An Optimality Approach to the Application Ratio for the Matching Adjustment in the Solvency II Regime

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Abstract

We refer to the Technical Specifications provided by EIOPA to implement the package of long-term guarantees measures which shall be included into the Solvency II Framework Directive. One of these regulatory measures concernes the *Application Ratio*, a coefficient defining what portion of the *Maximum Matching Adjustment* an insurance company can apply to the risk-free rates for discounting her obligations, given the matching properties of the assigned asset portfolio. In this paper we propose an optimization algorithm providing a reliable assessment of the Application Ratio. The Application Ratio provided by the algorithm is optimal in the sense that it has the maximum value given the structure of the asset-liability portfolio. This value corresponds to the minimum attainable level for the losses incurred from forced sales of defaultable bonds with mispriced market value. We show that under natural assumptions this optimality problem has the form of a linear programming problem, which can be easily solved using standard numerical procedures. A matching criterion defined in stronger form can also be applied by imposing appropriate run-off constraints in the linear programming problem. The value of the optimal Application Ratio can be used by a Supervisor as an objective benchmark for checking the appropriateness of the Application Ratio adopted by

the undertaking. The optimal liquidation policy provided by the algorithm can also be used by an insurance undertaking which want to apply conservative management actions to her asset-liability portfolio.

Keywords. Matching adjustment, Asset-liability management, Optimal matching, Management actions, Credit risk.

1 Introduction

In the current negotiation between the European Parliament, the Council and the European Commission on a Directive (Omnibus II) amending and implementing the Solvency II Framework Directive (EC, 2009), the “trilogue parties” have agreed to include a *Long-Term Guarantees Package* (LTGP), that is a package of regulatory measures concerning insurance products with long-term guarantees.

One of the most relevant measures in LTGP is based on the following basic idea:

· if an insurance company holds defaultable bonds in an asset portfolio designed for backing the expected insurance liabilities; and

· if it is Supervisor's opinion that these assets are currently underpriced by the market with respect to their correct default probability,

then the company is allowed to value these assets at the correct price – which is indicated by the Supervisor – net of the losses for forced sales caused by mismatching with the outstanding liabilities. The “Matching Adjustment” is the value adjustment, expressed or as a premium on the assets or as a discount on the liabilities, which is appropriate taking into account the matching properties of the asset-liability insurance portfolio.

The idea is a relevant one since in the current conditions of market distress it allows to protect from distorting short term perceptions and artificial volatility important quantities of government bonds, leaving the judgement on their credit standing to European supervisory agencies (the European Insurance and Occupational Pensions Authority – EIOPA, and the European Systemic Risk Board – ESRB), which could produce stabilising and countercyclical effects.

In order to provide a level playing field to all market participants however, the Matching Adjustment must be determined in a verifiable and not distorting manner. While the "maximum matching adjustment" (the value adjustment without any early forced sale of assets) for a given asset-liability portfolio can be easily identified, it could be very difficult to specify the correct "application ratio", that is the fraction of the maximum adjustment which is appropriate to apply given the portfolio structure.

In the proposal emerging from the current negotiation the Matching Adjustment is defined as a premium that (re)insurance undertakings will apply to the risk-free interest rates when the present value of the outstanding liabilities is computed. The calculation details for deriving the maximum matching adjustment and the application ratio are illustrated in a document prepared by EIOPA (2012). This document provides the technical specifications for an impact assessment which will be run for collecting information from the insurance industry on the effects of the LTG package. In our opinion, however, some aspects of the current proposal for defining and computing the Matching Adjustment are questionable.

In this paper an objective and verifiable methodology for determining the Matching Adjustment of an asset-liability insurance portfolio is proposed. The methodology is derived in a semideterministic model for the portfolio management, where the current and future asset prices are determined using typical market practices. The application ratio is obtained as the solution of an optimization problem subject to natural portfolio constraints. The optimization problem takes the form of a linear programming problem which can be easily solved using standard algorithms. The obtained solution (and also the possible infeasibility of the problem) is immediately understandable in economic terms, clearly reflecting the matching properties of the portfolio.

2 Formulation of the reference model

We consider a given portfolio of outstanding insurance policies and an *assigned* portfolio of assets. It is assumed for the moment that all the assets held are bonds with fixed cash flows.

2.1 The asset-liability portfolio: basic definitions

The current date is $t = 0$. All the projected cash flows of both assets and liabilities are scheduled on the same time grid:

$$\boldsymbol{t} := \{1, 2, \dots, m\} .$$

We denote by δ the length in years of the grid time step.

Remark. In order to obtain a good precision when mapping the cash flows on the grid time points the choice $\delta = 1/12$ (monthly grid) is usually the most appropriate. It turns out however that a monthly time grid can be demanding by a computational point of view. Then quarterly, semi-annual or annual grids ($\delta = 0.25, 0.5, 1$) can also be considered. ∎

The risk-free discount factor observed on the market at time 0 for maturity date k is given by:

$$v(0, k) := (1 + i_k)^{-k\delta} = e^{-k\delta\, r_k} ,$$

where i_k and r_k is the risk-free interest rate and the yield-to-maturity, respectively, on annual basis for maturity $k\delta$ years.

The stream of the expected net liability cash flows (benefits minus premiums) is denoted by:

$$\boldsymbol{y} := \{y_1, y_2, \dots, y_m\} , \quad \text{with} \quad y_k \in \mathbb{R} , \quad k = 1, 2, \dots, m .$$

We denote by m_y the terminal date of the liability stream, i.e. $m_y := \max\{k : \, y_k > 0\}$.

