each state variable is expressed in terms of the state variables $x_1(t),\dots,x_n(t)$ and the system inputs $u_1(t),\dots,u_r(t)$. In the general case, the form of the n-state equations is:

$$\begin{aligned} \dot{x}_1 &= f_1(\boldsymbol{x},\boldsymbol{u},t) \\ \dot{x}_2 &= f_2(\boldsymbol{x},\boldsymbol{u},t) \\ \vdots &= \quad \vdots \\ \dot{x}_n &= f_n(\boldsymbol{x},\boldsymbol{u},t) \end{aligned} \tag{1}$$

where $\dot{x}_i = \frac{dx_i}{dt}$ and each of the functions $f_i(\boldsymbol{x},\boldsymbol{u},t)$, $(i = 1,\dots,n)$ may be a general nonlinear, time varying function of the state variables, the system inputs, and time. It is common to express the state equations in a vector form, in which the state variables are collected into the vector $\boldsymbol{x}(t) = [x_1(t),x_2(t),\dots,x_n(t)]^T$, and the set of r inputs into the vector $\boldsymbol{u}(t) = [u_1(t),u_2(t),\dots,u_r(t)]^T$.

Given a set of initial conditions (the values of the x_i at a given time t_0) and the inputs for $t \geq t_0$, the evolutionary equation for the state is:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) \tag{2}$$

where $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)$ is a vector function with n components $f_i(\boldsymbol{x},\boldsymbol{u},t)$. The system (2) can represent, for example, a mechanical system, written as first order differential equation where $\boldsymbol{u}$ is the control action.

Note that in many cases, the inputs undergo some restrictions, that is $\boldsymbol{u} \in C$ where C represents a subset of $\mathbb{R}^r$, is the set of admissible controls. This restriction is always encountered in technical applications, and also in our devices, since, whatever the physical nature of the control systems, actuators have limits in their performances like forces, currents, voltages that cannot exceed prescribed values.

In general, the optimal control problems, $opt\, J$, considered in the variational approach can be summarized as a control algorithm that minimizes or maximizes, for a given dynamic system, an objective function (also called cost function) of the form $J = \int_{t_0}^{t_f} L(\boldsymbol{x},\boldsymbol{u},t)dt$ that may depend on the status of the system and the vector of inputs. In this way a control algorithm minimizing or maximizing a given cost function with possible constraints is introduced and in particular, in the present work, we state the problem as:

$$\begin{gathered} opt\, J = \int_{t_0}^{t_f} L(\boldsymbol{x},\boldsymbol{u},t)dt \\ subjected\ to \\ \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) \end{gathered} \tag{3}$$

In order to encompass the constraint conditions, the cost function can be completed by using Lagrangian multipliers $\boldsymbol{\lambda}(t) = [\lambda_1(t),\lambda_2(t),\dots,\lambda_n(t)]^T$, which are also called adjoint variables. The modified cost function becomes:

$$J = \int_{t_0}^{t_f} L(\boldsymbol{x},\boldsymbol{u},t) + \boldsymbol{\lambda}^T\big(\dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)\big)dt \tag{4}$$

In general, these types of problems of the calculus of variations are called bounded problems, or Lagrange-Pontryagin problems that can be generalized in this way:

$$\begin{cases} J = \int_{t_0}^{t_f} L(\boldsymbol{x}(t),\boldsymbol{u}(t),t)dt \; : \; \underset{\boldsymbol{u} \in \mathrm{C}}{opt} J \\ \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_{t_0} \end{cases} \tag{5}$$

that is it specifies physical limits for the control variables $u_1(t),u_2(t),\dots,u_r(t)$.

The problem (5) is called *optimal control* and its solution $\boldsymbol{u}^*$, in the maximum case, specifies the condition:

$$J(\boldsymbol{u}) \leq J(\boldsymbol{u}^*)\,, \qquad \boldsymbol{u}^* \in \mathrm{C}\ and \quad \forall \quad \boldsymbol{u} \in \mathrm{C} \tag{6}$$

$\boldsymbol{x}^*$ associated to $\boldsymbol{u}^*$ is called optimal trajectory.

The approach to determine the optimal control leads to the Euler-Lagrange equations, that permits conceptually to determine simultaneously $\boldsymbol{x}^*$ and $\boldsymbol{u}^*$. The form and characteristics of these equations, and additionally the chance of finding a closed form for $\boldsymbol{u}^*$ in terms of $\boldsymbol{x}^*$, much depends on the considered problem and related constraints.

