Flaviano Battelli, Matteo Franca & Kenneth J. Palmer (2022) Exponential dichotomy for noninvertible linear difference equations: block triangular systems, Journal of Difference Equations and Applications, 28:8, 1054-1086.

The final published version is available online at: https://doi.org/10.1080/10236198.2022.2113072

Exponential Dichotomy for Noninvertible Linear Difference Equations: Block Triangular Systems

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Abstract

In this paper block upper triangular systems of linear difference equations are considered, in which the coefficient matrices are not assumed invertible. The relationship between the exponential properties of such a system and its associated block diagonal system is studied. The reason it is important to study triangular systems is that any system of linear difference equations is kinematically similar to an upper triangular system. In the bounded invertible case, it is known that for equations on the intervals $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$, a block upper triangular system has an exponential dichotomy if and only if the associated block diagonal system has one. However when $J = \mathbb{Z}$, only the sufficiency holds. The sufficiency extends to the noninvertible case, provided

*Partially supported by GNAMPA-INdAM (Italy) and PRIN project *Equazioni Differenziali Ordinarie e Applicazioni*(Italy)

†Partially supported by GNAMPA-INdAM (Italy) and PRIN project *Equazioni Differenziali Ordinarie e Applicazioni*(Italy)

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the off-diagonal matrices are bounded. However the necessity does not hold even when $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$. Nevertheless if certain conditions are added, then the necessity does hold and it is also shown that these conditions are needed since it turns out that if both the triangular and diagonal systems have dichotomies, then these extra conditions must hold.

Mathematics Subject Classification. 34D05, 34C25.
Key words.

1 Introduction

In this paper we consider a block upper triangular system

$$x(k+1) = A(k)x(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) & \cdots & A_{1p}(k) \\ 0 & A_{22}(k) & \cdots & A_{2p}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix} x(k), \qquad (1)$$

where $A_{ij}(k)$ is $n_i \times n_j$ with $\sum_{i=1}^{p} n_i = n$, and its associated block diagonal system

$$x(k+1) = D(k)x(k) = \begin{pmatrix} A_{11}(k) & 0 & \cdots & 0 \\ 0 & A_{22}(k) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix} x(k) \qquad (2)$$

on an interval J, which we take to be $\mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$. For a system $x(k+1) = A(k)x(k)$, we define its *transition matrix* $\Phi(k, m)$ as $A(k-1)\cdots A(m)$ for $k > m$ in J and as I for $k = m$ in J. (Note when $J = \mathbb{Z}_-$, it is to be understood that the equation $x(k+1) = A(k)x(k)$ holds for $k \leq -1$ but that the transition matrix $\Phi(k, m)$ is defined for $m \leq k \leq 0$.) It is not assumed that $A(k)$ is invertible or that its norm is uniformly bounded. The question we are concerned with is the relation between the exponential dichotomy properties of (1) and (2).

The reason it is important to study triangular systems is that *any system of linear difference equations is kinematically similar to an upper triangular system.* To prove this, consider the system

$$x(k+1) = A(k)x(k), \quad k \in J. \qquad (3)$$

Suppose first that $A(k)$ is invertible for all k. Choose a fixed $m \in J$. Then for all $k \in J$, its transition matrix $\Phi(k, m)$ is invertible and applying Gram-Schmidt to its columns we may write $\Phi(k, m) = S(k)R(k)$, where $S(k)$ is orthogonal and $R(k)$ is upper triangular. Note that $\Phi(k+1, m) = A(k)\Phi(k, m) = A(k)S(k)R(k)$ but also $\Phi(k+1, m) = S(k+1)R(k+1)$. So $S^*(k+1)A(k)S(k) = R(k+1)R^{-1}(k)$. This means the kinematic similarity $x = S(k)y$ takes equation (3) into the equation $y(k+1) = B(k)y(k)$, where $B(k)$ is the upper triangular

$R(k+1)R^{-1}(k)$. This completes the proof for the invertible case. Consider now the general case. For all $k \in J$ and $i = 1, 2, ...$, there exists an invertible matrix $A^i(k)$ such that $|A^i(k) - A(k)| < 1/i$. We know there exist $S^i(k)$ orthogonal and $R^i(k)$ upper triangular such that $(S^i(k+1))^* A^i(k) S^i(k)$ is upper triangular. Since $|S^i_k| \leq 1$ for all k and i, we may use Cantor's diagonalization method to find a subsequence $S^{j_i}(k) \to S(k)$ for all k. Then $(S^{j_i}(k))^* \to S^*(k)$ for all k. Then since $(S^{j_i}(k))^* S^{j_i}(k) = I$ for all i and k it follows that $S^*(k)S(k) = I$ for all k. So $S(k)$ is orthogonal for all k. However for all i and k we also know that $(S^{j_i}(k+1))^* A^{j_i}(k) S^{j_i}(k)$ is upper triangular. So the limit $S^*(k+1)A(k)S(k)$ is also upper triangular. Hence the proof is complete.

In the invertible case, some results relating the dichotomy properties of (1) and (2) have already been proved. When $A(k)$ is invertible and uniformly bounded, it follows from Theorem 4.1 in [7] that when $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$, (1) has an exponential dichotomy if and only if (2) has one. However when $J = \mathbb{Z}$, if (2) has an exponential dichotomy then (1) has one but the converse is not true in general. We are not aware of any results in the noninvertible case.

When $A(k)$ is neither invertible nor bounded, it turns out that in one direction the results are much more complicated. First we look at the less complicated direction. Consider system (1) on $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ or $\mathbb{Z}$, where $A_{ij}(k)$ is bounded for $i \neq j$. Then in Theorem 1 in Section 3, we use the roughness theorem to show that if the diagonal system (2) has an exponential dichotomy on J, system (1) also has one and with projection of the same rank. We also give an example to show that the boundedness condition is necessary, even for invertible systems.

In Section 4 we first consider invertible systems and show even when they are unbounded, (1) having an exponential dichotomy on $\mathbb{Z}_+$ or $\mathbb{Z}_-$ implies (2) has an exponential dichotomy also. However we show by example that the same result does not hold for $\mathbb{Z}$ even if $A(k)$ is bounded.

When $A(k)$ is not invertible, in Section 5 we show by examples that it can happen that (1) has an exponential dichotomy on $\mathbb{Z}_+$ or $\mathbb{Z}_-$, but (2) does not have an exponential dichotomy. In the invertible case, we were able to show that if (1) has an exponential dichotomy on $\mathbb{Z}_+$ or $\mathbb{Z}_-$, then the projection can be taken as upper triangular. However one of the examples in Section 5 shows that, in general, this is not true in the noninvertible case.

In Theorem 3 in Section 6, we show that for $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ and $\mathbb{Z}$, exponential dichotomy on J for (1) implies the same for (2) provided that for the exponential dichotomy of (1), the projection can be taken in upper triangular form with each diagonal block having rank independent of k. For $\mathbb{Z}$ this seems to be a new result even in the bounded invertible case. In Section 7 we prove a converse to this. In Theorems 4, 5 and 6 we show for $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ and $\mathbb{Z}$ respectively that if both (1) and (2) have an exponential dichotomy on J, then the projection for (1) can be taken in upper triangular form with each diagonal block having rank independent of k with the projection for (2) being the diagonal part. It seems that for $\mathbb{Z}$ this result is new even in the bounded, invertible case. In Section

8 we specialize our results to upper triangular systems, that is, block upper triangular systems where the blocks are scalars.

In Section 2 we prove some results about exponential dichotomies in block diagonal systems which are needed later.

Note that for differential equations, exponential dichotomy of triangular systems have been studied in [1] and [4]. Also a result for exponential trichotomy for triangular systems of linear difference equations is given in [5].

In all that follows, we denote the transition matrix of (1) or (2) by $\Phi(k,m)$ and by $\Phi_i(k,m)$ the transition matrix of $x_i(k+1) = A_{ii}(k)x_i(k)$.

2 Block diagonal systems

In this section we study exponential dichotomies in block diagonal systems. First we recall the definition of exponential dichotomy from [2].

Definition. We say system (3) with transition matrix $\Phi(k,m)$ has an *exponential dichotomy* on an infinite interval J of integers if there is a projection function $P(k)$ of constant rank such that

$$\Phi(k,m)P(m) = P(k)\Phi(k,m)$$

for $k \geq m$ in J and $A(k) : \mathcal{N}P(k) \to \mathcal{N}P(k+1)$ is invertible for $k, k+1$ in J so that $\Phi(k,m) : \mathcal{N}P(m) \to \mathcal{N}P(k)$ is invertible for $k \geq m$ in J and either
(i) there exist positive constants K and α such that for $k \geq m$ in J

$$|\Phi(k,m)P(m)| \leq Ke^{-\alpha(k-m)}, \quad |\Phi(m,k)(I - P(k))| \leq Ke^{\alpha(k-m)},$$

where $\Phi(m,k)$ is the inverse of $\Phi(k,m) : \mathcal{N}P(m) \to \mathcal{N}P(k)$ or, equivalently,
(ii) there are positive constants M, K and α such that

$$|P(k)|, \ |I - P(k)| \leq M \quad \text{for } k \in J,$$

and for $k \geq m$ in J

$$|\Phi(k,m)\xi| \leq Ke^{-\alpha(k-m)}|\xi|, \quad \xi \in \mathcal{R}P(m),$$

$$|\Phi(k,m)\xi| \geq K^{-1}e^{\alpha(k-m)}|\xi|, \quad \xi \in \mathcal{N}P(m).$$

When $x(k+1) = A(k)x(k)$ has a dichotomy on $\mathbb{Z}$ or $\mathbb{Z}_+$, we refer to $\mathcal{R}P(m)$ as the *stable subspace* at m. It follows from Lemma 3.1 in [2] that this is just the subspace of initial values at m of solutions bounded in $k \geq m$. Similarly, when the dichotomy is on $\mathbb{Z}$ or $\mathbb{Z}_-$, we refer to $\mathcal{N}P(m)$ as the *unstable subspace* at m and by Lemma 3.1 in [2], this is just the subspace of initial values at m of solutions bounded in $k \leq m$. If we just say stable subspace or unstable subspace, it is understood that $m = 0$.

The following proposition is the main result of this section. Note that (ii) is a strengthening of what is proved in (i), as in (ii) it is not assumed that the dichotomy for (2) has a projection in block diagonal form.

Proposition 1. (i) *System (2) has an exponential dichotomy on* $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ *or* $\mathbb{Z}$ *with projection* $P(k)$ *in the block diagonal form*

$$P(k) = \begin{pmatrix} P_1(k) & 0 & 0 & \cdots & \cdot & 0 \\ 0 & P_2(k) & 0 & \cdots & \cdot & 0 \\ 0 & 0 & P_3(k) & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1}(k) & 0 \\ 0 & 0 & 0 & \cdots & \cdot & P_p(k) \end{pmatrix}, \tag{4}$$

if and only if for all i,

$$x_i(k+1) = A_{ii}(k)x_i(k) \tag{5}$$

has an exponential dichotomy on J *with projection* $P_i(k)$.

(ii) *System (2) has an exponential dichotomy on* $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ *or* $\mathbb{Z}$ *if and only if for all* i, (5) *has an exponential dichotomy on* J *with projection, say,* $P_i(k)$. *Moreover the projection* $P(k)$ *of the dichotomy for* (2) *can be taken as in* (4). *Hence when* $J = \mathbb{Z}_+$ *or* $\mathbb{Z}$, *the stable subspace for* (2) *is the Cartesian product* $\mathcal{R}P_1(0) \times \mathcal{R}P_2(0) \times \cdots \times \mathcal{R}P_p(0)$; *when* $J = \mathbb{Z}_-$ *or* $\mathbb{Z}$, *the unstable subspace for* (2) *is the Cartesian product* $\mathcal{N}P_1(0) \times \mathcal{N}P_2(0) \times \cdots \times \mathcal{N}P_p(0)$.

