

Mass condensation on networks

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Abstract. We construct classical stochastic mass transport processes for stationary states which are chosen to factorize over pairs of sites of an undirected, connected, but otherwise arbitrary graph. For the special topology of a ring we derive static properties such as the critical point of the transition between the liquid and the condensed phase, the shape of the condensate and its scaling with the system size. It turns out that the shape is not universal, but determined by the interplay of local and ultralocal interactions. In two dimensions the effect of anisotropic interactions of hopping rates can be treated analytically, since the partition function allows a dimensional reduction to an effective one-dimensional zero-range process. Here we predict the onset, shape and scaling of the condensate on a square lattice. We indicate further extensions in the outlook.¹

1. Introduction

A variety of stochastic processes out-of-equilibrium may be summarized under the name of stochastic mass transport models, mass transport in the sense of microscopic or macroscopic dynamics that involves spatial stochastic transport of some generic conserved quantity called 'mass'. The meaning of mass in this context ranges from cars in macroscopic traffic, to ants, to molecular motors in intracellular traffic. The transport is classically described in terms of probabilities for hopping events from one site to another. Such processes are of much interest from a theoretical as well as from a practical point of view. Since they are in general out-of-equilibrium and specified in terms of dynamical rules without any energy concept, it is difficult to predict and classify the possible stationary states and to identify transitions between phases like a liquid phase or a phase with a condensate that are associated with the different stationary states. In particular one observes the phenomenon of spontaneous symmetry breaking already in one-dimensional systems, in which the symmetry breaking manifests itself in the spontaneous formation of a condensate: a finite fraction of constituent particles condenses onto a finite extension in space (sometimes even on a single site) in the thermodynamic limit, in which the number of particles along with the volume is sent to infinity.

The practical relevance of these stochastic processes comes from the variety of condensation transitions which are ubiquitous in nature. To these belong the formation of large aggregates in aggregation and fragmentation processes [1], coalescence of shaken balls in granular flow [2], the formation of hubs in networks [3], or, last but not least, the formation of jams in traffic on various scales [4].

¹ Talk presented by H Meyer-Ortmanns

One well-known stochastic process is the zero-range process (ZRP) in which hopping rates are specified for particles via the probabilities to hop from one site to a neighboring site. The rates depend only on the departure site (and are therefore of zero range). The stationary states and the phase structure can be predicted analytically. The ZRP has been generalized in various directions, treating mass, space and time as continuous variables or allowing for internal degrees of freedom of what is called the mass, or adding annihilation and creation of particles in addition to their diffusion (for a review see for example [5]).

From a systematic point of view a natural generalization is towards the inclusion of next-neighbor interactions in the sense that the hopping rates not only depend on the departure site, but also on the neighboring sites, more precisely on the difference in occupation numbers $m \geq 1$ of such pairs. Evans *et al.* [6] formulated such a model that leads to pair-factorized steady states on a ring topology. From the exact knowledge of the stationary states they predicted the formation of a condensate above a certain critical mass density with the characteristic feature that the condensate was extended over a range that scales with \sqrt{N} if N denotes the system size.

In this paper we review results of our work [7, 8, 9] which focusses on stochastic transport that leads to pair-factorized steady states on arbitrary graphs. More precisely, we look for dynamical rules that lead to stationary probability distributions of occupation numbers, which fully factorize over the links of an arbitrary graph. The rules are specified in terms of weight functions that finally determine the hopping rates and include the ZRP as special case. Once we know the stationary states in the whole parameter space, the next step is to derive the static and (if possible) the dynamic features which characterize the different phases. Here we start with the simplest topology, that of a ring, and analyze there the transition from a 'liquid', characterized by a uniform particle distribution, to a so-called 'condensed' phase, that is a phase with a condensate. We then study in detail how the shape of the condensate can be tuned by an appropriate variation of the local and ultralocal contributions to the hopping rate. We also predict the scaling of the condensate's extension with the system size and observe properties that remind us to finite-size scaling of first- and second-order phase transitions. For two dimensions we report on numerical results for isotropic interactions and analytical results for anisotropic ones. The analytical treatment becomes possible due to the dimensional reduction of the partition function to an effective one-dimensional ZRP. We conclude with a summary and outlook to possible extensions.

