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Integrable chains with Jordan - Schwinger representations

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Abstract. The restiction to the class of Jordan - Schwinger representations of $s\ell(n+1)$ results in simple relations for the L matrices and in explicit expressions for the general Yang-Baxter operators as products of two parameter permutation operators. Limits are studied which are related to the finite dimensional representations and to degenerate Yangians. The analogy to the $s\ell(2)$ case leads to analogous forms of global spin chain operators.

1. Introduction

The algebra generated by $n+1$ Heisenberg pairs $x_s, \partial_s, s = 1, ..., n+1$ contains a $g\ell(n+1)$ subalgebra. This was noticed early in the context of angular momentum [1, 2, 3]. Representations of $g\ell(n+1)$ can be constructed by combining a representation of $g\ell(n)$ with such an algebra [4, 5, 6, 7, 8]. This iteration in rank can be conveniently formulated in terms of L matrices [9] and used in constructing Yang-Baxter operators [10].

In this contribution we focus on the $g\ell(n+1)$ algebra built merely from $n+1$ Heisenberg pairs and consider representations spanned by monomials in x_s. One encounters therein lowest weight representations, called Jordan-Schwinger representations, which are characterised by just one representation parameter, to be compared with $n+1$ parameters for generic representations. The complete iteration would take $\frac{1}{2}n(n+1)+1$ Heisenberg pairs to construct the generic generators, whereas $n+1$ are sufficient for the first elementary building block.

This means that the restriction to the Jordan-Schwinger class of representations allows the restriction to $n+1$ degrees of freedom (per site of a spin chain), and this restriction goes along with simplicity in structure, which we would like to emphasize here.

In physical applications of higher rank symmetries and related integrable structures the Jordan-Schwinger case plays an important role, because the composition of the symmetry generators by more elementary operators related to canonical variables is natural.

The studies of the generalised chiral Potts model provides an example where the features of factorisation and permutations have been observed in the construction of Yang-Baxter solutions [11, 12, 13, 14]. The context and the formulation differs much from our discussion and involves deformation at roots of unity, which is avoided here.

The example having attracted much interest recently is the observation of Yangian symmetry based on the superalgebra $s\ell(4|4)$ in the calculation of $\mathcal{N}=4$ super Yang-Mills amplitudes [15].

The generators of $g\ell(n+1)$ representations can be written in terms of the $n+1$ Heisenberg pairs in several ways. We select two versions, write the related L matrices and study relations for their products. This provides an easy way to operators obeying Yang-Baxter RLL relations.

We obtain explicit expressions of the two R operators permuting the first and the second representation parameters in the product of two L matrices. Then the R operator intertwining two Jordan-Schwinger representations is obtained as the product of these two parameter permutation operators. We study the limit, where the representation parameter $2\ell_1$ of the first tensor factor approaches a non-negative integer value and a finite-dimensional invariant subspace emerges. We study also the asymptotics of large representation parameters, where degenerate Yangian algebra representations appear.

The various R operators as well as the L matrices provide building units of global operators of integrable spin chains. The relations between the building units imply relations of factorisation and commutativity between the global chain operators.

In this way we review some basic points of [16] and previous work by presenting the non-trivial extension of the $s\ell(2)$ case discussed there to the Jordan-Schwinger restricted higher rank case. In these papers the results on the R operators have been used to construct Baxter operators, prove Baxter relations and analyse the relations between different approaches to Baxter operators [17].

Baxter relations for the generic higher rank case have been considered recently in [18, 19]. They are not compatible in a simple way with the restriction to Jordan-Schwinger representations; the tensor product of such restricted representations does not decompose into restricted ones only.

2. Jordan-Schwinger representations of $g\ell(n+1)$

Choosing $n+1$ canonical pairs $x_s, \partial_s, s = 1, ..., n+1$ one can construct operators obeying the $g\ell(n+1)$ algebra relations. The straightforward form $E_{ij} = x_i\partial_j$ can be modified in several ways:

$$E^J_{ij} = \frac{x_i}{x_j}(N_j + \delta_j), \quad E^{-J}_{ij} = -\frac{x_j}{x_i}(N_j + \delta_j), \quad E^{TJ}_{ij} = -(N_i + \delta_i)\frac{x_j}{x_i}, \quad E^{-TJ}_{ij} = (N_i + \delta_i)\frac{x_i}{x_j}, \tag{1}$$

for $i, j = 1, ..., n+1$. $N_i = x_i\partial_i$ acts as infinitesimal dilatation operator on the coordinate operator x_i. In each case the algebra relations are

$$[E^C_{ij}, E^C_{jk}] = E^C_{ik}, \ i \neq k, \quad [E^C_{ij}, E^C_{ji}] = N_i + \delta_i - N_j - \delta_j \tag{2}$$

