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Yangian symmetric correlators, $\mathbb{R}$ operators and amplitudes

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Abstract. Yangian symmetric correlators can be constructed by the action of Yang-Baxter R operators on trivial basic correlators. The example of a four-point correlator is given in two representations and the construction of the completely connected N point correlator is described. The helicity representation is dicussed and the relation of the four-point correlator to tree-level scattering amplitudes is shown.

1. Spin chains with $g\ell_n$ symmetry

Let us first formulate the representations used as the quantum space at a site of the chain (compare [15]). Jordan-Schwinger type (JS) representations are based on the Heisenberg algebra, i.e. on n canonically conjugated pairs,

$$[p_a, x_b] = \delta_{ab}, \ a, b, = 1, .., n. \tag{1}$$

Generators of $g\ell_n$ representations can be built, e.g. as

$$\mathrm{L}^+_{ab} = p_a x_b, \quad \text{or} \ \ \mathrm{L}^-_{ab} = -x_a p_b \tag{2}$$

and functions of x_a are the elements of the representation space. The two representations $L^{\pm}$ are dual to each other. If we distinguish the variables in one of the representations by the notation y_a, then the duality can be formulated as

$$[\mathrm{L}^+_{x,ab} + \mathrm{L}^-_{y,ab}, (\mathbf{xy})] = 0, \quad (\mathbf{xy}) = \sum_1^n x_a y_a. \tag{3}$$

The representation in terms of functions of x_a decomposes into the ones of definite degree of homogeneity,

$$(\mathbf{xp}) \cdot \psi(\mathbf{x}) = 2\ell\psi(\mathbf{x}). \tag{4}$$

Such a representation can also be described by functions of $n-1$ coordinate ratios, e.g. $x'_a = \frac{x_a}{x_n}, a = 1, ...n-1$.

$$\psi(\mathbf{x}) = x_n^{2\ell}\phi(\mathbf{x}'), \ \mathrm{L}_x \cdot \psi(\mathbf{x}) = x_n^{2\ell}\mathrm{L}_{(x',2\ell)} \ \phi(\mathbf{x}').$$

The matrix elements of $\mathrm{L}_{(x',2\ell)}$ generate a lowest weight representation with the constant representing the lowest weight vector. For generic values of 2ℓ all polynomials in x'_a form an irreducible representation. For the special values of non-negative integer 2ℓ the lowest weight modul is finite-dimensional.

We use the basic ingredients of the Quantum Inverse Scattering Method (QISM) [1, 2, 3, 4, 5]. The generators of JS representations enter the L operators depending on a spectral parameter.

$$\mathrm{L}^{+}(u) = \mathrm{I}(u) + \mathrm{L}^{+} = \mathrm{I}(u) + \mathbf{p} \otimes \mathbf{x}, \quad \mathrm{L}^{-}(u) = \mathrm{I}u + \mathrm{L}^{-} = \mathrm{I}(u) - \mathbf{x} \otimes \mathbf{p}. \tag{5}$$

In general, if L_{ab} generate a representation of $g\ell_n$ on V then the L matrix operator, $\mathrm{L}(u) = \mathrm{I}u + \mathrm{L}_{ab}$, acting on the tensor product of this space with the one of the fundamental representation,$V \otimes V_f$, obeys the fundamental Yang-Baxter (YB) relation,

$$\mathcal{R}_{12}(u-v)\mathrm{L}(u) \otimes \mathrm{L}(v) = \mathrm{L}(v) \otimes \mathrm{L}(u)\mathcal{R}_{12}(u-v),$$

$$\mathcal{R}_{ab,ef}(u-v)\,\mathrm{L}_{ec}(u)\,\mathrm{L}_{fd}(v) = \mathrm{L}_{bf}(v)\,\mathrm{L}_{ae}(u)\,\mathcal{R}_{ef,cd}(u-v), \tag{6}$$

with the $n^2 \times n^2$ matrix $\mathcal{R}_{12}(u) = \mathrm{I} + \mathrm{P}_{12}$ involving the permutation in $V_f \otimes V_f$. The simple form of the $\mathrm{L}^{\pm}$ operators of JS type (5) allows to derive easily the following relations:

- elementary canonical transform defined as $\mathcal{C}^{-1}(\mathbf{x},\mathbf{p})\mathcal{C} = (\mathbf{p}, -\mathbf{x})$:

$$\mathcal{C}^{-1}\mathrm{L}^{+}(u)\mathcal{C} = \mathrm{L}^{-}(u) \tag{7}$$

- matrix transposition:

$$\left(\mathrm{L}^{+}(u)\right)^{t} = -\mathrm{L}^{-}(-u-1) \tag{8}$$

- operator transposition:

$$\mathrm{L}^{+T}(u) = -\mathrm{L}^{+}(-1-u) \tag{9}$$

- inversion:

$$\left(\frac{\mathrm{L}^{+}(u)}{u}\right)^{-1} = \frac{\mathrm{L}^{+}(-u-1-(\mathbf{xp}))}{-u-1-(\mathbf{xp})} \tag{10}$$

- basic actions:

$$\mathrm{L}^{+}(u)\cdot 1 = (u+1)\mathrm{I}\cdot 1, \quad \mathrm{L}^{-}(u)\cdot 1 = u\mathrm{I}\cdot 1$$

$$\mathrm{L}^{+}(u)\cdot \delta(\mathbf{x}) = u\mathrm{I}\cdot \delta(\mathbf{x}), \quad \mathrm{L}^{-}(u)\cdot \delta(\mathbf{x}) = (u+1)\mathrm{I}\cdot \delta(\mathbf{x}). \tag{11}$$

Now we consider the representations and operators referring to the entire chain of N sites. The fundamental Yang-Baxter relation (6) is not only equivalent to the Lie algebra relation but can serve also as the starting point to formulate the related co-algebra, i.e. the underlying structures of the tensor product representations.

