Ultraviolet divergences, repulsive forces and a spherical plasma shell

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Abstract. We discuss the vacuum energy of the electromagnetic field interacting with a spherical plasma shell together with a model for the classical motion of the shell. We discuss the ultraviolet divergences in terms of the heat kernel coefficients. Using these, we carry out the renormalization by redefining the parameters of the classical model. It turns out that this is possible and that the resulting model has a vacuum energy which changes sign in dependence on the parameters of the plasma shell. In the limit of the plasma shell becoming an ideal conductor the vacuum energy found by Boyer in 1968 is reproduced.

1. Introduction

In general, the repulsive forces found for the Casimir energy of a conducting sphere and a conducting cube remain a still not finally understood problem. It was a quite big surprise for the community when Boyer in 1968 [1] found the Casimir effect for the conducting sphere to be repulsive. Not only that this invalidated Casimir's electron model [2], this result was to some extend also counterintuitive. The background for that is in the relation between Casimir and van der Waals forces. From the one hand side, the Casimir forces follow from the zero point energy of the electromagnetic field confined between material bodies and on the other hand side the same formula emerges as a limiting case for becoming ideal conductors from the Lifshitz formula for the same material bodies. It is common understanding that this limiting case where the surfaces become ideal conductors is just the bridge between quantum field theory with its vacuum energy and the dispersion forces in many particle theory represented by the Lifshitz formula. These forces generalize the interaction between two molecules, which is given by the Casimir-Polder potential and which is always attractive. For this attraction there exists even a very general reason; it is a second order perturbation to a ground state energy which in quantum mechanics is always negative. Because of this deep connection Casimir's assumption that the attractive force he found for parallel planes should keep its sign when considering a sphere instead, was so very natural. For basically the same reasons the repulsive character of the force for the sphere was so unexpected.

During the last years with the development of nanotechnology, the discussion of repulsive forces became actual again. The reason is in the stickyness of the attractive forces. Because their increase at small separations any configuration of closely nearby located nanoobjects is potentially instable with respect to these bodies sticking together. This is a limitation in a number of applications. The question was discussed whether it is possible to use the repulsive force on a sphere for reducing the stickyness. Here one must remember that this repulsive

force was found for an ideal configuration, namely an infinitely thin conducting spherical shell. This thinness is necessary because otherwise the ultraviolet divergences do not go away. In fact these cancel each other when taking the vacuum energies from inside the sphere and from outside together. But this compensation does not work for any finite thickness. In this sense the repulsive force must be considered a pure theoretical phenomenon. In this connection it must also be mentioned that recently in [3] quite general arguments were given in favor of the statement that the forces between separated bodies areattractive.

There is another situation with a repulsive force which makes the puzzle even worse. This is a dielectric ball. In general, this is nothing else than a macroscopic body characterized by a permittivity and one should expect the vacuum energy of the electromagnetic field in its presence to be finite. This is, however, not the case, except for the dilute approximation [4]. This means that the permittivity ϵ of the ball is close to its vacuum value such that $\epsilon - 1 = 4\pi N\alpha$, where N is the density and α is the static polarizability of the molecules, is small. Then the expansion of the vacuum energy up to second order in this small parameter is called the dilute approximation. There are two remarkable facts. The first is that in this approximation the vacuum energy is always finite in zetafunctional regularization or can be made finite in a unique manner. This was found for the first time in the beginning of the 80ies and mathematically confirmed in the end of the 90ies (see [4] and the citations therein). The second fact is that this vacuum energy can be calculated in two ways. The first way is the standard one by calculating the zero point energy of the electromagnetic field in the presence of the ball, see [5]. The second one is to sum up the pairwise Casimir-Polder potential

$$V(r_1, r_2) = -\frac{23}{(4\pi)^3 N^2} \frac{(\epsilon_1 - 1)^2}{|r_1 - r_2|^7},$$