Proof. (i) First we prove the necessity. So suppose that (2) has an exponential dichotomy on $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ or $\mathbb{Z}$ with constants K, α and projection $P(k)$ in the block diagonal form (4). Since

$$\begin{aligned} &\operatorname{diag}(A_{11}(k)P_1(k), \ldots, A_{pp}(k)P_p(k)) = A(k)P(k) = P(k+1)A(k) \\ &= \operatorname{diag}(P_1(k+1)A_{11}(k), \ldots, P_p(k+1)A_{pp}(k)), \end{aligned}$$

the invariance of $P_i(k)$ with respect to (5) follows from the invariance of $P(k)$ with respect to (2). Next suppose for some i, $\Phi_i(k,m)y = 0$ for some $y \in \mathcal{N}P_i(m)$ and $k \geq m$. Then $x = (0, \ldots, y, \ldots, 0)$ with y in the ith place is in $\mathcal{N}P(m)$ and, if Φ is the transition matrix for (2),

$$\Phi(k,m)x = \operatorname{diag}(\Phi_1(k,m), \ldots, \Phi_p(k,m))x = (0, \ldots, \Phi_i(k,m)y, \ldots, 0) = 0.$$

Since $\Phi(k,m)$ is one to one on $\mathcal{N}P(m)$, it follows that $x = 0$ and hence $y = 0$. So for each i, $\Phi_i(k,m)$ is one to one on $\mathcal{N}P_i(m)$ when $k \geq m$. By invariance, $\Phi_i(k,m)$ maps $\mathcal{N}P_i(m)$ into $\mathcal{N}P_i(k)$. So the rank of $P_i(k)$ is less than or equal to the rank of $P_i(m)$ for all i. However, the sums of the ranks are equal and so the rank of $P_i(k)$ is equal to the rank of $P_i(m)$ for all i. Hence $\Phi_i(k,m)$ maps $\mathcal{N}P_i(m)$ bijectively onto $\mathcal{N}P_i(k)$ and we denote by $\Phi_i(m,k) : \mathcal{N}P_i(k) \to \mathcal{N}P_i(m)$ the inverse of $\Phi_i(k,m)$ restricted to $\mathcal{N}P_i(m)$. Then we see that the inverse $\Phi(m,k) : \mathcal{N}P(k) \to \mathcal{N}P(m)$ of $\Phi(k,m)$ restricted to $\mathcal{N}P(m)$ satisfies

$$\Phi(m,k) = \operatorname{diag}(\Phi_1(m,k), \ldots, \Phi_p(m,k)).$$

Next since for $k \geq m$,

$$\Phi(k,m)P(m) = \operatorname{diag}(\Phi_1(k,m)P_1(m), \ldots, \Phi_p(k,m)P_p(m))$$

and

$$\Phi(m,k)(\mathbf{I}-P(k)) = \operatorname{diag}(\Phi_1(m,k)(\mathbf{I}-P_1(k)),\ldots,\Phi_p(m,k)(\mathbf{I}-P_p(k))),$$

it follows from the inequalities

$$|\Phi(k,m)P(m)| \le Ke^{-\alpha(k-m)}, \quad |\Phi(m,k)(\mathbf{I}-P(k))| \le Ke^{-\alpha(k-m)}$$

that for each i

$$|\Phi_i(k,m)P_i(m)| \le Ke^{-\alpha(k-m)}, \quad |\Phi_i(m,k)(\mathbf{I}-P_i(k))| \le Ke^{-\alpha(k-m)},$$

provided we are using a matrix norm which has the property that if A is a partitioned matrix $[A_{ij}]$, then for each i,j, $|A_{ij}| \le |A|$. Hence, for all i, (5) has an exponential dichotomy on J with projection $P_i(k)$. Thus the necessity is proved.

The sufficiency is proved by reversing these arguments.

(ii) Note first that it follows from (i) that if for each i, (5) has an exponential dichotomy on J with projection $P_i(k)$, then (2) has an exponential dichotomy on J with projection $P(k)$ as in (4).

Now assume that (2) has an exponential dichotomy on J with projection $P(k)$, where we are not assuming $P(k)$ is in block diagonal form. We first suppose $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$.

First we prove the case $p = 2$. Consider first the case of $\mathbb{Z}_+$. For $m \ge 0$, define $V_i(m)$ as the subspace of initial values of solutions of (5) which are bounded in $k \ge m$. Since a solution $(x_1(k), x_2(k)))$ of (2), with $p = 2$, is bounded on $k \ge m$ if and only if for each i, $x_i(k)$ is a bounded solution of (5) on $k \ge m$, and since $\mathcal{R}P(m)$ is the subspace of initial values of solutions of (2) which are bounded in $k \ge m$, it follows that

$$\mathcal{R}P(m) = V_1(m) \times V_2(m).$$

For $i = 1,2$ we let W_i be subspaces such that $\mathbb{R}^{n_i} = V_i(0) \oplus W_i$ and for $m \ge 0$ define

$$W_i(m) = \Phi_i(m,0)(W_i),$$

so that

$$W(m) := \Phi(m,0)(W_1 \times W_2) = \Phi_1(m,0)(W_1) \times \Phi_2(m,0)(W_2) = W_1(m) \times W_2(m).$$

Since $W_1 \times W_2$ is a complementary subspace to $\mathcal{R}P(0)$, it follows from the proof of Proposition 3.2 in [2], that (2) has an exponential dichotomy on $\mathbb{Z}_+$ with projection $Q(k)$ having the same range as $P(k)$ and nullspace $W(k)$. In particular, for $m \ge 0$

$$\mathcal{R}P(m) \oplus W(m) = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

It follows that

$$V_1(m) \oplus W_1(m) = \mathbb{R}^{n_1}, \quad V_2(m) \oplus W_2(m) = \mathbb{R}^{n_2}.$$

For $i = 1, 2$, let $Q_i(m)$ be the projection with range $V_i(m)$ and nullspace $W_i(m) = \Phi_i(m, 0)(W_i)$. Then

$$Q(m) = \begin{pmatrix} Q_1(m) & 0 \\ 0 & Q_2(m) \end{pmatrix},$$

since it is a projection with the same range as $P(m)$ and nullspace $W(m)$. Then it follows from (i) that for $i = 1, 2$, (5) has an exponential dichotomy with projection $Q_i(k)$. Since we already observed above that that (2) has an exponential dichotomy on $\mathbb{Z}_+$ with projection $Q(k)$, this completes the proof of (ii) for $\mathbb{Z}_+$ when $p = 2$.

Now consider the case of $\mathbb{Z}_-$. For $m \leq 0$ and $i = 1, 2$, define $V_i(m)$ as the subspace of those $\xi_i \in \mathbb{R}^{n_i}$ for which there is a solution $x_i(k)$ of (5) which is bounded in $k \leq m$ with $x_i(m) = \xi_i$. Since $(x_1(k), x_2(k)))$ is a solution of (2), with $p = 2$, which is bounded on $k \leq m$ if and only if for each i, $x_i(k)$ is a solution of (5) bounded in $k \leq m$, and since $\mathcal{N}P(m)$ is the subspace of initial values at m of solutions of (2) which are bounded in $k \leq m$, we have

$$\mathcal{N}P(m) = V_1(m) \times V_2(m).$$

Next for $i = 1, 2$ define

$$W_i(m) = \{\xi : \Phi_i(0, m)\xi \in W_i\} = \Phi_i(0, m)^{-1}(W_i),$$

where W_i is a fixed complement of $V_i(0)$ in $\mathbb{R}^{n_i}$. Then

$$\begin{aligned} W(m) &:= \Phi(0, m)^{-1}(W_1 \times W_2) \\ &= \Phi_1(0, m)^{-1}(W_1) \times \Phi_2(m, 0)^{-1}(W_2) \\ &= W_1(m) \times W_2(m). \end{aligned}$$

From the proof of Proposition 3.2 in [2], since $W_1 \times W_2$ is a complement to $\mathcal{N}P(0) = V_1 \times V_2$, we know that (2) has an exponential dichotomy on $\mathbb{Z}_-$ with projection $Q(k)$ having the same nullspace as $P(k)$ and range $W(k)$. In particular

$$W(m) \oplus \mathcal{N}P(m) = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

It follows that

$$W_1(m) \oplus V_1(m) = \mathbb{R}^{n_1}, \quad W_2(m) \oplus V_2(m) = \mathbb{R}^{n_2}.$$

For $i = 1, 2$ define $Q_i(m)$ as the projection with range $W_i(m)$ and nullspace $V_i(m)$. Then

$$Q(m) = \begin{pmatrix} Q_1(m) & 0 \\ 0 & Q_2(m) \end{pmatrix}$$

is the projection with the same nullspace as $P(m)$ and range $V(m)$. It follows from (i) that for $i = 1, 2$, (5) has an exponential dichotomy with projection $Q_i(k)$. This completes the proof of (ii) for $\mathbb{Z}_-$ when $p = 2$.

The proof of (ii) for general $p \geq 2$ follows by induction on p using the $p = 2$ case. Thus the proof of (ii) is completed for $\mathbb{Z}_+$ and $\mathbb{Z}_-$.

Finally, suppose that (2) has an exponential dichotomy on $\mathbb{Z}$. By the $\mathbb{Z}_+$ and $\mathbb{Z}_-$ cases, the stable subspace for (2) is the Cartesian product $\mathcal{R}P_1^+(0) \times \cdots \times \mathcal{R}P_p^+(0)$ and the unstable subspace for (2) is the Cartesian product $\mathcal{N}P_1^-(0) \times \cdots \times \mathcal{N}P_p^-(0)$, where $P_i^{\pm}(k)$ is the projection of the dichotomy of (5) on $\mathbb{Z}_{\pm}$. We know that

$$(\mathcal{R}P_1^+(0) \times \cdots \times \mathcal{R}P_p^+(0)) \oplus (\mathcal{N}P_1^-(0) \times \cdots \times \mathcal{N}P_p^-(0)) = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}.$$

From this it follows that $\mathcal{R}P_i^+(0) \oplus \mathcal{N}P_1^-(0) = \mathbb{R}^{n_i}$ for all i. Hence, by the proof of Corollary 3.3 in [2], each equation (5) has an exponential dichotomy on $\mathbb{Z}$ with projection $P_i(k)$ (say) such that $\mathcal{R}P_i(0) = \mathcal{R}P_i^+(0)$ and $\mathcal{N}P_i(0) = \mathcal{N}P_i^-(0)$. Then it follows from (i) that the projection for the dichotomy of (2) on $\mathbb{Z}$ is as in (4). □

3 Diagonal exponential dichotomy implies upper triangular exponential dichotomy

In Theorem 1, we show if (2) has an exponential dichotomy on $\mathbb{Z}_+$, $\mathbb{Z}_-$ or $\mathbb{Z}$, then (1) has one also provided a certain boundedness condition holds.

Theorem 1. *Consider system* (1) *on $J = \mathbb{Z}_+, \mathbb{Z}_-$ or $\mathbb{Z}$, where $A_{ij}(k)$ is bounded for $i \neq j$. Then if the diagonal system* (2) *has an exponential dichotomy, system* (1) *also has one and with projection of the same rank.*

Proof. We define the matrix

$$S = \begin{pmatrix} \mathbf{I}_{n_1} & 0 & \cdots & 0 \\ 0 & \beta \mathbf{I}_{n_2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \beta^{p-1}\mathbf{I}_{n_p} \end{pmatrix},$$

where $\beta > 0$. A simple calculation shows that with $A(k)$ as in (1),

$$S^{-1}A(k)S = A_\beta(k) = \begin{pmatrix} A_{11}(k) & \beta A_{12}(k) & \cdots & \beta^{p-1}A_{1p}(k) \\ 0 & A_{22}(k) & \cdots & \beta^{p-2}A_{2p}(t) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{kk}(t) \end{pmatrix},$$

that is, the submatrices $A_{ij}(k)$ for $i < j$ are multiplied by β^{j-i}. We see that the constant kinematic similarity $x = Sv$ takes (1) into the system $v(k+1) =$

$A_\beta(k)v(k)$. However, the latter system is a small perturbation of (2) when β is small. Then it follows from the roughness theorem in [3] for perturbed systems of the form $A(k)+B(k)$ that $v(k+1) = A_\beta(k)v_k$ has an exponential dichotomy if β is sufficiently small. Since (1) is kinematically similar to $v(k+1) = A_\beta(k)v_k$, it follows from Remark 3.4.13 (4) in [6] that (1) has an exponential dichotomy also. □

The following example shows that boundedness is essential in this result, even for invertible systems.

Example. Consider the system

$$x(k+1) = \frac{1}{2}x(k) + 2^{2k}y(k), \quad y(k+1) = \frac{1}{2}y(k).$$

This is an upper triangular system, for which the diagonal part has an exponential dichotomy on $\mathbb{Z}_+$ with projection of rank 2. The solution with initial value $(2/3, 1)$ is $((2/3)2^k, (1/2)^k)$ and hence is unbounded. On the other hand $((1/2)^k, 0)$ is a bounded solution. Hence if the upper triangular system has an exponential dichotomy, the corresponding projection $P(k)$ must be of rank 1. However it follows from the second Remark after Theorem 2[1] that if both the upper triangular and the diagonal system have an exponential dichotomy, the projections have the same rank. So the upper triangular system cannot have an exponential dichotomy.

Remark. If $A_{ii}(k)$ is invertible for all i and k, we can replace the boundedness of $A_{ij}(k)$ in Theorem 1 by the boundedness of $A_{ii}^{-1}(k)A_{ij}(k)$ for $i \neq j$. This would be proved by using Theorem 5.2 in [2] which establishes the roughness of exponential dichotomy for perturbed systems of the form $A(k)[I + B(k)]$.

4 Invertible systems

Now we consider the converse of Theorem 1 for invertible systems. Note that Theorem 2 below does not hold for equations on $\mathbb{Z}$, as we see from the example at the end of this section. To prove Theorem 2, we use the following lemma from [1].

Lemma 1. *Consider the Cartesian product of vector spaces*

$$U = U_1 \times U_2 \times \cdots \times U_p,$$

where $p \geq 2$. If V is a subspace of U, there is a projection P on U with range

[1] Matteo: with some documentclass the Remarks have a number and they are labelled, it depends on the journal. So if they are numbered I would use a label. Ken: We can fix this up when we decide which journal.

V, which has the form

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots & \cdot & P_{1p} \\ 0 & P_2 & P_{23} & \cdots & \cdot & P_{2p} \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & P_{p-1,p} \\ 0 & 0 & 0 & \cdots & 0 & P_p \end{pmatrix}, \tag{6}$$

where P_i is a projection on U_i for $i = 1, \ldots, p$.