The monodromy matrix is defined as acting on $V_1 \otimes V_2 ... \otimes V_N \otimes V_f$ and constructed from the matrix product of the corresponding L operators.

$$\mathrm{T}^{\sigma}_{1,\ldots,N}(\mathbf{u}) = \mathrm{L}^{\sigma_1}_1(u_1)...\mathrm{L}^{\sigma_1}_N(u_N), \qquad \sigma = (\sigma_1, ...\sigma_N), \; \mathbf{u} = (u_1, ...u_N). \tag{12}$$

It is well known that the monodromy matrices obey the fundamental YB relation, i.e. the relation (6) with $L(u)$ substituted by $\mathrm{T}(u)$.

We need still another kind of YB operators, called general because they act on the tensor product of generic representations, in our case with JS type generators acting on functions of the coordinate components. These general YB operators appear as product of two factors. The basic factor can be derived from the L operators by considering a YB relation in the following form as the defining condition,

$$\mathrm{R}^{\sigma_1,\sigma_2}_{12}(u_1-u_2)\mathrm{L}^{\sigma_1}_1(u_1)\mathrm{L}^{\sigma_2}_2(u_2) = \mathrm{L}^{\sigma_1}_1(u_2)\mathrm{L}^{\sigma_2}_2(u_1)\mathrm{R}^{\sigma_1,\sigma_2}_{12}(u_1-u_2). \tag{13}$$

Here L_1 and L_2 act on variables $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, i.e. on the tensor product of generic representations. They enter the relation in matrix product (unlike the fundamental YB relation where their tensor product enters). We find the explicit solutions

$$\mathrm{R}_{12}^{+-}(u) = (\mathbf{x}_1\mathbf{x}_2)^u, \tag{14}$$

$$\mathrm{R}_{12}^{++}(u) = \int \frac{dc}{c^{1+u}} e^{-c(\mathbf{x}_1\mathbf{p}_2)}. \tag{15}$$

If the action is restricted to homogeneous functions of degree $2\ell_1$ and $2\ell_2$, repectively,

$$\psi(\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2) = \lambda_1^{2\ell_1}\lambda_2^{2\ell_2}\ \psi(\mathbf{x}_1, \mathbf{x}_2), \tag{16}$$

then additional YB relations can be written,

$$\mathrm{R}_{21}^{\sigma_2,\sigma_1}(u_1^{\sigma_1} - u_2^{\sigma_2})\mathrm{L}_1^{\sigma_1}(u_1)\mathrm{L}_2^{\sigma_2}(u_2) = \mathrm{L}_1^{\sigma_1}(u_1)\mathrm{L}_2^{\sigma_2}(u_2)\mathrm{R}_{21}^{\sigma_2,\sigma_1}(u_1^{\sigma_1} - u_2^{\sigma_2}). \tag{17}$$

These are obtained form the above ones by using the inversion relation (10) obeyed by the L matrix operators and replacing the operators of dilatations $(\mathbf{x}_i, \mathbf{p}_i)$ by their eigenvalues $2\ell_i$. Here and in the following we use the notations $u_i^+ = u_i + 2\ell_i,\ u_i^- = u_i - 2\ell_i - n$.

Now the general Yang-Baxter operator is the product of both of the above two types,

$$\mathrm{R}_{12}(u_1, u_1^{\sigma_1}, u_2, u_2^{\sigma_2}) = \mathrm{P}_{12}\mathrm{R}_{12}^{\sigma_1,\sigma_2}(u_1 - u_2)\mathrm{R}_{21}^{\sigma_2,\sigma_1}(u_1^{\sigma_1} - u_2^{\sigma_2}).$$

P_{12} denotes the operator of permutation of the tensor factors in $V_1 \otimes V_2$. The corresponding YB relation reads

$$\mathrm{R}_{12}(u_1, u_1^{\sigma_1}, u_2, u_2^{\sigma_2})\mathrm{L}_1^{\sigma_1}(u_1)\mathrm{L}_2^{\sigma_2}(u_2) = \mathrm{L}_2^{\sigma_2}(u_2)\mathrm{L}_1^{\sigma_1}(u_1)\mathrm{R}_{12}(u_1, u_1^{\sigma_1}, u_2, u_2^{\sigma_2}). \tag{18}$$

The restriction to particular degrees of homogeneity (16) is assumed.

2. Yangian symmetric correlators

We study the matrix-operator eigenvalue condition with the monodromy operators (12) for functions of N points in a space of n dimensions,

$$\mathrm{T}_{1,\ldots,N}^{\sigma}(\mathbf{u})\Phi = E(\mathbf{u})\mathrm{I}\ \Phi, \tag{19}$$

and *define* [16] : N point Yangian symmetric correlators $\Phi = \Phi(\mathbf{x}_1, ..., \mathbf{x}_N)$ are homogeneous eigenfunctions of the matrix operator $\mathrm{T}(\mathbf{u})$.

Each n-dimensional point $\mathbf{x}_i, i = 1, ..., N$, is marked with a spectral parameter u_i, a dilatation weight $2\ell_i$ and a signature σ_i.

The notion of Yangian algebras has been introduced in [6]. Its relation to the QISM has been explained in [7]. The Yangian symmetry of scattering amplitudes was understood on the basis of [8, 9].

Symmetric correlators can be generated by R operators [17]. If $\Phi(1, .., N; \sigma, \mathbf{u})$ is a solution of the monodromy eigenvalue relation (19) then also

$$\mathrm{R}_{i,i+1}^{\sigma_i,\sigma_{i+1}}(u_i - u_{i+1}) \cdot \Phi(1, 2, .., N; \sigma, \mathbf{u}) \quad \text{and} \quad \mathrm{R}_{i+1,i}^{\sigma_{i+1},\sigma_i}(u_i^{\sigma_i} - u_{i+1}^{\sigma_{i+1}}) \cdot \Phi(1, 2, .., N; \sigma, \mathbf{u}) \tag{20}$$

obey (19) with parameter permutations.

