Research Article

A New Correlation Coefficient for T-Spherical Fuzzy Sets and Its Application in Multicriteria Decision-Making and Pattern Recognition

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The goal of this paper is to design a new correlation coefficient for T-spherical fuzzy sets (TSFSs), which can accurately measure the nature of correlation (i.e., positive and negative) as well as the degree of relationship between TSFS. In order to formulate our proposed idea, we had taken inspiration from the statistical concept of the correlation coefficient. While doing so, we firstly introduce the variance and covariance of two TSFS and then constructed our scheme using these two newly defined notions. The numerical value of our proposed correlation coefficient lies within the interval $[-1,+1]$, as it should be from a statistical point of view, whereas the existing methods cannot measure the negative correlation between TSFS, as their numerical value falls within the interval $[0, 1]$, which is not reasonable both statistically and intuitively. This aspect has also been thoroughly demonstrated using some numerical examples. The comparison results witnessed the dominance and upper hand of our proposed method over the existing definitions, with reliable and better results. In order to demonstrate the feasibility, usefulness, and practical application, we applied our proposed scheme to solve technical and scientific problems of multicriteria decision-making and pattern recognition. The numerical results show that our proposed scheme is practically suitable, technically applicable, and intuitively reasonable.

1. Introduction

The notion of fuzzy sets (FS) was initially proposed by Zadeh [1] to deal with imprecise and vague situations where the membership $M_A(x)$ is assigned to each element [2, 3], and it is a generalization of the crisp set [4]. In crisp set theory, we consider only deterministic and precise situations that could not handle imprecise and vague situations. Alternatively, fuzzy sets provide better solutions for real-life problems. Various researchers have developed certain novel techniques related to fuzzy sets and demonstrated their application in other fields such as [5–14] that are some practical examples. In a fuzzy set, however, only the degree of membership $M_A(x)$ is of relevance to research, and

nonmembership $N_A(x)$ is not an option. In light of this, Atanassov [15] developed the notion of intuitionistic fuzzy sets (IFS), which takes into account both membership and nonmembership functions under a particular condition: $0 \leq M_A(x) + N_A(x) \leq 1$, where $0 \leq M_A(x) \leq 1$ and $0 \leq N_A(x) \leq 1$, and the term $\tau_A(x) = 1 - M_A(x) - N_A(x)$ is called degree of hesitancy. IFS has a wide range of application in many fields, e.g., decision-making [16], logic programming [17], pattern recognition [18–20] medical diagnosis [21], information retrieval [22] and cluster analysis [23], and communication [24–26]. Although the theory of IFS has been successfully applied in different fields, there are some real-life situations where human opinions involve more than two independent statements, like yes, no, abstain, and refusal. Therefore, to deal with these types of situations, Cuong and Kreinovich [27] extended Zadeh FS and Atanassov's IFS into picture fuzzy sets (PFS) with the following condition $(M_A(x) + I_A(x) + N_A(x) \leq 1)$, where $M_A(x)(\in [0,1])$ is the degree of positive membership, $I_A(x)(\in [0,1])$ which represents the degree of neutral membership, and $N_A(x)(\in [0,1])$ is a degree of negative membership. The proposed idea of Cuong and Kreinovich [27] has great importance and application to deal with human opinion efficiently as we have discussed earlier but in some particular scenarios, the sum of membership degrees exceeds 1, i.e., $(M_A(x) + I_A(x) + N_A(x) > 1)$. In these situations, IFS is insufficient to provide a satisfactory result. To address this problem, Mahmood et al. [28] developed the spherical fuzzy set (SFS) by adding new constraints: $(M_A^2(x) + I_A^2(x) + N_A^2(x) \leq 1$ for SFSs) and $(M_A^n(x) + I_A^n(x) + N_A^n(x) \leq 1$ for TSFSs) which is a generalization of FS, IFS, and PFS, which satisfies the sum of membership degrees is less than or equal to 1. The concept of TSFS is a recent development in fuzzy set theory and successfully applied to medical diagnosis and multiattribute decision-making. In addition, the extensions of IFS method are also found in literature abundantly such as fuzzy multisets [29], type-2 fuzzy sets [30], hesitant fuzzy sets [31], and Pythagorean fuzzy sets, [32].