Remark. If one adopts the point of view of the replicating portfolio for modelling the insurance obligations (see Daul et al., 2009, or Sandström, 2011 chapter 8), $\boldsymbol{y}$ can be interpreted as the cash flows stream generated by that portfolio. Since we interpret y_k as an expected cash flow, we refer to the replicating portfolio in the "central" scenario. However we could also use a different representations of the obligations, considering the liability stream $\boldsymbol{y}^{stress}$ corresponding to "stressed" scenarios, i.e. stemming from applying the stresses prescribed by the Supervisor to the underwriting risks to which the insurance portfolio is exposed. ∎

Remark. Generally, the liability cash flow Y_k to be paid at date k is exposed to both actuarial ("technical") and financial uncertainty, where usually the second type of uncertainty is generated by profit-sharing clauses[1]. In the framework of the arbitrage-free stochastic pricing models, if Y_k is affected only by financial uncertainty y_k should be interpreted as the expectation of Y_k with respect to the *forward risk-neutral* probabilities. ∎

At the current date n assets are held in the assigned portfolio for backing the liabilities. These assets are bonds with fixed cash flows exposed to credit risk. The notional cash flows generated by asset i are denoted as:

$$\overline{\boldsymbol{x}}_i := \{\overline{x}_{i,1}, \overline{x}_{i,2}, \ldots, \overline{x}_{i,m}\}\,, \quad i = 1, 2, \ldots, n\,, \quad \text{with} \quad \overline{x}_{i,k} \geq 0 \; \forall i, k\,.$$

We denote by m_i the maturity date of asset i, i.e. $m_i := \max\{k : \overline{x}_{i,k} > 0\}$. It is natural to pose $m = \max\{m_y, m_i,\, i = 1, 2, \ldots, n\}$. The quantity of asset i held at time 0 is:

$$q_i \geq 0\,, \quad i = 1, 2, \ldots, n\,.$$

The current market price of one unit of asset i is denoted as:

$$P_{i,0}\,, \quad i = 1, 2, \ldots, n\,,$$

and the market credit spread of asset i is given by:

$$\sigma_i\,, \quad i = 1, 2, \ldots, n\,.$$

We also consider the *fundamental spread* of asset i given at time 0 by EIOPA:

$$\widehat{\sigma}_i\,, \quad i = 1, 2, \ldots, n\,.$$

Remark. Here σ_i and $\widehat{\sigma}_i$ represent the constant spread which is equivalent to term structure of credit spreads corresponding to rating, sector and time-to-maturity of asset i. ∎

[1]For a joint modelling of the two types of uncertainties see e.g. De Felice et al., 2005.

2.2 The current definition of the Matching Adjustment

In the current Solvency II regulation the Matching Adjustment is defined as a premium applied to the basic risk-free interest rates when computing the present value of the insurance liabilities. The details for calculation are contained in a technical specifications document provided by EIOPA (2012), where the prescriptions of articles 77 c ("Classical Matching Adjustment") and 77 e ("Extended Matching Adjustment") of the Directive are implemented. This specifications, which can be referred to as "the current proposal", can be summarized as follows.

At time 0 the *Best Estimate* of the liabilities is defined as:

$$B := \sum_{k=1}^{m} y_k \, v(0,k)\,,$$

and the market value of the asset portfolio, the *assigned portfolio*, is given by:

$$A := \sum_{i=1}^{n} q_i \, P_{i,0}\,.$$

For simplicity sake we assume that all the assets held are *admissible*. The *maximum matching adjustment*, defined as a premium on the rates for discounting liabilities, is obtained as:

$$\Delta r^{max} := r^{(A)} - r - \sigma^{avg}\,,$$

where:

$$r^{(A)} : \sum_{k=1}^{m} y_k \, e^{-r^{(A)}\,k\,\delta} = A\,,$$

$$r : \sum_{k=1}^{m} y_k \, e^{-r\,k\,\delta} = B\,,$$

and where:

$$\sigma^{avg} := \sum_{i=1}^{n} \widehat{\sigma}_i \, \frac{q_i \, P_{i,0}}{A}$$

is the *average fundamental spread*.

Remark. The rationale of the definition of the maximum matching adjustment is not completely clear. For example, if $A > B$ one has $r^{(A)} < r$ and then $\Delta r^{max} < 0$, which seems to be counterintuitive. Actually Δr^{max} has the expected positivity property only when the asset value A is sufficiently lower than the best estimate B, in order to have $r^{(A)} > r + \sigma^{avg}$. Some additional consistency conditions required in EIOPA's document do not seam sufficient to avoid these drawbacks. ■

The corresponding *Matching Adjustment* Δr is equal to the maximum matching adjustment multiplied by the *Application Ratio* α:

$$\Delta r := \alpha \, \Delta r^{max} \, ,$$

where the Application Ratio is given by:

$$\alpha := \max \left\{ 0, \, 1 - \frac{L^{stress}}{B} \right\} \, ,$$

L^{stress} being the *discounted-cash-flow-shortfall*, which "*reflects the losses through forced sales caused by the incidence of lapse risk, mortality risk, disability-morbidity risk and/or life catastrophe risk*" (EIOPA, 2012 p. 28). The shortfall L^{stress} is obtained by aggregating, via a prescribed correlation matrix, the individual shortfalls computed under prescribed stressed conditions on each of the four risk drivers considered.

Remark. One can define the *adjusted best estimate* of the liabilities as:

$$\widehat{B} := \sum_{k=1}^{m} y_k \, v(0,k) \, e^{-\Delta r \, k\delta} \, .$$

In terms of Net Asset Value, applying this Matching Adjustment Δr is the same as increasing the value of the asset by $\Delta B = B - \widehat{B}$. If $\alpha = 1$ one should have:

$$\widehat{B} \approx \sum_{k=1}^{m} y_k \, e^{-(r^{(A)} - \sigma^{avg}) \, k\delta} \, .$$

■

2.3 Matching Adjustment defined on the asset prices

Some aspects of the current proposal on the Matching Adjustment are questionable. In particular, the definition of the Application Ratio does not correctly take into account the actual matching structure between assets and liabilities. In order to better implement prescriptions of article 77 e, we propose a formulation of the Matching Adjustment derived as a premium on the price of the assets, instead of on the rates for discounting liabilities (a premium-based approach has also been proposed in CRO-CFO, 2012). Using a simple model with very natural assumptions, the matching properties of the asset-liability portfolio can be defined and measured in a reliable and objective manner. However, all the results of the approach can be easily reformulated in terms of adjustment on rates instead of prices.