The Yang-Baxter RLL relations (13, 17) allow to permute the R operators acting on cyclically adjacent points and having particular arguments with the monodromy operator. The modification of the monodromy operator and the appropriate argument of R can be read off from the *parameter permutation rules* [14] :

$$\begin{aligned} &\mathrm{R}_{12}^{\sigma_1,\sigma_2}(u_1-u_2): \quad (u_1,u_1^\sigma,u_2,u_2^\sigma,...,u_N,u_N^\sigma) \to (u_2,u_1^\sigma,u_1,u_2^\sigma...,u_N,u_N^\sigma), \\ &\mathrm{R}_{21}^{\sigma_2,\sigma_1}(u_1^\sigma-u_2^\sigma): \quad (u_1,u_1^\sigma,u_2,u_2^\sigma,...,u_N,u_N^\sigma) \to (u_1,u_2^\sigma,u_2,u_1^\sigma...,u_N,u_N^\sigma). \end{aligned} \tag{21}$$

The generation of non-trivial solutions can be started from (trivial) basic correlators.

$$\Omega(\sigma) = \prod_{j:\sigma_j=-} \delta(\mathbf{x}_j), \quad \mathrm{T}^{\sigma_{\mathbf{T}}}_{1,...,N}(\mathbf{u})\ \Omega(\sigma_{\mathbf{o}}) = \prod_1^N (u_i + \frac{1}{2}(1+\sigma_{T,i}\sigma_{o,i}))\mathrm{I}\ \Omega(\sigma_{\mathbf{o}})$$

We resort to two standard signature configurations

- regular case : $\sigma_0 = (+,+,...,+), \quad \Omega = 1,$
- uniform case : $\sigma_T = (+,+,...,+), \quad \Omega_{I,J} = \prod_{i\in I}\delta(\mathbf{x}_i).$

Signature cases can be related by elementary canonical transformation $\mathcal{C}$ acting uniformly on the components of the site variables $x_{s,a}, p_{s,a}, s = 1,...,n, a = 1,...N,\ \mathcal{C}^{-1}(\mathbf{x}_{s,a},\mathbf{p}_{s,a})\mathcal{C} = (\mathbf{p}_{s,a}, -\mathbf{x}_{s,a})$.

Let us construct an example of a 4-point correlator in both representations: In the uniform case we have

$$\begin{aligned} &\mathrm{R}_{32}^{++}(u_2^+-u_3^+)\mathrm{R}_{12}^{++}(u_4-u_2)\mathrm{R}_{34}^{++}(u_3-u_1)\mathrm{R}_{14}^{++}(u_1-u_4)\delta(\mathbf{x}_2)\delta(\mathbf{x}_4) \\ &= \Phi^{+-+-}(u_2,u_1^+,u_4,u_3^+,u_1,u_2^+,u_3,u_4^+) = \\ &\int \frac{dc_{23}dc_{21}dc_{43}dc_{41}}{c_{32}^{1+u_2-u_3-n}c_{21}^{1+u_4-u_2}c_{43}^{1+u_3-u_1}c_{41}^{1+u_1-u_4}}\delta(\mathbf{x}_2-c_{23}\mathbf{x}_3-c_{21}\mathbf{x}_1)\delta(\mathbf{x}_4-c_{41}\mathbf{x}_1-c_{43}\mathbf{x}_3). \end{aligned} \tag{22}$$

In the regular case we have

$$\begin{aligned} &\mathrm{R}_{32}^{+-}(u_2^+-u_3^+)\mathrm{R}_{12}^{+-}(u_4-u_2)\mathrm{R}_{34}^{+-}(u_3-u_1)\mathrm{R}_{14}^{+-}(u_1-u_4)\cdot 1 = \\ &(\mathbf{x}_3\mathbf{x}_2)^{u_2-u_3-n}(\mathbf{x}_1\mathbf{x}_2)^{u_4-u_2}(\mathbf{x}_3\mathbf{x}_4)^{u_3-u_1}(\mathbf{x}_1\mathbf{x}_4)^{u_1-u_4}. \end{aligned}$$

The expression is an eigenfunction of the monodromy with the spectral parameters permuted to (u_2,u_4,u_1,u_3).

3. The generic N point correlator

We consider the particular signature configuration $-...-+...+$ and use the uniform representation

$$\Omega_{N,K} = \prod_1^K \delta(\mathbf{x}_i) = \Phi_0(u_1,u_1-n;...u_K,u_K-n;u_{K+1},u_{K+1};...;u_N,u_N). \tag{23}$$

The last notation shows the spectral parameters u_s as in the monodromy operator and also their combination with the weights u_s^+. As shown above the action of R operators is accompanied by parameter permutations and writing the resulting parameter configuration allows easily to fix the arguments of the R operators such that the result is a symmetric correlator.

Each R_{ji}^{++} operation brings in an integration over c_{ij}. The completely connected correlator of interest involves $K\times(N-K)$ integrations, there is an integral with the variable c_{ij} for each pair of sites i,j of different signature. The integration variables can be regarded as the elements of a

$K \times (N-K)$ matrix $\hat{c}$ and are related to the points of the Grassmannian $G_{N,K}$. The integration contour lies in the maximal Schubert cell.

$$M_{N,K} = \int d^{K(N-K)}c \ \ \phi(c) \ \delta(\hat{C}\mathbf{x}), \tag{24}$$

$$d^{K(N-K)}c = d^{N-K}c_1 ... d^{N-K}c_N, \ d^{N-K}c_i = dc_{i,K+1}...dc_{i,N},$$

$$\hat{C} = (I_{K\times K}, -\hat{c}), \quad \delta(\hat{C}\mathbf{x}) = \prod_1^K \delta^{(n)}(\mathbf{x}_i - \sum_{j=K+1}^N c_{ij}\mathbf{x}_j).$$

We shall see how the symmetry condition (19) fixes ϕ.