earson proposed the concept of correlation in 1895, and it has since become one of the most widely used indices in statistics. Correlation is a statistical term that describes how two variables are related in a linear pattern. Murthy et al. [33] introduced the concept of fuzzy set correlation for the first time in 1985. The correlation coefficient for IFS was proposed by Gerstenkorn and Manko [34] who introduced the correlation coefficient for IFS. Later on, Hong and Hwang [35] went on to extend intuitionistic fuzzy sets' association in probability space. Mitchell [36] illustrated new formula for intuitionistic fuzzy sets and interpreted two intuitionistic fuzzy sets as ensembles of ordinary fuzzy sets. Several novel expansions, such as [23, 37–41], have also been proposed in the literature. Ullah et al. [42] recently discovered correlation coefficients for T-spherical fuzzy sets, which they used in clustering and multiattribute decision-making. Guleria and Bajaj [43] also presented a similar idea of the correlation coefficient for TSFSs and illustrated its application in pattern recognition and medical diagnosis. These newly developed ideas are applicable in many areas but their proposed method only measures [0, 1] ranging correlations and is unable to deal with negative correlation.

Keeping this drawback in mind, we propose a new correlation coefficient for TSFSs that can easily address the highlighted weaknesses in the existing methods. The numerical value of our proposed correlation coefficient lies within the interval [−1,+1] as it should be from a statistical point of view, whereas the existing methods cannot measure the negative correlation between TSFSs, as their numerical value falls within the interval [0, 1], which is not reasonable both statistically and intuitively.

The organization of this manuscript has been furnished as follows. Section 2 entails basic concepts of IFS, PFS, SFS, and TSFS. It also reviews some existing ideas of the CC of TSFS. Section 3 details our proposed scheme (CC of TSFS), and it also validates and verifies our method with the help of some theorems and propositions. In Section 4, we demonstrate numerical comparisons regarding the performance of our method and the other existing methods to show the strengths and superiority of the proposed idea. Section 5 illustrates the feasibility and usefulness of our idea in the thematic areas of MCDM and PR with the help of some real-world problems. In Section 6, we finalize this manuscript with concluding remarks.

2. Preliminaries

This section has been dedicated to recalling some essential theories, functions, and characteristics of TSFSs as well as the coefficient of correlation to facilitate the understanding of our research study. In the entire paper, we use $M(x)$, $N(x)$, $I(x)$, and $\Lambda(x)$ to represent the degree of membership, nonmembership, abstinence, and refusal in a unit interval [0, 1], respectively.

2.1. Generalizations of Fuzzy Sets

Definition 1. (see [15]). Let F be an intuitionistic fuzzy set (IFS) on a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, and then it can be algebraically defined as $F = \{\langle x, M_F(x), N_F(x)\rangle / x \in X\}$, with a constraint of $0 \leq M_F(x) + N_F(x) \leq 1$, whereas $H = 1 - \{M_F(x) + N_F(x)\}$ is known as the degree of hesitancy for an element $x \in X$ to be belonging to F.

Definition 2. (see [44]). A mathematical expression for a PFS P defined over a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$ can be presented as $J = \{\langle x, M_J(x), I_J(x), N_J(x)\rangle / x \in X\}$, with a constraint of $0 \leq M_J(x) + I_J(x) + N_J(x) \leq 1$, where the degree of refusal for an element $x \in X$ to be in J is represented as $\Lambda(x) = 1 - \{M_J(x) + I_J(x) + N_J(x)\}$.

Definition 3. (see [45]). The standard negation set of the PFS J can be expressed as $N(J) = \{\langle x, N_J(x), \Lambda_J(x), M_J(x)\rangle / x \in X\}$.