2. Pair-factorized steady states on arbitrary graphs

Usually in transport phenomena one would try to describe a given physical process by a set of hopping rates and predict the stationary states as solution of the corresponding master equation. If at all, this is often only approximately analytically possible. Here we proceed in the reverse way. Roughly speaking, we start from a probability distribution which is assumed to factorize over the edges of a given graph and ask which hopping rates lead to such a distribution as their stationary state. More precisely, let us consider connected, undirected, but otherwise arbitrary graphs with sites i , with k_i the degree of site i , i.e. the number of edges emerging from node i , and edges $\langle i, j \rangle$. Let M be the total number of particles of unit mass which is assumed to be conserved during the particle's diffusion, and m_i denote the occupation number of site i , $m_i \geq 0$. Let us split a hopping event into a product of a rate $u_i(m_i, m_{j_1}, m_{j_2}, \dots, m_{j_{k_i}})$ which determines the probability that one particle leaves site i and depends on nodes j_1, j_2, \dots, j_{k_i} adjacent to i , and a function $W(j \rightarrow i) \equiv W_{ji}$ that is proportional to the probability to choose site i as new target when coming from site j . For example, on a one-dimensional ring, one chooses for the particle's hopping a target to the right site with probability r , and to the left site with probability $1 - r$. If we denote by $w(\vec{m})$ the probability for a distribution of occupation numbers $\{\vec{m}\} = \{m_1, \dots, m_N\}$, $m_i \in \{0, 1, 2, \dots, M\}$, $i \in \{1, \dots, N\}$, we now ask which hopping rates $u_i(m_i, m_{j_1}, m_{j_2}, \dots, m_{j_{k_i}})$

lead to a probability distribution $w(\vec{m})$ that factorizes over the pairs (edges) of a graph according to

$$w(\vec{m}) = \prod_{\langle i,j \rangle} g_{ij}(m_i, m_j) \prod_{i=1}^N k_i^{m_i} \delta \left(\sum_{i=1}^N m_i - M \right). \quad (1)$$

Here $\langle i, j \rangle$ denotes pairs of nearest neighbors, $g_{ij}(m_i, m_j)$ is a symmetric but otherwise arbitrary weight function that may depend on the link $\langle i, j \rangle$; the δ -function ensures conservation of the overall mass M (it will be dropped below). The factors $k_i^{m_i}$ are split off for convenience. Our answer is a possible class of hopping rates given as

$$u_i(m_i, m_{j_1}, m_{j_2}, \dots, m_{j_{k_i}}) = \prod_{j \in \{j_1, j_2, \dots, j_{k_i}\}} \frac{g_{ij}(m_i - 1, m_j)}{g_{ij}(m_i, m_j)}, \quad (2)$$

where $\{j_1, j_2, \dots, j_{k_i}\}$ denotes the set of neighbors of node i , provided the transition matrix W_{ji} satisfies the condition

$$k_i = \sum_j W_{ji} k_j, \quad (3)$$

so that the probability current is site-wise conserved. (This choice is certainly not the only possible one, since, for example, the symmetry condition on $g(m, n)$ was released in [6].) For the proof one considers the balance equation for a conserved probability current and shows that the pair-factorized steady state (1) solves this equation provided u is given by (2) and W satisfies (3). For details we refer to [9]. Although this solution is not the most general one, it contains important special cases like the ZRP, and regular topologies like all-to-all coupling or the square lattice which we shall consider below for anisotropic hopping. It should be noticed that the weight factors and therefore the hopping rates may depend on the edge $\langle i, j \rangle$, a fact which we shall exploit in section 4, where we choose $g_{ij}(m, n) = g_y(m, n)$ for all edges in the y -direction and $g_{ij}(m, n) = g_x(m, n) = \sqrt{f(m)f(n)}$ for links $\langle i, j \rangle$ in x -direction with $f(m)$ in general an arbitrary function depending on one site (so that we would separately expect fully factorized states in x -direction and pair-factorized states in y -direction), in section 4 chosen to be $f(m) = 1$ for simplicity.