This symmetric correlator can be obtained by a $K(N-K)$-fold R operator action on the basic one (23) in several ways. We indicate as a standard way for $N \geq 2K$ the following procedure.

$$M_{N,K} = \Phi_K(u_1, u_{K+1}; ...; u_K, u_{2K}; ...; u_{N-K+1}, u_1^+; ...; u_N, u_K^+), \tag{25}$$

where Φ_K is the result of R operators action in the Kth step starting from the basic correlator (23). The first step is the action by $N-K$ R operators

$$\mathrm{R}_{N,N-1}(u_1^+ - u_N)...\mathrm{R}_{K+2,K+1}(u_1^+ - u_{K+2})\mathrm{R}_{K+1,1}(u_1^+ - u_{K+1})\Phi_0 =$$

$$\Phi_1(u_1, u_{K+1}; ...; u_K, u_K^+; u_{K+1}, u_{K+2}; ...; u_N, u_1^+) =$$

$$\int \frac{dc^1_{1,K+1} dc^1_{K+1,K+2} ... dc^1_{N-1,N}}{(c^1_{1,K+1})^{1+u_1^+ - u_{K+1}} (c^1_{K+1,K+2})^{1+u_1^+ - u_{K+2}} ... (c^1_{N-1,N})^{1+u_1^+ - u_N}}$$

$$\delta\left(\mathbf{x}_1 - c^1_{1,K+1}\mathbf{x}_{K+1} + \sum_{K+2}^N (-1)^{K+j} c^1_{K+1,K+2}...c^1_{j-1,j}\ \mathbf{x}_j\right) \ \prod_2^K \delta(\mathbf{x}_i).$$

We observe that the u^+ parameters of the sites $1, K+1, K+1, ...N$ are shifted cyclically leaving the ones at $2, ...K$ untouched. At the Lth step the action is by

$$\mathrm{R}_{N,N-1}(u_L^+ - v_N)...\mathrm{R}_{K+2,K+1}(u_L^+ - v_{K+2})\mathrm{R}_{K+1,L}(u_L^+ - v_{K+1}),$$

where $v_{K+1}, ..., v_N$ is the sequence of u^+ parameters appearing at the step $L-1$ at the sites $K+1, ..., N$. The action produces the cyclic shift of the sequence of u^+ parameters at the sites $L, K+1, ...N$ leaving the sites $1, ..., L-1, L+1, ...K$ untouched.

By these R actions a completely connected correlator is generated, i.e. there are links between all pairs of sites with different signature. We consider the details for the case $K=2$ where

$$M_{N,2} = \Phi_2(u_1, u_3; u_2, u_4; u_3, u_5; ...; u_{N-1}, u_1 - n; u_N, u_2 - n) =$$

$$\int \frac{dc^1_{1,3} dc^1_{3,4} ... dc^1_{N-1,N}}{\mathcal{N}_1} \frac{dc^2_{2,3} dc^2_{3,4} ... dc^2_{N-1,N}}{\mathcal{N}_2} \times$$

$$\delta(\mathbf{x}_1 - c^1_{1,3}\mathbf{x}_3 + \sum_4^N (-1)^j S^{(1,2)}_{3,j}\ \mathbf{x}_j)\delta(\mathbf{x}_2 - c^2_{2,3}\mathbf{x}_3 + \sum_4^N (-1)^j c^2_{3,4}...c^2_{j-1,j}\ \mathbf{x}_j),$$

$$\mathcal{N}_1 = (c^1_{1,3})^{1+u_1^+ - u_3} (c^1_{3,4})^{1+u_1^+ - u_4} ... (c^1_{N-1,N})^{1+u_1^+ - u_N},$$

$$\mathcal{N}_2 = (c^2_{2,3})^{1+u_2^+ - u_4} (c^1_{3,4})^{1+u_2^+ - u_5} ... (c^1_{N-1,N})^{1+u_2^+ - u_1^+}. \tag{26}$$

In the general case we end up with

$$M_{N,K} = \int \frac{d^{N-K}c^1}{\mathcal{N}_1} ... \int \frac{d^{N-K}c^K}{\mathcal{N}_K} \times$$

$$\delta(\mathbf{x}_1 - \sum_1^{N-K} (-1)^j S^{1,K}_{K+1,K+j} \mathbf{x}_{K+j}) ... \delta(\mathbf{x}_K - \sum_1^{N-K} (-1)^j S^{K,K}_{K+1,K+j} \mathbf{x}_{K+j}), \tag{27}$$

where

$$d^{N-K}c^L = dc^L_{L,K+1} dc^L_{K+1,K+2} ... dc^L_{N-1,N},$$

$$\mathcal{N}_L = (c^L_{L,K+1})^{1+u^+_L - u_{K+L}} (c^L_{K+1,K+2})^{1+u^+_L - u_{K+L+1}} ...$$

$$...(c^L_{K-L,K-L+1})^{1+u^+_L - u_N} (c^L_{K-L+1,K-L+2})^{1+u^+_L - u^+_1} ... (c^L_{N-1,N})^{1+u^+_L - u^+_{L-1}}.$$

It is convenient to define $S^{(i_0,i)}_{j_0,j}$ for $1 \leq i_0 < i \leq K < j_0 < j \leq N$ as a sum of products of the integration variables $c^L_{j_1-1,j_1}$ entering by the action of R_{j_1,j_1-1} at the Lth step, $i_0 \leq L \leq i$ by the following iteration

$$S^{(i_0,i)}_{j_0,j} = S^{(i_0,i)}_{j_0,j-1} c^i_{j-1,j} + S^{(i_0,i-1)}_{j_0,j} \tag{28}$$

$$S^{(i,i)}_{j_0,j} = c^i_{i,j_0} c^i_{j_0,j_0+1} ... c^i_{j-1,j}, \; S^{(i_0,i)}_{j_0,j_0} = c^i_{ij_0}.$$

The second relation results in

$$c^i_{j-1,j} = \frac{S^{(i,i)}_{j_0,j}}{S^{(i,i)}_{j_0,j-1}},$$

and then the frist relation can be solved as

$$S^{(i_0,i-1)}_{j_0,j} = \begin{vmatrix} S^{i,i}_{j_0,j-1} & S^{i,i}_{j_0,j} \\ S^{i_0,i}_{j_0,j-1} & S^{i_0,i}_{j_0,j} \end{vmatrix} \frac{1}{S^{i,i}_{j_0,j-1}}.$$

We shall transform the integral (27) with respect to $c^i_{j-1,j}$ into the link form (24) where the integration variables are $c_{i,j} = (-1)^j S^{i,K}_{K+1,K+j}$.