Definition 4. (see [28]). Let S be an SFS on a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, and then it can be mathematically defined as $S = \{\langle x, M_S(x), I_S(x), N_S(x)\rangle / x \in X\}$, with a constraint of $0 \leq M_S^2(x) + I_S^2 + N_S^2(x) \leq 1$, while the degree of refusal for an element $x \in X$ to be belonging to S can be expressed as $\Lambda(x) = \sqrt{1 - \{M_S^2(x) + I_S^2 + N_S^2(x)\}}$.

Definition 5. (see [28]). For a TSFS T on a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, its mathematical representation can be expressed as $T = \{\langle x, M_T(x), I_T(x), N_T(x)\rangle / x \in X\}$, with a constraint of $0 \le M_T^n(x) + I_T^n + N_T^n(x) \le 1$. The degree of refusal for an element $x \in X$ to be in TSFS T is defined as $\Lambda_T(x) = \sqrt[n]{1 - \{M_T^n(x) + I_T^n(x) + N_T^n(x)\}}$.

2.2. Existing Methods (CC of TSFSs)

Definition 6. (see [42]). Let S_1 and S_2 be two TSFSs in a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, and the coefficient of correlation between TSFSs S_1 and S_2 is

$$R(S_1, S_2) = \frac{\sum_{i=1}^{m}\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\sqrt{\begin{array}{l}\sum_{i=1}^{m}\left[\left(M_{S_1}^2(x_i)\right)^n + \left(I_{S_1}^2(x_i)\right)^n + \left(N_{S_1}^2(x_i)\right)^n + \left(\Lambda_{S_1}^2(x_i)\right)^n\right] \times \\ \sum_{i=1}^{m}\left[\left(M_{S_2}^2(x_i)\right)^n + \left(I_{S_2}^2(x_i)\right)^n + \left(N_{S_2}^2(x_i)\right)^n + \left(\Lambda_{S_2}^2(x_i)\right)^n\right]\end{array}}}, \tag{1}$$

while the correlation coefficient for weighted TSFSs is

$$R_w(S_1, S_2) = \frac{\sum_{i=1}^{m} w_i\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\sqrt{\begin{array}{l}\sum_{i=1}^{m} w_i\left[\left(M_{S_1}^2(x_i)\right)^n + \left(I_{S_1}^2(x_i)\right)^n + \left(N_{S_1}^2(x_i)\right)^n + \left(\Lambda_{S_1}^2(x_i)\right)^n\right] \times \\ \sum_{i=1}^{m} w_i\left[\left(M_{S_2}^2(x_i)\right)^n + \left(I_{S_2}^2(x_i)\right)^n + \left(N_{S_2}^2(x_i)\right)^n + \left(\Lambda_{S_2}^2(x_i)\right)^n\right]\end{array}}}. \tag{2}$$

Both of the above definitions (1) and (2) satisfy the following features:

(a) $R(S_1, S_2) = R(S_2, S_1)$

(b) $0 \le R(S_1, S_2) \le 1$

(c) $R(S_1, S_2) = 1$, iff $S_1 = S_2$

Definition 7. (see [42]). Another form of correlation coefficient between two TSFSs S_1 and S_2 in $X = \{x_1, x_2, x_3, \cdots, x_m\}$ is

$$R^*(S_1, S_2) = \frac{\sum_{i=1}^{m}\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\max\left\langle\begin{array}{l}\sum_{i=1}^{m}\left[\left(M_{S_1}^2(x_i)\right)^n + \left(I_{S_1}^2(x_i)\right)^n + \left(N_{S_1}^2(x_i)\right)^n + \left(\Lambda_{S_1}^2(x_i)\right)^n\right], \\ \sum_{i=1}^{m}\left[\left(M_{S_2}^2(x_i)\right)^n + \left(I_{S_2}^2(x_i)\right)^n + \left(N_{S_2}^2(x_i)\right)^n + \left(\Lambda_{S_2}^2(x_i)\right)^n\right]\end{array}\right\rangle}, \tag{3}$$