Theorem 2. *If the invertible block upper triangular system (1) has an exponential dichotomy on $\mathbb{Z}_+$ or $\mathbb{Z}_-$, then the projection for the exponential dichotomy of (1) can be taken as*

$$P(k) = \begin{pmatrix} P_1(k) & P_{12}(k) & P_{13}(k) & \cdots & \cdot & P_{1p}(k) \\ 0 & P_2(k) & P_{23}(k) & \cdots & \cdot & P_{2p}(k) \\ 0 & 0 & P_3(k) & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1}(k) & P_{p-1,p}(k) \\ 0 & 0 & 0 & \cdots & \cdot & P_p(k) \end{pmatrix}, \tag{7}$$

where the $P_i(k)$ are projections. Moreover, the block diagonal system (2) has an exponential dichotomy with projection given by

$$\tilde{P}(k) = \begin{pmatrix} P_1(k) & 0 & 0 & \cdots & \cdot & 0 \\ 0 & P_2(k) & 0 & \cdots & \cdot & 0 \\ 0 & 0 & P_3(k) & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1}(k) & 0 \\ 0 & 0 & 0 & \cdots & \cdot & P_p(k) \end{pmatrix}. \tag{8}$$

Proof. Note the invertible case occurs exactly when $A_{ii}(k)$ is invertible for all i and k. In the case of $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$), denote by V the stable (resp. unstable) subspace for (1) at $k = 0$. Then it follows from Lemma 1 that there is a projection P of the upper triangular form (6) with range V. In the case of $\mathbb{Z}_-$, we replace P by $\mathbf{I} - P$ so that P has nullspace V. By Proposition 3.2 in [2], (1) has an exponential dichotomy on $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) with projection $P(k)$ with $P(0) = P$.

We recall that $\Phi_i(k, m)$ is the transition matrix for $x_i(k + 1) = A_{ii}(k)x_i(k)$. Then, for all k and m, the transition matrix for (2) is

$$\tilde{U}(k, m) = \begin{pmatrix} \Phi_1(k, m) & 0 & \cdots & 0 \\ 0 & \Phi_2(k, m) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Phi_p(k, m) \end{pmatrix},$$

whereas that for (1) is

$$U(k,m) = \begin{pmatrix} \Phi_1(k,m) & \Phi_{12}(k,m) & \cdots & \Phi_{1p}(k,m) \\ 0 & \Phi_2(k,m) & \cdots & \Phi_{2m}(k,m) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Phi_p(k,m) \end{pmatrix}.$$

Note that because of the invertibility, the identity $U(k,\ell)U(\ell,m) = U(k,m)$ holds for all k,ℓ,m and similarly for $\tilde{U}$. Then we take

$$\tilde{P}(0) = \begin{pmatrix} P_1 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & P_2 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdot & P_p \end{pmatrix}, \quad \tilde{P}(k) = \tilde{U}(k,0)\tilde{P}(0)\tilde{U}(0,k).$$

Now, by invariance,

$$P(k) = U(k,0)P(0)U(0,k),$$

which has the upper triangular form as given in the statement of the theorem. It is clear from its definition that $\tilde{P}(k)$ is invariant with respect to (2), that it is block diagonal and that it has the same diagonal blocks as $P(k)$.

Next we see that $U(k,0)P(0)U(0,m) = U(k,m)P(m)$ and $\tilde{U}(k)\tilde{P}(0)\tilde{U}^{-1}(m) = \tilde{U}(k,m)\tilde{P}(m)$ only differ in the (i,j)-entries, $i < j$, with those in $\tilde{U}(k,m)\tilde{P}(m)$ being zero. It follows that

$$|\tilde{U}(k,m)\tilde{P}(m)| \leq |U(k,m)P(m)| \leq Ke^{-\alpha(k-m)}, \quad m \leq k$$

where we have used the fact that the exponential dichotomy of (1) implies the existence of positive constants K and α such that

$$|U(k,m)P(m)|,\ |U(m,k)(\mathbf{I} - P(k))| \leq Ke^{-\alpha(k-m)}, \quad m \leq k.$$

(Note we use a matrix norm with the property that if $A = [a_{ij}]$ and $B = [b_{ij}]$ and $|a_{ij}| \leq |b_{ij}|$ for all i,j, then $|A| \leq |B|$.) Similarly

$$|\tilde{U}(m,k)(\mathbf{I} - \tilde{P}(k))| \leq |U(m,k)(\mathbf{I} - P(k))| \leq Ke^{-\alpha(k-m)}, \quad m \leq k.$$

The Theorem follows. □

Remark. It follows from Theorems 1 and 2 that when $A(k)$ in (1) is invertible and bounded, (1) has an exponential dichotomy on $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$ if and only if (2) has an exponential dichotomy on $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$. Note this follows from Theorem 4.1 in Pötzsche [7].

Remark. Note that it follows from Lemma 2 below that $P(k)$ in (7) and $\tilde{P}(k)$ in (8) have the same rank. Hence if (1) and (2) are invertible and both have exponential dichotomies on $\mathbb{Z}_+$ or $\mathbb{Z}_-$, then the corresponding projections have the same rank. Clearly this holds for $\mathbb{Z}$ also. Here we show this holds also in the noninvertible case. Choosing ε_{ik} arbitrarily small such that $A_{ii}(k) + \varepsilon_{ik}\mathbf{I}_{n_i}$ is invertible for $i = 1, \dots, p$ and all k, we can ensure that the coefficient matrices in (1) and (2) with $A_{ii}(k)$ replaced by $A_{ii}(k) + \varepsilon_{ik}\mathbf{I}_{n_i}$ are invertible. Then it follows from the roughness theorem, that if the ε_{ik} are small enough, the perturbed systems have exponential dichotomies with projections of the same rank as for the unperturbed systems. However for the perturbed invertible systems, these ranks are equal. So they are equal for the unperturbed systems. This applies to all of $\mathbb{Z}_+$, $\mathbb{Z}_-$ and $\mathbb{Z}$.

Example. Now we present an example for $\mathbb{Z}$ which shows that Theorem 2 does not hold even for invertible systems. It is based on an example for differential equations in [1]. *When $\delta \neq 0$, the system*

$$x(k+1) = A(k)x(k), \quad k \in \mathbb{Z},$$

where

$$A(k) = \begin{cases} \begin{pmatrix} 2 & \delta \\ 0 & 1/2 \end{pmatrix} & (k \geq 0) \\ \begin{pmatrix} 1/2 & \delta \\ 0 & 2 \end{pmatrix} & (k \leq -1) \end{cases}$$

has an exponential dichotomy on $\mathbb{Z}$ but the corresponding diagonal system does not.

Note that the diagonal system has exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, both with projections of rank 1. Then it follows from Theorem 1 that the upper triangular system also has exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, both with projections of rank 1. The solution $(x(k), y(k))$ of this system with value (x_0, y_0) at $k = 0$ is given by

$$(x(k), y(k)) = \begin{cases} \left((x_0 + 2\delta y_0/3)2^k - (2\delta y_0/3)2^{-k}, y_0 2^{-k}\right) & (k \geq 0) \\ \left((x_0 - 2\delta y_0/3)2^{-k} + (2\delta y_0/3)2^k, y_0 2^k\right) & (k \leq 0). \end{cases}$$

So the stable subspace is spanned by the vector $(-2\delta, 3)$ and the unstable subspace by $(2\delta, 3)$. Since these are independent, it follows from Corollary 3.3 in [2] that the upper triangular system has an exponential dichotomy on $\mathbb{Z}$. However the diagonal system does not have an exponential dichotomy on $\mathbb{Z}$ because it has the nontrivial bounded solution $(0, 2^{-|k|})$. Note that the projection at $k = 0$ for the dichotomy of the upper triangular system is

$$\begin{pmatrix} 1/2 & -\delta/3 \\ -3/(4\delta) & 1/2 \end{pmatrix}$$

which is not upper triangular.

5 Some examples

In this section we give three examples which show that Theorem 2 does not hold in general for noninvertible systems on $\mathbb{Z}_+$ or $\mathbb{Z}_-$. Note in the previous section we gave an example which shows that Theorem 2 does not hold even for invertible systems in the case of $\mathbb{Z}$. The first example below shows that if (1) has an exponential dichotomy on $\mathbb{Z}_+$, it does not follow that, unlike the invertible case, the projection can be chosen upper triangular. The second and third examples show, both for $\mathbb{Z}_+$ and $\mathbb{Z}_-$, that even if the projection can be chosen upper triangular, it does not follow that the diagonal system has a dichotomy.

(i) *Consider the two-dimensional system*

$$x(k+1) = A(k)x(k), \quad k \geq 0, \tag{9}$$

where

$$A(0) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \quad A(k) = \begin{pmatrix} a_1(k) & 0 \\ 0 & a_2(k) \end{pmatrix}, \quad k \geq 1,$$

with $ab \neq 0$*,* $a_2(1) = 0$*, such that*

$$x(k+1) = A(k)x(k)$$

has an exponential dichotomy on $k \geq 1$ *with stable subspace spanned by* $e_2 = (0,1)$ *(*$a_1(k) = 2$*,* $a_2(k) = 0$ *for* $k \geq 1$ *is an example of this). Then* (9) *has an exponential dichotomy on* $k \geq 0$ *but the corresponding diagonal system does not. Moreover* (9) *does not have an exponential dichotomy on* $k \geq 0$ *with respect to an upper triangular projection.*

We show first that the diagonal system does not have an exponential dichotomy on $\mathbb{Z}_+$. Indeed all the solutions $x(k)$, $k \geq 0$, of the diagonal system satisfy $x(k) = 0$ for $k \geq 2$. So $\mathcal{N}\Phi(2,0) = \mathbb{R}^2$. From Remark 2.1 (v) in [2] we see that the stable space at $k = 0$ has dimension 2. On the other hand, the subspace of initial values of bounded solutions at $k = 1$ is the span of e_2. Hence the diagonal system cannot have an exponential dichotomy because the rank of the projection would not be constant.

Now we prove that (9) has an exponential dichotomy on $\mathbb{Z}_+$. By assumption (9) has an exponential dichotomy on $k \geq 1$ with stable subspace e_2. Next we see that $\Phi(1,0)^{-1}(\text{span}\{e_2\}) = A(0)^{-1}(\text{span}\{e_2\}) = \text{span}\{e_1\}$, where $e_1 = (1,0)$. So by Theorem 4.3 in [2] we can extend the exponential dichotomy from $k \geq 1$ to $k \geq 0$. Hence (9) has an exponential dichotomy on $k \geq 0$. Note also from (11) in [2] it follows that the stable subspace at $k = 0$ is spanned by e_1.

Finally we show that (9) does not have an exponential dichotomy on $k \geq 0$ with an upper triangular projection. Suppose $P(k)$ is such an upper triangular projection. Since $\mathcal{R}P(0)$ is the span of e_1 and $\mathcal{R}P(1)$ is the span of e_2, we have

$$P(0) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}, \quad P(1) = \begin{pmatrix} 0 & 0 \\ \beta & 1 \end{pmatrix}$$

for some α, β. Since $P(k)$ is upper triangular, $\beta = 0$. Then from $P(1)A(0) = A(0)P(0)$ we get

$$\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

a contradiction since $ab \neq 0$.

(ii) Here is another example for $\mathbb{Z}_+$. *System* (1) *with*

$$A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(k) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad k \geq 1,$$

has an exponential dichotomy on $k \geq 0$ with an upper triangular projection but the diagonal system does not have an exponential dichotomy on $k \geq 0$.

The system defined above has an exponential dichotomy on $k \geq 1$ with projection

$$P(k) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad k \geq 1.$$

Note that $A(0)^{-1}(\mathcal{R}P(1))$ is the span of e_1. So dim $(A(0)^{-1}(\mathcal{R}P(1))) = 1$. Therefore, by Theorem 4.3 in [2], (or Remark 4.1 following it), we can extend the exponential dichotomy to $k \geq 0$ with the projection unchanged for $k \geq 1$. Further it follows from the same Remark 4.1 in [2] that $\mathcal{R}P(0)$ has to be equal to $A(0)^{-1}\mathcal{R}P(1) = \text{span}\{e_1\}$. So $P(0) = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}$ for some $c \in \mathbb{R}$ so that $P(0)$ is upper triangular.

However the diagonal system does not have an exponential dichotomy on $k \geq 0$ since in the diagonal system $A(0) = 0$. Hence at $k = 0$ the subspace of initial values of bounded solutions has dimension 2 whereas at $k = 1$ the dimension is 1.

(iii) Here is an example on $\mathbb{Z}_-$. *The system*

$$x(k+1) = A(k)x(k), \quad k \leq -1$$

with

$$A(-1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(k) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}, \quad k \leq -2,$$

has an exponential dichotomy on $k \leq 0$ with upper triangular projection but the diagonal system does not have an exponential dichotomy on $k \leq 0$.

The system $x(k+1) = A(k)x(k)$, $k \leq -2$ has an exponential dichotomy on $k \leq -1$ with projection

$$P(k) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad k \leq -1.$$

$A(-1)$ is one to one on $\mathcal{N}P(-1)$ and $\mathcal{N}(A(-1))$ is the span of e_1 which coincides with the range of $P(-1)$. So by Theorem 4.4 in [2] and its proof, the dichotomy can be extended to $k \leq 0$ with $P(k)$ unchanged for $k \leq -1$ and with $P(0)$ any projection with nullspace equal to $A(-1)(\mathcal{N}P(-1)) = \text{span}\{e_1\}$ since $A(-1)(\mathcal{R}P(-1)) = \{0\}$. So we can take

$$P(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we can extend the exponential dichotomy to $k \leq 0$. Note also the projection $P(k)$ is upper triangular for all $k \leq 0$.

However the diagonal system does not have an exponential dichotomy on $k \leq 0$. In fact, the diagonal system has an exponential dichotomy with projection of rank 1 on $k \leq -1$. So if the system did have an exponential dichotomy on $k \leq 0$, the projection would be of rank 1; since in the diagonal system $A(-1) = 0$, this is impossible because $A(-1)$ must be one to one on the nullspace of the projection at $k = -1$.