In order to apply our approach we need to model the management of the asset portfolio during the lifetime of the outstanding policies, then we need to slightly extend the asset-liability model. The approach we propose can be defined *semi-deterministic*, in the sense that the default probabilities are properly modelled, while the interest rates are deterministic, the future interest rates being given by the forward rates implied by the current term structure.

2.3.1 The semi-deterministic model: additional definitions

When considering the management actions on the asset portfolio, we denote by:

$$s_{i,k}\,, \quad i=1,2,\ldots,n\,,\ k=0,1,\ldots,m\,, \quad \text{with} \quad 0 \leq s_{i,k} \leq q_i\,,$$

the amount of asset i sold at date $k < m_i$.

We then consider the price $\widehat{P}_{i,0}$ of asset i associated at time 0 to the fundamental spread $\widehat{\sigma}_i$. Using the market best practice, this *fundamental price*, or *EIOPA price*, is computed as:

$$\widehat{P}_{0,i} := \sum_{k=1}^{m} \overline{x}_{i,k}\, v(0,k)\, e^{-k\delta\,\widehat{\sigma}_i}\,, \quad i=1,2,\ldots,n\,.$$

We also consider the stream $\widetilde{\boldsymbol{x}}_i := \{\widetilde{x}_{i,1}, \widetilde{x}_{i,2}, \ldots, \widetilde{x}_{i,m}, \}$ of the *de-risked cash flows* of asset i consistent with the market spread, which are given by:

$$\widetilde{x}_{i,k} := \overline{x}_{i,k}\, e^{-k\delta\,\sigma_i}\,, \quad k = 1, 2, \ldots, m\,,$$

and the stream $\widehat{\boldsymbol{x}}_i := \{\widehat{x}_{i,1}, \widehat{x}_{i,2}, \ldots, \widehat{x}_{i,m}, \}$ of the de-risked cash flows of asset i consistent with the fundamental spread:

$$\widehat{x}_{i,k} := \overline{x}_{i,k}\, e^{-k\delta\,\widehat{\sigma}_i}\,, \quad k = 1, 2, \ldots, m\,.$$

Remark. In standard stochastic models for credit risk (see e.g. Duffie et al., 2003), the factors $e^{-k\delta\,\sigma_i}$ and $e^{-k\delta\,\widehat{\sigma}_i}$ provide the *risk-neutral* "probability of survival" (i.e., not-default) of the bond i consistent with the credit spread σ_i and $\widehat{\sigma}_i$, respectively. For simplicity sake, assume that the credit spreads include only the default event. Let τ_i denote the default time of asset i and let $\mathbf{1}_{\mathcal{E}}$ be the indicator function of the event $\mathcal{E}$. If one considers the stochastic cash flow $X_{i,k} := \overline{x}_{i,k}\,\mathbf{1}_{\tau_i > k\delta}$, one finds that the de-risked cash flow $\widetilde{x}_{i,k}$ (or $\widehat{x}_{i,k}$) is the expectation of $X_{i,k}$ with respect to the risk-neutral probabilities corresponding to the spread σ_i (or $\widehat{\sigma}_i$, resp.). Since we assumed that $\overline{x}_{i,k}$ is deterministic, this is also the expectation with respect to the forward risk-neutral probabilities. Therefore the price of $X_{i,k}$ is obtained by discounting $\widetilde{x}_{i,k}$ (or $\widehat{x}_{i,k}$) with the risk-neutral rate (see Appendix 1). ■

Using the de-risked cash flows, we can compute the future market price and the future EIOPA price of asset i, which are defined as, respectively:

$$P_{i,k} = \sum_{h=k+1}^{m} \widetilde{x}_{i,h}\, m(k,h)\,, \qquad \widehat{P}_{i,k} = \sum_{h=k+1}^{m} \widehat{x}_{i,h}\, m(k,h)\,, \qquad k = 1, 2, \ldots, m\,.$$

where:

$$m(k,h) := \frac{v(0,h)}{v(0,k)}\,, \quad h \geq k\,,$$

is the forward risk-free capitalization factor from date k to date h.

Remark. Therefore the future prices $P_{i,k}$ and $\widehat{P}_{i,k}$ $(k > 0)$ are computed under forward rate assumption: at date k the residual (ex-coupon) cash flow stream, de-risked under $P_{i,0}$ and

$\widehat{P}_{i,0}$ respectively, is discounted with the forward risk-free rates. Of course, for "non eligible assets" one has $\widehat{P}_{i,k} = P_{i,k}$. By the computational point of view, it is useful to observe that for $i = 1, 2, \ldots, n$ one has $P_{i,k} = \widehat{P}_{i,k} = 0$ for $k \geq m_i$. ∎

Using the previous definitions we can introduce our proposal for a Matching Adjustment based on prices.

2.3.2 The alternative Matching Adjustment definition

In our proposal, the maximum matching adjustment (as a premium on the market value of assets) is defined as:

$$\Delta A^{max} := \sum_{i=1}^{n} q_i \left(\widehat{P}_{i,0} - P_{i,0}\right).$$

Remark. This definition of the maximum matching adjustment can be expressed equivalently in terms of a discount rate applied to the liabilities. One has $\Delta A^{max} = \widehat{A} - A$. If ΔA^{max} would be applied, one would obtain a Net Asset Value $\widehat{S} = \widehat{A} - B$ instead of $S = A - B$. The Net Asset Value $\widehat{S}$ can also be obtained as $S = A - \widehat{B}$ with $\widehat{B} := B - \Delta A^{max}$. Then one can define a discount rate spread $\Delta\widehat{r}^{max}$ as the solution of:

$$\sum_{k=1}^{m} y_k \, v(0,k) \, e^{-\Delta\widehat{r}^{max} \, k \, \delta} = \widehat{B}\,.$$

This is conceptually analogous to Δr^{max}, without having the same inconsistencies however. Observe in particular that $\Delta\widehat{r}^{max} > 0 \, (< 0)$ if and only if $\Delta A^{max} > 0 \, (< 0)$, independently of whether A is greater or less than B. ∎

The maximum matching adjustment ΔA^{max} can be applied only in the case of "perfect matching", i.e. if it is not necessary to liquidate any amount of the assets held at time 0 during the lifetime of the liabilities. Any early liquidation of assets forced by liquidity problems will cause a loss and a consequent reduction of ΔA^{max}.