(1)

where r_1 and r_2 are the locations of these molecules, over all pairs of molecules constituting the ball. This was done in [6] carrying out the integrations in

$$E = -\frac{23}{8\pi} \frac{(\epsilon_1 - 1)^2}{(4\pi)^2} \int_V dr_1 \int_V dr_2 \frac{1}{|r_1 - r_2|^\gamma}.$$

(2)

Here the power in the denominator was changed to γ in order to make the integrations convergent. This is a regularization and to make the integrals convergent one needs to have $\gamma < 3$. In the end one has to perform the continuation to $\gamma = 7$. One arrives at

$$E = -\frac{23}{2^{2+\gamma}\pi}(\epsilon_1 - 1)^2 \frac{\Gamma\left(2 - \frac{\gamma}{2}\right)}{\Gamma\left(4 - \frac{\gamma}{2}\right)(3 - \gamma)} \frac{1}{R^{\gamma-6}}.$$

(3)

This expression has a unique analytic continuation to $\gamma = 7$ where it takes a finite value and one comes the the known result

$$E_0 = \frac{23}{384\pi R}(c_1 - c_2)^2 + O\left[(c_1 - c_2)^3\right].$$

(4)

As seen it is repulsive despite the fact that the pairwise potential is attractive. A deeper physical understanding of this fact is missing. Of course one can trace back mathematically the sign change, but the answer that it happens during an analytic continuation which is necessary to do in order to work with correct defined quantities, is not especially illuminating.

Besides the puzzle with the sign, another motivation for the present talk is to contribute to the discussion of the renormalization of vacuum energy in the presence of boundaries or singular background fields in application to the Casimir effect.

As it was pointed out in [7], there is a gap between the two well understood situations. These are, on the one side, the Casimir force between distinct objects which is always finite and, on

the other side, the vacuum energy in smooth background fields which can be renormalized by standard methods of quantum field theory. In between these two, the situation is not finally settled. The mentioned paper, it was questioned whether boundaries can be incorporated at all into a well posed renormalization program. For instance, it was argued that the process of making the background field concentrated on a surface is not physical.

The aim of the present contribution is to discuss an example of a background field concentrated on a surface having both, a well posed renormalization procedure for the vacuum energy and a meaningful physical interpretation. As model we take a spherical plasma shell interacting with the electromagnetic field and we allow for a classical vibrational motion of the shell. The investigation of the plasma shell model was pioneered by Barton [8] and it is aimed to describe the π-electrons in a C_{60}-molecule.

2. The plasma shell model and its renormalization

In the plasma shell model the electrons are described by an electrically charged fluid whose motion is confined the shell. Further, the model contains an immobile, overall electrically neutralizing background aimed to describe the carbon atoms and the remaining electrons. The fluid is allowed a non-relativistic motion. Of course, this model is a quite crude simplification, especially because the motion of the electrons should rather follow a relativistic dispersion relation [9, 10]. On the other hand side it appears to be physically meaningful and should therefore result in physically meaningful results for the vacuum energy. For instance, it should allow for a treatment of the vacuum fluctuation of the electromagnetic field coupled to the plasma shell.

The interaction of the plasma shell with the electromagnetic field results in matching conditions on the electromagnetic field across the shell as shown in [8] (and earlier, for a plane sheet, in [11]). These conditions do not depend on the state of the excitations of the fluid. The vacuum energy can be calculated from the fluctuations of the electromagnetic field whereas the fluctuations of the fluid must not be taken into account as shown in [12] (or vice verse). In this setup, the polarizations of the electromagnetic field separate into TE- and TM-modes. For the corresponding amplitudes the following matching conditions hold ([8], [13]),

$$
\begin{aligned}
\lim_{r \to R+0} f_{l,m}(kr) - \lim_{r \to R-0} f_{l,m}(kr) &= 0, \\
\lim_{r \to R+0} (r f_{l,m}(k,r))' - \lim_{r \to R-0} (r f_{l,m}(k,r))' &= \Omega R f_{l,m}(kR),
\end{aligned}
\tag{5}
$$