The L matrix with the matrix elements

$$L^C_{ij} = u\delta_{ij} + E^C_{ji} \tag{3}$$

obeys the fundamental RLL relation with Yang's $(n+1)^2 \times (n+1)^2$ R-matrix $\mathcal{R}_{12}(u) = u\mathrm{I} + \mathrm{P}_{12}$, where P_{12} denotes the the permutation matrix.

$$\mathcal{R}_{12}(u-v)L_1(u) \otimes L_2(v) = L_2(v) \otimes L_1(u)\mathcal{R}_{12}(u-v) \tag{4}$$

We shall use matrix notation. The canonical pairs are put into diagonal matrices, $X = diag(x_1, ..., x_{n+1}), \partial^x = diag(\partial_1, ..., \partial_{n+1})$. M is the matrix with all elements equal to 1. In the first relations to be discussed the latter matrix is relevant and also its relation involving diagonal matrices,

$$M \ XY \ M = M \ (X \cdot Y), \tag{5}$$

where $X \cdot Y = tr(XY)$. Y, ∂^y stand for the diagonal matrices involving another set of $n+1$ canonical pairs.

We restrict to a special case in the shifts δ and write the L matrices in matrix notation

$$L^J(u) = I(u-1) + \hat{\partial}M\hat{X}, \quad L^{TJ}(v) = Iv - \hat{Y}M\hat{\partial}^y. \tag{6}$$

Now we write down products of these matrices

$$L_x^J(u)L_y^{TJ}(v) = I(u-1)v + v\partial^x MX - uYM\partial^y - (X\cdot Y)\partial^x M\partial^y,$$

$$L_y^{TJ}(v)L_x^J(u) = I(u-1)v + v\partial^x MX - uYM\partial^y - YMX(\partial^y\cdot\partial^x),$$

and study the similarity transformation by $(X\cdot Y)^\lambda$ leading to the extension $\partial^x \to \partial^x + \lambda\frac{Y}{YX}, \partial^y \to \partial^y + \lambda\frac{X}{YX}$. We notice that the result can be written in terms of L matrix products again as

$$(X\cdot Y)^{-\lambda}L_x^J(u)L_y^{TJ}(v)(X\cdot Y)^\lambda = L_x^J(u+\lambda)L_y^{TJ}(v-\lambda) - \lambda(1-\lambda+v-u)(I - \frac{YMX}{Y\cdot X}), \quad (7)$$

$$(X\cdot Y)^{-\lambda}L_y^{TJ}(v)L_x^J(u)(X\cdot Y)^\lambda = L_y^{JS}(v)L_x^J(u) - \frac{YMX}{Y\cdot X}\lambda(\lambda-1+u-v+N^y+N^x), \quad (8)$$

$N^x = X\cdot\partial^x, N^y = \partial^y\cdot Y$. Corresponding relations for the product of L matrices of one and the same representation can be obtained by noticing that (6) implies

$$L^{TJ}(u;x,\partial_x) = L^J(u-1;\partial_x,-x)$$

and performing the canonical transformation interchanging the canonical pairs

$$K_0(x_s,\partial_s)K_0 = (\partial_s,-x_s).$$

The interesting particular case appears at $\lambda = u-v-1$ where the second term of one relation vanishes. With a simple parameter substitution we obtain the Yang-Baxter relation

$$(X\cdot\partial_y)^{u-v}L_x^J(u)L_y^J(v) = L_x^J(v)L_y^J(u)(X\cdot\partial_y)^{u-v}, \quad (9)$$

$$(\partial_x\cdot Y)^{u-v}L_x^{TJ}(u)L_y^{TJ}(v) = L_x^{TJ}(v)L_y^{TJ}(u)(\partial_x\cdot Y)^{u-v}.$$

The vanishing of the remainder in the other relation can happen at $\lambda \neq 0$ only if the representation space is restricted to eigenfunctions of $N^x + N^y$.

The simple explicit expressions of the L matrices allows to write an explicit expression for the ordinary transfer matrix for a corresponding spin chain. The definition of the general transfer matrix as a trace of a product of e.g. with $R_{i0}(u) = (\partial_i X_0)^u$ can be given only after specifying the representation space labelled by 0.

As suggested by the notation one may consider representations on functions of $x_i, i = 1,...,n+1$ spanned by $\prod x_i^{\alpha_i+m_i}$ with m_i running over all integers and $\alpha_1,...,\alpha_{n+1}$ fixed. Such a representation decomposes into ones where the $U(1)$ representation generated by $N^x = \sum_1^{n+1} x_s\partial_s$ is irreducible. In the particular case where for a distinguished index value i we set $\alpha_i = 2\ell,\ \sum_s m_s = 0$ and $\alpha_s = 0, s\neq i$ it can be reduced to the subspace spanned by the monomials with non-negative integer $m_s, s\neq i$, which has the lowest weight vector $x_i^{2\ell}$. Regarding the general labelling of $g\ell(n+1)$ representations of lowest weight by the eigenvalues of E_{rr}^J in their action on the lowest weight vector the considered Jordan-Schwinger case has the feature of all these labels vanishing but one. The involved representation of $s\ell(n+1)$ is irreducible for a generic value of 2ℓ but contains a $2\ell+1$ dimensional invariant subspace for non-negative integer values of 2ℓ, just as in the rank $n=1$ case. This known fact can be checked easily. The separation of the $s\ell(n+1)$ subalgebra representation will be done explicitly in the next section by transformation to $x'_s, \partial'_s, s\neq i$, where the new coordinates are defined by the ratios $x'_s = \frac{x_s}{x_i}$.