We introduce an estimate of the present value of the losses caused by forced sales by defining:

$$L := \sum_{i=1}^{n} \sum_{k=1}^{m} s_{i,k} \, (\widehat{P}_{i,k} - P_{i,k}) \, v(0,k) \, . \tag{1}$$

Once a criterion for defining a reference liquidation pattern $\{s^*_{i,k}\}$ has been chosen, the Application Ratio is naturally defined as:

$$\alpha^* := \max \left\{ 0, \, 1 - \frac{L^*}{\Delta A^{max}} \right\} ,$$

where L^* is the loss through forced sales generated by $\{s^*_{i,k}\}$. Hence the Matching Adjustment (in term of value) is given by:

$$\Delta A^* := \alpha^* \, \Delta A^{max} = \Delta A^{max} - L^* \, .$$

Remark. In terms of value, one can compare ΔA^* with the quantity ΔB considered in section (2.2). For a comparison in terms of rates, one can derive:

$$r^* : \; \sum_{k=1}^{m} y_k \, e^{-r^* \, k \, \delta} = B - \Delta A^* \, ,$$

and then consider $\Delta r^* := r^* - r$. This difference is conceptually analogous to the difference Δr obtained with the current proposal, in the framework of article 77 e. ∎

2.3.3 An optimality criterion

We define α by choosing $\{s^*_{i,k}\}$ (and L^*) as the solution of the following optimization problem:

$$\min_{s_{i,k}} \sum_{i=1}^{n} \sum_{k=1}^{m} s_{i,k} \, (\widehat{P}_{i,k} - P_{i,k}) \, v(0,k) \, ,$$

under a set of appropriate constraints. The objective function L is linear in the liquidation pattern. Moreover, as it will be shown in section (2.4), a natural choice of the optimization constraints produce inequalities which also involve only linear functions of $\{s_{i,k}\}$. Then the optimization problem is a linear programming problem.

Remark. We choose L as the objective function to be minimized in order to derive an upper bound for α, given the matching properties of assets and liabilities. If one considers the liability stream $\boldsymbol{y}^{stress}$ corresponding to the stressed scenarios prescribed by the Supervisor for computing the discounted-cash-flow-shortfall, then L^* provides an objective estimate of L^{stress}, consistent with the requirement of the technical specifications in EIOPA (2012). ■

2.4 The optimization problem

In order to formulate the optimization problem in a slightly more general form, we denote by $\boldsymbol{b} := \{b_1, b_2, \ldots, b_m\}$ the liability stream and by $\boldsymbol{a}_i := \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$ the cash flow stream generated by the asset i, $i = 1, 2, \ldots, n$. We can assume $\boldsymbol{b} = \boldsymbol{y}$ or $\boldsymbol{b} = \boldsymbol{y}^{stress}$ according to whether the central or the stressed scenario for the liabilities is assumed. Moreover we pose $\boldsymbol{a}_i = \widehat{\boldsymbol{x}}_i$, since in our model we are interested in considering the de-risked cash flows of asset i consistent with the fundamental spread $\widehat{\sigma}_i$. The objective function (1) must be minimized with respect to $\{s_{i,k}\}$, for $i = 1, 2, \ldots, n$, $1 \leq k < m_i$, under appropriate constraints.

2.4.1 The optimization constraints

The constraints to be considered in the optimization problem are quite natural. Of course we have:

· Non-negativity constraints:

$$s_{i,k} \geq 0\,, \quad i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m\,,$$

since short-selling are precluded, and:

· Quantity constraints:

$$\sum_{k=1}^{m} s_{i,k} \leq q_i\,, \quad i = 1, 2, \ldots, n\,. \tag{2}$$

Moreover at each date k of the time grid we have to include a matching constraint, requiring that there is enough liquidity to pay-out the corresponding insurance obligations. These requirements are obviously specified as follows.

· Matching constraint at date $k = 1$: $z_1 \geq 0$, with:

$$z_1 := \sum_{i=1}^{n} q_i \, a_{i,1} + \sum_{i=1}^{n} s_{i,1} \, P_{i,1} - b_1 \, .$$

Remark. Since $P_{i,k} = 0$ for $k \geq m_i$, double counting of a_{i,m_i} is avoided. ■

· Matching constraint at date $k = 2$: $z_2 \geq 0$, with:

$$z_2 := z_1 \, m_{1,2} + \sum_{i=1}^{n} (q_i - s_{i,1}) \, a_{i,2} + \sum_{i=1}^{n} s_{i,2} \, P_{i,2} - b_2 \, .$$

· Matching constraint at date $k = 3$: $z_3 \geq 0$, with:

$$z_3 := z_2 \, m_{2,3} + \sum_{i=1}^{n} (q_i - s_{i,1} - s_{i,2}) \, a_{i,3} + \sum_{i=1}^{n} s_{i,3} \, P_{i,3} - b_3 \, .$$

Therefore the following recursive equation holds:

$$\begin{cases} z_0 = 0 \\ z_k = m_{k-1,\,k} \, z_{k-1} + \displaystyle\sum_{i=1}^{n} \left[(q_i - s_{i,[k-1]}) \, a_{i,k} + P_{i,k} \, s_{i,k} \right] - b_k \, , \qquad k = 1, 2, \ldots, m \, , \end{cases} \tag{3}$$

where $s_{i,[k]} := \sum_{h=1}^{k} s_{i,h}$.