6 Noninvertible systems: upper triangular exponential dichotomy implies diagonal exponential dichotomy

The examples given in the previous two sections show that the upper triangular system (1) may have an exponential dichotomy on either $\mathbb{Z}_+$, $\mathbb{Z}_-$ or $\mathbb{Z}$ but the corresponding diagonal system (2) does not. These examples suggest that we need to add the condition that the projection can be chosen block upper triangular and examples (ii) and (iii) in the previous section suggest that we also need to add the condition that the rank of the projections along the diagonal do not vary with k. This we do in Theorem 3 below. First we prove a Lemma.

Lemma 2. *Let P be a block upper triangular projection*

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots & \cdot & P_{1p} \\ 0 & P_2 & P_{23} & \cdots & \cdot & P_{2p} \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & P_{p-1,p} \\ 0 & 0 & 0 & \cdots & \cdot & P_p \end{pmatrix}.$$

Then

$$\text{rank}\, P = \sum_{i=1}^{p} \text{rank}\, P_i$$

and for $i = 1, \ldots, p$, if $\eta \in \mathcal{N}P_i$, there exist $\eta_j \in \mathcal{R}P_j$, $j = 1, \ldots, i-1$, such that $(\eta_1, \ldots, \eta_{i-1}, \eta, 0, \ldots, 0) \in \mathcal{N}P$.

Proof. First we prove the lemma for $p = 2$ so that

$$P = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_2 \end{pmatrix}.$$

Then $P^2 = P$ implies that

$$P_1 P_{12} + P_{12} P_2 = P_{12}.$$

So

$$P_2 \eta = 0 \Longrightarrow P_{12}\eta = P_1 P_{12}\eta. \tag{10}$$

Hence if $P_2\eta = 0$, then $P_{12}\eta \in \mathcal{R}P_1$ and $P_1\xi + P_{12}\eta = 0$ if and only if $P_1(\xi + P_{12}\eta) = 0$. Then

$$\begin{aligned} \mathcal{N}P &= \{(\xi, \eta) : P_2\eta = 0, P_1\xi + P_{12}\eta = 0\} \\ &= \{(\xi, \eta) : P_2\eta = 0, P_1(\xi + P_{12}\eta) = 0\} \\ &= \{(\xi_1 - P_{12}\eta, \eta) : \xi_1 \in \mathcal{N}P_1, \eta \in \mathcal{N}P_2\}. \end{aligned} \tag{11}$$

So $(\xi_1, \eta) \to (\xi_1 - P_{12}\eta, \eta)$ is a one to one linear mapping of $\mathcal{N}P_1 \times \mathcal{N}P_2$ onto $\mathcal{N}P$. It follows that

$$\dim \mathcal{N}P = \dim \mathcal{N}P_1 + \dim \mathcal{N}P_2$$

and hence that

$$\operatorname{rank} P = \operatorname{rank} P_1 + \operatorname{rank} P_2.$$

We see next that if $\eta \in \mathcal{N}P_2$, then $(-P_{12}\eta, \eta) \in \mathcal{N}P$. So $(\eta_1, \eta) \in \mathcal{N}P$ with $\eta_1 = -P_{12}\eta$, where $\eta_1 = -P_{12}\eta \in \mathcal{R}P_1$, using (10). This proves the lemma for $p = 2$. Next suppose the lemma holds for some $p - 1 > 1$. We prove it for p.

Suppose P has p blocks. Let

$$Q_i = \begin{pmatrix} P_i & P_{i,i+1} & P_{i,i+2} & \cdots & P_{i,p} \\ 0 & P_{i+1} & P_{i+1,i+2} & \cdots & P_{i+1,p} \\ 0 & 0 & P_{i+2} & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \cdot \\ 0 & 0 & 0 & P_{p-1} & P_{p-1,p} \\ 0 & 0 & 0 & \cdot & P_p \end{pmatrix}, \quad i = 1, \ldots, p$$

and write

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots & \cdot & P_{1p} \\ 0 & P_2 & P_{23} & \cdots & \cdot & P_{2p} \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & P_{p-1,p} \\ 0 & 0 & 0 & \cdots & \cdot & P_p \end{pmatrix} = \begin{pmatrix} P_1 & \bar{P}_{12} \\ 0 & Q_2 \end{pmatrix}.$$

By the $p = 2$ case,

$$\operatorname{rank} P = \operatorname{rank} P_1 + \operatorname{rank} Q_2$$

and by the induction hypothesis,

$$\operatorname{rank} Q_2 = \sum_{i=2}^{p} \operatorname{rank} P_i.$$

It follows that

$$\operatorname{rank} P = \sum_{i=1}^{p} \operatorname{rank} P_i.$$

Next

$$\mathcal{N}P = \{(\xi_1 - \bar{P}_{12}\eta, \eta) : \xi_1 \in \mathcal{N}P_1, \eta \in \mathcal{N}Q_2\}$$

and $\bar{P}_{12}\eta \in \mathcal{R}P_1$. By the induction hypothesis, given $\eta_p \in \mathcal{N}P_p$ there exist $\eta_i \in \mathcal{R}P_i$, $i = 2, \dots, p-1$ such that

$$(\eta_2, \dots, \eta_{p-1}, \eta_p) \in \mathcal{N}Q_2.$$

Then, with

$$\eta_1 = -\bar{P}_{12}(\eta_2, \dots, \eta_{p-1}, \eta_p) \in \mathcal{R}P_1$$

we find that

$$(\eta_1, \eta_2, \dots, \eta_{p-1}, \eta_p) \in \mathcal{N}P.$$

Finally, if $\eta_i \in \mathcal{N}P_i$ then $(\eta_i, 0 \dots, 0) \in \mathcal{N}Q_i$, and then applying the previous part to the projection P in the form

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots & \cdot & * \\ 0 & P_2 & P_{23} & \cdots & \cdot & * \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{i-1} & * \\ 0 & 0 & 0 & \cdots & \cdot & Q_i \end{pmatrix}$$

we see that $\eta_j \in \mathcal{R}P_j$, $j = 1, \dots, i-1$ exist such that

$$(\eta_1, \dots, \eta_{i-1}, \eta_i, 0 \dots, 0) \in \mathcal{N}P.$$

This completes the proof of the lemma by induction. □

Remark. Replacing P with $\mathbf{I} - P$ we see that for any $i = 1, \dots, p$ and $\xi_i \in \mathcal{R}P_i$ there exists $\xi_j \in \mathcal{N}P_j$, $j = 1, \dots, i-1$ such that $(\xi_1, \dots, \xi_{i-1}, \xi_i, 0, \dots, 0) \in \mathcal{R}P$. In particular, if

$$P = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_2 \end{pmatrix},$$

we have

$$\mathcal{R}\begin{pmatrix} P_1 & P_{12} \\ 0 & P_2 \end{pmatrix} = \{(\xi + P_{12}\eta, \eta) : \xi \in \mathcal{R}P_1, \eta \in \mathcal{R}P_2\}$$

and $P_{12}\eta \in \mathcal{R}(\mathbf{I} - P_1) = \mathcal{N}P_1$.

Now we can prove the following theorem, where the conditions imposed are suggested by the examples in the previous two sections.

Theorem 3. *Suppose the block upper triangular system (1) has an exponential dichotomy on $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ or $\mathbb{Z}$, such that the projection for the exponential dichotomy of (1) can be taken in the block upper triangular form* (7)*, where for each i the rank of $P_i(k)$ does not depend on k. Then the block diagonal system (2) has an exponential dichotomy with projection in the diagonal form* (8)*.*

Proof. First we consider the cases $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with $p = 2$. We assume

$$x(k+1) = A(k)x(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{pmatrix} x(k) \tag{12}$$

has an exponential dichotomy with projection $P(k)$ which has the form

$$P(k) = \begin{pmatrix} P_1(k) & P_{12}(k) \\ 0 & P_2(k) \end{pmatrix}. \tag{13}$$

Corresponding to the diagonal system,

$$x(k+1) = D(k)x(k) = \begin{pmatrix} A_{11}(k) & 0 \\ 0 & A_{22}(k) \end{pmatrix} x(k), \tag{14}$$

we take

$$\tilde{P}(k) = \begin{pmatrix} P_1(k) & 0 \\ 0 & P_2(k) \end{pmatrix}. \tag{15}$$

We divide the rest of the proof into several steps.

Step 1: Invariance of $\tilde{P}(k)$.

It follows from

$$P(k+1)A(k) = A(k)P(k), \quad k, k+1 \in J,$$

that

$$\tilde{P}(k+1)D(k) = D(k)\tilde{P}(k), \quad k, k+1 \in J.$$

So $\tilde{P}(k)$ is invariant with respect to (2).

Step 2: For $k, k+1 \in J$, $D(k)$ maps the nullspace of $\tilde{P}(k)$ one to one onto the nullspace of $\tilde{P}(k+1)$.

This is clearly the case if and only if for $i = 1, 2$, $A_{ii}(k)$ maps the nullspace of $P_i(k)$ one to one onto the nullspace of $P_i(k+1)$. By the invariance we know that $P(k+1)A(k) = A(k)P(k)$ and hence $P_i(k+1)A_{ii}(k) = A_{ii}(k)P_i(k)$. Hence $A_{ii}(k)$ maps the nullspace of $P_i(k)$ into the nullspace of $P_i(k+1)$. By hypothesis, we know that these two nullspaces have the same dimension. So we just have to prove that $A_{ii}(k)$ is either one to one on the nullspace of $P_i(k)$ or onto the nullspace of $P_i(k+1)$. We begin by proving the claim for $i = 1$.

Since $A(k)$ is one to one on the nullspace of $P(k)$, it follows that $A_{11}(k)$ is one to one on the nullspace of $P_1(k)$. In fact if $A_{11}(k)x_1 = 0$ for some x_1 in the nullspace of $P_1(k)$, then $A(k)(x_1, 0) = 0$; this means $(x_1, 0) \in \mathcal{N}P(k) \cap \mathcal{N}A(k)$.

Since $A(k)$ is one to one on $\mathcal{N}P(k)$, it follows that $(x_1, 0)$ is zero and hence that $x_1 = 0$. So $A_{11}(k)$ maps the nullspace of $P_1(k)$ one to one onto the nullspace of $P_1(k+1)$.

Let $\eta \in \mathcal{N}P_2(k+1)$. By the case $p = 2$ of Lemma 2, there exists η_1 such that (η_1, η) is in the nullspace of $P(k+1)$. Then there exists (ξ_0, η_0) in the nullspace of $P(k)$ which is mapped by $A(k)$ to (η_1, η); notice that $\eta_0 \in \mathcal{N}P_2(k)$ and $A_{22}(k)\eta_0 = \eta$. So $A_{22}(k)$ maps the nullspace of $P_2(k)$ onto the nullspace of $P_2(k+1)$. Hence $A_{22}(k)$ maps the nullspace of $P_2(k)$ one to one onto the nullspace of $P_2(k+1)$. Thus we have shown that $D(k)$ maps the nullspace of $\tilde{P}(k)$ one to one onto the nullspace of $\tilde{P}(k+1)$.

Step 3: First dichotomy inequality.

The transition matrix for (12) is

$$U(k,m) = \begin{pmatrix} \Phi_1(k,m) & \Phi_{12}(k,m) \\ 0 & \Phi_2(k,m) \end{pmatrix}$$

and the transition matrix for (14) is

$$\tilde{U}(k,m) = \begin{pmatrix} \Phi_1(k,m) & 0 \\ 0 & \Phi_2(k,m) \end{pmatrix}.$$

We see that $U(k,m)P(m)$ and $\tilde{U}(k,m)\tilde{P}(m)$ only differ in the $(1,2)-$entry with that in $\tilde{U}(k,m)\tilde{P}(m)$ being zero. It follows that

$$|\tilde{U}(k,m)\tilde{P}(m)| \le |U(k,m)P(m)| \le Ke^{-\alpha(k-m)}, \quad m \le k$$

where K and α are the constants for the exponential dichotomy of (12). (Note we use a matrix norm with the property that if $A = [a_{ij}]$ and $B = [b_{ij}]$ and $|a_{ij}| \le |b_{ij}|$ for all i, j, then $|A| \le |B|$.)