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3. Restriction to $s\ell(n+1)$

As the first step towards the reduction to $s\ell(n+1)$ we rewrite the L matrices and their relations in coordinate ratios and separate the dilatation operators $N^x = \sum_1^{n+1} x_s \partial_s^x, \tilde{N}^y = \sum_1^{n+1} \partial_s^y y_s$. This takes distinguishing one direction in the $n+1$ dimensional space, e.g. the one related to the index i or j respectively:

$$x'_s = \frac{x_s}{x_i}, \quad N_s^x = N_s^{x\prime} = x'_s \partial'_s, s \neq i, \quad N_i = N^x - \hat{m}_x, \quad \hat{m}_x \sum_{s \neq i} N'_s.$$

$$y'_s = \frac{y_s}{y_j}, \quad \tilde{N}_s^y = \tilde{N}_s^{y\prime} = \partial_d^y y'_s, s \neq j, \quad \tilde{N}_j^y = \tilde{N}^y - \hat{m}_y, \quad \hat{m}_y = \sum_{s \neq j} \tilde{N}'_s$$

The distinguished component of the derivative has to be transformed with care:

$$\partial_i = x_i^{-1} N_i = x_i^{-1}(N^x - \hat{m}_x) = \mathcal{N}^x\ x_i^{-1}, \quad \mathcal{N}^x = N^x + 1 - \hat{m}_x$$

$$\partial_j = \tilde{N}_j y_j^{-1} = y_j^{-1} \mathcal{N}^y, \quad \mathcal{N}^y = \tilde{N}^y - 1 - \hat{m}_y$$

We introduce diagonal matrices $X', N', \partial'^x, \tilde{\partial}^x$ with the corresponding components $x'_s, N'_s, \partial_s'^x, \tilde{\partial}_s^x, s \neq i$ on the diagonal and with 1 as the ith diagonal element. Correpondingly, the diagonal matrices $Y', \tilde{N}', \partial'^y, \tilde{\partial}^y$ are introduced. The definitions have been put in such a way to have

$$X = X' \cdot x_i, \ \partial^x = \tilde{\partial}^x \cdot \mathcal{N}^x x_i^{-1}, \quad Y = Y' \cdot x_j, \ \partial^y = y_i^{-1} \mathcal{N}^y \tilde{\partial}^y.$$

With these notations we write factorised forms of the L matrices

$$L_x^J(u) = L_x^{J/i}(u-1, u+N^x) = X'^{-1} m_i^{-1} K_i(u-1, u+N^x; N') m_i X' = \tag{10}$$

$$\tilde{\partial}^x m_i^t K_i^t(u-1, u+N^x; N') m_i^{t-1} \tilde{\partial}^{-1\ x},$$

$$L_y^{TJ}(v) = L_y^{TJ/j}(v, v+1-\tilde{N}^y) = Y' m_j^t K_j^t(v, v+1-\tilde{N}^y; -\tilde{N}') m_j^{t-1} Y'^{-1} = \tag{11}$$

$$\tilde{\partial}^{y\ -1} m_j^{-1} K_j(v, v+1-\tilde{N}^y; -\tilde{N}^y) m_j \tilde{\partial}^y,$$

$$K_i(u_1, u_2; N) = (I + I_i(u_2 - 1))\ (I + N c_i)\ (I u_1 - I_i(u_1 - 1)), \quad m_i = I + r_i.$$

Auxiliary matrices I_i, c_i, r_i have been introduced here. c_i, r_i have most elements vanishing besides of the elements on the column i or row i respectively. These elements are equal 1, but the corresponding diagonal element is zero too. Further we denote $I_i = \hat{e}_{ii}$. These standard matrices obey the algebraic relations

$$c_i^T = r_i, \quad c_i^2 = 0, \quad I_i c_i = 0, \quad r_i I_i = 0, \quad c_i I_i = c_i, \quad I_i r_i = r_i, \tag{12}$$

$$r_i c_i = n I_i, \quad r_i \hat{Z} c_i = I_i \sum_{s \neq i} Z_s, \quad c_i r_i = M - I_i - r_i - c_i = M_i.$$

Here Z stands for any diagonal matrix. The dependence on the $U(1)$ or dilatation generators N^x or N^y has been separated; it enters the coordinate factorized form only by the diagonal matrix factor $I + I_i(u_2 - 1)$ or $I + I_j(v_2 - 1)$ via $u_2 = u + N^x$ or $v_2 = v + 1 - \tilde{N}^y$. These generators are substituted by corresponding numbers $N^x \to 2\ell_x$ or $\tilde{N}^y \to -2\ell_y$ upon restriction to irreducible $U(1)$ representations. In physical terms, only the n degrees of freedom expressed by the ratio coordinates and their conjugate momenta are left active now. The ∂ factorised form plays an auxiliary role in establishing the relation between the J and TJ forms of L matrices. Notice that in the latter form the dilatation generator enters also via the definition of $\tilde{\partial}$.