As it is shown in Appendix 2, the solution of recursion (3) can be written as:

$$z_k = \varphi_k + \sum_{i=1}^{n} \sum_{h=1}^{k} \lambda_{i,h,k} \, s_{i,h} - \beta_k \, , \quad k = 1, 2, \ldots, m \, , \tag{4}$$

where:

$$\varphi_k := \sum_{i=1}^{n} q_i \sum_{h=1}^{k} m_{h,k} \, a_{i,h} \tag{5}$$

are the proceeds at date k from the bond cash in-flows,

$$\beta_k := \sum_{h=1}^{k} m_{h,k} \, b_h \tag{6}$$

is the cumulated cash out-flows at date k from the insurance obligations and:

$$\lambda_{i,h,k} := m_{h,k}\, P_{i,h} - \sum_{j=h+1}^{k} m_{j,k}\, a_{i,j} \tag{7}$$

are the sales net proceeds at date k from one unit of asset i sold at date h.

2.4.2 Formulation of the linear programming problem

Taking into account that the liquidation of asset i only makes sense previously to the date m_i or m_y, the linear programming problem can be formulated as:

$$\begin{cases} \quad \min\limits_{s_{i,k}} \sum_{i=1}^{n} \sum_{k=1}^{m_i^y} s_{i,k}\,(\widehat{P}_{i,k} - P_{i,k})\, v(0,k)\,, \\ \textit{under}: \\ \textit{1) Matching constraints} \\ \quad \sum_{i=1}^{n} \sum_{h=1}^{k_i} \lambda_{i,h,k}\, s_{i,h} \geq (\beta_k - \varphi_k)\ , \\ \quad k = 1, 2, \ldots, m_y\,, \\ \textit{2) Quantity constraints} \\ \quad \sum_{k=1}^{m_i^y} s_{i,k} \leq q_i\,, \quad i = 1, 2, \ldots, n\,, \\ \textit{3) Non-negativity constraints} \\ \quad s_{i,k} \geq 0\,, \quad i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m_i\,, \end{cases} \tag{8}$$

where, for $i = 1, 2, \ldots, n$:

$$m_i^y := \min\{m_i - 1\,, m_y\}\,, \quad k_i = \min\{m_i - 1\,, k\}\,, \quad k = 1, 2, \ldots, m_y\,,$$

and when the coefficients φ_k, β_k and $\lambda_{i,h,k}$ are given by (5), (6) and (7) (consistently changing the upper limit of the summations).

Remark. Once the solution $\{s_{i,k}^*\}$ of problem (8) has been obtained, the value of α^* is immediately derived. However the solution $\{s^*_{i,k}\}$ itself, and the corresponding dual solution, can have useful interpretations and applications. ∎

2.4.3 The feasibility issue

While constraints 2) and 3) are obvious consistency requirements, the matching constraint 1) has a fundamental solvency meaning, requiring that the asset portfolio provides sufficient cash flows to meet the insurance obligations. So it is interesting to consider the feasibility of the optimization problem assuming that constraints 2) and 3) are fulfilled. In this case problem (8) is feasible if:

$$\sum_{i=1}^{n} q_i \, P_{i,0} \geq \sum_{k=1}^{m} b_k \, v(0,k) \, ,$$

i.e. if the market value of the asset portfolio is not lower that the best estimate of the liabilities. The weaker condition:

$$\sum_{i=1}^{n} q_i \, \widehat{P}_{i,0} \geq \sum_{k=1}^{m} b_k \, v(0,k)$$

should be sufficient for the feasibility only if liquidation of mispriced assets (i.e. with $\widehat{P}_{i,0} > P_{i,0}$) is not forced, that is if $s^*_{i,k} = 0$ for all $k < m_i$ and for i such that $\widehat{P}_{i,0} > P_{i,0}$.

2.5 Including the run-off constraints

Constraints in problem (8) are quite natural in our model. One can consider however to set also a *run-off condition* to the Matching Adjustment. By "run-off condition" we mean the requirement that only the assets in the assigned portfolio which are actually backing the liabilities are allowed to be valued at price $\widehat{P}_{i,k}$ instead of $P_{i,k}$. Therefore assets eventually held in the portfolio after the liability run-off (i.e. after all the liabilities have been settled) must be priced at $P_{i,k}$. The run-off condition is clearly inspired by a definition in stronger form of the asset-liability matching. It can be easily included in our model by modifying the quantity constraints 2), requiring that the constraint for asset i holds as an equality, instead of an inequality, if the maturity date m_i of the asset is greater than the terminal date m_y of the liability stream. Therefore under the run-off condition the linear programming problem is re-formulated as follows.

$$
\left\{
\begin{array}{l}
\qquad \min\limits_{s_{i,k}} \sum\limits_{i=1}^{n} \sum\limits_{k=1}^{m_i^y} s_{i,k} \, (\widehat{P}_{i,k} - P_{i,k}) \, v(0,k) \, , \\[2ex]
\textit{under}: \\[2ex]
\textit{1) Matching constraints} \\[1ex]
\qquad \sum\limits_{i=1}^{n} \sum\limits_{h=1}^{k_i} \lambda_{i,h,k} \, s_{i,h} \geq (\beta_k - \varphi_k) \ , \\[1ex]
\qquad k = 1, 2, \ldots, m_y \, , \\[2ex]
\textit{2) Quantity/run-off constraints} \\[1ex]
\qquad \textit{For } i = 1, 2, \ldots, n \ : \\[1ex]
\qquad \sum\limits_{k=1}^{m_i^y} s_{i,k} \leq q_i \quad \textit{if} \quad m_i \leq m_y \, , \\
\qquad \sum\limits_{k=1}^{m_i^y} s_{i,k} = q_i \quad \textit{if} \quad m_i > m_y \, , \\[2ex]
\textit{3) Non-negativity constraints} \\[1ex]
\qquad s_{i,k} \geq 0 \, , \quad i = 1, 2, \ldots, n, \; k = 1, 2, \ldots, m_i \, .
\end{array}
\right.
\tag{9}
$$

2.6 Including stocks in the assigned portfolio

A portfolio containing also a stock component could be considered by including an additional asset, indexed as $i = 0$. If $P_{0,0}$ is the market price at time 0 of one unit of stock and if $q_0 \geq 0$ is the corresponding quantity, the stock component is modelled in the programming problem by posing:

$$
a_{0,k} = 0 \, , \quad k = 1, 2, \ldots, m \, ,
$$

and:

$$
\left\{
\begin{array}{l}
P_{0,k} = \widehat{P}_{0,k} = P_{0,0} \, m(0,k) \, , \quad k = 1, 2, \ldots, m_y \, , \\[1ex]
P_{0,k} = \widehat{P}_{0,k} = 0 \, , \quad k = m_y + 1, \ldots, m \, .
\end{array}
\right.
$$

All the previous relations hold extending the formulae to $i = 0$.