Step 4: Second dichotomy inequality

It follows from Step 2 that $\Phi_i(k,m) : \mathcal{N}P_i(m) \to \mathcal{N}P_i(k)$ is invertible when $k \ge m$ for $i = 1, 2$ and hence that $\tilde{U}(k,m) : \mathcal{N}\tilde{P}(m) \to \mathcal{N}\tilde{P}(k)$ is invertible when $k \ge m$. Then if $m \le k$, we can define $\tilde{U}(m,k)$ as the inverse of the map $\tilde{U}(k,m)$ from the nullspace of $\tilde{P}(m)$ to the nullspace of $\tilde{P}(k)$ and, similarly $\Phi_i(m,k) : \mathcal{N}P_i(k) \to \mathcal{N}P_i(m)$ will be the inverse of $\Phi_i(k,m) : \mathcal{N}P_i(m) \to \mathcal{N}P_i(k)$. Note that $\Phi_i(m,k)$ are the diagonal terms of the block diagonal matrix $\tilde{U}(m,k)$. Now we show that for $i = 1, 2$,

$$|\Phi_i(m,k)\xi| \le Ke^{-\alpha(k-m)}|\xi|, \quad \xi \in \mathcal{N}P_i(k), \quad m \le k, \tag{16}$$

where K and α are the positive constants involved in the dichotomy of (12). First we derive a formula for the inverse of $U(k,m) : \mathcal{N}P(m) \to \mathcal{N}P(k)$. Note it follows from (11) in the proof of Lemma 2 that

$$\mathcal{N}P(k) = \{(\xi_1 - P_{12}(k)\eta, \eta) : \xi_1 \in \mathcal{N}P_1(k), \eta \in \mathcal{N}P_2(k)\}. \tag{17}$$

So

$$\begin{pmatrix} I & -P_{12}(k) \\ 0 & I \end{pmatrix}$$

maps $\mathcal{N}P_1(k) \times \mathcal{N}P_2(k) = \mathcal{N}\tilde{P}(k)$ bijectively onto $\mathcal{N}P(k)$. Then

$$\hat{U}(k,m) = \begin{pmatrix} I & P_{12}(k) \\ 0 & I \end{pmatrix} U(k,m) \begin{pmatrix} I & -P_{12}(m) \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Phi_1(k,m) & W(k,m) \\ 0 & \Phi_2(k,m) \end{pmatrix}$$

maps $\mathcal{N}P_1(m) \times \mathcal{N}P_2(m) = \mathcal{N}\tilde{P}(m)$ bijectively onto $\mathcal{N}P_1(k) \times \mathcal{N}P_2(k) = \mathcal{N}\tilde{P}(k)$, where

$$W(k,m) = \Phi_{12}(k,m) + P_{12}(k)\Phi_2(k,m) - \Phi_1(k,m)P_{12}(m)$$

maps $\mathcal{N}P_2(m)$ into $\mathcal{N}P_1(k)$ so that $W(k,m)\Phi_2(m,k)$ maps $\mathcal{N}P_2(k)$ into $\mathcal{N}P_1(k)$. Hence $\hat{U}(k,m) : \mathcal{N}\tilde{P}(m) \to \mathcal{N}\tilde{P}(k)$ and, by direct multiplication, we see that

$$\hat{U}(m,k) := \begin{pmatrix} \Phi_1(m,k) & -\Phi_1(m,k)W(k,m)\Phi_2(m,k) \\ 0 & \Phi_2(m,k) \end{pmatrix} : \mathcal{N}\tilde{P}(k) \to \mathcal{N}\tilde{P}(m),$$

is the inverse of $\hat{U}(k,m)$. It follows that $U(k,m) : \mathcal{N}P(m) \to \mathcal{N}P(k)$ has the inverse $U(m,k)$ given by

$$\begin{pmatrix} I & -P_{12}(m) \\ 0 & I \end{pmatrix} \begin{pmatrix} \Phi_1(m,k) & -\Phi_1(m,k)W(k,m)\Phi_2(m,k) \\ 0 & \Phi_2(m,k) \end{pmatrix} \begin{pmatrix} I & P_{12}(k) \\ 0 & I \end{pmatrix}$$

so that if $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathcal{N}P(k)$,

$$U(m,k) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \Phi_1(m,k)\xi + V(k,m)\eta \\ \Phi_2(m,k)\eta \end{pmatrix},$$

where

$$V(k,m) = -\Phi_1(m,k)W(k,m)\Phi_2(m,k) + \Phi_1(m,k)P_{12}(k) - P_{12}(m)\Phi_2(m,k).$$

Now we know there exist positive constants K and α such that for $m \le k$,

$$\left| U(m,k) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right| \le K e^{-\alpha(k-m)} \left| \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right|.$$

Taking $\eta = 0$ so that, by (17), $\xi \in \mathcal{N}P_1(k)$, we conclude that for $m \le k$,

$$|\Phi_1(m,k)\xi| \le K e^{-\alpha(k-m)} |\xi|, \quad \xi \in \mathcal{N}P_1(k),$$

assuming our norm has the properties

$$\left| \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \right| = |x_1| \le \left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|.$$

Next taking $\xi = 0$ so that, by (17), $\eta \in \mathcal{N}P_2(k)$, we conclude that for $m \leq k$,

$$|\Phi_2(m,k)\eta| \leq Ke^{-\alpha(k-m)}|\eta|, \quad \eta \in \mathcal{N}P_2(k),$$

assuming our norm has the properties

$$\left|\begin{pmatrix} 0 \\ x_2 \end{pmatrix}\right| = |x_2| \leq \left|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right|.$$

Thus we have proved (16). Using the boundedness of the projection $P(k)$ and hence that of $P_1(k)$ and $P_2(k)$, it follows there exists a constant L such that

$$|\tilde{U}(m,k)(\mathbf{I} - \tilde{P}(k))| \leq Le^{-\alpha(k-m)}, \quad m \leq k.$$

This completes the proof of Theorem 3 for the cases $J = \mathbb{Z}_+$ and $\mathbb{Z}_-$ when $p = 2$.

Now we prove Theorem 3 for the cases $J = \mathbb{Z}_+$ and $\mathbb{Z}_-$ for general $p \geq 2$. Suppose we have proved it for $p - 1 \geq 2$ and now we want to prove it for p. To do this, we partition

$$A(k) = \begin{pmatrix} A_{11}(k) & \bar{A}_{12}(k) \\ 0 & B(k) \end{pmatrix}$$

where

$$\bar{A}_{12}(k) = \begin{pmatrix} A_{12}(k) & \cdots & A_{1p}(k) \end{pmatrix}, \quad B(k) = \begin{pmatrix} A_{22}(k) & A_{23}(k) & \cdots & A_{2p}(k) \\ 0 & A_{33}(k) & \cdots & A_{34}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix},$$

with correspondingly

$$P(k) = \begin{pmatrix} P_1(k) & \bar{P}_{12}(k) \\ 0 & R(k) \end{pmatrix}.$$

Then, setting $x = (x_1, \bar{x}_2)$ where $\bar{x}_2 = (x_2, \ldots, x_p)$, it follows from the $p = 2$ case that

$$x_1(k+1) = A_{11}(k)x_1(k)$$

has an exponential dichotomy on J with projection $P_1(k)$ and that

$$\bar{x}_2(k+1) = B(k)\bar{x}_2(k)$$

has an exponential dichotomy on J with projection $R(k)$. However then it follows by the induction hypothesis that for $i = 2, \ldots, p$,

$$x_i(k+1) = A_{ii}(k)x_i(k)$$

has an exponential dichotomy on J with projection $P_i(k)$. This completes the induction proof for the cases $J = \mathbb{Z}_+$, $\mathbb{Z}_-$.

Now we consider the case $J = \mathbb{Z}$. Suppose first $p = 2$. Then (12) has an exponential dichotomy on $\mathbb{Z}$ with projection (13) of rank r where, for $i = 1, 2$.

the rank of $P_i(k)$ is r_i and $r_1 + r_2 = r$ (by Lemma 2). By the $\mathbb{Z}_+$ and $\mathbb{Z}_-$ cases, the diagonal system (14) has exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projection (15) of rank r. By Proposition 1, for $i = 1, 2$, the equation $x_i(k+1) = A_{ii}(k)x_i(k)$ has exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projection $P_i(k)$. Since the stable subspace on $\mathbb{Z}_+$ is $\mathcal{R}P_i(0)$ and the unstable subspace on $\mathbb{Z}_-$ is $\mathcal{N}P_i(0)$ and these intersect in $\{0\}$ since $\mathcal{R}P(0) \cap \mathcal{N}P(0) = \{0\}$, it follows from Corollary 3.3 in [2] that for $i = 1, 2$, the equation $x_i(k+1) = A_{ii}(k)x_i(k)$ has an exponential dichotomy on $\mathbb{Z}$ with projection $P_i(k)$.

This proves the case $p = 2$. The case for general $p \geq 2$ follows easily by induction using the $p = 2$ case. □

7 A converse theorem

In this section we prove a converse to Theorem 3. We start with the $\mathbb{Z}_+$ case. For this we need the following

Lemma 3. *For any $m \in \mathbb{Z}_+$, let $V(m) \subset \mathbb{R}^n$ be the subspace of initial values at $k = m$ of solutions of (1) that are bounded on $k \geq m$ and, for $i = 1, \dots, p$, let $V_i(m) \subset \mathbb{R}^{n_i}$ be the subspace of initial values at $k = m$ of solutions of $x_i(k+1) = A_{ii}(k)x_i(k)$ that are bounded on $k \geq m$. Then*

$$\dim V(m) \leq \sum_{i=1}^{p} \dim V_i(m).$$

Proof. Clearly the lemma holds for $p = 1$. Assuming it holds for $p - 1 \geq 1$, we prove it holds for p. We write (1) as

$$x(k+1) = \begin{pmatrix} \tilde{B}_{11}(k) & \tilde{B}_{1p}(k) \\ 0 & A_{pp}(k) \end{pmatrix} x(k), \tag{18}$$

where

$$\tilde{B}_{11}(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) & \cdots & A_{1,p-1}(k) \\ 0 & A_{22}(k) & \cdots & A_{2,p-1}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{p-1,p-1}(k) \end{pmatrix}, \quad \tilde{B}_{1p}(k) = \begin{pmatrix} A_{1p}(k) \\ A_{2p}(k) \\ \cdots \\ A_{p-1,p}(k) \end{pmatrix} \tag{19}$$

and determine the bounded solutions of (18) on $k \geq m$. Let $\tilde{V}_1(m)$ be the subspace of initial values at m of the bounded solutions in $k \geq m$ of $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ and S_m be the subspace of $V_p(m)$ such that if $\eta \in S_m$, then the equation

$$\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k) + \tilde{B}_{1p}(k)\Phi_p(k, m)\eta$$

admits a bounded solution. Choose a complement $W_1(m)$ to $\tilde{V}_1(m)$ in $\mathbb{R}^{n-n_p}$. Then there is a unique such bounded solution for which $\tilde{x}_1(m)$ is in $W_1(m)$. Let $\tilde{x}_1(m) = L(m)\eta$; then $L(m)$ is a linear mapping from S_m to $W_1(m)$ and

$$V(m) = \{(\xi + L(m)\eta, \eta) : \xi \in \tilde{V}_1(m),\ \eta \in S_m\}$$

is the subspace of initial values at $k = m$ of solutions of (18) bounded on $k \geq m$. Hence

$$\dim V(m) = \dim \tilde{V}_1(m) + \dim S_m \leq \dim \tilde{V}_1(m) + \dim V_p(m).$$

By the induction hypothesis we get: $\dim \tilde{V}_1(m) \leq \sum_{i=1}^{p-1} \dim V_i(m)$. The required conclusion follows. □

Note that in Lemma 3 we do not need that the systems considered have an exponential dichotomy on $\mathbb{Z}_+$. We only state a relation between the dimensions of the spaces of initial conditions at $k = m$ such that the corresponding solutions are bounded for $k \geq m$. In particular the dimensions of $V_i(m)$ and $V(m)$ may depend on m. The exponential dichotomy of (1) and of the corresponding blocks $x_i(k+1) = A_{ii}(k)x_i(k)$ are taken into account in the next theorem.

Theorem 4. *Suppose* (1) *has an exponential dichotomy on* $\mathbb{Z}_+$ *with projection of rank* r *and that for all* i, $x_i(k+1) = A_{ii}(k)x_i(k)$ *has an exponential dichotomy on* $\mathbb{Z}_+$ *with projection* $P_i(k)$ *of rank* r_i *(by Proposition 1, this is equivalent to the exponential dichotomy of* (2) *with rank* $\sum_{i=1}^{p} r_i$*). Then* $\sum_{i=1}^{p} r_i = r$ *and the projection* $P(k)$ *for the dichotomy of* (1) *can be taken in upper triangular form with* $P_i(k)$ *as diagonal blocks.*

Proof. Note it follows from the second remark after the proof of Theorem 2 that $r = \sum_{i=1}^{p} r_i$. We prove the theorem by induction on p. It is trivial for $p = 1$. Assuming it is true for $p - 1 \geq 1$, we prove it for p. We write (1) as in (18) where $\tilde{B}_{11}(k), \tilde{B}_{1p}(k)$ are as in (19). Let $x(k) = (\tilde{x}_1(k), x_p(k))$ where $\tilde{x}_1(k) = (x_1(k), \ldots, x_{p-1}(k))$; we rewrite the transition matrix $\Phi(k,m)$ of (18) for $k \geq m$ as follows

$$\Phi(k,m) = \begin{pmatrix} \tilde{\Phi}_1(k,m) & \tilde{\Phi}_{1p}(k,m) \\ 0 & \Phi_p(k,m) \end{pmatrix} \tag{20}$$

so that $\tilde{\Phi}_1(k,m)$ is the transition matrix of the equation $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$, and $\Phi_p(k,m)$ is the transition matrix of the equation $x_p(k+1) = A_{pp}(k)x_p(k)$.

Let $V(m)$, $V_i(m) = \mathcal{R}P_i(m)$, $\tilde{V}_1(m)$ be the subspaces of initial values at $k = m$ of bounded solutions on $k \geq m$ of (18), $x_i(k+1) = A_{ii}(k)x_i(k)$, $i = 1, \ldots, p$ and $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ respectively. From Lemma 3 it follows that

$$r = \dim V(m) \leq \dim \tilde{V}_1(m) + \dim V_p(m) = \dim \tilde{V}_1(m) + r_p.$$

However, by the same lemma, $\dim \tilde{V}_1(m) \leq \sum_{i=1}^{p-1} r_i = r - r_p$. Hence

$$\dim \tilde{V}_1(m) = r - r_p \tag{21}$$

and $\dim V(m) = \dim \tilde{V}_1(m) + \dim V_p(m)$. As a consequence

$$V(m) = \{(\xi + L(m)\eta, \eta) : \xi \in \tilde{V}_1(m),\ \eta \in \mathcal{R}P_p(m)\}, \tag{22}$$

where $L(m)$ is as in the proof of Lemma 3 with S_m replaced by $\mathcal{R}P_p(m)$.