for the TE-mode, and

$$
\begin{aligned}
\lim_{r \to R+0} (r g_{l,m}(k,r))' - \lim_{r \to R-0} (r g_{l,m}(k,r))' &= 0, \\
\lim_{r \to R+0} g_{l,m}(kr) - \lim_{r \to R-0} g_{l,m}(kr) &= -\frac{\Omega}{k^2 R} (R g_{l,m}(k,R))',
\end{aligned}
\tag{6}
$$

for the TM mode. The parameter

$$
\Omega = \frac{4\pi n e^2}{m c^2}
\tag{7}
$$

carries information on the properties of the fluid like its density n and mass m.

Considered as a scalar problem, the matching conditions (5) of the TE mode are equivalent to a delta function potential $\Omega \delta(r - R)$ in the wave equation and the conditions (6) of the TM mode loosely speaking correspond to the derivative of a delta function. A difference is that in the scalar problems the zeroth orbital momentum, $l = 0$, or s-wave contribution is present whereas in the electromagnetic case it is absent and the sum over l in the vacuum energy starts from $l = 1$. In the limit $\Omega \to \infty$ which is formally the ideal conductor limit the boundary conditions

(5) and (6) became Dirichlet boundary conditions for TE polarization and Neumann for TM polarization.

We extend this model by allowing for radial vibrations (breathing mode) of the plasma shell. In C_{60} these are determined by the elastic forces acting between the carbon atoms. Without going here in any detail we describe these vibrations phenomenologically by a Hamilton function

$$H_{\text{class}} = \frac{p^2}{2m} + \frac{m}{2}\,\omega_b^2\,(R - R_0)^2 + E_{\text{rest}} \tag{8}$$

with a momentum $p = m\dot{R}$. Here m is the mass of the shell, ω_b is the frequency of the breathing mode, R_0 is the radius at rest and E_{rest} is the energy which is required to bring the pieces of the shell apart, i.e., it is some kind of ionization energy.

Now we consider a system consisting of the classical motion of the shell as described by H_{class} and the vacuum energy E_{vac} of the electromagnetic field interacting with the shell by means of the matching conditions (5) and (6). We assume the classical motion adiabatically slow such that the vacuum energy can be taken as a function of the momentarily radius of the shell, $E_{\text{vac}} = E_{\text{vac}}(R)$, and we neglect the backreaction of the electromagnetic field on the shell. Under these assumptions the energy of the classical system, $E_{\text{class}}(R) = H_{\text{class}}$, and the vacuum energy add up to the total energy of the considered system,

$$E_{\text{tot}} = E_{\text{class}}(R) + E_{\text{vac}}(R). \tag{9}$$

Next we consider the ultraviolet divergences of the vacuum energy. These are given in general terms by the heat kernel coefficients a_n (we use the notations of [14] and we can define a 'divergent part' of the vacuum energy which is, as known, not uniquely defined. It depends on the kind of regularization one has to introduce. For instance, in zeta functional regularization, the regularized vacuum energy reads

$$E_{\text{vac}}(s) = \frac{\mu^{2s}}{2} \sum_n \omega_n^{1-2s}, \tag{10}$$

where μ is an arbitrary parameter with the dimension of a mass and with a frequency damping function it is,

$$E_{\text{vac}}(\delta) = \frac{1}{2} \sum_n \omega_n\, e^{-\delta\omega_n}, \tag{11}$$

where ω_n are the frequencies of the quantum fluctuations of the electromagnetic field. In our problem the spectrum is continuous, but for the moment it is more instructive to keep the notations of a discrete spectrum. In zeta functional regularization, the divergent part reads

$$E_{\text{vac}}^{\text{div}}(s) = -\frac{a_2}{32\pi^2}\left(\frac{1}{s} + \ln\mu^2\right), \tag{12}$$

where we used the notations of [14] in which the heat kernel expansion reads

$$K(t) \sim \frac{1}{(4\pi t)^{3/2}}\left(a_0 + a_{\frac{1}{2}}\sqrt{t} + a_1 t + \dots\right). \tag{13}$$