In the case $K = 2$ we have to consider

$$S^{(2,2)}_{3,j} = c^2_{3,4} ... c^2_{j-1,j}, S^{(1,1)}_{3,j} = c^1_{3,4} ... c^1_{j-1,j}, S^{(1,2)}_{3,4} = c^1_{34} + c^2_{34},$$

$$S^{(1,2)}_{3,j} = S^{(1,2)}_{3,j-1} c^2_{j-1,j} + c^1_{3,4} ... c^1_{j-1,j}.$$

Comparing the R operator expression with the expected maximal cell link integral expression (24) we see that the link variables $c_{2j}, c_{1j}, j > 3$ have to be proportional correspondingly to $S^{(2,2)}_{3,j}$ and $S^{(1,2)}_{3,j}$. Therefore we find the expression of the R operation variables in terms of these sums.

$$c^2_{j-1,j} = \frac{S^{(2,2)}_{3,j}}{S^{(2,2)}_{3,j-1}}, \; c^1_{j-1,j} = \frac{S^{(1,1)}_{3,j}}{S^{(1,1)}_{3,j-1}}, \; S^{(1,2)}_{3,j} = S^{(1,2)}_{3,j-1} c^2_{j-1,j} + S^{(1,1)}_{3,j},$$

$$S^{(1,1)}_{3,j} = \frac{1}{S^{(2,2)}_{3,j-1}} \begin{vmatrix} S^{(2,2)}_{3,j-1} & S^{(1,2)}_{3,j-1} \\ S^{(2,2)}_{3,j} & S^{(1,2)}_{3,j}. \end{vmatrix}$$

We identify the link variables $c_{ij}, i = 1, 2; j = 3, ..., N$.

$$c_{13} = c^1_{1,3}, \; c_{14} = -c^1_{1,3} S^{(12)}_{34} = -c^1_{13}(c^1_{34} + c^2_{34}), c_{1j} = (-1)^{j+1} c^1_{13} S^{(1,2)}_{3j}$$

$$c_{23} = c^2_{2,3},\ c_{24} = -c^2_{1,3} S^{(2,2)}_{34} = -c^2_{13} c^2_{34}, c_{2j} = (-1)^{j+1} c^2_{23} S^{(2,2)}_{3j}.$$

Now we can relate the expressions $S^{i,2}_{3j}$ with the link variables.

$$S^{(1,2)}_{3j} = (-1)^{j+1} \frac{c_{1j}}{c_{13}},\quad S^{(2,2)}_{3j} = (-1)^{j+1} \frac{c_{2j}}{c_{23}}.$$

Finally we express the original R integration variables in terms of the link variables.

$$c^2_{23} = c_{23},\ c^2_{j-1,j} = -\frac{c_{2,j}}{c_{2,j-1}}, j \geq 4. \tag{29}$$

$$c^1_{13} = c_{13},\ c^1_{j-1,j} = \frac{S^{(1,1)}_{3,j}}{S^{(1,1)}_{3,j-1}}. \tag{30}$$

We substitute the result for $S^{(11)}_{3j}$ in terms of $S^{(i2)}_{3j}$ and obtain

$$c^1_{j-1,j} = -\frac{c_{2,j-2}}{c_{2,j-1}} \frac{\begin{vmatrix} c_{2,j-1} & c_{1,j-1} \\ c_{2,j} & c_{1,j} \end{vmatrix}}{\begin{vmatrix} c_{2,j-2} & c_{1,j-2} \\ c_{2,j-1} & c_{1,j-1} \end{vmatrix}}.$$

The last expression works well for $j \geq 5$. For $j = 4$ one finds the correct result with the substitution $c_{22} = 1, c_{12} = 0$.

We calculate the denominator of the integrand in $M_{N,2}$.

$$\mathcal{N}_2 = c_{23}^{1+u_2^+ - u_4} \left(\frac{c_{24}}{c_{23}} \right)^{1+u_2^+ - u_5} \cdots \left(\frac{c_{2,N-1}}{c_{2,N-2}} \right)^{1+u_2^+ - u_N} \left(\frac{c_{2N}}{c_{2,N-1}} \right)^{1+u_2^+ - u_1^+} =$$

$$c_{23}^{u_5 - u_4} c_{24}^{u_6 - u_5} \ldots c_{2,N-1}^{u_1^+ - u_N} c_{2N}^{1+u_2^+ - u_1^+},$$

$$\mathcal{N}_1 = c_{13}^{1+u_1^+ - u_3} \left(\frac{m_4}{c_{23} c_{13}} \right)^{1+u_1^+ - u_4} \left(\frac{c_{23} m_5}{c_{24} m_4} \right)^{1+u_1^+ - u_5} \cdots$$

$$\cdots \left(\frac{c_{2,n_2} m_{N-1}}{c_{2,N-2} m_{N_2}} \right)^{1+u_1^+ - u_{N-1}} \left(\frac{c_{2,N-2} m_N}{c_{2,N-1} m_{N-1}} \right)^{1+u_1^+ - u_N} =$$

$$c_{13}^{u_4 - u_3} m_4^{u_5 - u_4} \ldots m_{N-1}^{u_N - u_{N-1}} m_N^{1+u_1^+ - u_N} \quad c_{23}^{u_4 - u_5} c_{24}^{u_5 - u_6} \ldots c_{2,N-2}^{u_{N-1} - u_N} c_{2,N_1}^{-1+u_N - u_1^+}.$$