Claim. *Let $W_1(0)$ be a complement to $\tilde{V}_1(0)$ and set*

$$W(k) = \Phi(k,0)(W_1(0) \times \mathcal{N}P_p(0)).$$

Then (18) has an exponential dichotomy on $\mathbb{Z}_+$ with projection $P(k)$ having range $V(k)$ and nullspace $W(k)$. Moreover, $P(k)$ has the form

$$P(k) = \begin{pmatrix} \tilde{P}_1(k) & \tilde{P}_{1p}(k) \\ 0 & P_p(k) \end{pmatrix},$$

where $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy with projection $\tilde{P}_1(k)$, which has range $\tilde{V}_1(k)$ and nullspace $W_1(k) = \tilde{\Phi}_1(k,0)(W_1(0))$.

To prove the Claim, we first observe that (18) has a dichotomy on $\mathbb{Z}_+$. We start with proving that

$$W(0) = W_1(0) \times \mathcal{N}P_p(0) \text{ is a complement to } V(0). \tag{23}$$

In fact by construction and (21),

$$\dim W(0) = n - n_p - (r - r_p) + n_p - r_p = n - r$$

so that dim $V(0)$+dim $W(0) = n$. Next, if $(\xi, \eta) \in V(0) \cap W(0)$ then by (22) we immediately have $\eta \in \mathcal{N}P_p(0) \cap \mathcal{R}P_p(0)$; hence $\eta = 0$, and $(\xi, 0) \in V(0)$. Then again from (22) we get $\xi \in \tilde{V}_1(0)$ and $\xi \in W_1(0)$ by construction; so $\xi = 0$ and (23) is proved. Then it follows from the proof of Proposition 3.2 in [2] that $\Phi(k,0)$ is one to one on $W(0)$, that $V(k) \oplus W(k) = \mathbb{R}^n$ for $k \geq 0$ and (18) has an exponential dichotomy on $\mathbb{Z}_+$ with respect to the projection $P(k)$ with range $V(k)$ and nullspace $W(k)$.

Next we prove that $P(k)$ has the form given in the Claim. We first construct the projection $P_1(k)$ and to this end we show that for $m \geq 0$, $\tilde{V}_1(m) \oplus W_1(m) = \mathbb{R}^{n-n_p}$, where, using (20),

$$\Phi(m,0) = \begin{pmatrix} \tilde{\Phi}_1(m,0) & \tilde{\Phi}_{1p}(m,0) \\ 0 & \Phi_p(m,0) \end{pmatrix}.$$

Since $\Phi(m,0)$ is one to one on $W(0)$, $\tilde{\Phi}_1(m,0)$ is one to one on $W_1(0)$. So $W_1(m) = \tilde{\Phi}_1(m,0)(W_1(0))$ has the same dimension $n - n_p - (r - r_p)$ as $W_1(0)$, where we have used (21). Now suppose $\tilde{x}_1 \in \tilde{V}_1(m) \cap W_1(m)$. Then $\tilde{x}_1 = \tilde{\Phi}_1(m,0)\tilde{y}_1$ where $\tilde{y}_1 \in W_1(0)$. Then $\tilde{\Phi}_1(k,0)\tilde{y}_1 = \tilde{\Phi}_1(k,m)\tilde{x}_1$ is bounded in $k \geq 0$ so that $\tilde{y}_1 \in \tilde{V}_1(0)$. Hence $\tilde{y}_1 = 0$ and therefore $\tilde{x}_1 = 0$. So $\tilde{V}_1(m) \cap W_1(m) = \{0\}$. Thus $\tilde{V}_1(m) \oplus W_1(m) = \mathbb{R}^{n-n_p}$ and we may define $\tilde{P}_1(m)$ as the projection with range $\tilde{V}_1(m)$ and nullspace $W_1(m)$.

Now by the argument we used to get (22), taking $\mathcal{N}\tilde{P}_1(m) = W_1(m)$ as the complement to $\tilde{V}_1(m) = \mathcal{R}\tilde{P}_1(m)$, we can show that

$$\mathcal{R}P(m) = V(m) = \{(\xi + L^+(m)\eta, \eta) : \xi \in \mathcal{R}\tilde{P}_1(m), \eta \in \mathcal{R}P_p(m)\},$$

where $L^+(m) : \mathcal{R}P_p(m) \to \mathcal{N}\tilde{P}_1(m)$. Next

$$\begin{aligned}
\mathcal{N}P(m) &= W(m) \\
&= \Phi(m,0)(\mathcal{N}\tilde{P}_1(0) \times \mathcal{N}P_p(0)) \\
&= \{(\tilde{\Phi}_1(m,0)\xi_1 + \tilde{\Phi}_{12}(m,0)\eta_1, \tilde{\Phi}_2(m,0)\eta_1) : \xi_1 \in \mathcal{N}\tilde{P}_1(0), \eta_1 \in \mathcal{N}P_p(0)\} \\
&= \{(\xi + L^-(m)\eta, \eta) : \xi \in \mathcal{N}\tilde{P}_1(m), \eta \in \mathcal{N}P_p(m)\},
\end{aligned}$$

where $L^-(m) : \mathcal{N}P_p(m) \to \mathbb{R}^{n-n_p}$ is given by

$$L^-(m)\eta = \tilde{\Phi}_{12}(m,0)\tilde{\Phi}_2(0,m)\eta.$$

Now we need a lemma.

Lemma 4. *Let U_1 and U_2 be vector spaces. Suppose there is a projection P on $U_1 \times U_2$ and projections P_1, P_2 on U_1, U_2 such that*

$$\begin{aligned}
\mathcal{R}P &= \{(\xi_1 + L_1\xi_2, \xi_2) : \xi_1 \in \mathcal{R}P_1, \xi_2 \in \mathcal{R}P_2\} \\
\mathcal{N}P &= \{(\eta_1 + L_2\eta_2, \eta_2) : \eta_1 \in \mathcal{N}P_1, \eta_2 \in \mathcal{N}P_2\},
\end{aligned}$$

where $L_1 : \mathcal{R}P_2 \to U_1$ and $L_2 : \mathcal{N}P_2 \to U_1$ are linear mappings. Then P has the form

$$P = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_2 \end{pmatrix}.$$

Proof. Given $x_1 \in U_1$ and $x_2 \in U_2$, we can write

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \xi_1 + L_1\xi_2 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \eta_1 + L_2\eta_2 \\ \eta_2 \end{pmatrix}$$

where $\xi_1 \in \mathcal{R}P_1$, $\xi_2 \in \mathcal{R}P_2$, $\eta_1 \in \mathcal{N}P_1$ and $\eta_2 \in \mathcal{N}P_2$. It follows first that $x_2 = \xi_2 + \eta_2$ so that

$$\xi_2 = P_2x_2, \quad \eta_2 = (I - P_2)y.$$

Next we see that $x_1 = \xi_1 + L_1\xi_2 + \eta_1 + L_2\eta_2$ so that

$$P_1x_1 = \xi_1 + P_1L_1\xi_2 + P_1L_2\eta_2.$$

[2] Then

$$\begin{aligned}
P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \xi_1 + L_1\xi_2 \\ \xi_2 \end{pmatrix} \\
&= \begin{pmatrix} P_1x_1 - P_1L_1\xi_2 - P_1L_2\eta_2 + L_1\xi_2 \\ \xi_2 \end{pmatrix} \\
&= \begin{pmatrix} P_1x_1 - P_1L_1P_2x_2 - P_1L_2(I - P_2)x_2 + L_1P_2x_2 \\ P_2x_2 \end{pmatrix}
\end{aligned}$$

so that

$$P = \begin{pmatrix} P_1 & -P_1L_1P_2 - P_1L_2(I - P_2) + L_1P_2 \\ 0 & P_2 \end{pmatrix}.$$

□

[2] The red P_1 was P_2

Applying this lemma to $U_1 = \mathbb{R}^{n-n_p}$, $U_2 = \mathbb{R}^{n_p}$, $P = P(m)$, $P_1 = \tilde{P}_1(m)$, $P_2 = P_p(m)$, $L_1 = L^+(m)$ and $L_2 = L^-(m)$, we deduce that $P(m)$ has the form given in the Claim.

Finally we show $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy with projection $\tilde{P}_1(k)$. From Theorem 3, we deduce that the diagonal system corresponding to (18) has an exponential dichotomy with projection

$$P_{diag}(k) = \begin{pmatrix} \tilde{P}_1(k) & 0 \\ 0 & P_p(k) \end{pmatrix},$$

provided that the ranks of $\tilde{P}_1(k)$ and $P_p(k)$ do not depend on k. However recall that $P_p(k)$ is the projection associated with the dichotomy of $x_p(k+1) = A_{pp}(k)x_p(k)$ and so it has rank r_p. Next the rank of $\tilde{P}_1(k)$ is the dimension of $\tilde{V}_1(k)$ which we know by (21) to be $r - r_p$. So the diagonal system

$$x(k+1) = \begin{pmatrix} \tilde{B}_{11}(k) & 0 \\ 0 & A_{pp}(k) \end{pmatrix} x(k) \tag{24}$$

has an exponential dichotomy with projection $P_{diag}(k)$. Then, by Lemma 1 (i), $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy on $\mathbb{Z}_+$ with projection $\tilde{P}_1(k)$. This completes the proof of the Claim.

Applying the inductive hypothesis to $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$, we see that the system

$$\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) & \cdots & A_{1,p-1}(k) \\ 0 & A_{22}(k) & \cdots & A_{2,p-1}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{p-1,p-1}(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ \cdots \\ x_{p-1}(k) \end{pmatrix}, \tag{25}$$

has an exponential dichotomy with projection

$$\hat{P}_1(k) = \begin{pmatrix} P_1(k) & P_{12}(k) & \cdots & \cdot & P_{1,p-1}(k) \\ 0 & P_2(k) & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & P_{p-2}(k) & P_{p-2,p-1}(k) \\ 0 & 0 & \cdots & 0 & P_{p-1}(k) \end{pmatrix}. \tag{26}$$

Then we apply the Claim again, but this time under the additional assumption that $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy with projection $\hat{P}_1(k)$ of rank $r - r_p$. Then $\tilde{P}_1(k)$ defined in the Claim is $\hat{P}_1(k)$ if we take $W_1(0)$ to be $\mathcal{N}\hat{P}_1(0)$. Then we find that the new projection has the form

$$\hat{P}(m) = \begin{pmatrix} \hat{P}_1(m) & \tilde{P}_{1p}(m) \\ 0 & P_p(m) \end{pmatrix}$$

with perhaps a different $\tilde{P}_{1p}(m)$. So $\hat{P}(m)$ is block upper triangular with the diagonal blocks $P_1(m), \ldots, P_p(m)$, as required. □

Now we consider the $\mathbb{Z}_-$ case.

Theorem 5. *Suppose* (1) *has an exponential dichotomy on $\mathbb{Z}_-$ with projection of rank r and that for all i, $x_i(k+1) = A_{ii}(k)x_i(k)$ has an exponential dichotomy on $\mathbb{Z}_-$ with projection $P_i(k)$ of rank r_i (by Proposition 1, this is equivalent to the exponential dichotomy of* (2) *with rank $\sum_{i=1}^p r_i$). Then $\sum_{i=1}^p r_i = r$ and the projection $P(k)$ for the dichotomy of* (1) *can be taken in upper triangular form with $P_i(k)$ as diagonal blocks.*

Proof. Note it follows from the second remark after the proof of Theorem 2 that $r = \sum_{i=1}^p r_i$. We prove this theorem by induction on p. It is trivial for $p = 1$. Assuming it is true for $p - 1 \geq 1$, we prove it for p. We write (1) as

$$x(k+1) = \begin{pmatrix} \tilde{B}_{11}(k) & \tilde{B}_{1p}(k) \\ 0 & A_{pp}(k) \end{pmatrix} x(k), \tag{27}$$

where $\tilde{B}_{11}(k)$ and $\tilde{B}_{1p}(k)$ are as in (19). To proceed further, we need a lemma but before that we need a definition.

Definition. The difference equation $x(k+1) = A(k)x(k)$ on $\mathbb{Z}_-$ is said to have the *backward unique bounded* (BUB) property if for all $m \leq 0$, the only solution $x(k)$ which is bounded in $k \leq m$ and satisfies $x(m) = 0$ is the trivial solution.

Remark. If $x(k+1) = A(k)x(k)$ has an exponential dichotomy on $\mathbb{Z}_-$ with projection $P(k)$, then it has the BUB property. For let $x(k)$ be a solution bounded in $k \leq m$ with $x(m) = 0$. From Lemma 3.1 in [2], we have $x(k) \in \mathcal{N}P(k)$ so that $0 = x(m) = \Phi(m,k)x(k)$. Since $\Phi(m,k)$ is one to one on $\mathcal{N}P(k)$, this implies that $x(k) = 0$ for $k \leq m$.