Moreover it should be noticed that local hopping rates $u_i(m; m_{j_1}, m_{j_2}, \dots, m_{j_{k_i}})$ can go along with non-local target functions $W(j \rightarrow i)$, this means that hopping events with non-local hopping are also covered by the proof. This fact we have actually exploited in our numerical simulations of the condensed phase. When we are only interested in the stationary states, we may use an algorithm to converge to this state as fast as possible. Here it turned out that a direct simulation of the hopping events usually takes longer by an order of magnitude than the direct simulation of the probability distributions of occupation numbers via a Metropolis algorithm. In the direct simulation of the hopping rates we select randomly a site i with occupation number m_i and nearest neighbors $n(i)$ and calculate the rate u_i for the given choice, next we leave the site with a probability proportional to this u_i and choose the target site with a probability proportional to the target function W_{ji} . In the indirect method we update the occupation number probabilities $w(\vec{m})$ via the Metropolis algorithm instead, that is, we start from a configuration of occupation numbers $\vec{\alpha} = \{m_1, \dots, m_i, \dots, m_k, \dots, m_N\}$, choose randomly a site j and let a particle jump from i to j to end up at the configuration $\vec{\beta} = \{m_1, \dots, m_i - 1, \dots, m_j + 1, \dots, m_N\}$. The new configuration is then accepted according to the Metropolis rule with probability $\text{prob}(\vec{\alpha} \rightarrow \vec{\beta}) = \min\{1, w(\vec{\beta})/w(\vec{\alpha})\}$. If the update is accepted, it corresponds to the non-local hopping from i to j and a choice of $W_{ji} = 1/(N - 1)$ for a random selection of j among all sites (different from i) of the graph. The Metropolis algorithm ensures detailed balance, in this updating current conservation is therefore realized in a special way as equilibrium process, while a nonvanishing

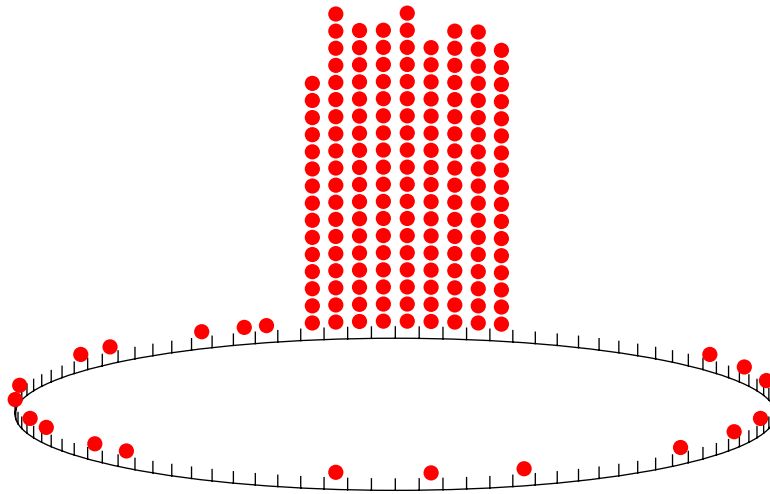


Figure 1. Schematic: Condensate of rectangular shape on a ring

conserved current would be the generic case, as it can be implemented by directly simulating the hopping events. The advantage is that the simulation of weights via the Metropolis algorithm amounts to non-local hopping, since i and j need not be nearest neighbors; it favors the melting of metastable 'precursors' of local condensates and therefore facilitates their merging to one big final condensate, but the stationary distribution of particles in the end is the same for a vanishing current or a current of nonzero constant velocity.

Exact knowledge of the stationary states of our dynamical rules is a useful first step similarly to the knowledge of the partition function in equilibrium dynamics. It remains to characterize the features of these states, or more precisely, the phases which they characterize (liquid or condensed) and inherent correlations between occupation numbers at different sites. While the liquid phase here turns out to be accessible in the grand-canonical formulation, the condensed phase is not. In addition, a reformulation in terms of a transfer matrix is available for one dimension, and there the model can be mapped to an SOS model [6], but to our knowledge the transfer-matrix formalism is not available in higher dimensions. Therefore, in the next section, we restrict our analytical study of the phase structure and the static properties of the condensate to a ring topology.

3. Condensation on a ring

Imagine we have a ring with N sites, and M balls available to distribute over the sites of the ring. We choose dynamical rules for their hopping between the sites which depend on the number of balls on the departure site as well as on the neighboring sites. Let us choose homogeneous rules over the ring so that no site is distinguished. What we were able to predict [7, 9] is not only the onset of condensation of balls over a limited extension of the ring above some critical density, but also the shape of the condensate, whether rectangular, bell-shaped or delta-like (cf. Figures 1, 2, and 3, respectively), and the scaling with the size N of the ring were predicted and numerically confirmed. The actual calculations are rather involved and cannot be presented here in detail, but we shall summarize the results and point on the main assumptions entering the analytical calculations.