Remark. To be conservative, the market price $P_{0,0}$ could be reduced by applying the relevant equity shock, as prescribed in the Standard Formula for the 5-th Quantitative Impact Study (QIS5, see EC, 2010). ■

3 Numerical example

Numerical examples can be easily produced using the software MatchRace, which we have specifically designed for solving the programming problems (8) and (9).

3.1 The data

We provide here a simple example where the stream $\boldsymbol{b}$ of the expected net liabilities is defined on an annual time grid with 20 years length ($\delta = 1$, $m_y = 20$). The liability cash flows are reported in the third column of Table 1 and illustrated in the first figure of Panel 1. In the fourth column of the table the corresponding cash flows under the stressed scenario are also reported; this liability pattern is illustrated in the third figure of Panel 1. In the second column of Table 1 the current term structure of risk-free interest rates is provided. With these discount rates the best estimate of the liabilities is $B = 100\,000$. To simplify the exposition the stream of stressed liabilities $\boldsymbol{b}^{stress}$ has been chosen in order that one also has $B^{stress} = 100\,000$.

The characteristics of the assigned asset portfolio are illustrated in Table 2. A number of $n = 18$ assets are held in the portfolio, one zero-coupon bond and 17 coupon bonds. The principal of all the assets is 100 and the coupon rate is 4% for all the coupon bonds. The asset maturities are given in the third column of the table. We assume a market spread σ_i of 200 basis points and an EIOPA spread $\widehat{\sigma}_i$ of 30 basis points for all the assets. The corresponding market and EIOPA prices at time zero are reported in the sixth and seventh column, respectively (the future prices are not reported for brevity). Given the quantities q_i provided in the last column of Table 2, the corresponding streams of asset cash flows on the time grid $\boldsymbol{t}$ are as in columns 5-7 of Table 1. The nominal cash flows are given in column 5, while the market and EIOPA de-risked cash flows are reported in column 6 and 7, respectively.

As one can see, the assets have been chosen in order to have some degree of matching between the de-risked asset cash flows $x_k := \sum_i \widehat{x}_{i,k}$ (illustrated in the second figure of Panel 1) and the liability cash flows (under the best estimate scenario) b_k. However the asset portfolio also includes a coupon bond maturing two years later than the liability run-off m_y ($m = m_{19} = 22$).

3.2 Matching adjustment according to the current proposal

It results that the market value of the assets is $A = 91\,500$ (NAV $= -8\,500$). Given the liability structure and the risk-free rates, one obtains $r = 1.46\%$ and, given the asset value A, one has $r^{(A)} = 2.93\%$. Since $\sigma^{avg} = 0.30\%$ we have $\Delta r^{max} = 1.17\%$. In order to derive the Application Ratio α according to the current EIOPA proposal, we need to compute L^{stress}. Using the figures in the first row of Table 1 (columns 4, 3, 5, respectively) one has $L^{stress} = 20\,817 - 9\,799 = 11\,018$. Then one gets:

$$\alpha := \max\left\{0,\ 1 - \frac{L^{stress}}{B}\right\} = \max\left\{0,\ 1 - \frac{11\,018}{100\,000}\right\} = 88.98\%\,,$$

and therefore $\Delta r = \alpha\,\Delta r^{max} = 1.043\%$. This corresponds to an adjusted value of the liabilities $\widehat{B} = 93\,962$, with a monetary adjustment $\Delta\widehat{B} = 6\,038$.

k	i_k	b_k	b_k^{stress}	$\sum_i \overline{x}_{i,k}$	$\sum_i \widetilde{x}_{i,k}$	x_k	z_k
1	0.805%	10615	20817	9799	9607	9770	0
2	0.504%	10615	9414	10299	9899	10237	0
3	0.593%	10474	9289	9519	8970	9434	0
4	0.759%	10332	9163	9799	9053	9682	0
5	0.953%	9907	8787	9519	8622	9377	0
6	1.151%	9270	8222	9239	8204	9074	0
7	1.332%	8492	7531	8459	7364	8283	0
8	1.493%	7784	6904	8199	6998	8005	21
9	1.637%	7077	6276	7899	6609	7689	436
10	1.761%	6369	5649	6024	4942	5847	0
11	1.866%	5378	4770	4529	3643	4383	0
12	1.972%	4246	3766	5800	4573	5595	0
13	2.038%	3185	2824	4600	3556	4424	0
14	2.105%	2123	1883	4440	3365	4258	0
15	2.173%	1415	1255	3280	2437	3136	0
16	2.186%	1062	941	2160	1573	2059	900
17	2.200%	708	628	1080	771	1026	1203
18	2.215%	354	314	40	28	38	878
19	2.229%	142	126	40	27	38	759
20	2.245%	71	63	40	27	38	707
21	2.245%	0	0	40	26	38	0
22	2.245%	0	0	1040	673	974	0