Lemma 5. *Suppose in* (1)*, $x_i(k+1) = A_{ii}(k)x_i(k)$ has the BUB property on $\mathbb{Z}_-$ for all i. Then* (1) *also has the BUB property on $\mathbb{Z}_-$. Moreover, for all $m \leq 0$, let $V(m)$ be the subspace of those $\xi \in \mathbb{R}^n$ for which there is a solution $x(k)$ of* (1) *which is bounded in $k \leq m$ and such that $x(m) = \xi$ and for each i, let $V_i(m)$ be the subspace of those $\xi_i \in \mathbb{R}^{n_i}$ for which there is a solution $x_i(k)$ of $x_i(k+1) = A_{ii}(k)x_i(k)$ which is bounded in $k \leq m$ and such that $x_i(m) = \xi_i$. Then*

$$\dim V(m) \leq \sum_{i=1}^p \dim V_i(m).$$

Proof. Clearly the lemma holds for $p = 1$. Assuming it for $p - 1 \geq 1$, we prove it for p. Write (1) as in (27) and set $\tilde{x}_1 = (x_1, \ldots, x_{p-1})$. Denote by $\tilde{V}_1(m)$ the set of ξ for which there is a solution $\tilde{x}_1(k)$ of $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ which is bounded in $k \leq m$ and such that $\tilde{x}_1(m) = \xi$. We determine the bounded solutions of

$$\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k) + \tilde{B}_{1p}(k)x_p(k), \quad x_p(k+1) = A_{pp}(k)x_p(k) \tag{28}$$

on $k \leq m$. $x_p(k)$ is a bounded solution of the second equation on $k \leq m$ if and only if $x_p(m) \in V_p(m)$. Also because $x_p(k+1) = A_{pp}(k)x_p(k)$ has the BUB property, the solution $x_p(k)$ is determined by $x_p(m)$. For $\eta \in V_p(m)$, let $x_p(k)$ be the unique bounded solution of $x_p(k+1) = A_{pp}(k)x_p(k)$ with $x_p(m) = \eta$, and let S_m be the subspace of $V_p(m)$ such that if $\eta \in S_m$, then the equation

$$\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k) + \tilde{B}_{1p}(k)x_p(k),$$

admits a bounded solution on $k \leq m$. Since by the induction hypothesis, $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has the BUB property, there is a unique such bounded solution such that $\tilde{x}_1(m)$ is in $W(m)$, where $W(m)$ is a complement to $\tilde{V}_1(m)$ in $\mathbb{R}^{n-n_p}$. The existence of such a solution is clear and the uniqueness follows from the fact that the difference between two such bounded solutions would be a bounded solution of $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ vanishing at $k = m$. Let $\tilde{x}_1(m) = L(m)x_p(m)$; then $L(m)$ is a linear mapping from S_m to $W(m)$. Hence

$$V(m) = \{(\xi + L(m)\eta, \eta) : \xi \in \tilde{V}_1(m),\ \eta \in S_m\} \tag{29}$$

is the subspace of initial values at $k = m$ of solutions of (28) bounded on $k \leq m$. It is clear also that (28) has the BUB property since both $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ and $x_p(k+1) = A_{pp}(k)x_p(k)$ have. Also from (29), we have

$$\dim V(m) = \dim \tilde{V}_1(m) + \dim S_m \leq \dim \tilde{V}_1(m) + \dim V_p(m). \tag{30}$$

By the induction hypothesis $\dim \tilde{V}_1(m) \leq \sum_{i=1}^{p-1} \dim V_i(m)$. The required conclusion follows. □

Now we go back to the proof of Theorem 5. In this case, by Lemma 3.1 in [2], we have $V(m) = \mathcal{N}P(m)$ and $V_i(m) = \mathcal{N}P_i(m)$. Hence (30) reads

$$n - r \leq \dim \tilde{V}_1(m) + n_p - r_p \tag{31}$$

where $\tilde{V}_1(m)$ has been defined in the proof of Lemma 5. Next from Lemma 5 we know that

$$\dim \tilde{V}_1(m) \leq \sum_{i=1}^{p-1} \dim V_i(m) = \sum_{i=1}^{p-1} \dim \mathcal{N}P_i(m) = \sum_{i=1}^{p-1} (n_i - r_i). \tag{32}$$

Putting (31) and (32) together we get

$$\dim \tilde{V}_1(m) = n - n_p - (r - r_p) \tag{33}$$

and then (30) gives:

$$n - r = n - n_p - (r - r_p) + \dim S_m.$$

So $\dim S_m = n_p - r_p$ and hence $S_m = \mathcal{N}P_p(m)$ since $S_m \subset \mathcal{N}P_p(m)$ and both spaces have the same dimension. As a consequence, from (29),

$$V(m) = \{(\xi + L(m)\eta, \eta) : \xi \in \tilde{V}_1(m),\ \eta \in \mathcal{N}P_p(m)\}, \tag{34}$$

where $L(m)$ is as defined in the proof of Lemma 5.

Claim. *Let $W_1(0)$ be a complement to $\tilde{V}_1(0)$ in $\mathbb{R}^{n-n_p}$ and set*

$$W(k) = \Phi(0,k)^{-1}(W_1(0) \times \mathcal{R}P_p(0)) := \{\xi \in \mathbb{R}^n : \Phi(0,k)\xi \in W_1(0) \times \mathcal{R}P_p(0)\}.$$

Then (18) has an exponential dichotomy on $\mathbb{Z}_-$ with projection $P(k)$ having nullspace $V(k)$ and range $W(k)$. Moreover, $P(k)$ has the form

$$P(k) = \begin{pmatrix} \tilde{P}_1(k) & \tilde{P}_{1p}(k) \\ 0 & P_p(k) \end{pmatrix},$$

where $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy with projection $\tilde{P}_1(k)$, which has nullspace $\tilde{V}_1(k)$ and range $\tilde{\Phi}_1(0,k)^{-1}(W_1(0))$.

To prove the Claim, we first see that when $m \le \ell \le 0$, the transition matrix $\tilde{\Phi}_1(\ell, m)$ of $x(k+1) = \tilde{B}_{11}(k)x(k)$ maps $\tilde{V}_1(m)$ bijectively onto $\tilde{V}_1(\ell)$. Indeed suppose $\xi \in \tilde{V}_1(m)$. Then, by (34), $\begin{pmatrix} \xi \\ 0 \end{pmatrix} \in V(m)$ and so

$$\begin{pmatrix} \tilde{\Phi}_1(\ell,m)\xi \\ 0 \end{pmatrix} = \Phi(\ell,m) \begin{pmatrix} \xi \\ 0 \end{pmatrix} \in V(\ell)$$

and then $\tilde{\Phi}_1(\ell,m)\xi \in \tilde{V}_1(\ell)$, by (34) again. Moreover, if $\tilde{\Phi}_1(\ell,m)\xi = 0$ then $\Phi(\ell,m)\begin{pmatrix} \xi \\ 0 \end{pmatrix} = 0$ and hence $\xi = 0$ because $\Phi(\ell,m) : V(m) \to V(\ell)$ is invertible. So $\tilde{\Phi}_1(\ell,m) : \tilde{V}_1(m) \to \tilde{V}_1(\ell)$ is one to one and hence invertible because both spaces have the same dimension $n - n_p - (r - r_p)$, as we see in (33).

Now we see from (34) with $m = 0$ that

$$W(0) = W_1(0) \times \mathcal{R}P_p(0)$$

is a complement to $V(0)$ in $\mathbb{R}^n$. Then it follows from the proof of Proposition 3.2 in [2] that $W(k) \oplus V(k) = \mathbb{R}^n$ for $k \le 0$ and (27) has an exponential dichotomy with respect to the projection $P(k)$ with range $W(k)$ and nullspace $V(k)$.

We prove that $P(k)$ has the form given in the Claim. To this end, we first prove that $\mathbb{R}^{n-n_p} = \tilde{V}_1(k) \oplus W_1(k)$, where

$$W_1(k) = \tilde{\Phi}_1(0,k)^{-1}(W_1(0)).$$

Let $x \in \tilde{V}_1(k) \cap W_1(k)$. Then $\tilde{\Phi}_1(0,k)x \in W_1(0)$. Since $x \in \tilde{V}_1(k)$ and $\tilde{\Phi}_1(0,k)$ maps $\tilde{V}_1(k)$ onto $\tilde{V}_1(0)$, we have $\tilde{\Phi}_1(0,k)x \in \tilde{V}_1(0)$. So $\tilde{\Phi}_1(0,k)x = 0$ and hence $x = 0$ since $\tilde{\Phi}_1(0,k)$ is one to one on $\tilde{V}_1(k)$, as noted above in the first paragraph of the proof of the Claim. Therefore $\tilde{V}_1(k) \cap W_1(k) = \{0\}$.

Next let $x \in \mathbb{R}^{n-n_p}$. Then $\tilde{\Phi}_1(0,k)x = v_1 + w_1$, where $v_1 \in \tilde{V}_1(0)$ and $w_1 \in W_1(0)$. However $v_1 = \tilde{\Phi}_1(0,k)v_k$, for some $v_k \in \tilde{V}_1(k)$. So $\tilde{\Phi}_1(0,k)(x - v_k) = w_1 \in W_1(0)$. Hence $x - v_k \in W_1(k)$ and therefore $x \in \tilde{V}_1(k) + W_1(k)$. So

$\mathbb{R}^{n-n_p} = \tilde{V}_1(k) \oplus W_1(k)$. Then, the projection with range $W_1(k)$ and nullspace $\tilde{V}_1(k)$ is well-defined and we denote it by $\tilde{P}_1(k)$ as in the statement of the Claim. Note that since for $k \le -1$, $A(k)(W_1(k)) \subset W_1(k+1)$ (by definition) and $A(k)(\tilde{V}_1(k)) = \tilde{V}_1(k+1)$ (see the first part of the proof of this Claim), it follows that $\tilde{P}_1(k)$ is invariant with respect to $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$.

Now by the argument we used to get (34), taking $\mathcal{R}\tilde{P}_1(m)$ as the complement to $\tilde{V}_1(m)$, we can show that

$$\mathcal{N}P(m) = V(m) = \{(\xi + L^-(m)\eta, \eta) : \xi \in \tilde{V}_1(m), \eta \in \mathcal{N}P_p(m)\},$$

where $L^-(m) : \mathcal{N}P_p(m) \to \mathcal{R}\tilde{P}_1(m)$. Next

$$\begin{aligned}
&W(m)\\
&= \{x = (\xi, \eta) \in \mathbb{R}^{n-n_p} \times \mathbb{R}^{n_p} : \Phi(0,m)x \in \mathcal{R}\tilde{P}_1(0) \times \mathcal{R}P_p(0)\}\\
&= \{(\xi,\eta) : \tilde{\Phi}_1(0,m)\xi + \tilde{\Phi}_{12}(0,m)\eta \in \mathcal{R}\tilde{P}_1(0), \tilde{\Phi}_2(0,m)\eta \in \mathcal{R}P_p(0)\}\\
&= \{(\xi,\eta) : (\mathbf{I} - \tilde{P}_1(0))[\tilde{\Phi}_1(0,m)\xi + \tilde{\Phi}_{12}(0,m)\eta] = 0, \eta \in \mathcal{R}P_p(m)\}\\
&\qquad \text{since} \quad \mathcal{R}P_p(m) = \tilde{\Phi}_2^{-1}(0,m)(\mathcal{R}P_p(0))\\
&= \{(\xi,\eta) : \tilde{\Phi}_1(0,m)(\mathbf{I} - \tilde{P}_1(m))\xi + (\mathbf{I} - \tilde{P}_1(0))\tilde{\Phi}_{12}(0,m)\eta = 0, \eta \in \mathcal{R}P_p(m)\}\\
&\qquad \text{by invariance of } \tilde{P}_1(m)\\
&= \{(\xi,\eta) : (\mathbf{I} - \tilde{P}_1(m))\xi = L^+(m)\eta, \eta \in \mathcal{R}P_p(m)\},
\end{aligned}$$

where $L^+(m) : \mathcal{R}P_p(m) \to \mathcal{N}\tilde{P}_1(m)$ is given by

$$L^+(m)\eta = -\tilde{\Phi}_1(m,0)(\mathbf{I} - \tilde{P}_1(0))\tilde{\Phi}_{12}(0,m)\eta,$$

$\tilde{\Phi}_1(m,0)$ being the inverse of $\tilde{\Phi}_1(0,m) : \tilde{V}_1(m) \to \tilde{V}_1(0)$. Thus $\tilde{P}_1(m)\xi$ is arbitrary but $(\mathbf{I} - \tilde{P}_1(m))\xi = L^+(m)\eta$. So

$$\mathcal{R}P(m) = W(m) = \{(\xi + L^+(m)\eta, \eta) : \xi \in \mathcal{R}\tilde{P}_1(m), \eta \in \mathcal{R}P_p(m)\}.$$

Applying Lemma 4 to $U_1 = \mathbb{R}^{n-n_p}$, $U_2 = \mathbb{R}^{n_p}$, $P = P(m)$, $P_1 = \tilde{P}_1(m)$, $P_2 = P_p(m)$, $L_1 = L^+(m)$ and $L_2 = L^-(m)$, we deduce that $P(m)$ has the form given in the Claim.

Now according to Theorem 3, we can further deduce that the diagonal system corresponding to (27) has an exponential dichotomy with projection $\operatorname{diag}(\tilde{P}_1(k), P_p(k))$ provided that the ranks of $\tilde{P}_1(k)$ and $P_p(k)$ do not depend on k. However recall that $P_p(k)$ is the projection associated with the dichotomy of $x_p(k+1) = A_{pp}(k)x_p(k)$ and so it has rank r_p. Moreover from Lemma 2 we know that

$$\operatorname{rank} P(k) = \operatorname{rank} \tilde{P}_1(k) + \operatorname{rank} P_p(k).$$

Hence $\operatorname{rank} \tilde{P}_1(k) = r - r_p$ is independent of k. So the diagonal system corresponding to (27) has an exponential dichotomy with projection $\operatorname{diag}(\tilde{P}_1(k), P_p(k))$.

Then by Lemma 1, $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy with projection $\tilde{P}_1(k)$. Thus the proof of the claim is completed.