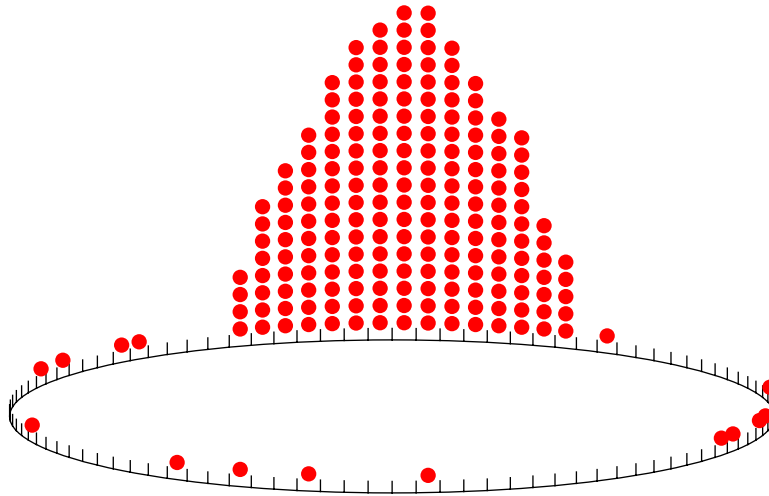


Figure 2. Schematic: Condensate of bell shape on a ring

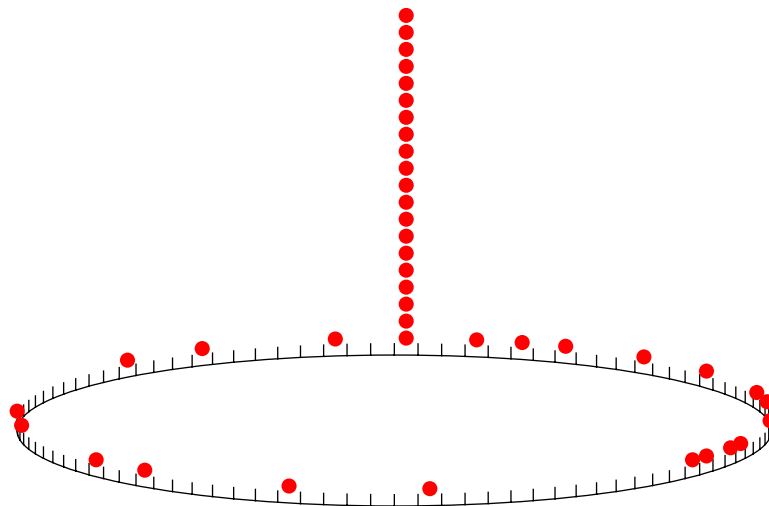


Figure 3. Schematic: Localized condensate on a ring as it also occurs for a ZRP

As special case of (2), when the hopping rate has the form

$$u(m_i|m_{i-1}, m_{i+1}) = \frac{g(m_i - 1, m_{i-1}) g(m_i - 1, m_{i+1})}{g(m_i, m_{i-1}) g(m_i, m_{i+1})}, \quad (4)$$

with $g(m, n)$ being a symmetric non-negative but otherwise arbitrary function, the steady state factorizes over pairs of sites according to

$$P(m_1, \dots, m_N) = \prod_{i=1}^N g(m_i, m_{i+1}) \delta\left(\sum_{i=1}^N m_i - M\right), \quad (5)$$

where the ring geometry implies $m_{N+1} \equiv m_1$. The difference as compared to the model of Ref. [6] is that there $r = 1$ and no assumption on the symmetry of $g(m, n)$ was made. The parameter

r changes only the net current of particles. When $r = 1/2$, the current is zero and the system is in equilibrium. Since the steady state (5) does not depend on r , its static properties can be calculated by formally treating the system as if it was in equilibrium with the probability of a microstate given by Eq. (5).