Here we have abbreviated the determinants

$$m_j = \begin{vmatrix} c_{2,j-1} & c_{1,j-1} \\ c_{2,j} & c_{1,j} \end{vmatrix}, \quad j \geq 4.$$

The result for the integrands denominator is

$$\mathcal{N}_1 \mathcal{N}_2 = c_{13}^{u_4 - u_3} m_4^{u_5 - u_4} \ldots m_{N-1}^{u_N - u_{N-1}} m_N^{1+u_1^+ - u_N} \quad c_{2,N-1}^{-1} c_{2N}^{1+u_2^+ - u_1^+}.$$

Still the transformation is incomplete, the Jacobian must be taken into account. Consider first the factor in the Jacobian

$$\frac{\partial(c^2_{23} c^2_{34} \ldots)}{\partial(c_{23} c_{24} \ldots)}.$$

The matrix comes out triangualar by using (29) and the product of the diagonal elements is

$$\prod_3^N \frac{\partial c_{2j}}{\partial c^2_{j-1,j}} = \frac{1}{c_{23}c_{24}...c_{2N}}.$$

The triangularity simplifies the calculation also in

$$\frac{\partial(c^1_{13}c^1_{34}...)}{\partial(c_{13}c_{14}...)}.$$

Using (30) we see that it equals to

$$\prod \frac{\partial c^1_{j-1,j}}{\partial c_{1j}} = \frac{1}{c_{13}}\frac{c_{23}}{m_4}...\frac{c_{2,N-2}}{m_{N-1}}.$$

The Jacobian of the transformation to the link integration is the product of the above partial Jacobians due to trangularity.

$$\frac{\partial(c^1_{13}c^1_{34}...c^1_{N-1,N}c^2_{23}c^2_{34}...c^2_{N-1,N})}{\partial(c_{13}c_{14}...c_{1N}c_{23}c_{24}...c_{2,N}} = \frac{1}{c_{2,N-1}c_{14}}\frac{1}{m_4...m_{N-1}}.$$

We obtain the completely connected symmetric correlator for $K=2$ in form of the maximal cell link integral.

$$M_{N,2} = \int \frac{dc^1_{1,3}dc^1_{3,4}...dc^1_{N-1,N}dc^2_{2,3}dc^2_{3,4}...dc^2_{N-1,N}}{(c_{1,3})^{1+u_4-u_3}(34)^{1+u_5-u_4}...(N-1,N)^{1+u_1^+-u_N}c_{2N}^{1+u_2-u_1}}\times$$

$$\delta(\mathbf{x}_1 - \sum_3^N c_{1j}\ \mathbf{x}_j)\delta(\mathbf{x}_2 - \sum_3^N c_{2j}\ \mathbf{x}_j). \tag{31}$$

Here $(j-1,j) = m_j$ denotes the minor of the matrix $\hat{C}$ of the columns $j-1, j$,

$$\hat{C} = \begin{pmatrix} 1 & 0 & -c_{13} & ... & -c_{1N} \\ 0 & 1 & -c_{23} & ... & -c_{2N} \end{pmatrix}.$$

The transformation can be done for general K relying on the iterative relation (28). The result can be written as (24) with

$$\phi(c) = (2,...,K+1)^{-1-u_{K+2}+u_{K+1}}(3,...,K+2)^{-1-u_{K+3}+u_{K+2}}...(N-K,...,N-1)^{-1-u_N+u_{N-1}}$$

$$(N-K+1,...,N)^{-1-u_1^++u_N}(N-K+2,...,N,1)^{-1-u_2^++u_1^+}...(N,1...,K-1)^{-1-u_K^++u_{K-1}^+}. \tag{32}$$

The factors are powers of the subsequent $K \times K$ minors of the rectangular matrix $\hat{C}$ (24).

In application to scattering amplitudes the special case of this form without parameters was proposed in [10] and used extensively in the related literature. The parameter dependent expression has been discussed recently in [13].

The result is by construction an eigenfunction of the monodromy $T(u_1, u_2, ..., u_N)$. The action of the R operators has resulted in permutations of the weight dependent parameters u_i^+; the resulting configuration can be characterized by the permutation

$$\begin{pmatrix} u_1 & u_2 & ... & u_K & u_{K+1} & ... & u_{N-K} & u_{N-K+1} & ... & u_N \\ u_{K+1} & u_{K+2} & ... & u_{2K} & u_{2K+1} & ... & u_N & u_1 - n & ... & u_K - n \end{pmatrix}.$$

The values of v_i^+ are on the second row and the weights attributed to the points are calculated as $2\ell_i = v_i^+ - u_i$.

In the case $K = 2$ we have the list of weights and exponents as

$$2\ell_1 = u_3 - u_1, 2\ell_2 = u_4 - u_2, ..., 2\ell_i = u_{i+2} - u_i, ...2\ell_{N-2} = u_N - u_{N-2},$$

$$2\ell_{N-1} = u_1 - n - u_{N-1}, 2\ell_N = u_2 - n - u_N,$$

$$\alpha_3 = u_4 - u_3, \alpha_4 = u_5 - u_4, ..., \alpha_i = u_{i+1} - u_i, ..., \alpha_{N_1} = u_N - u_{N-1}, \alpha_1 = u_1 - n - u_1, \alpha_2 = u_2 - u_1.$$

The weights are not independent, in any case

$$\sum_{s=1}^{N} 2\ell_s = -2n.$$

The weights do not fix all the N spectral parameters, e.g. u_N is independent. In the case of even $N = 2M$ there are actually two relations,

$$\sum_{k=1}^{M} 2\ell_{2k} = -n, \ \sum_{k=1}^{M} 2\ell_{2k-1} = -n.$$

Therefore the weights fix only $N - 2$ of the spectral parameters, e.g. u_N and u_{N-1} are free. Thus for even N the exponents of the minors in the final result (31) involve besides of the $N-2$ independent weights the extra parameter $u_N - u_{N-1}$. The particular case with $N = 4$ will be discussed below.