Applying the induction hypothesis, we see that the system

$$\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) & \cdots & A_{1,p-1}(k) \\ 0 & A_{22}(k) & \cdots & A_{2,p-1}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{p-1,p-1}(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ \cdots \\ x_{p-1}(k) \end{pmatrix} \tag{35}$$

has an exponential dichotomy with projection

$$\hat{P}_1(k) = \begin{pmatrix} P_1(k) & P_{12}(k) & \cdots & \cdot & P_{1,p-1}(k) \\ 0 & P_2(k) & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & P_{p-2}(k) & P_{p-2,p-1}(k) \\ 0 & 0 & \cdots & 0 & P_{p-1}(k) \end{pmatrix}. \tag{36}$$

Then we apply the Claim again, but this time under the additional assumption that $\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k)$ has an exponential dichotomy with projection $\hat{P}_1(k)$ of rank $r - r_p$. Then it turns out that the $\tilde{P}_1(k)$ defined in the Claim is $\hat{P}_1(k)$ if we take $W_1(0)$ to be $\mathcal{R}\hat{P}_1(0)$. Then we find that the new projection has the form

$$\hat{P}(m) = \begin{pmatrix} \hat{P}_1(m) & \tilde{P}_{1p}(m) \\ 0 & P_p(m) \end{pmatrix},$$

which is block upper triangular as required, but possibly with a different $\tilde{P}_{1p}(m)$. □

Now we consider the $\mathbb{Z}$ case.

Theorem 6. *Suppose* (1) *has an exponential dichotomy on $\mathbb{Z}$ with projection of rank r and that for all i, $x_i(k+1) = A_{ii}(k)x_i(k)$ has an exponential dichotomy on $\mathbb{Z}$ with projection $P_i(k)$ of rank r_i (by Proposition 1, this is equivalent to the exponential dichotomy of* (2) *with rank $\sum_{i=1}^p r_i$). Then $\sum_{i=1}^p r_i = r$ and the projection $P(k)$ for the dichotomy of* (1) *is in upper triangular form with $P_i(k)$ as diagonal blocks.*

Proof. Note it follows from the second remark after the proof of Theorem 2 that $r = \sum_{i=1}^p r_i$. We prove this theorem by induction on p. First we prove it for $p = 2$. Note that for any $m \in \mathbb{Z}$, (1) has dichotomies on $[m, \infty)$ and $(-\infty, m]$ with projection $P(k)$, where we note that $P(k)$ is uniquely defined for the dichotomy of (1) on $\mathbb{Z}$ by Proposition 3.2 in [2]. Then we claim that

$$\mathcal{R}P(m) = \{(\xi_1 + L_+(m)\xi_2, \xi_2)) : \xi_1 \in \mathcal{R}P_1(m), \xi_2 \in \mathcal{R}P_2(m)\} \tag{37}$$

and

$$\mathcal{N}P(m) = \{(\eta_1 + L_-(m)\eta_2, \eta_2)) : \eta_1 \in \mathcal{N}P_1(m), \eta_2 \in \mathcal{N}P_2(m)\}, \tag{38}$$

where $L_+(m) : \mathcal{R}P_2(m) \to \mathcal{N}P_1(m)$ and $L_-(m) : \mathcal{N}P_2(m) \to \mathcal{R}P_1(m)$ are linear mappings.

To prove (37), let S_m be the subspace of $\mathcal{R}P_2(m)$ such that if $\xi_2 \in S_m$, then the equation

$$x_1(k+1) = A_{11}(k)x_1(k) + A_{12}(k)\Phi_2(k,m)\xi_2$$

admits a bounded solution. There is a unique such bounded solution with $x_1(m) \in \mathcal{N}P_1(m)$. Let $x_1(m) = L_+(m)\xi_2$; then $L_+(m)$ is a linear mapping from S_m to $\mathcal{N}P_1(m)$ and

$$\mathcal{R}P(m) = \{(\xi_1 + L_+(m)\xi_2, \xi_2) : \xi_1 \in \mathcal{R}P_1(m),\ \xi_2 \in S_m\}.$$

Hence

$$r_1 + r_2 = \dim \mathcal{R}P(m) = \dim \mathcal{R}P_1(m) + \dim S_m \le r_1 + r_2.$$

It follows that $\dim S_m = r_2$ and hence $S_m = \mathcal{R}P_2(m)$. Then (37) follows. (38) is proved similarly.

Applying Lemma 4 to $U_1 = \mathbb{R}^{n_1}$, $U_2 = \mathbb{R}^{n_2}$, $P = P(m)$, $P_1 = P_1(m)$, $P_2 = P_2(m)$, $L_1 = L^+(m)$ and $L_2 = L^-(m)$, we deduce the theorem for $p = 2$.

Now suppose the theorem holds for $p - 1 \ge 2$. Then we prove it for p. We write (1) as in (18) where $\tilde{B}_{11}(k), \tilde{B}_{1p}(k)$ are as in (19). By Theorems 4 and 5, we know that (18) has exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projections

$$P(k) = \begin{pmatrix} \tilde{P}_1(k) & \tilde{P}_{1p}(k) \\ 0 & P_p(k) \end{pmatrix} \qquad Q(k) = \begin{pmatrix} \tilde{Q}_1(k) & \tilde{Q}_{1p}(k) \\ 0 & Q_p(k) \end{pmatrix}$$

respectively, where $\tilde{P}_1(k)$ and $\tilde{Q}(k)$ have rank $r - r_p$ and $P_p(k)$ and $Q_p(k)$ have rank r_p. Moreover,

$$\tilde{x}_1(k+1) = \tilde{B}_{11}(k)\tilde{x}_1(k) \tag{39}$$

has exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projections $\tilde{P}_1(k)$ and $\tilde{Q}_1(k)$ respectively. Notice that if $\tilde{x}_1(k)$ is a nontrivial bounded solution on $\mathbb{Z}$ of (39) then $(\tilde{x}_1(k), 0)$ is a nontrivial bounded solution on $\mathbb{Z}$ of (18). Since (18) has no nontrivial bounded solution on $\mathbb{Z}$ because it has a dichotomy on $\mathbb{Z}$, it follows that (39) has no nontrivial bounded solution on $\mathbb{Z}$. Since, in addition, $\tilde{P}_1(k)$ and $\tilde{Q}_1(k)$ have the same rank, it follows from Corollary 3.3 in [2] that (39) has a dichotomy on $\mathbb{Z}$. So we may apply the induction hypothesis to deduce that the projection for the dichotomy of (39) on $\mathbb{Z}$ is in upper triangular form with $P_i(k)$, $i = 1, \dots, p-1$ as diagonal blocks. However, by the $p = 2$ case, we know that the projection for the dichotomy of (18) on $\mathbb{Z}$ has the form

$$P(k) = \begin{pmatrix} \tilde{P}_1(k) & \tilde{P}_{1p}(k) \\ 0 & P_p(k) \end{pmatrix},$$

where now $\tilde{P}_1(k)$ is the projection for the dichotomy of (39) on $\mathbb{Z}$. Thus the induction proof is completed. □

8 Upper triangular systems

In this section we prove two theorems about upper triangular systems, that is, block upper triangular systems where the blocks are scalars. First we give a necessary and sufficient condition that a diagonal system have an exponential dichotomy.

Theorem 7. *The diagonal system*

$$x(k+1) = \begin{pmatrix} a_{11}(k) & 0 & \cdots & 0 \\ 0 & a_{22}(k) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_{nn}(k) \end{pmatrix} x(k) \tag{40}$$

has an exponential dichotomy on $J = \mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$ with rank r if and only if there exists $I \subset \{1, \dots, n\}$, where $\#I = r$, and constants K and $\alpha > 0$ such that

$$|a_{ii}(k-1) \cdots a_{ii}(m)| \le K e^{-\alpha(k-m)}, \quad \text{for any } m, k \text{ in } J, \quad m \le k,$$

for $i \in I$, and

$$|a_{ii}(k-1) \cdots a_{ii}(m)| \ge K^{-1} e^{\alpha(k-m)}, \quad \text{for any } m, k \text{ in } J, \quad m \le k,$$

for $i \notin I$.

Remark. Note that the second inequality implies that $a_{ii}(k) \neq 0$ for $i \notin I$ and $k, k+1 \in J$.

Proof. Suppose (40) has an exponential dichotomy on $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$). Then by Proposition 1, there exists I such that the stable (resp. unstable) subspace is $\text{span}\{e_i : i \in I\}$ (resp. $\text{span}\{e_i : i \notin I\}$), where the e_i form the standard basis in $\mathbb{R}^n$. Then we may take the unstable (resp. stable) subspace as $\text{span}\{e_i : i \notin I\}$ (resp. $\text{span}\{e_i : i \in I\}$). If (40) has an exponential dichotomy on $\mathbb{Z}$, then these must of course be the stable and unstable subspaces. Then $a_{ii}(k) \neq 0$ for $i \notin I$ and $k, k+1 \in J$ and there exist positive constants K and α such that

$$|a_{ii}(k-1) \cdots a_{ii}(m)| \le K e^{-\alpha(k-m)}, \quad m \le k \text{ in } J,$$

for $i \in I$, and

$$|a_{ii}(k-1) \cdots a_{ii}(m)| \ge K^{-1} e^{\alpha(k-m)}, \quad m \le k \text{ in } J,$$

for $i \notin I$.

Suppose conversely that the inequalities hold so that in particular $a_{ii}(k) \neq 0$ for $i \notin I$ and $k, k+1 \in J$. Let P be the projection with range $\text{span}\{e_i : i \in I\}$ and nullspace $\text{span}\{e_i : i \notin I\}$. Suppose $k \ge m$ in J. Then if $\Phi(k, m)$ is the transition matrix for (40), $\Phi(k, m) e_i = a_{ii}(k-1) \cdots a_{ii}(m) e_i$. Hence $\Phi(k, m)$ maps $\mathcal{N}P$

one to one onto itself so that the inverse $\Phi(m,k)$ of $\Phi(k,m) : \mathcal{N}P \to \mathcal{N}P$ exists. Furthermore $\Phi(k,m)P$ is a diagonal matrix with $a_{ii}(k-1)\cdots a_{ii}(m)$ in the ith position when $i \in I$ and 0 otherwise and $\Phi(m,k)(\mathbf{I}-P)$ is a diagonal matrix with $[a_{ii}(k-1)\cdots a_{ii}(m)]^{-1}$ in the ith position when $i \notin I$ and 0 otherwise. Clearly we have

$$\Phi(k,m)P = P\Phi(k,m), \quad k \geq m.$$

If we use the maximum norm for matrices, then for all $k \geq m$ there exist $i \in I$ such that

$$|\Phi(k,m)P| = |a_{ii}(k-1)\cdots a_{ii}(m)| \leq Ke^{-\alpha(k-m)}$$

and $i \notin I$ such that

$$|\Phi(m,k)(\mathbf{I}-P)| = |[a_{ii}(k-1)\cdots a_{ii}(m)]^{-1}| \leq Ke^{-\alpha(k-m)}.$$

So we have an exponential dichotomy with constant projection P. □

Finally we examine the relation between the exponential dichotomy of an upper triangular system and its associated diagonal system.

Theorem 8. *(i) If the diagonal system* (40) *has an exponential dichotomy on* $J = \mathbb{Z}$, $\mathbb{Z}_+$ *or* $\mathbb{Z}_-$ *and* $|a_{ij}(k)|$ *is bounded for* $i < j$, *then the upper triangular system*

$$x(k+1) = \begin{pmatrix} a_{11}(k) & a_{12}(k) & \cdots & a_{1n}(k) \\ 0 & a_{22}(k) & \cdots & a_{2n}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_{nn}(k) \end{pmatrix} x(k) \tag{41}$$

has an exponential dichotomy on $J = \mathbb{Z}$, $\mathbb{Z}_+$ *or* $\mathbb{Z}_-$ *with projection of the same rank.*

(ii) If the upper triangular system (41) *is invertible and has an exponential dichotomy on* $J = \mathbb{Z}_+$ *or* $\mathbb{Z}_-$, *then the projection can be taken in the form*

$$P(k) = \begin{pmatrix} p_1(k) & p_{12}(k) & \cdots & p_{1n}(k) \\ 0 & p_2(k) & \cdots & p_{2n}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & p_n(k) \end{pmatrix} \tag{42}$$

and the diagonal system (40) *has an exponential dichotomy on* $J = \mathbb{Z}_+$ *or* $\mathbb{Z}_-$ *with projection*

$$\tilde{P}(k) = \begin{pmatrix} p_1(k) & 0 & \cdots & 0 \\ 0 & p_2(k) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & p_n(k) \end{pmatrix}. \tag{43}$$

(iii) If the upper triangular system (41) *has an exponential dichotomy on* $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ *or* $\mathbb{Z}$ *with projection* (42) *where for each* i, $p_i(k) = 1$ *for all* k *or* 0 *for all* k, *then the diagonal system* (40) *has an exponential dichotomy on* $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ *or* $\mathbb{Z}$ *with projection* (43).

(iv) If both (41) *and* (40) *have an exponential dichotomy on* $J = \mathbb{Z}_+$ *or* $\mathbb{Z}_-$ *with projections (say)* $P(k)$ *and* $\tilde{P}(k)$ *respectively, then we may choose* $P(k)$ *as in* (42) *and* $\tilde{P}(k)$ *as in* (43)*, that is, they coincide on the diagonal. Furthermore, for each* i*,* $p_i(k) = 1$ *for all* k *or* 0 *for all* k*.*

(v) If both (41) *and* (40) *have an exponential dichotomy on* $J = \mathbb{Z}$ *with projections (say)* $P(k)$ *and* $\tilde{P}(k)$ *respectively, then* $P(k)$ *has the form* (42) *and* $\tilde{P}(k)$ *the form* (43)*, that is, they coincide on the diagonal. Furthermore, for each* i*,* $p_i(k) = 1$ *for all* k *or* 0 *for all* k*.*

Proof. Immediate from Theorems 1, 2, 3, 4, 5 and 6.

□

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