In this way we obtain the similarity transformations of the L matrix products with $(X' \cdot Y')^\lambda$ by argument shift on the l.h.s. from the relations (7), (8),

$$(X' \cdot Y')^{-\lambda} L_x^{J/i}(u-1, u+N^x+\lambda) L_y^{TJ/j}(v, v+1-\tilde{N}^y-\lambda)(X' \cdot Y')^\lambda = \tag{13}$$

$$L_x^{J/i}(u-1+\lambda, u+N^x+\lambda) L_y^{TJ/j}(v-\lambda, v+1-\tilde{N}^y-\lambda) - \lambda(1-\lambda+v-u)(I - \frac{Y'MX'}{Y' \cdot X'}),$$

$$(X' \cdot Y')^{-\lambda} L_y^{TJ/j}(v, v+1-\tilde{N}^y-\lambda) L_x^{J/i}(u-1, u+N^x+\lambda)(X' \cdot Y')^\lambda = \tag{14}$$

$$L_y^{TJ/j}(v, v+1-\tilde{N}^y) L_x^{J/i}(u-1, u+N^x) - \frac{Y'MX'}{Y' \cdot X'} \lambda(\lambda-1+u-v+\tilde{N}^y+N^x).$$

Notice that the exponent λ in the similarity operator may depend on $N^x, \tilde{N}^y$ since the latter commute with the ratio coordinates. Therefore the case of the vanishing of the extra term can be considered for both relations now and this results in parameter permutation relations.

4. Yang-Baxter relations

The similarity of the factorised representations of the Lax matrices (10), (11) of different types, in particular the coordinate TJ and the derivative J forms lead to

$$L^{TJ/j}(v, v+1-\tilde{N}; Y', \tilde{N}') = L^{J/j}(v, v+1-\tilde{N}, \tilde{\partial} \to Y', N' \to -\tilde{N}'). \tag{15}$$

The representation parameters coincide and the indicated substitutitons have to be done. As pointed out above the derivative factorised expressions depend on the dilatation $N = \sum_1^{n+1} N_s$ not only via the representation parameter argument but also by the factor $\mathcal{N}$ involved in the definition of $\tilde{\partial}$. In the case J

$$\mathcal{N} = N + 1 - \sum N_s' = \hat{u}_2 - u_1 - \sum N_s'.$$

Thus $\mathcal{N}$ is connected to the representation parameter arguments of the L matrix, not in universal way. In the substitution of the L^{TJ} matrix (15) we have

$$\mathcal{N} = \hat{v}_2 - v_1 - \sum N_s' = 1 - \tilde{N}^y - \sum N_s'$$

One should be careful to read $\tilde{N}$ according to the TJ convention and N_s' in the last term according to the J convention.

The indicated substitution can be expressed as

$$(\tilde{\partial}, N') = (\Gamma(\mathcal{N}))^{-1}(\partial', \quad N')\Gamma(\mathcal{N}),$$

and

$$(Y, -\tilde{N}') = K_0'(\partial', N')K_0'.$$

The latter denotes the elementary canonical transformation of the reduced n degrees of freedom. Thus,

$$L^{TJ/j}(v, v+1-\tilde{N}; Y', \tilde{N}') = K_0'(\Gamma(\mathcal{N}))\tilde{\partial} m_j^t K_j^t(v, v+1-\tilde{N}^y; N') m_j^{t-1} \tilde{\partial}^{-1}(\Gamma(\mathcal{N}))^{-1} K_0' =$$

$$K_0'(\Gamma(\mathcal{N})) L^{J/j}(v, v+1-\tilde{N}^y; Y', N') \ (\Gamma(\mathcal{N}))^{-1} K_0'$$

This relation can be used to derive from the permutation relations (13), (14) involving L matrices of different form (J and TJ) the ones with only J . Substituting L^{TJ} the operator K_0' can be carried to the other operators leading to the substitution $Y' \to \partial'_y$ in the similarity operator and the remainder. It is the important consequence of the above discussion that the substituted similarity transformations $\Gamma(\mathcal{N})$ carry different arguments $\mathcal{N}$ on both sides.