In the scheme with the frequency damping we have

$$E_{\text{vac}}^{\text{div}}(\delta) = \frac{3a_0}{2\pi^2}\frac{1}{\delta^4} + \frac{a_{1/2}}{4\pi^{3/2}}\frac{1}{\delta^3} + \frac{a_1}{8\pi^2}\frac{1}{\delta^2} + \frac{a_2}{16\pi^2}\ln\delta. \tag{14}$$

The regularizations are removed by $s \to 0$ resp. $\delta \to 0$. These formulas follow, for example, from section 3.4 in [14] for $m = 0$.

The idea of the renormalization is to have in the classical energy E_{class} parameters which can be changed in a way to absorb $E_{\text{vac}}^{\text{div}}$. In the considered model such parameters are the mass m of the shell, the frequency ω_b of the breathing mode, the radius at rest R_0, and the energy E_{rest}. Now, whether this is possible, is a matter of the dependence of the heat kernel coefficients, especially of a_2, on the radius R which is the dynamical variable of the classical system. In the considered, very simple model we have only a polynomial dependence on R up to second order in (8). Since we assumed adiabaticity for the motion of the shell we do not have a time dependence in a_2 so that it cannot contain $\dot{R}$. Hence, the kinetic energy remains unchanged and, together with it, the mass m. Only the remaining parameters, ω_b, R_0 and E_{rest} can be used to accommodate the divergent part. In fact, this turns out to be sufficient for the considered model. As it will be seen below, the heat kernel coefficients $a_0, \ldots, a_2$ entering the divergent part, depend on the radius polynomial and at most quadratically. In this way this model is renormalizable.

It should be mentioned that this scheme follows closely the corresponding one in quantum field theory with $E_{\text{vac}}^{\text{div}}$ in place of the counterterms. Also the interpretation of the renormalization is similar. Namely, we argue that the vacuum energy in fact cannot be switched off and what we observe are parameters like, for example in QED, electron mass and charge, after renormalization.

Within this scheme of renormalization, the specific form of the heat kernel coefficients is insignificant. One has to bother of its dependence on R in order to fit into the freedom of redefining the parameters in E_{class}. If this is the case, one may define a renormalized vacuum energy by means of

$$E_{\text{vac}}^{\text{ren}} = \lim_{s \to 0} \left(E_{\text{vac}}(s) - E_{\text{vac}}^{\text{div}}(s) \right) \tag{15}$$

(and the same with δ in place of s) and one has now to consider

$$E_{\text{tot}} = E_{\text{class}} + E_{\text{vac}}^{\text{ren}} \tag{16}$$

in place of (9). In this way, the question on how to remove the ultraviolet divergences is answered.

It remains, however, the question about the uniqueness of his procedure which comes in from the parameter μ in the zeta functional scheme or from the possibility of a redefinition $\delta \to c\delta$, where c is an arbitrary constant, in the other scheme.

In the case of QED at this place one imposes conditions on the analog of $E_{\text{vac}}^{\text{ren}}$ in a way, that the mass and the charge take the values one observes experimentally.

In our case a similar scheme is conceivable too. A different scheme, suggested in [4], using the large mass expansion to fix the ambiguity does not work here since the electromagnetic field is massless. A way out could be to look for a minimum of the total energy, E_{tot}, (9), which however would imply to take the model (8) seriously. This is not the aim of the present paper. Instead, as a normalization condition we demand that in the limit of the plasma frequency $\Omega \to \infty$, where the matching conditions (5) and (6) turn into that of an ideal conductor, we shall recover the vacuum energy of a conducting spherical shell, i.e., just the quantity which was first calculated by Boyer in [1]. Indeed, as we will see in the next section, this is possible using the freedom of a finite renormalization.