This paper is specifically devoted to establish an optimal control algorithm acting on a variable damping of a mechanical system. In fact, the existing technology, especially that based on the use of magneto-rheological dampers, allows for a very effective modification of the viscosity of a fluid through the control of electric currents in solenoids.

The following section 2.1 is therefore devoted to the statement of the prototype equations of this type of control. In section 2.2, the explicit form of Euler-Lagrange equations, is derived, focusing on closed form solution for $\boldsymbol{u}^*$, indeed an important aspect in engineering. In fact, the controller is required to act in real-time, without the chance of solving complicated differential equations at any instant t to determine the control action at the following time step $t + \Delta t$. Δt, in practical cases of interest, should be as small as possible, since the frequency of the controllable mechanical vibration is of order $1/\Delta t$: the larger Δt, the slower the dynamic of the control system.

2.1 *Prototype equations of the controlled system*

In this section we specialize the calculus of variation to the case of semi-active control by the Euler-Lagrange equations. Before to start with a detailed analysis, the benefits that this method provides are briefly illustrated.

First of all, the evolutionary equations of the system are very general and any nonlinear effect can be included, so frequently appearing in mechanical systems, such as those due to the kinematic linkages in the suspension architecture or/and those related to constitutive relationships of elastic and visco-elastic components. Moreover, the optimal control may include any type of nonlinear control laws.

Based on these observations we can start from a general approach of one degree of freedom system (Figure 1) that consists of a damper located between a sprung mass and the incoming noise signal $y(t)$.

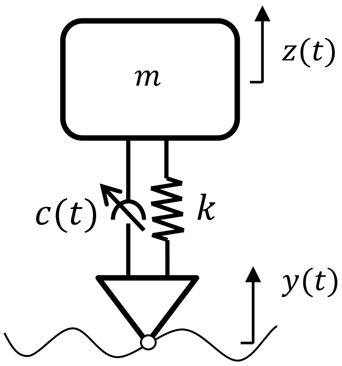

Figure 1: A sprung mass with a damper controlled suspension system.

As a very general statement, that will be specified later for different required control strategy, we let:

$$
\begin{gathered}
J=\int_{t_0}^{t_f} E(z,\dot z,y,\dot y,c)dt \\
\underset{c\in C}{Opt}\, J \\
subjected\ to \\
\begin{cases}
m\ddot z + c(t)\, f^c(\dot z-\dot y) + k\, f^k(z-y)=0\\
z(t_0)=z_{t_0}\ ;\ \dot z(t_0)=\dot z_{t_0}\\
c(t)\in[c_{min},c_{max}]\ , c_{min}>0\ ,\ c_{max}>c_{min}
\end{cases}
\end{gathered}
\tag{7}
$$

where:

- $E(z,\dot z,y,\dot y,c)$ is an objective function that will be chosen in function of some physical intuition;
- $f^c(\dot z-\dot y)$ and $f^k(z-y)$ are nonlinear functions of the argument $(\dot z-\dot y)$ and $(z-y)$ respectively, typically such nonlinearity include geometric or constitutive nonlinearity of the suspension system;
- $c(t)$ describes the control action and $c(t)\in[c_{min},c_{max}]$, $c_{min}\geq 0$, which implies the device is semi-active;
- k is the spring constant;
- m is the sprung mass;
- $z(t)$ is the absolute sprung mass displacement;
- $y(t)$ the noise signal.

Note that the initial conditions $z(t_0)=z_{t_0}$ and $z(t_f)=z_{t_f}$ are substituted by $z(t_0)=z_{t_0}$ and $\dot z(t_0)=\dot z_{t_0}$.

The procedure to extract the optimal control law is described in the following.

Considering the equation (4) we have:

$$
\delta J=\delta\int_{t_0}^{t_f} E(z,\dot z,y,\dot y,c) + \lambda\big(m\ddot z + c(t)\, f^c(\dot z-\dot y) + k\, f^k(z-y)\big)dt = 0 \tag{8}
$$

In the problem of actual interest, we are faced with the restraint $c(t)\in[c_{min},c_{max}]$. This is not explicitly included in the minimization analysis that leads to a solution in terms of $z^*(t)$ and $c^*(t)$ that does not satisfy the condition $c(t)\in[c_{min},c_{max}]$ at any t. However, the actual value of $c^*(t)$ is managed by the simple clipping procedure:

$$
\begin{cases}
c(t)=c^*(t) & \forall\quad c_{min}\geq c^*(t)\geq c_{max}\\
c(t)=c_{min} & \forall\quad c^*(t)<c_{min}\\
c(t)=c_{max} & \forall\quad c^*(t)>c_{max}
\end{cases}
\tag{9}
$$

If equation (8) admits a solution for z^* and c^*, then (9) provide the desired result.