4. Scattering amplitudes

For applications to scattering amplitudes we introduce a further kind of representation. It is obtained from the uniform representation by partial canonical transformations after dividing the index set labelling the components at each site. $s = 1, ..., n \to \dot{\alpha} = 1, ..., p;\ \alpha = p+1, ..., n$. The elementary canonical transformation is applied to the dotted components only as $\mathcal{C}^{-1}(\mathbf{x}_{\dot{\alpha}}, \mathbf{p}_{\dot{\alpha}})\mathcal{C}$. The matrix in the L operator splits into blocks transforming in different ways as

$$\mathbf{p} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{p}_{\dot{\alpha}}\mathbf{x}_{\dot{\alpha}} & \mathbf{p}_{\dot{\alpha}}\mathbf{x}_{\alpha} \\ \mathbf{p}_{\alpha}\mathbf{x}_{\dot{\alpha}} & \mathbf{p}_{\alpha}\mathbf{x}_{\alpha} \end{pmatrix} \to \begin{pmatrix} -\mathbf{x}_{\dot{\alpha}}\mathbf{p}_{\dot{\alpha}} & -\mathbf{x}_{\dot{\alpha}}\mathbf{x}_{\alpha} \\ \mathbf{p}_{\alpha}\mathbf{p}_{\dot{\alpha}} & \mathbf{p}_{\alpha}\mathbf{x}_{\alpha} \end{pmatrix}.$$

We reconsider the basic symmetry condition (19). By expansion of $T(\mathbf{u} + \Delta)$ we obtain at the $N-1$st power of Δ the well known result that the sums over the chain sites of the off-diagonal $g\ell_n$ generators annihilate the symmetric correlator. In this way the conserved quantities of the global chain symmetry are expressed.

In the helicity representation a part of the off-diagonal operators (the ones in the upper right block in L) acts multiplicatively on functions of the coordinate variables:

$$\sum_{s=1}^{N} k_{s,\dot{\alpha},\alpha}\Phi = 0, \qquad k_{s,\dot{\alpha},\alpha} = \mathbf{x}_{s,\dot{\alpha}}\mathbf{x}_{s,\alpha}.$$

If $k_{s,\dot{\alpha},\alpha}$ are the components of the momenta of scattering particles the particular conservation law is just the energy-momemtum conservation. It works for massless particles where the mass-shell condition is the one for the factorisability. We recall that a light-like four vector can be represented by the product of Weyl spinor components of left and right chirality. In the following we rename the coordinate components introducing different symbols for the ones in the two index ranges, $\bar{\lambda}_{\dot{\alpha}}$ and λ_α. We conclude that a generic Yangian symmetric correlator in the helicity representation appears with the delta distribution as a factor expressing this conservation law.

$$\Phi(\bar{\lambda}_1, \lambda_1, ..., \bar{\lambda}_N, \lambda_N) = \delta(\sum_1^N k_{s,\dot{\alpha},\alpha}) \times \phi(\bar{\lambda}_1, \lambda_1, ..., \bar{\lambda}_N, \lambda_N). \tag{33}$$

In this representation the scalar product becomes

$$(\mathbf{x}_1\mathbf{p}_2) \to (\lambda_1\pi_2) - (\bar{\pi}_1\bar{\lambda}_2)$$

and the action of the R operator has the form

$$\mathrm{R}_{12}^{++}(u)F(\bar{\lambda}_{1,\dot{\alpha}}, \lambda_{1,\alpha}, \bar{\lambda}_{2,\dot{\alpha}}, \lambda_{2,\alpha}) = \int \frac{dc}{c^{1+u}} F(\bar{\lambda}_{1,\dot{\alpha}} + c\bar{\lambda}_{2,\dot{\alpha}}, \lambda_{1,\alpha}, \bar{\lambda}_{2,\dot{\alpha}}, \lambda_{2,\alpha} - c\lambda_{1,\alpha}).$$

We consider the transformation of the dilatation operator from the uniform to the helicity representation

$$(\mathbf{x}_i \cdot \mathbf{p}_i) \to \lambda_{i,\alpha}\frac{\partial}{\partial\lambda_{i,\alpha}} - \bar{\lambda}_{i,\dot{\alpha}}\frac{\partial}{\partial\bar{\lambda}_{i,\dot{\alpha}}} - 2.$$

The helicity of a scattering particle state h_i is calculated as one half of the difference of the weights in λ and $\bar{\lambda}$, therefore

$$2\ell_i = 2h_i - 2. \tag{34}$$

On the other hand we have $2\ell_i = v_i^+ - v_i$ for the weights of a correlator $\Phi(v_1, v_1^+; v_2, v_2^+, ..., v_N, v_N^+)$. Thus a part of the spectral parameter dependence can be expressed in terms of the particle helicities.