3. The heat kernel coefficients and the renormalized vacuum energy
The electromagnetic field interacting with the plasma shell is defined in the whole space and it has a continuous spectrum. In that case the vacuum energy, after the subtraction of the

contribution of the empty space, can be represented in the form (see Eq.(3.43) in [14])

$$E_0(s) = -\frac{\cos \pi s}{\pi} \mu^{2s} \sum_{l=1}^{\infty} \nu \int_0^{\infty} dk \, k^{1-2s} \frac{\partial}{\partial k} \ln f_l(ik) \qquad (17)$$

with $\nu = l + 1/2$. The arbitrary parameter μ has the dimension of a mass and $f_l(k)$ is the Jost function of the corresponding scattering problem. With argument rotated to the imaginary axis these read

$$
\begin{aligned}
f_l^{\mathrm{TE}}(ik) &= 1 + \frac{\Omega}{k} s_l(kR) e_l(kR), \\
f_l^{\mathrm{TM}}(ik) &= k^2 \left(1 - \frac{\Omega}{k} s_l'(kR) e_l'(kR) \right),
\end{aligned}
\qquad (18)
$$

where we used the modified Riccati-Bessel functions

$$s_l(x) = \sqrt{\frac{\pi x}{2}}\, I_{l+1/2}(x), \qquad e_l(x) = \sqrt{\frac{2x}{\pi}}\, K_{l+1/2}(x). \qquad (19)$$

By these formulas, the heat kernel coefficients, including that for double poles, which are related to the vacuum energy by means of

$$2\mu^{-2s}(4\pi)^{3/2}\Gamma\left(s - \frac{1}{2}\right) E_0(s) = \sum_{k\geq 0} \frac{a_{k/2}}{s - 2 + \frac{k}{2}} + \sum_{k\geq 3} \frac{a'_{k/2}}{\left(s - 2 + \frac{k}{2}\right)^2} + \dots, \qquad (20)$$

can be calculated. For the TE mode this repeats earlier calculations, for the TM mode is was done for the first time in [13]. The result is shown in the table. It must be mentioned that for the TM-mode double poles in s appear which were included in Eq. (20) and the corresponding coefficients are marked with a prime. These result in logarithmic terms in the heat kernel asymptotics. In fact, in our case these appear starting with $k = 5/2$. Therefore these do not affect the renormalization.

$k/2$	TE-mode	TM-mode
0	0	0
1/2	0	$8\pi^{3/2}R^2$
1	$-4\pi\Omega R^2$	$-\frac{4\pi}{3}\Omega R^2$
3/2	$\pi^{3/2}\Omega^2 R^2$	$-\frac{10}{3}\pi^{3/2}$
2	$-\frac{2}{3}\pi\Omega^3 R^2 + 4\pi\Omega$	$-4\pi\Omega + \frac{2\pi}{15}\Omega^3 R^2$

Table 1. The first few heat kernel coefficients for the plasma shell as defined in Eq. (20).

Using these heat kernel coefficients, the renormalization program discussed in the preceding section was carried out in [13]. The vacuum energy can be divided into asymptotic and numeric parts. The asymptotic parts were found to be

$$
\begin{aligned}
E_{\mathrm{vac}}^{\mathrm{TE, as}}(s) &= -\frac{a_2^{\mathrm{TE}}}{32\pi^2}\left[\frac{1}{s} - 2\ln\frac{\Omega}{2\mu}\right] + \frac{\Omega^3 R^2}{72\pi} + \frac{\Omega}{180\pi} + E_{\mathrm{vac}}^{\mathrm{TE, an}} + O(s), \\
E_{\mathrm{vac}}^{\mathrm{TM, as}}(s) &= -\frac{a_2^{\mathrm{TM}}}{32\pi^2}\left[\frac{1}{s} - 2\ln\frac{\Omega}{2\mu}\right] + \frac{7\Omega^3 R^2}{1800\pi} - \frac{29\Omega}{36\pi} + E_{\mathrm{vac}}^{\mathrm{TM, an}} + O(s).
\end{aligned}
\qquad (21)
$$

These expressions are sums of a divergent part (it is proportional to the heat kernel coefficient a_2 as expected), two terms growing with Ω and an 'analytical' part which has a finite limit for $s \to 0$.