For some choices of the objective function $E(z,\dot z,y,\dot y,c)$, it may happen that the optimal solution does not exist. In this cases, the control is abandoned. These cases, under some conditions, could be approached by the Pontryagin's theorem. However, because the application of the variational approach becomes very difficult, we prefer to skip the analysis of these cases in this engineering context.

We can further manipulate equations (8):

$$
\begin{aligned}
\delta J=\int_{t_0}^{t_f}\Bigg[&\frac{\partial E}{\partial z}\,\delta z+\frac{\partial E}{\partial \dot z}\,\delta\dot z+\frac{\partial E}{\partial c}\,\delta c+\lambda m\,\delta\ddot z\\
&+\lambda f^c(\dot z-\dot y)\delta c\\
&+\lambda c\,\frac{\partial f^c(\dot z-\dot y)}{\partial\dot z}\delta\dot z\\
&+\lambda k\,\frac{\partial f^k(z-y)}{\partial z}\delta z\Bigg]\ dt\\
&=0
\end{aligned}
\tag{10}
$$

Using the variational method and assuming that $\delta z(t_0)=0$, $\delta z(t_f)=0$, $\delta c(t_0)=0$, $\delta c(t_f)=0$, we obtain:

$$
\begin{cases}
\ddot\lambda m-\dfrac{d}{dt}\left[\dfrac{\partial E}{\partial\dot z}+\lambda c\,\dfrac{\partial f^c(\dot z-\dot y)}{\partial\dot z}\right]+\dfrac{\partial E}{\partial z} +\lambda k\ \dfrac{\partial f^k(z-y)}{\partial z}=0\\
m\ddot z + c(t)\, f^c(\dot z-\dot y) + k\, f^k(z-y)=0\\
\dfrac{\partial E}{\partial c}+\lambda f^c(\dot z-\dot y)=0\\
z(t_0)=z_{t_0}\ ;\ \dot z(t_0)=\dot z_{t_0}
\end{cases}
\tag{11}
$$

an algebraic-differential problem in terms of z, c, λ.

2.2 Master equation of the optimal controller

It is important to remark the way equations (11) are used in this context. In fact, although they represent a system of algebraic-differential equations, our goal is not their solutions. The engineering goal is to determine how $c(t)$ should be modified depending on some measured quantities during the operating conditions. Therefore, equation (11) must be used considering they provides an implicit relationship between $c(t)$ and the variables $z(t)$, $y(t)$. Once these last quantities are measured by suitable sensors, equations (11) permit to determine the best values of $c(t)$ at any instant t to minimize the desired objective function J. Part of the following analysis is devoted to find explicit form of $c(t)$ in terms of $z(t)$ and $y(t)$, by suitable selection of the form of the function E.

From equation (11), for $f^c(\dot{z}-\dot{y}) \neq 0$, we have:

$$\begin{cases} \ddot{\lambda} m - \dfrac{d}{dt}\left[\dfrac{\partial E}{\partial \dot{z}} + \lambda c \dfrac{\partial f^c(\dot{z}-\dot{y})}{\partial \dot{z}}\right] + \dfrac{\partial E}{\partial z} + \lambda k \dfrac{\partial f^k(z-y)}{\partial z} = 0 \\ \lambda = -\dfrac{\partial E}{\partial c}\dfrac{1}{f^c(\dot{z}-\dot{y})} \end{cases} \tag{12}$$

When substituting the second equation into the first one equation:

$$\ddot{\lambda} m - \frac{d}{dt}\left[\frac{\partial E}{\partial \dot{z}} - \frac{\partial E}{\partial c}\frac{1}{f^c(\dot{z}-\dot{y})} c \frac{\partial f^c(\dot{z}-\dot{y})}{\partial \dot{z}}\right] + \frac{\partial E}{\partial z} - \frac{\partial E}{\partial c}\frac{k}{f^c(\dot{z}-\dot{y})}\frac{\partial f^k(z-y)}{\partial z} = 0 \tag{13}$$

and integrating two times

$$\lambda m = \int \left[\frac{\partial E}{\partial \dot{z}} - \frac{\partial E}{\partial c}\frac{1}{f^c(\dot{z}-\dot{y})} c \frac{\partial f^c(\dot{z}-\dot{y})}{\partial \dot{z}}\right] d\tau - \int\int \left[\frac{\partial E}{\partial z} - \frac{\partial E}{\partial c}\frac{k}{f^c(\dot{z}-\dot{y})}\frac{\partial f^k(z-y)}{\partial z}\right] d\tau\, d\tau' + c_0\, t + c_1 \tag{14}$$