$$\tilde{R}(\lambda)L_x^{J/i}(u-1, u+N^x+\lambda)L_y^{J/j}(v, v+1-\tilde{N}^y-\lambda) =$$

$$L_x^{J/i}(u-1+\lambda, u+N^x+\lambda)L_y^{J/j}(v-\lambda, v+1-\tilde{N}^y-\lambda)\tilde{R}(\lambda)$$

$$-\lambda(1-\lambda+v-u)(I-\frac{\partial' M X'}{\partial'\cdot X'})\tilde{R}(\lambda),$$

$$\tilde{R}(\lambda)L_y^{J/j}(v, v+1-\tilde{N}^y-\lambda)L_x^{J/i}(u-1, u+N^x+\lambda) =$$

$$L_y^{J/j}(v, v+1-\tilde{N}^y)L_x^{J/i}(u-1, u+N^x)\tilde{R}(\lambda)$$

$$-\frac{\partial'_y M X'}{\partial'_y\cdot X'}\lambda(\lambda-1+u-v+\tilde{N}^y+N^x)\tilde{R}(\lambda),$$

with

$$\tilde{R}(\lambda) = \Gamma(1-\tilde{N}^y-\sum N'_s)(X'\cdot\partial')^{-\lambda}\Gamma^{-1}(1-\tilde{N}^y-\lambda-\hat{m}_y).$$

We abbreviate $\hat{m}_y = \sum_{s\neq j}\tilde{N}_s^{y\prime}$. In the first relation we choose $\lambda = v-u+1$ to remove the remainder and denote the second representation arguments on both sides by u_2 and v_2 and put finally $u_1 = u-1, v_1 = v$. In this way the arguments of the Γ functions become $1-\tilde{N}^y = v_2-u_1,\ 1-\tilde{N}^y-v+u-1 = v_2-v_1$. We obtain the standard first parameter permutation relation (compare [20])

$$\mathrm{R}^1_{xy}(u_1|v_1,v_2)L_x^J(u_1,u_2)L_y^J(v_1,v_2) = L_x^J(v_1,u_2)L_y^J(u_1,v_2)\mathrm{R}^1_{xy}(u_1|v_1,v_2),$$

where

$$\mathrm{R}^1_{xy}(u_1|v_1,v_2) = \Gamma^{-1}(v_2-u_1-\hat{m}_y)(X'\cdot\partial')^{u_1-v_1}\Gamma(v_2-v_1-\hat{m}_y). \tag{16}$$

In the second relation we choose $\lambda = v-u+1-\tilde{N}^y-N^x$ to remove the remainder. We denote now $u_2 = u+N^x, v_2 = v+1-\tilde{N}^y, u_1 = u-1, v_1 = v$. Then the arguments of the Γ functions are $1-\tilde{N}^y = v_2-v_1, 1-\tilde{N}^y-v+u-1 = u_2-v_2$. However the first arguments in the L matrices appear reversed compared to the standard parameter permutation (compare [20]). Renaming v_1, u_1 by u_1, v_1 leads to the standard form of the second parameter permutation relation

$$\mathrm{R}^2_{yx}(u_1,u_2|v_2)L_y^J(u_1,u_2)L_x^J(v_1,v_2) = L_y^J(u_1,v_2)L_x^J(v_1,u_2)\mathrm{R}^2_{yx}(u_1,u_2|v_2),$$

where

$$\mathrm{R}^2_{yx}(u_1,u_2|v_2) = \Gamma^{-1}(v_2-u_1-\hat{m}_y)(X'\cdot\partial')^{u_2-v_2}\Gamma(u_2-u_1-\hat{m}_y). \tag{17}$$

The general Yang-Baxter operator $\mathrm{R}_{12}(u-v;\ell_1,\ell_2) = \mathrm{R}_{12}(u_1,u_2|v_1,v_2)$ acting on the tensor product of $s\ell(n+1)$ representation modules of the considered JS type with weights ℓ_1,ℓ_2 $\mathcal{U}_{1,\ell_1}\otimes\mathcal{U}_{2,\ell_2}\to\mathcal{U}_{1,\ell_2}\otimes\mathcal{U}_{2,\ell_1}$ and obeying the RLL relation with Jordan-Schwinger type L matrices

$$\mathrm{R}_{12}(u_1,u_2|v_1,v_2)L_1^{J/i}(u_1,u_2)L_2^{J/j}(v_1,v_2) = L_1^{J/i}(v_1,v_2)L_2^{J/j}(u_1,u_2)\mathrm{R}_{12}(u_1,u_2|v_1,v_2),$$

$u_1 = u-1, u_2 = u+2\ell_1$, factorises into the constructed parameter permutation operators,

$$\mathrm{R}_{12}(u_1,u_2|v_1,v_2) = \mathrm{R}^1_{12}(u_1|v_1,u_2)\mathrm{R}^2_{12}(u_1,u_2|v_2) = \mathrm{R}^2_{12}(v_1,u_2|v_2)\mathrm{R}^1_{12}(u_1|v_1,v_2). \tag{18}$$

This factorisation is a specific feature of the considered Jordan-Schwinger type representations. The general Yang-Baxter operator acting on generic $s\ell(n+1)$ representations decomposes into

$n+1$ factors corresponding to the $n+1$ representation labels of the L matrix of generic representation (cf. [18]).