All quantities entering $E_{\text{vac}}^{\text{TE, an}}$ and $E_{\text{vac}}^{\text{TM, an}}$ are defined in [13]. The analytical parts have a limit for $\Omega \to \infty$,

$$\begin{aligned} \lim_{\Omega \to \infty} E_{\text{vac}}^{\text{TE, an}} &= \frac{17}{128R}, \\ \lim_{\Omega \to \infty} E_{\text{vac}}^{\text{TM, an}} &= -\frac{11}{128R} \end{aligned} \qquad (22)$$

which matches the corresponding expressions known from the conducting sphere.

With Eqs. (21) and the property (22) we have all information we need to complete the renormalization. We remind the discussion the preceding section that all terms which are proportional to R, R^2 or which do not depend on R can be removed by a redefinition of the parameters in the classical part. In (21) this concerns all except the last ones, $E_{\text{vac}}^{\text{TE, an}}$ and $E_{\text{vac}}^{\text{TM, an}}$. That means, that we not only can remove the contribution proportional to a_2, but also the terms growing with Ω. For this reasons we define the renormalized vacuum energies by

$$\begin{aligned} E_{\text{vac}}^{\text{TE, ren}} &= E_{\text{vac}}^{\text{TE, num}} + E_{\text{vac}}^{\text{TE, an}}, \\ E_{\text{vac}}^{\text{TM, ren}} &= E_{\text{vac}}^{\text{TM, num}} + E_{\text{vac}}^{\text{TM, an}}. \end{aligned} \qquad (23)$$

With these formulas we completed the model consisting of the classical energy and the vacuum energy which is the sum of the two contributions in (23). The main merit of this vacuum energy is that it turns for $\Omega \to \infty$ into the ideal conductor limit. Using the formulas for the Jost functions and also the formulas given in [13], it is possible to evaluate this vacuum energy numerically. The results are shown in the figures 1 and 2 for the dimensionless function $\mathcal{E}$ defined by

$$E_{\text{vac}}^{\text{ren}} = \frac{\mathcal{E}(\Omega R)}{R} \qquad (24)$$
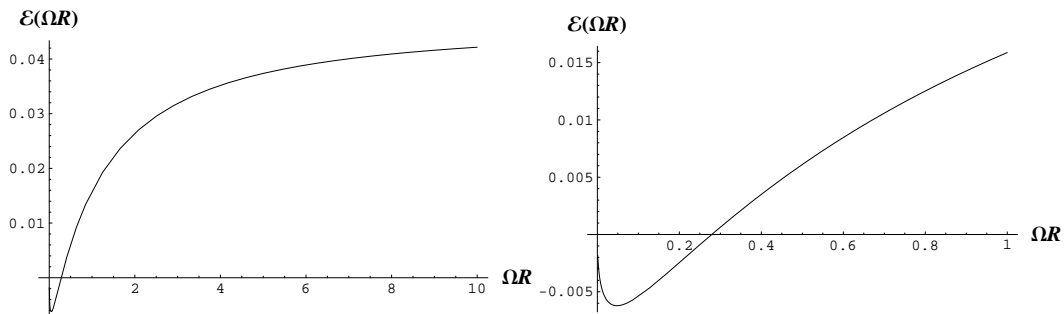
as functions of their arguments $x = \Omega R$.