Substituting this last expression into the second of (12),

$$\frac{\partial E}{\partial c}\frac{m}{f^c(\dot{z}-\dot{y})} + \int \left[\frac{\partial E}{\partial \dot{z}} - \frac{\partial E}{\partial c}\frac{1}{f^c(\dot{z}-\dot{y})} c \frac{\partial f^c(\dot{z}-\dot{y})}{\partial \dot{z}}\right] d\tau - \int\int \left[\frac{\partial E}{\partial z} - \frac{\partial E}{\partial c}\frac{k}{f^c(\dot{z}-\dot{y})}\frac{\partial f^k(z-y)}{\partial z}\right] d\tau\, d\tau' + c_0\, t + c_1 = 0 \tag{15}$$

This equation, together with equation (11), is the basis to derive several control laws for different choices of the objective function E extracting the control function $c(t)$ from the equation (15), as in the following examples.

However, it is useful to remark that this equation does not always provide the desired optimal control solution. The first obvious case is met when none relative maximum or minimum exists for the functional J. A second case is met when not a unique solution can be determined.

2.3 Semi-active active implementation and the clipping technique

One must observe we adopted a Lagrangian approach that skipped the use of the Pontryagin's theorem, in that the condition $c(t) \in [c_{min}, c_{max}]$ is not directly included into the problem formulation, but indeed introduced *a posteriori* through the clipping operation. Although this simplifies the method, a legitimated question arises about the characteristic of the solution we found. It is possible to show easily, that (i) if the solution is unique, in the sense that a single relative minimum or maximum exists for J, and (ii) under some reasonable regularity conditions for J, then the clipping procedure produces the best solution, compatible with the restraints on the control function. This means $c^*(t)$ found solving equations (11) or (15), and subjected to the clipping restriction (9) produces the smallest (or the largest) possible value for J within the admissible range for $c(t)$, i.e. an absolute minimum (or maximum) is obtained for J.

This can be easily understood from Figure 2 where a simplified illustration of the question in two dimensions is represented.

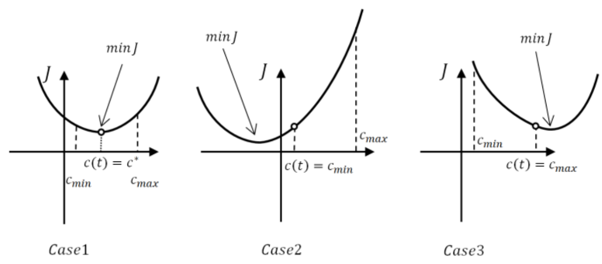

Figure 2:Control law in the case of limited ranges

J is on the vertical axis, while $c(t)$ is on the horizontal one. If there is a unique relative minimum for J determined by an unconstrained Lagrangian approach, then we can distinguish three cases. If the minimum is located within the interval $[c_{min}, c_{max}]$, (Figure 2: *case 1*), then the Lagrangian approach keeps directly $c(t) = c^*(t)$, that is the minimum for J. If the minimum falls outside of the interval $[c_{min}, c_{max}]$, (Figure 2: *case 2 and case 3*), then the Lagrangian approach identifies again $c(t) = c^*(t)$ as the minimum solution, but the clipping operation puts the solution on the boundary of the interval $[c_{min}, c_{max}]$ closest to $c^*(t)$. However, this is still the best minimum solution compatible with the control function restraint.

2.4 Analysis of special class of objective functions

Let us specify the objective function as:

$$E(z, \dot{z}, y, \dot{y}, c) = \alpha \ddot{z}^2 + \beta \dot{z}^2 + \gamma z^2 \tag{16}$$

In this case, a combination of acceleration, speed and displacement are considered for the optimization. The coefficients α, β and γ are responsible for the weights the designer desires to assign to each of the three terms.