The criterion for condensation is non-analyticity of the grand-canonical partition function

$$Z_N(z) = \sum_{\{m_i\}} z^{\sum_i m_i} \prod_i g(m_i, m_{i+1}) = \text{Tr } T^N, \quad (6)$$

where $T_{mn} \equiv z^{(m+n)/2} g(m, n)$ and z is the fugacity. Assuming z_c is the finite radius of convergence of $Z_N(z)$, the critical value of the density $\rho = M/N$ then follows from

$$\rho_c = \lim_{N \rightarrow \infty, z \rightarrow z_c^-} \frac{z}{N} \frac{\partial \ln Z_N(z)}{\partial z} = \frac{\sum_m m \phi_m^2}{\sum_m \phi_m^2}, \quad (7)$$

where ϕ is an eigenvector of T_{mn} associated with the largest eigenvalue λ_{\max} , for $z = z_c$. At the critical point, $Z_N \cong \lambda_{\max}^N$ for large N .

To investigate how the introduction of site-site interactions influences the properties of condensation, we stick to the following choice:

$$g(m, n) = K(|m - n|) \sqrt{p(m)p(n)}. \quad (8)$$

When $K(x) = 1$, $g(m, n)$ factorizes and we recover the ZRP. Here we assume that both $K(x)$ and $p(m)$ are some positive, decaying functions of x and m , respectively. The choice (8) is motivated by studies on the SOS model in the context of surface roughening [10, 11] and corresponds to the energy $E = -\ln K(|m - n|) - (1/2) [\ln p(m) + \ln p(n)]$ of an interface within a 1+1-dimensional surface (where the surface refers to the envelope of occupation numbers). In Ref. [6], the following choice was proposed:

$$K(x) = e^{-Jx}, \quad p(m) = e^{U\delta_{m0}}, \quad (9)$$

with parameters J and U generating an effective surface stiffness and a pinning potential, respectively. For the weights (9), we obtain the critical density as a function of J and U as follows:

$$\rho_c = \frac{e^{J_0} - 1}{(e^{J_0} - e^{-2(J-J_0)})(e^{2(J-J_0)} - 1)}, \quad (10)$$

with $J_0 = U - \ln(e^U - 1)$. If $J < J_0$, the critical density is infinite.

The generic case — In order to calculate the width W and the envelope of the condensate, one cannot use the partition function (6), because it is not defined in the condensed phase. Instead, for the weight functions (8), we use an approach which allows to calculate both quantities in the case when $K(x)$ decays exponentially or faster in x , and $p(m) \rightarrow 1$ for $m \rightarrow \infty$, that is for local interactions with exponential decay and an ultralocal part that approaches a constant above some occupation number m_{\max} . We make the assumption that the system factorizes into two terms, one coming from the condensate and one from the background, filled up to the critical particle density. The condensate then extends over W sites occupied by $M' = M - N\rho_c$ particles on the average, average occupation numbers $\langle m_i \rangle$ grow with N , and the uniform background is characterized by $\langle m_i \rangle = \rho_c$. Since fluctuations in the background are finite and have no long-range correlations, the mass M' cannot fluctuate more than $\sim \sqrt{N}$ so that we treat it as constant. We can therefore assume that the probability of having the condensate extended over W sites is given by

$$\begin{aligned} P(W) &\approx Z_b(W) Z_c(W) \\ &\propto \exp(-W \ln \lambda_{\max} + \ln Z_c(W)), \end{aligned} \quad (11)$$

where $Z_b(W) = \lambda_{\max}^{N-W}$ is the partition function for the background at the critical point $\rho = \rho_c$, λ_{\max} the maximal eigenvalue of the transfer matrix, and $Z_c(W)$ is the partition function of the condensate extended over W sites and having exactly M' particles. In the calculation of $Z_c(W)$ we have furthermore extended the summation over negative occupation numbers, although this is physically not meaningful, but it allows doing the calculation, and the induced error is negligible in the end if we compare with results of the numerical simulations. The factor Z_c can be explicitly calculated when it is independent of the ultralocal part of the weight, i.e., independent of $p(m)$. This happens in particular for the choice of weight functions as it was used in [6]. Otherwise we use what we called the fixed-envelope approximation. Here we approximate Z_c via the probability $P(W)$ of having a condensate extended over W sites with unknown but fixed envelope $h(t)$, for which we use a form that is inspired by numerical simulations of the condensation. The condensate's extension follows in these cases from the maximum of the probability $P(W)$.