We return to the example of a 4-point correlator (22) and study its application to $2 \to 2$ scattering. We put the parameters back to the standard ordering.

$$\Phi(u_1, u_1^+; u_2, u_2^+; u_3, u_3^+, u_4, u_4^+) =$$

$$\mathrm{R}_{32}(u_2^+ - u_3^+)\mathrm{R}_{12}(u_2 - u_1)\mathrm{R}_{34}(u_4 - u_3)\mathrm{R}_{14}(u_3 - u_2)\ \delta(\mathbf{x}_2)\delta(\mathbf{x}_4) =$$

$$\int \frac{dc_{23}dc_{21}dc_{43}dc_{41}}{c_{23}^{1+u_1-u_4-n} c_{21}^{1+u_2-u_1} c_{43}^{u_4-u_3} c_{41}^{u_3-u_2}} \times$$

$$\delta(\bar{\lambda}_1 + c_{41}\bar{\lambda}_4 + c_{21}\bar{\lambda}_2)\delta(\bar{\lambda}_3 + c_{43}\bar{\lambda}_4 + c_{23}\bar{\lambda}_2)\delta(\lambda_2 - c_{21}\lambda_1 - c_{23}\lambda_3)\delta(\lambda_4 - c_{41}\lambda_1 - c_{43}\lambda_3),$$

$$u_1^+ = u_3, u_2^+ = u_4, u_3^+ = u_1 - n, u_4^+ = u_2 - n. \tag{35}$$

Now we specify to the case $n = 4, p = 2$. We shall transform to the form (33) following [11]. We count 8 delta distributions. We intend to transform the expression in such a way that the 4 deltas of the energy momentum conservation appear explicitly and the remaining 4 are used to do the 4 link integrals. First we address the 4 linear equations related to the deltas involving λ components.

$$\lambda_2 - c_{21}\lambda_1 - c_{23}\lambda_3 = \lambda_2', \quad \lambda_4 - c_{41}\bar{\lambda}_1 - c_{43}\lambda_3 = \lambda_4'. \tag{36}$$

We do projections by performing (antisymmetric) spinor products,

$$< 1, 2 >=< \lambda_1, \lambda_2 >= \lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1}.$$

Note the difference to the above products (3) denoted by ordinary brackets $(...,...)$ which are defined in the euclidean way. The conditions (36) with $\lambda'_2 = \lambda'_4 = 0$ are solved by

$$c^*_{21} = \frac{<2,3>}{<1,3>},\ c^*_{23} = \frac{<2,1>}{<3,1>},\ c^*_{41} = \frac{<4,3>}{<1,3>},\ c^*_{43} = \frac{<4,1>}{<3,1>},$$

and the Jacobi determinant of the transformation $\lambda'_{2,\alpha}, \lambda'_{4,\alpha} \to c_{21}, c_{23}, c_{41}, c_{43}$ is

$$\frac{\partial(\lambda'_{2,1}\lambda'_{2,2})}{\partial(c_{21},c_{23})}\frac{\partial(\lambda'_{4,1}\lambda'_{4,2})}{\partial(c_{41},c_{43})} =<1,3>^2 .$$

In this way we have obtained

$$\delta(\lambda_2 - c_{21}\lambda_1 - c_{23}\lambda_3)\delta(\lambda_4 - c_{41}\lambda_1 - c_{43}\lambda_3) =<1,3>^{-2}\ \delta(c_{21}-c^*_{21})\delta(c_{23}-c^*_{23})\delta(c_{41}-c^*_{41})\delta(c_{43}-c^*_{43}).$$

The deltas involving $\bar\lambda$ components can be transformed by multiplying the matrix of the involved equations

$$\tilde C = \begin{pmatrix} 1 & c_{21} & 0 & c_{41} \\ 0 & c_{23} & 1 & c_{43} \end{pmatrix}$$

from the left by a 2×2 matrix A taking into account the appropriate Jacobian. We choose this matrix in such a way that

$$A\ \tilde C = \begin{pmatrix} \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} & \lambda_{4,1} \\ \lambda_{1,2} & \lambda_{2,2} & \lambda_{3,2} & \lambda_{4,2} \end{pmatrix}.$$

It works indeed with

$$A = \begin{pmatrix} \lambda_{1,1} & \lambda_{3,1} \\ \lambda_{2,1} & \lambda_{3,2} \end{pmatrix},$$

if we substitute simultaneously the link variables $c_{i,j}$ by the solutions $c^*_{i,j}$ obtained above.

We observe that the Jacobi determinant is the inverse of the one in the first transformation. Finally all factors of $<1,3>$ cancel. We obtain the form (33) in our case.

$$\Phi(u_1,u_1^+;u_2,u_2^+;u_3,u_3^+,u_4,u_4^+) = \delta^{(4)}(\sum_1^4 \lambda_{i,\alpha}\bar\lambda_{i,\dot\alpha})\ \ \phi(\lambda,\bar\lambda),$$

$$\phi(\lambda,\bar\lambda) = \frac{<1,2>^4}{<1,2>^{1+u_1-u_4}<2,3>^{1+u_2-u_1}<4,1>^{1+u_4-u_3}<3,4>^{1+u_3-u_2}}.$$

We reconsider the relations for the spectral parameters (35) and calculate the scaling weights as $2\ell_i = u_i^+ - u_i$,

$$2\ell_1 = u_3 - u_1,\ 2\ell_2 = u_4 - u_2,\ 2\ell_3 = u_1 - n - u_3, 2\ell_4 = u_2 - n - u_4. \qquad (37)$$

We see that they are pairwise connected, $2\ell_1 = -2\ell_3 - n, 2\ell_2 = -2\ell_4 - n$. The relation between the weights implies for the helicities ($n=4$) $h_1 = -h_3,\ h_2 = -h_4$. The obtained expression implies the spinor-helicity expressions of $2\to 2$ helicity amplitudes of several cases by appropriate choices of the helicity values, $h = \pm 1, \pm\frac{1}{2}, 0$.

We can substitute the spectral parameters partially by the helicities.

$$\phi(\lambda,\bar\lambda) = \left(\frac{<2,3><4,1>}{<1,2><3,4>}\right)^{u_3-u_4-1}\ \frac{<1,2>^{2h_1}<3,4>^{-2h_2}}{<2,3>^{2h_1-2h_2}}$$

The amplitude contributions still involve $u_3 - u_4$ as a free parameter. The parameter extension of amplitudes has been proposed in [12]. The analytic structure expected from physical unitarity (dispersion relations) must be used to fix it.

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