Therefore:

$$Opt\ J$$

$$J = \int_{t_0}^{t_f} \alpha \left(\frac{c(t)}{m} f^c(\dot{z} - \dot{y}) + \frac{k}{m} f^k(z - y) \right)^2 + \beta \dot{z}^2 + \gamma z^2 dt \tag{17}$$

$$\begin{cases} m\ddot{z} + c(t)\, f^c(\dot{z} - \dot{y}) + k\, f^k(z - y) = 0 \\ z(t_0) = z_{t_0}\ ;\ \dot{z}(t_0) = \dot{z}_{t_0} \end{cases}$$

where the equation of motion is used to replace $\ddot{z}$ in (16). Using equation (15) with $c_0 = 0$, we obtain, for $f^c(\dot{z} - \dot{y}) \neq 0$,

$$c^*(t) = -k \frac{f^k(z - y)}{f^c(\dot{z} - \dot{y})} + \frac{g_1}{f^c(\dot{z} - \dot{y})} \int_{t_0}^{t} \dot{z}\, d\tau + \frac{g_2}{f^c(\dot{z} - \dot{y})} \int_{t_0}^{t} \int_{t_0}^{t} z d\tau\, d\tau' + g_3 \tag{18}$$

The coefficients g_1, g_2 and g_3 are tuning parameters:

$$\begin{cases} g_1 = -\dfrac{m\beta}{\alpha} \\ g_2 = \dfrac{m\gamma}{\alpha} \\ g_3 = -c_1 \end{cases} \tag{19}$$

In the section 3 the control law (18) is analyzed by using numerical simulations.

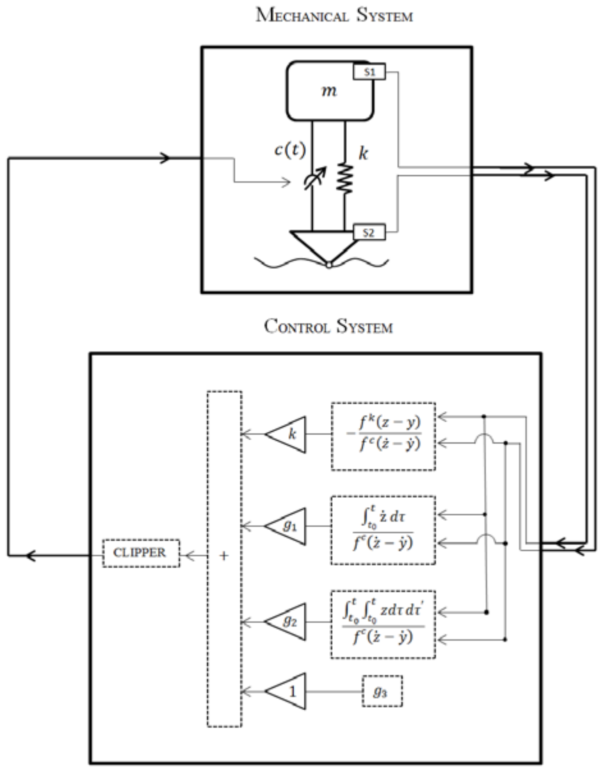

Figure 3: Diagram of the control system

The synthesis of determined controller is depicted in the block scheme of Figure 3, emphasizing the structure of the determined feedback filter.

3 NUMERICAL RESULTS

The non-linear control law (18) is an explicit solution for $c(t)$ in terms of $z(t)$, $y(t)$ and their derivatives, that are measured by suitable sensors. In fact, the general control system is equipped, at least, with two sensors, as illustrated in Figure 3, to determine the best values of $c(t)$ that minimize the desired objective function E specified by equation (16). The sensors $S1$ and $S2$ of Figure 3 are generally accelerometers. The sensors are connected to an electronic controller implementing the structure depicted in Figure 3. The performance of the actual controller is applied to a quarter-car model and compared with the sky-hook technique, synthesized as follows:

$$c_{sky-hook}(t) = \begin{cases} c_{min} & if \quad \dot{z}(\dot{z} - \dot{y}) \leq 0 \\ c_{max} & if \quad \dot{z}(\dot{z} - \dot{y}) > 0 \end{cases} \tag{20}$$

The numerical experiments reported are obtained using ISO8608, using a power spectral density (PSD):

$$G(n) = G(n_0) \left(\frac{n}{n_0} \right)^{-2} \tag{21}$$

where n is the spatial frequency $cycle/meters$, and n_0 and $G(n_0)$ are constant, for the following simulation, equal to:

$$n_0 = 0.1 \frac{cycle}{m} \tag{22}$$

$$G(n_0) = 32 \cdot 10^{-6} \frac{m^3}{cycle}$$

In the figures 4 and 5 the displacement and acceleration of the sprung mass are reported. Note that the variational control law tested with the gain $g_1 = g_2 = g_3 = 0$ produces a very low acceleration, although, a higher value for the sprung mass displacement.