As it turns out, within the same fixed-envelope approximation scheme, when local and ultralocal weights are "short-ranged"², we can tune the extension to scale with the system size as N^α with $0 \leq \alpha \leq 1/2$. When both parts of the weights are long-ranged, the condensate gets localized to a single site as in the ZRP, cf. Figure 3. For a long-range local part and a short-range ultralocal part, the condensate takes a rectangular form (Figure 1), its height scales proportionally to the system size, while its extension remains constant: features that remind us to finite-size scaling of a first-order phase transition. Therefore hopping rates, leading to pair-factorized steady states, do not necessarily lead to extended condensates, and if they do so, the shape of the condensate is non-universal. For any details of the calculation we refer to [9].

4. Anisotropic hopping

A generalization of the methods used in the previous section to higher dimensions is not straightforward. Let us combine a ZRP in one dimension with a process that in isolation would lead to a pair-factorized steady state in the second dimension, but both processes coupled via the constraint of overall mass conservation. This implies that $g(m, n)$ lives on edges only in one, say vertical, direction. On horizontal links, the weight is assumed for simplicity to be constant and equal to 1, but in general one may choose any weight $f(m_{i,j})$ depending only on a single site. An extrapolation of our results in one dimension suggests a localized condensate in one direction and an extended condensate in the second direction, although the competition between the different interactions could in principle lead to a spreading of a condensate in both directions, or a uniform distribution in one direction and a condensate in the other direction. The former scenario is actually numerically observed for a certain range of mass density values $\rho_1 < \rho < \rho_2$, cf. Figure 4. Below ρ_1 we find a 'liquid' uniformly spread over the lattice, and above ρ_2 we find a wall of uniform height in one direction, but localized in the other direction, cf. Figure 5. These qualitative features can be understood and analytically reproduced by means of dimensional reduction of the partition function in two dimensions to an effective ZRP in one dimension. Let us assume that we have a two-dimensional lattice with $N = L \times L$ sites, periodic boundary conditions and the following steady state:

$$w(\vec{m}) = \prod_{i=1}^L \prod_{j=1}^L g(m_{i,j}, m_{i,j+1}) \delta \left(\sum_{i,j} m_{i,j} - M \right). \quad (12)$$

² We call short-range interactions local weight factors $K(x) = K(|m - n|)$ and ultralocal factors $p(m)$ which decay faster than any power in their argument, and long-range interactions those which decay as a power of their argument (x or m , respectively). So the relevant characteristics for both local and ultralocal weight factors is the range of interactions in occupation-number space.

The hopping rate assumes now the form

$$u(m_{i,j}|\dots) = \frac{g(m_{i,j} - 1, m_{i,j-1}) g(m_{i,j} - 1, m_{i,j+1})}{g(m_{i,j}, m_{i,j-1}) g(m_{i,j}, m_{i,j+1})}, \quad (13)$$

the particles can jump either in horizontal or vertical direction. The current is defined by any W obeying equations (3). Although we will not address dynamical issues here, for definiteness let us assume that the particles can jump only to the right with probability $W(i, j \rightarrow i, j + 1) = p$ or to the top with probability $W(i, j \rightarrow i + 1, j) = 1 - p$, while $W(i, j \rightarrow i, j - 1) = W(i, j \rightarrow i - 1, j) = 0$. Moreover, let the weight $g(m, n)$ be chosen according to (9). We will argue that ρ_1 is the critical density of the corresponding pair-factorized state in one dimension and also calculate ρ_2 .

First we observe the following. In the steady state we can treat the probability $w(\vec{m})$ as the weight of a microstate of a system being in equilibrium, therefore we can formally write the partition function of the system in the canonical formulation as

$$\begin{aligned} Z_{c2d}(N = L^2, M) &= \sum_{\{m_{i,j}\}} \prod_{i=1}^L \prod_{j=1}^L g(m_{i,j}, m_{i,j+1}) \delta\left(\sum_{i,j} m_{i,j} - M\right) \\ &= \sum_{M_1=0}^M \cdots \sum_{M_L=0}^M \prod_{i=1}^L Z_{c1d}(L, M_i) \delta\left(M - \sum_i M_i\right), \end{aligned} \quad (14)$$

where

$$Z_{c1d}(L, \tilde{M}) = \sum_{m_1=0}^{\tilde{M}} \cdots \sum_{m_L=0}^{\tilde{M}} \prod_{i=1}^L g(m_i, m_{i+1}) \delta\left(\tilde{M} - \sum_i m_i\right) \quad (15)$$

is the partition function of a one-dimensional pair-factorized steady state. Since (14) has the same functional form as the partition function for the zero-range process [12], the partition function Z_{c1d} may be seen as the weight $p(m) \equiv Z_{c1d}(L, m)$ that is now associated with the total mass $m = M_i$ along the i th vertical line.