In the case of coinciding distinguished directions $i=j$ the obtained expressions for the factor operators (16), (17) can be rewritten as

$$\mathrm{R}^1_{12}(u_1|v_1,v_2) = U_2^{-1}(v_2-u_1)\ D_{21}^{(u_1-v_1)}\ U_2(v_2-v_1), \tag{19}$$

$$\mathrm{R}^2_{12}(u_1,u_2|v_2) = U_1^{-1}(v_2-u_1)\ D_{12}^{(u_2-v_2)}\ U_1(u_2-u_1),$$

where

$$D_{12} = \prod_{s\neq i} x'_{1s} - \sum_{r\neq i} x'_{2r} \prod_{s\neq i,r} x'_{1s},$$

$$U_1(\alpha) = (\prod_{s\neq i} x'_{1s})^{-\alpha} \prod_{s\neq i} \Gamma(N'_{1s}+1)\ \Gamma^{-1}(1+\sum_{s\neq i} N'_{1s}-\alpha) = (\prod_{s\neq i} x'_{1s})^{-\alpha} V_1(\alpha).$$

In the case of rank $n=1$ the expressions coincide with the ones in [20, 16].

The complete analogy of the factorisation and Yang Baxter relations between R_{12}, R^1_{12}, R^2_{12}, L in the Jordan-Schwinger $s\ell(n+1)$ case to the case of $s\ell(2)$ implies in both cases the same relations of factorisation and commutativity between the the corresponding operators describing a spin chain: the ordinary transfer matrix, the general transfer matrix,

$$\mathrm{T}_s(u) = tr_0(P_{10}R_{10}(u)...P_{N0}R_{N0}(u))$$

and two Baxter operators

$$Q_1(u) = tr_0(P_{10}R^1_{10}(u_1|v_1,0)...P_{N0}R^1_{N0}(u_1|v_1,0)),$$

$$Q_2(u) = tr_0(P_{10}R^2_{10}(u_1,u_2|0)...P_{N0}R^2_{N0}(u_1,u_2|0)).$$

The trace in the representation space labelled by 0 with the weight denoted by spin s is well defined for generic values of $2s$. In particular we have

$$\mathrm{PT}_s(u) = \mathrm{Q}_2(u-s)\mathrm{Q}_1(u+s+1) = \mathrm{Q}_1(u+s+1)\mathrm{Q}_2(u-s).$$

P denotes the cyclic shift in the chain.

Starting with the second parameter permutation relation involving R^2_{12} and its explicit expression (17) one can derive

$$(I+r_iX_1)\mathrm{R}^2_{12}(u_1,u_2|0)L_1^J(u_1,u_2)(I-r_iX_2) = \{...\}c_i+ \tag{20}$$

$$P_{X12}\mathrm{R}^2_{12}(u_1+1,u_2+1)|0) + u_1u_2I_i\mathrm{R}^2_{12}(u_1-1,u_2-1|0)+$$

$$\{u_1(I-I_i-P_{X12}) + P_{X12}B\}\,\mathrm{R}^2_{12}(u_1,u_2|0).$$

One-dimensional matrix projectors $I_i, P_{X12}, P_{X12}B$ are involved,

$$P_{X12} = \frac{X_1^{-1}M_iX_2}{(X_1^{-1}\cdot X_2)}, \quad B = (X_1^{-1}\cdot X_2)[X_1^{-1}M_iX_1N_1 - N_1X_2^{-1}M_iX_2](X_1^{-1}\cdot X_2)^{-1}.$$

In the lowest rank case $n=1$ this becomes the intermediate step towards the Baxter relation for the product of Q_2 with the ordinary transfer matrix [16]. Indeed, in this case $B=0$ and $P_{X12} = I - I_i$. The unspecified term proportional to c_i is irrelevant for this purpose.

It is not clear whether this relation can be exploited for deriving relations for global chain operators in the higher rank case in the framework of Jordan-Schwinger type representations. This could be of interest because the known treatments of Baxter relations expand over generic representations.

5. Integer limits

We study the Yang-Baxter operator in the case $u_2 - u_1 = 2\ell_1 + 1 = m + 1 + \varepsilon$ where m is a non-negative integer. We shall see how the appearence of a $m+1$ dimensional invariant subspace in the first tensor factor is reflected in the limiting behaviour of the Yang-Baxter operator and shall obtain its restriction to the finite-dimensional invariant subspace for the first tensor factor in the limit.