Figure 1. The function $\mathcal{E}(\Omega R) = R E_{\text{vac}}^{\text{ren}}$ plotted as function of Ω. For large Ω it tends to the ideal conductor limit, $\lim_{\Omega R \to \infty} = 0.0046$ (left panel). For small Ω (right panel) it takes negative values and decreases as $\mathcal{E}(\Omega R) \sim -0.0589\sqrt{\Omega R}$.
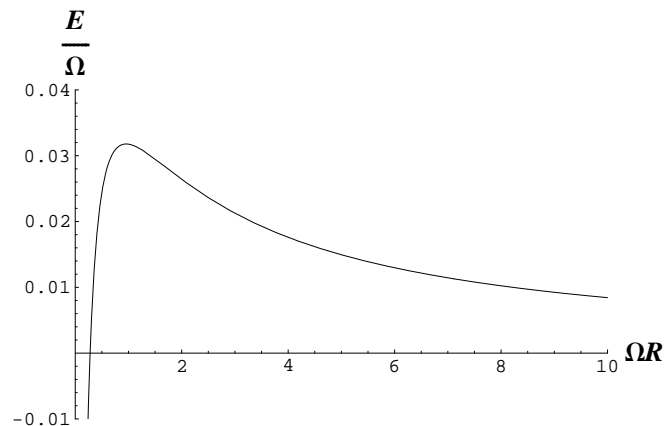
Figure 2. The renormalized vacuum energy divided by Ω, $E_{\text{vac}}^{\text{ren}}/\Omega$, as a function of R. For large R it approaches the ideal conductor limit and for small radii, $R \leq \Omega^{-1}$, it becomes attractive.

4. Conclusion

In the foregoing sections we considered the vacuum energy of the electromagnetic field interacting with a spherical plasma shell. We considered the heat kernel coefficients for both polarizations. It turned out that the vacuum energy in zeta functional regularization and, with it the corresponding zeta function, have double poles. This implies that the corresponding spectral problem is more complicated. On the other hand, at least based on the calculations carried out in this paper, there is nothing which would diminish the reasonableness of this model.

A basic concern of this paper is to construct a model allowing for a physically meaningful interpretation of the renormalization. We considered with the breathing mode of the shell the simplest model for the classical motion of the shell. It turned out that this model is able to accommodate all renormalizations which we were like to carry out. These are the removal of the pole in $s = 0$, i.e., of the ultraviolet divergence, and the removal of all contributions growing together with the plasma frequency Ω. It should be mentioned that this includes also the removal of the arbitrary constant μ which came in with the regularization. The nontrivial statement which allowed for doing so is that the dependence on the radius R of all these contributions is polynomial not exceeding R^2.

It should be mentioned that the same construction which we made in this paper for the plasma shell model interacting with the electromagnetic field can be done along the same lines also with a shell carrying a delta function and interacting with a scalar field which could be considered as a kind of acoustic field. All calculations for such a system were done in literature, the heat kernel coefficients were found in [4] and the renormalized vacuum energy was calculated in [15]. However, these fragments were not assembled into a consistent renormalization scheme.

It would be interesting to investigate the question whether the presented here renormalization procedure can be carried out also for more general deformations of the shell. In principle, most ingredients for such a calculation are available. Especially, the heat kernel coefficients for the TE modes can be taken from [16]. It would remain to calculate the coefficients for the TM modes.

The renormalized vacuum energy within our model is represented in the Figures 1 and 2. It smoothly interpolates between the limiting cases of large and small plasma frequency Ω. In the limit of $\Omega \to \infty$, where we recover the known result for the conducting sphere, the energy is positive, whereas for small Ω it becomes negative. This is in accordance with expectations coming from the weak coupling limit where the forces should become like van der Waals forces

which are attractive. It must be mentioned, however, that a simple perturbative expansion for small Ω would fail since the behavior for $\Omega \to 0$ is nonanalytic.

Concerning the arbitrariness of the normalization procedure we would like to mention that the removal of the contributions growing together with Ω can be considered as a normalization condition. The motivation of such a condition is twofold. First of all the fact must be emphasized that in this way a reasonable condition exists. This is nontrivial having in mind, for example, the situation with the dielectric ball which shows beyond the dilute approximation inevitably a divergence [4], even if dispersion is included [17] and no normalization condition was found so far. A second motivation comes from the physical sense of the limit $\Omega \to \infty$. A plasma shell becomes for growing Ω more and more like a conductor. If one starts for finite Ω from assuming that a finite vacuum energy makes physically sense, then an unbounded increase with increasing Ω would be unphysical. Hence a finite limit must be reached. From the calculations carried out in this paper, especially, from Eqs.(22), it turns out that this is just the ideal conductor limit. It should be underlined that there is no way to get any different finite value since all ambiguities in (21), especially the logarithmic term which is proportional to a_2, grow with powers of Ω. There is a further reason for our choice of the normalization condition. It follows from dimensional considerations suggesting the general form (24) of the vacuum energy. From these it follows, that a function $\mathcal{E}(\Omega R)$ growing with Ω would result in a vacuum energy which does not vanish for $R \to \infty$ (at fixed Ω). This is clearly unphysical.