In [7] we have shown that for $m > \rho_c L$, where ρ_c is the critical density for condensation in the one-dimensional system, $Z_{c1d}(L, m)$ behaves as $\sim \exp(-c\sqrt{m})$ with some $c > 0$. This means that for large m , the hopping rate of the corresponding ZRP, $u(m) = p(m-1)/p(m) = Z_{c1d}(L, m-1)/Z_{c1d}(L, m)$, behaves as $u(m) \cong 1 + c/(2\sqrt{m})$. For such a hopping rate it is known [12] that the ZRP exhibits a condensation transition. The condensate occupies a single site and the fluid-phase distribution is a stretched exponential distribution. Since m denotes now the mass along a vertical line, the mapping from the ZRP back to the anisotropic model allows us to predict the spontaneous symmetry breaking to the state with a condensate so that one of the masses M_i will grow to pick up all the difference $\Delta M = M - \rho_1 L^2 = L^2(\rho - \rho_1)$ between the total mass M and the mass $\rho_1 L^2$ in the critical background. Thus, not each line carries its own condensate, but the condensate is localized onto a single line if there is a condensate at all, while all other lines have a mass density close to the critical value. To predict the critical density ρ_1 we observe that (14) in the grand-canonical formulation,

$$\begin{aligned} Z_{g2d}(L, z) &= \sum_M Z_{c2d}(L, M) z^M = \sum_{\{M_i\}} \prod_{i=1}^L Z_{c1d}(L, M_i) z^{M_i} \\ &= \left(\sum_M Z_{c1d}(L, M) z^M \right)^L, \end{aligned} \quad (16)$$

becomes just a power of a grand-canonical partition function in one dimension. Since the

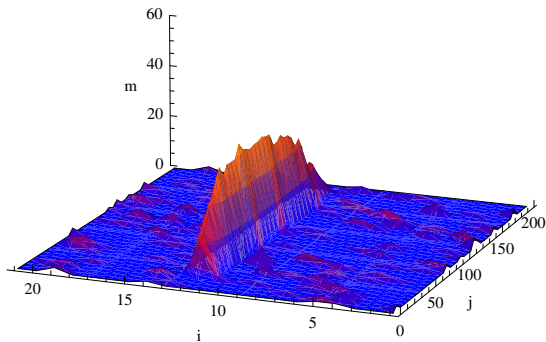


Figure 4. Condensate localized in x -direction and extended in y -direction for $\rho_1 < \rho < \rho_2$.

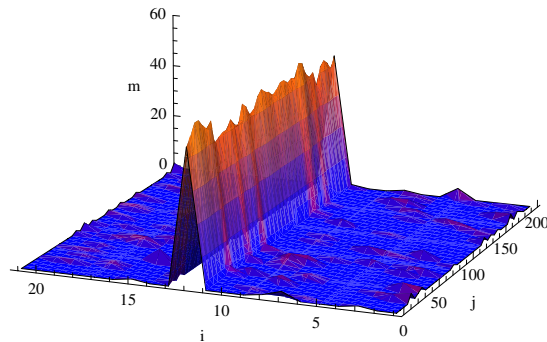


Figure 5. Condensate localized in x -direction, but uniformly spread out like a liquid in y -direction for $\rho > \rho_2$.

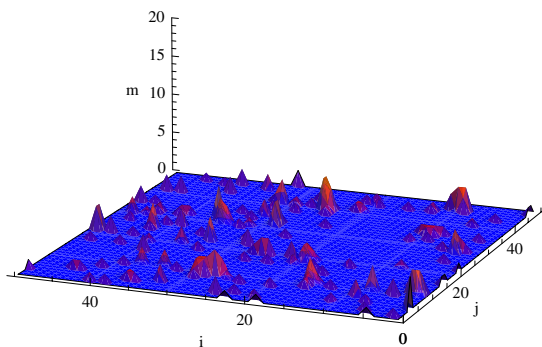


Figure 6. Liquid phase for isotropic hopping in two dimensions.