In the first factorised representation (18) we have the factors $U_2^{-1}(2\ell_1 + 1), U_1(2\ell_1 + 1)$. In the limit two contributions appear naturally.

$$U_1(2\ell_1 + 1) = \prod x_{1s}'^{-m-1}\Gamma(1 + N_{1s}')$$

$$\left\{\Pi(m \geq \hat{m}_1)\Gamma(1 + m + \varepsilon - \hat{m}_1)(-1)^{m+1-\hat{m}_1}\ \varepsilon + \Pi(\hat{m}_1 > m)\frac{1}{\Gamma(\hat{m}_1 - m - \varepsilon)}\right\},$$

$$U_2^{-1}(2\ell_1 + 1) = \left\{\Pi(m \geq \hat{m}_2)\frac{(-1)^{m+1-\hat{m}_2}}{\Gamma(1 + m + \varepsilon - \hat{m}_2)\ \varepsilon} + \Pi(\hat{m}_2 > m)\Gamma(\hat{m}_2 - m - \varepsilon)\right\}$$

$$\prod \frac{1}{\Gamma(1 + N_{2s}')}x_{1s}'^{m+1}.$$

Here we abbreviate $\hat{m}_1 = \sum_{s\neq i} N_{1s}'$ and $\Pi(...)$ stand for projectors with the projection rule indicated in the argument.

We obtain the Yang-Baxter operator $\mathbf{R}_{12}$ restricted to the finite dimensional subspace in the first tensor factor at $2\ell_1 = m$ as

$$\mathrm{P}_{12}\mathbf{R}_{12}(u - v|\frac{n}{2}, \ell_2) = \lim_{\varepsilon\to 0} \mathrm{R}_{12}(u - v|\frac{1}{2}(m + \varepsilon), \ell_2)\Pi_1^m = \tag{21}$$

$$\Pi_2^m \frac{(-1)^{\hat{m}_2}}{\Gamma(1 + m - \hat{m}_2)\prod\Gamma(1 + N_2 s)}(1 - \sum \frac{x_{\prime 1r}}{x_{\prime 2r}})^{u_1 - v_1} V_2(u_2 - v_1)$$

$$V_1^{-1}(v_2 u_1)(1 - \sum \frac{x_{2r}}{x_{1r}})^{u_2 - v_2} \prod \Gamma(1 + N_{1s})\ \Pi_1^m \Gamma(1 + m - \hat{m}_1)(-1)^{\hat{m}_1}.$$

Here Π_1^m denotes the projector on the subspace spanned by $\prod_{s\neq i} x_{1s}'^{m_s}$ with $\sum_{s\neq i} m_s \leq m$.

In the case $m = 1$ we expect that $\mathbf{R}_{12}$ reproduces the L matrix in the symmetry basis,

$$\mathbf{R}_{12}(u - v|\frac{1}{2}, \ell_2) = C \cdot L_2(u - v + \frac{1}{2}|\ell_2), \quad C = -\frac{\Gamma(u - v - \frac{1}{2} - \ell_2)}{\Gamma(\frac{1}{2} - u + v - \ell_2)} \tag{22}$$

We calculate easily the matrix element of $\mathbf{R}_{12}$ with the lowest weight state 1 and identify the coefficient C,

$$\mathbf{R}_{12} \cdot 1 = \frac{\Gamma(u_2 - v_2 - 1}{\Gamma(v_1 - u_1 - 1)} = (u_1 - v_1 + 1)\ C.$$

(22) can be checked by proceeding this calculation with the action on $1 \cdot x_{2s}^p$ and $x_{1r} \cdot x_{2s}^p$. In the calculation we notice that besides of the explicit projectors the factors $\prod \Gamma^{-1}(1 + N_s)$ restrict the powers of the monomials; negative powers are removed. The generalisation of the L matrix to the case of other values of m can be obtained in the same way.

We have seen that the restriction to the finite-dimensional subspace at integer values of 2ℓ goes in analogy to the rank $n = 1$ case. The analogy ends if regarding the complement to the invariant finite-dimensional subspace. Unlike the rank 1 case it is not simply related to just one Jordan-Schwinger type representation with $\ell = \frac{1}{2}m$ replaced by another value. This implies that the arguments for deriving Baxter relations cannot be copied from the rank 1 case in a straightforward way.

6. Degeneracy limits

Consider the Lax matrix (10) and identify $L^{J/i\pm}$.

$$L^{J/i}(u_1,u_2) = L^{J/i-)}(u_2)(Iu_1 - I_i(u_1-1))(I+r_iX') = (I-r_iX')(I+I_i(u_2-1))\, L^{J/i+}(u_1), \quad (23)$$

$$L^{J/i-}(u_2) = \lim_{u_1\to\infty} L^{(i)}(u_1,u_2)(Iu_1 - I_i(u_1-1))^{-1},$$

$$L^{J/i+}(u_1) = \lim_{u_2\to\infty} (I + I_i(u_2-1))^{-1} L^{(i)}(u_1,u_2).$$

By the limit relations it is clear that $L^{J/i\pm}$ obey the fundamental RLL relation (4) and thus their matrix elements generate algebra representations of the Yangian different from the ordinary one generated by the elements of L^J.