In fact for $g_1 = g_2 = g_3 = 0$ the objective function is:

$$E(z, \dot{z}, y, \dot{y}, c) = \alpha \ddot{z}^2 \tag{23}$$

that collapses into a simple acceleration requirement.

The same considerations are valid for the displacement and acceleration PSD represented in the figures 6 and 7. Finally a probability density function PDF of the two signals is presented in the figures 8 and 9.

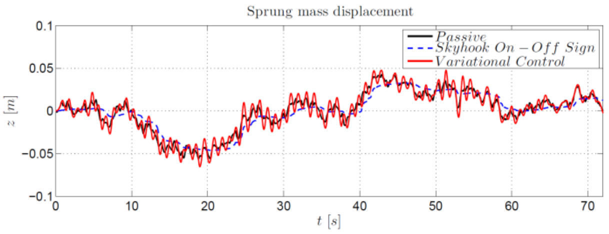

Figure 4: Displacement of the sprung mass $z(t)$.

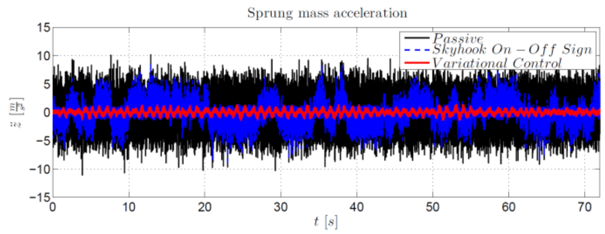

Figure 5: Acceleration of the sprung mass $\ddot{z}(t)$.

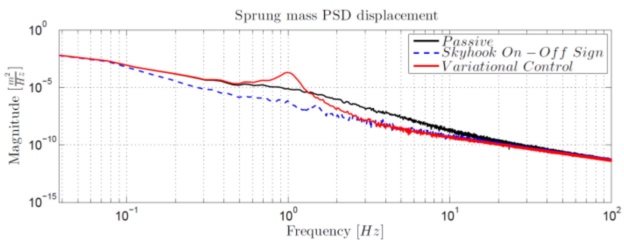

Figure 6: PSD displacement of the sprung mass $z(t)$.

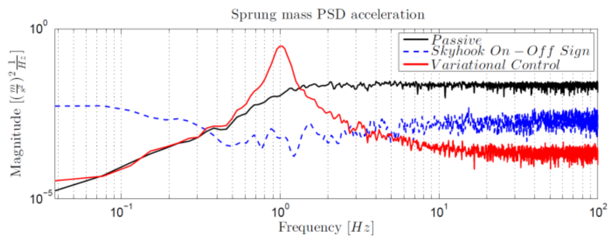

Figure 7: PSD acceleration of the sprung mass $\ddot{z}(t)$.

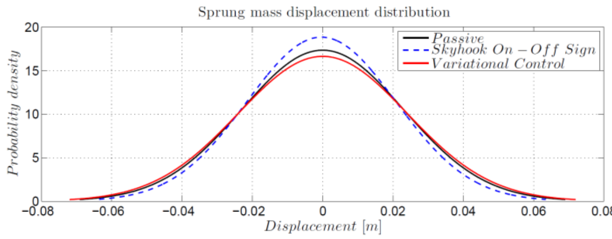

Figure 8: PDF displacement of the sprung mass $z(t)$.

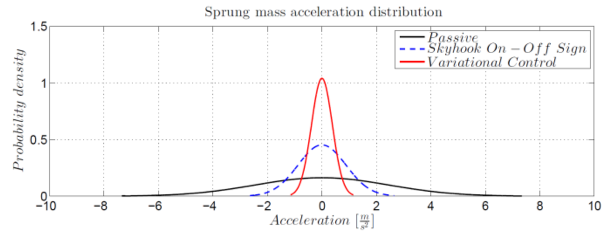

Figure 9: PDF acceleration of the sprung mass $\ddot{z}(t)$.

CONCLUSIONS

The presented control is based on a variational formulation and applied to a simple suspension model. It introduces a general control law that includes a variable damper with a semi-active device.

The obtained Euler-Lagrange equations provide the damping in a closed form in terms of the suspension measured state variables.

The new control strategy, minimizes the sprung mass acceleration. The numerical simulations show very good results and an excellent uniformity and robustness of the control. There are still many parameters to tune to improve the controller performances, the scope of future activities.

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