Table 1: Cash-flow streams of liabilities and assets

i	coupon	m_i	σ_i	$\widehat{\sigma}_i$	$P_{0,i}$	$\widehat{P}_{0,i}$	q_i
1	0%	1	2.00%	0.30%	97.271	98.907	65
2	4%	2	2.00%	0.30%	102.871	106.303	70
3	4%	3	2.00%	0.30%	104.009	109.155	65
4	4%	4	2.00%	0.30%	104.673	111.494	70
5	4%	5	2.00%	0.30%	104.905	113.344	70
6	4%	6	2.00%	0.30%	104.785	114.771	70
7	4%	7	2.00%	0.30%	104.448	115.909	65
8	4%	8	2.00%	0.30%	103.969	116.837	65
9	4%	9	2.00%	0.30%	103.394	117.600	65
10	4%	10	2.00%	0.30%	102.779	118.261	48
11	4%	11	2.00%	0.30%	102.174	118.879	35
12	4%	12	2.00%	0.30%	101.455	119.301	50
13	4%	13	2.00%	0.30%	100.958	119.944	40
14	4%	14	2.00%	0.30%	104.547	125.036	40
15	4%	15	2.00%	0.30%	99.787	120.854	30
16	4%	16	2.00%	0.30%	99.587	121.742	20
17	4%	17	2.00%	0.30%	99.378	122.582	10
18	4%	22	2.00%	0.30%	98.495	126.517	10

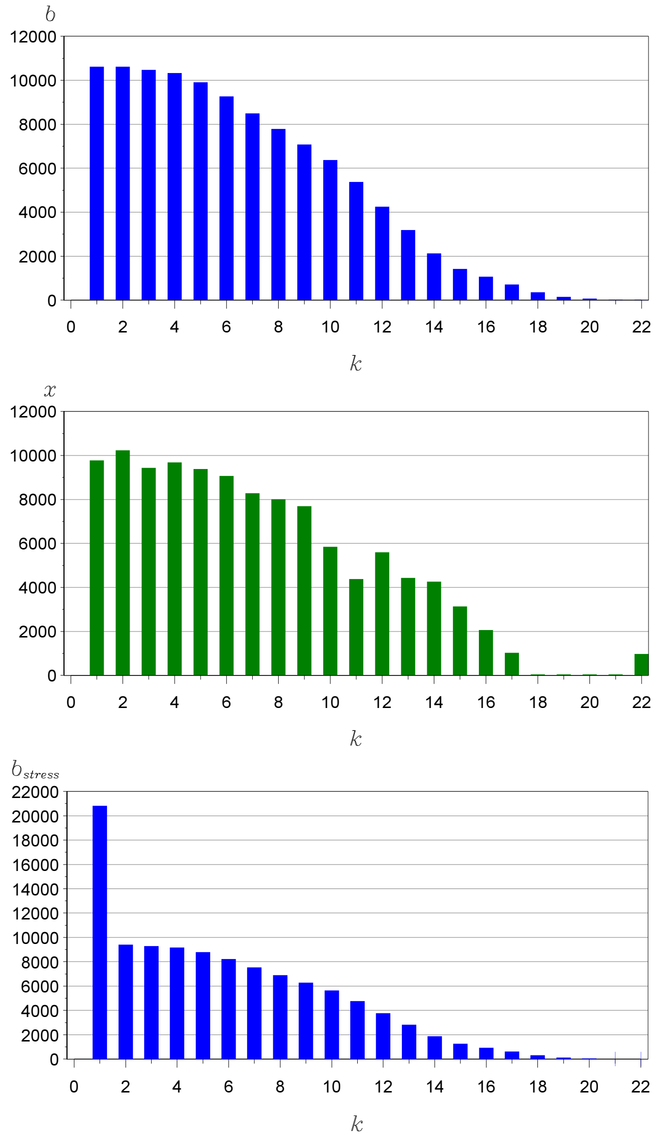
Table 2: Assigned asset portfolio (principal = 100)

3.3 Matching adjustment under the optimality approach

While the market price of the assets is $A = 91\,500$, using the EIOPA prices one has $\widehat{A} = \sum_{i=1}^{n} q_i\, \widehat{P}_{0,i} = 101\,867$. Therefore the maximum matching adjustment is $\Delta A^{max} = \widehat{A} - A = 10\,367$, which corresponds to a maximum interest rate spread $\Delta r^{max} = 1.85\%$.
Since with the EIOPA prices one has $\widehat{A} > B$ and since $B^{stress} = B$ (the NAV is $= 1\,867$ in both cases), the feasibility of the optimization problem is not precluded, both with respect to the central and the stressed liabilities. We study the linear programming problem (9), then requiring that also the run-off condition is fulfilled.

In a first step we consider the optimal matching problem with respect to the central liability stream $\boldsymbol{b}$. Running the problem solver MatchRace, the programming problem turns out to be feasible. The resulting values of z_k, which are reported in the last column of Table 1, show that all the quantity/run-off constraints have been satisfied, with the equality on 15 of the 22 dates of $\boldsymbol{t}$ and with the inequality on the remaining 7 dates.

By inspection of the problem solution, it results that in order to match the liabilities the assets have been sold according to Table 3. The minimum loss corresponding to these sales is $L^* = 1\,383$. Since the maximum matching adjustment is $\Delta A^{max} = \widehat{A} - A = 10\,367$ (corresponding to an interest rate spread $\Delta r^{max} = 1.85\%$), the matching adjustment is $\Delta A^* =$



Panel 1. Cash flow streams of assets and liabilities

$10\,367 - 1\,383 = 8.985$, corresponding to an interest rate spread $\Delta r^* = 1.59\%$. This corresponds to an Application Ratio $\alpha^* = 1 - 1\,383/10\,367 = 86.66\%$.

We consider then the optimization problem with respect to the stressed liability stream $\boldsymbol{b}^{stress}$. Under the constraints given by the different liability stream the linear programming

i	$k=1$	$k=2$	$k=2$	$k=4$	$k=5$	$k=6$	$k=7$	$k=10$	$k=11$
12	8.627	2.574							
13		1.802	8.752						
14							1.492	3.429	15.426
15			3.316	8.613	4.968				
16					0.597				
18					2.290	4.406	3.304		

Table 3: Quantity $s^*_{i,k}$ of asset i optimally sold at date k to match unstressed liabilities

problem results to be infeasible. This means that despite of the advantages given by the fundamental spread, the liability stream $\boldsymbol{b}^{stress}$ cannot be matched by the asset cash flows. As it seems natural, if one requires that the matching properties also hold under a stressed scenario, in these conditions no matching adjustment should be applied. This is different from the results of the current EIOPA approach which turns out to be insensitive to the actual matching properties of the asset-liability stream.