In this way, our normalization condition ensures the uniqueness of the renormalized vacuum energy and makes this model physically meaningful. In addition, the gap between the renormalization procedure in quantum field theory in smooth background fields and the removal of divergences of the Casimir energy in the background of boundaries, as suggested for example in [18] (section 6.5), is narrowed. At once, in this way, the much discussed vacuum energy of a conducting spherical shell now appears as a limiting case of a slightly more physical model.

Acknowledgments

References

[1] T H Boyer. Quantum Electromagnetic Zero-Point Energy of a Conducting Spherical Shell and Casimir Model for a Charged Particle. *Phys. Rev.*, 174:1764, 1968.

[2] H B G Casimir. Quantum electrodynamics - introductory remarks. *Physica*, 19(7):846–849, 1953.

[3] O Kenneth and I Klich. Opposites attract: A theorem about the Casimir force. *Physical Review Letters*, 97(16), 2006.

[4] M Bordag, K Kirsten and D Vassilevich. On the ground state energy for a penetrable sphere and for a dielectric ball. *Phys. Rev.*, D59:085011, 1999.

[5] G Lambiase, G Scarpetta and V V Nesterenko. Exact value of the vacuum electromagnetic energy of a dilute dielectric ball in the mode summation method. *Mod. Phys. Lett.*, A16:1983–1995, 2001.

[6] Kimball A Milton and Y Jack Ng. Observability of the bulk Casimir effect: Can the dynamical Casimir effect be relevant to sonoluminescence? *Phys. Rev.*, E57:5504–5510, 1998.

[7] N Graham et al. The Dirichlet Casimir problem. *Nucl. Phys.*, B677:379–404, 2004.

[8] G Barton. Casimir energies of spherical plasma shells. *J. Phys. A: Math. Gen.*, 37(3):1011–1049, 2004.

[9] S A Maksimenko and G Ya Slepyan. Electrodynamics of Carbon Nanotubes. *J. Comm. Tech. Electron.*, 47:235–252, 2002.

[10] Mikhail I Katsnelson. Graphene: carbon in two dimensions. 2006. arXiv:cond-mat/0612534.

[11] G Barton. Perturbative Casimir energies of dispersive spheres, cubes and cylinders. *J. Phys. A: Math. Gen.*, 34(19):4083–4114, 2001.

[12] M Bordag. On the interaction of a charge with a thin plasma sheet. *Phys.Rev.D*, 76:065011, 2007.

[13] M Bordag and N Khusnutdinov. On the vacuum energy of a spherical plasma shell. *Phys.Rev.D*, 77:085026, 2008.

[14] M Bordag, U Mohideen and V M Mostepanenko. New developments in the Casimir effect. *Phys. Rep.*, 353:1–205, 2001.

[15] Marco Scandurra. The ground state energy of a massive scalar field in the background of a semi-transparent spherical shell. *J. Phys. A: Math. Gen.*, 32:5679–5691, 1999.

[16] D V Vassilevich. Heat kernel expansion: User's manual. *Phys. Rept.*, 388:279–360, 2003.

[17] M Bordag and K Kirsten. Heat kernel coefficients and divergencies of the Casimir energy for the dispersive sphere. *Int. J. Mod. Phys. A*, 17:813–819, 2002.

[18] Steve Blau, Matt Visser and Andreas Wipf. Zeta Functions and the Casimir Energy. *Nucl. Phys.*, B310:163, 1988.