$L^{J/i-}$ coincide with the elementary degenerate L matrices used by Bazhanov et al. [19] as "partonic" building blocks. Their Baxter operator construction starts from the decomposition of a generic $s\ell(n+1)$ representation L matrix in terms of a product of $n+1$ L^{i-} matrices. The generic representation L matrix can be also obtained from the considered Jordan-Schwinger L matrices by iteration as discussed in [9, 10]. The relation (23) of L^J to L^{J-} shows how the connection of our approach to the one by Bazhanov et al. [17, 19] extends from the rank $n=1$ case discussed in [16] to higher rank.

R operators intertwining the degenerate $L^\pm$ with L and among each other can be obtained similar to the $s\ell(2)$ case. Consider the first parameter permutation RLL relation (16) in the case of coinciding distinguished directions in the asymptotics $v_1 \to \infty$.

$$\mathrm{R}^1_{12}(u_1|v_1,v_2)L^J_1(u_1,u_2)L^J_2(v_1,v_2) = L^J_1(v_1,u_2)L^J_2(u_1,v_2)\mathrm{R}^1_{12}(u_1|v_1,v_2),$$

$$L^{J/i}(v_1,v_2) = L^{J/i-}(v_2)(v_1 I - I_i(v_1-1))(1+\mathcal{O}(v_1^{-1}).$$

The symmetry of the L matrices implies

$$(v_1 I - I_i(v_1-1))L_2(u_1,v_2)(v_1 I - I_i(v_1-1))^{-1} = v_1^{-\hat m_2} L_2(u_1,v_2) v_1^{\hat m_2},\ \hat m_2 = \sum_{s\neq i} N'_{2s}.$$

In this way we obtain

$$\mathrm{r}^+_{12}(u_1-v_2)L^{J/i}_1(u_1,u_2)L^{J/i-}_2(v_2) = L^{J/i-}_1(u_2)L^J_2(u_1,v_2)\mathrm{r}^+_{12}(u_1-v_2),$$

where

$$v_1^{\hat m_2}\mathrm{R}^1_{12}(u_1|v_1,v_2) \to \mathrm{r}^+_{12}(u_1-v_2).$$

With the explicit expression (16) we obtain

$$\mathrm{r}^+_{12}(u) = \Gamma(1+\hat m_2 + u)\exp(\sum_{r\neq i} x_{1r}\partial_{2r}). \quad (24)$$

The result appears quite similar to the rank $n=1$ case and this analogy extends to the other degenracy limit Yang-Baxter operators and the relations among them.

$$\mathrm{R}^+_{12}(u_1,u_2|v_2)L^{J/i}_1(u_1,u_2)L^{J/i-}_2(v_2) = L^{J/i-}_1(v_2)L^J_2(u_1,u_2)\mathrm{R}^+_{12}(u_1,u_2|v_2),$$

where

$$v_1^{\hat m_2}\mathrm{R}_{12}(u_1,u_2|v_1,v_2) \to \mathrm{R}^+_{12}(u_1,u_2|v_2) = \mathrm{r}^+_{12}(u_1-u_2)\mathrm{R}^2_{12}(u_1,u_2|v_2).$$

$$\mathrm{R}^-_{12}(u_1,u_2|v_1)L^{J/i}_1(u_1,u_2)L^{J/i+}_2(v_1) = L^{J/i-}_1(v_1)L^J_2(u_1,u_2)\mathrm{R}^-_{12}(u_1,u_2|v_1),$$

where

$$\mathrm{R}_{12}(u_1,u_2|v_1,v_2)v_2^{-\hat{m}_1} \to \mathrm{R}^{-}_{12}(u_1,u_2|v_1) = \mathrm{R}^{1}_{12}(u_1|v_1,u_2)\mathrm{r}^{-}_{12}(u_1-u_2),$$

$$\mathrm{r}^{-}_{12}(u) = \exp(-\sum_{r\neq i} x_{2r}\partial_{1r})\frac{(-1)^{\hat{m}_1}}{\Gamma(1+\hat{m}_1+u)}.$$

Operators describing a spin chain can be built from these Yang-Baxter operators as well. The experience of the rank $n = 1$ case tells us that here the trace definition should include a regularisation,

$$\mathrm{Q}_{\pm}(u) = tr_0[q^{\hat{m}_0}\mathrm{P}_{10}\mathrm{R}^{\pm}_{10}(u_1,u_2|0)...\mathrm{P}_{N0}\mathrm{R}^{\pm}_{N0}(u_1,u_2|0)].$$

The commutativity and factorisation properties hold, in particular

$$\frac{1}{1-q^n}\mathrm{T}_s(u|q) = \mathrm{Q}_{+}(u-s-1)\mathrm{Q}_{-}(u+s) = \mathrm{Q}_{-}(u+s)\mathrm{Q}_{+}(u-s-1)$$

where $\mathrm{T}_s(u|q)$ is the q regularised version of the general transfer operator.

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