Appendix 1

Let $v_i(0,k)$ be the discount factor for date $k \geq 0$ corresponding to the credit rating of asset i. The constant spread σ_i is such that:

$$v_i(0,k) = v(0,k)\, e^{-k\delta\sigma_i} \,. \tag{10}$$

Let us denote by $V(0, X_{i,k})$ the market price at time 0 of $X_{i,k}$ to be paid at date k. In a standard stochastic model (see Duffie et al., 2003 pp. 100-118) one also has (assuming for simplicity sake a 0% recovery rate):

$$V(0, X_{i,k}) = \mathbf{E}^Q \left(e^{-\int_0^{k\delta} r_t dt}\ \mathbf{1}_{\tau_i > k\delta}\ X_{i,k} \right) \,,$$

where $\mathbf{E}^Q$ denotes the expectation with respect to the risk-neutral probabilities and r_t is the instantaneous risk-free interest rate. Under independence assumptions one has:

$$V(0, X_{i,k}) = \mathbf{E}^Q \left(\overline{x}_{i,k}\, e^{-\int_0^{k\delta} r_t dt} \right) \mathbf{E}^Q \left(\mathbf{1}_{\tau_i > k\delta} \right) = v(0,k)\ \mathbf{E}^{F_k} \left(\overline{x}_{i,k} \right)\ p_i(k) \,,$$

where $\mathbf{E}^{F_k}$ denotes the expectation with respect to the forward risk-neutral probabilities and $p_i(k) := \mathbf{E}^Q \left(\mathbf{1}_{\tau_i > k\delta} \right)$ is the risk-neutral probability that the asset survives at time $k\delta$. Since $\overline{x}_{i,k}$ is deterministic we simply have:

$$V(0, X_{i,k}) = \overline{x}_{i,k}\, v(0,k)\, p_i(k) \,.$$

Given that the expression $V(0, X_{i,k}) = \overline{x}_{i,k}\, v_i(0,k)$ also holds, by (10) one obtains $e^{-k\delta\sigma_i} = p_i(k)$. The same argument holds referring to the fundamental spread, i.e. one has $e^{-k\delta\,\widehat{\sigma}_i} = \widehat{p}_i(k)$, where $\widehat{p}_i(k)$ is the survive probability corresponding to $\widehat{\sigma}_i$.

Appendix 2

The solution (4) of recursion (3) can be found by induction.

Equation (3) is verified for $k = 1$. Given $k > 1$, suppose that the equation is verified for $k-1$; one has:

$$
\begin{aligned}
z_k &= m_{k-1,k} \left[\sum_{i=1}^{n} \sum_{h=1}^{k-1} \lambda_{i,h,k-1}\, s_{i,h} + \varphi_{k-1} - \beta_{k-1} \right] + \sum_{i=1}^{n} \left[(q_i - s_{i,[k-1]})\, a_{i,k} + P_{i,k}\, s_{i,k} \right] - b_k \\
&= \sum_{i=1}^{n} \left[\sum_{h=1}^{k-1} m_{k-1,k}\, \lambda_{i,h,k-1}\, s_{i,h} + P_{i,k}\, s_{i,k} - a_{i,k}\, s_{i,[k-1]} + q_i\, a_{i,k} \right] + m_{k-1,k}\, (\varphi_{k-1} - \beta_{k-1}) - b_k \\
&= \sum_{i=1}^{n} \left[\sum_{h=1}^{k-1} m_{k-1,k}\, \lambda_{i,h,k-1}\, s_{i,h} + P_{i,k}\, s_{i,k} - a_{i,k}\, s_{i,[k-1]} \right] + \varphi_k - \beta_k\, .
\end{aligned}
$$

Using (7), the term is square brackets is given by:

$$
\begin{aligned}
&= \sum_{h=1}^{k-2} \left[m_{h,k} P_{i,h} - \sum_{j=h+1}^{k-1} m_{j,k}\, a_{i,j} \right] s_{i,h} + m_{k-1,k} P_{i,k-1} s_{i,k-1} + P_{i,k}\, s_{i,k} - a_{i,k} \sum_{h=1}^{k-1} s_{i,h} \\
&= \sum_{h=1}^{k-2} \left[m_{h,k}\, P_{i,h} - \sum_{j=h+1}^{k-1} m_{j,k}\, a_{i,j} - a_{i,k} \right] s_{i,h} + (m_{k-1,k}\, P_{i,k-1} - a_{i,k}) s_{i,k-1} + P_{i,k}\, s_{i,k} \\
&= \sum_{h=1}^{k-2} \left[m_{h,k}\, P_{i,h} - \sum_{j=h+1}^{k} m_{j,k}\, a_{i,j} \right] s_{i,h} + (m_{k-1,k}\, P_{i,k-1} - a_{i,k}) s_{i,k-1} + P_{i,k}\, s_{i,k} \\
&= \sum_{h=1}^{k-1} \left[m_{h,k}\, P_{i,h} - \sum_{j=h+1}^{k-1} m_{j,k}\, a_{i,j} \right] s_{i,h} + P_{i,k}\, s_{i,k}\, ;
\end{aligned}
$$

therefore one has:

$$
\begin{aligned}
z_k &= \sum_{i=1}^{n} \left\{ \sum_{h=1}^{k-1} \left[m_{h,k}\, P_{i,h} - \sum_{j=h+1}^{k-1} m_{j,k}\, a_{i,j} \right] s_{i,h} + P_{i,k}\, s_{i,k} \right\} + \varphi_k - \beta_k \\
&= \sum_{i=1}^{n} \sum_{h=1}^{k} \lambda_{i,h,k}\, s_{i,h} + \varphi_k - \beta_k\, .
\end{aligned}
$$

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