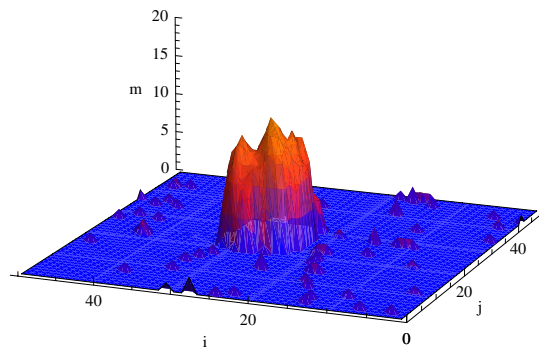


Figure 7. Condensate with quadratic support for isotropic hopping in two dimensions.

condensation transition is determined by the non-analytic behavior of Z_{g2d} , the critical density ρ_1 must be the same as ρ_c of the one-dimensional pair-factorized steady state. For $J = U = 1$ it reads $\rho_1 \approx 0.24$, as follows from [7].

It remains to determine how the particles are distributed along the lines which carry the condensate. Using results from [7] for the pair-factorized state in one dimension we know that the width W grows $\sim L\sqrt{\rho - \rho_1}$. This is the shape we see in Figure 4 for $\rho_1 < \rho < \rho_2$. When the density exceeds the values ρ_2 for which the density exceeds the width W and becomes equal to the linear size L , there are no empty sites left so that $g(m, n)$ effectively behaves as $\exp(-J|m - n|)$, a case for which we know from [7] that the system is always in a liquid state. This explains the uniform shape in y -direction in Figure 5, while it stays localized in x -direction. Both Figures 4 and 5 display only part of the lattice (20×200 sites) on which the actual simulations have been performed, these were lattices with 200×200 sites.

For interactions with isotropic weights $g_i(m, n)$, $i \in \{x, y\}$ in x - and y -direction we numerically observe the natural two-dimensional generalization of an extended condensate on a ring topology, but an analytic treatment was not possible so far, cf. Figure 6 for the liquid phase and Figure 7 for the condensed phase. It remains a challenge to derive the static features of the condensate on generic network topologies, in particular in higher dimensions.

5. Conclusions and outlook

The answer to our question which hopping rates lead to pair-factorized steady states on arbitrary graphs was given in terms of a class of weight functions. These weight functions may depend on the links of the graphs and need not be local, so that anisotropic hopping and hopping to remote target nodes are allowed. The topologies we have studied so far for deriving the phase structure were regular ones: the ring topology and a square lattice. The next step is to consider our dynamical rules on general network topologies and study the effect of explicit symmetry breaking due to the network topology, for example. For the ring and square topology it was possible to predict beyond the very onset of the phase transition several properties of the condensate: its shape as a function of the interaction between occupation numbers on different sites, its extension and scaling with the system size. For anisotropic interactions in two dimensions we derived and numerically confirmed two phase transitions: the first one from a liquid phase to a phase with a condensate localized in one direction and extended but still present in the second direction, the second transition from the former condensed phase to a phase with a condensate in one direction which is, however, uniformly spread out like a liquid in the second direction. From an experimental point of view our results are of interest for experiments in surface science [13, 14], in which condensates are observed of a certain shape and for which it may be tempting to trace back the underlying rules for particle hopping which are compatible with the observed shape and the scaling of the condensate's extension.

From a theoretical point of view it is challenging to generalize our pairwise factorizing steady states to states which fully factorize over triangles or plaquettes and to see which hopping rates lead to such steady states. Exactly known states like those we have considered in this article may always serve as starting point for approximations where weak correlations are involved, but may be neglected to zeroth order. Another direction of interest are dynamical features. We called the very formation of condensates on a finite ring spontaneous symmetry breaking. On the other hand, strictly speaking a symmetry cannot spontaneously break in a finite volume, neither here. Therefore any condensate will melt after some time and build up again at another site. To estimate this time as a function of the system size is a challenge for future calculations.

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