whereas its weighted form can be defined as

$$R_w^*(S_1, S_2) = \frac{\sum_{i=1}^{m} w_i\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\max\left\langle\begin{array}{l}\sum_{i=1}^{m} w_i\left[\left(M_{S_1}^2(x_i)\right)^n + \left(I_{S_1}^2(x_i)\right)^n + \left(N_{S_1}^2(x_i)\right)^n + \left(\Lambda_{S_1}^2(x_i)\right)^n\right], \\ \sum_{i=1}^{m} w_i\left[\left(M_{S_2}^2(x_i)\right)^n + \left(I_{S_2}^2(x_i)\right)^n + \left(N_{S_2}^2(x_i)\right)^n + \left(\Lambda_{S_2}^2(x_i)\right)^n\right]\end{array}\right\rangle}. \tag{4}$$

Equations (3) and (4) also satisfy the following properties:

(a) $R^*(S_1, S_2) = R^*(S_2, S_1)$

(b) $0 \leq R^*(S_1, S_2) \leq 1$

(c) $R^*(S_1, S_2) = 1$, iff $S_1 = S_2$

Definition 8. (see [43]). Coefficient of correlation between TSFSs S_1 and S_2 in $X = \{x_1, x_2, x_3, \cdots, x_m\}$ is defined as

$$K(S_1, S_2) = \frac{\sum_{i=1}^{m}\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\left[\sum_{i=1}^{m}\left(\left(M_{S_1}^n(x_i)\right)^2 + \left(I_{S_1}^n(x_i)\right)^2 + \left(N_{S_1}^n(x_i)\right)^2 + \left(\Lambda_{S_1}^n(x_i)\right)^2\right)\right]^{1/2} \times \left[\sum_{i=1}^{m}\left(\left(M_{S_2}^n(x_i)\right)^2 + \left(I_{S_2}^n(x_i)\right)^2 + \left(N_{S_2}^n(x_i)\right)^2 + \left(\Lambda_{S_2}^n(x_i)\right)^2\right)\right]^{1/2}}, \tag{5}$$

while the correlation coefficient for weighted TSFSs is

$$K_w(S_1, S_2) = \frac{\sum_{i=1}^{m} w_i\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\left[\sum_{i=1}^{m} w_i\left(\left(M_{S_1}^n(x_i)\right)^2 + \left(I_{S_1}^n(x_i)\right)^2 + \left(N_{S_1}^n(x_i)\right)^2 + \left(\Lambda_{S_1}^n(x_i)\right)^2\right)\right]^{1/2} \times \left[\sum_{i=1}^{m} w_i\left(\left(M_{S_2}^n(x_i)\right)^2 + \left(I_{S_2}^n(x_i)\right)^2 + \left(N_{S_2}^n(x_i)\right)^2 + \left(\Lambda_{S_2}^n(x_i)\right)^2\right)\right]^{1/2}}. \tag{6}$$

Definition 9. (see [43]). Another definition of the correlation coefficient between two TSFSs S_1 and S_2 is

$$K^*(S_1, S_2) = \frac{\sum_{i=1}^{m}\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\max\left\langle \begin{array}{l} \sum_{i=1}^{m}\left[\left(M_{S_1}^2(x_i)\right)^n + \left(I_{S_1}^2(x_i)\right)^n + \left(N_{S_1}^2(x_i)\right)^n + \left(\Lambda_{S_1}^2(x_i)\right)^n\right], \\ \sum_{i=1}^{m}\left[\left[\left(M_{S_2}^2(x_i)\right)^n + \left(I_{S_2}^2(x_i)\right)^n + \left(N_{S_2}^2(x_i)\right)^n + \left(\Lambda_{S_2}^2(x_i)\right)^n\right]\right] \end{array} \right\rangle}, \tag{7}$$

whereas its weighted form can be defined as

$$K_w^*(S_1, S_2) = \frac{\sum_{i=1}^{m} w_i\left[M_{S_1}^n(x_i) \times M_{S_2}^n(x_i) + I_{S_1}^n(x_i) \times I_{S_2}^n(x_i) + N_{S_1}^n(x_i) \times N_{S_2}^n(x_i) + \Lambda_{S_1}^n(x_i) \times \Lambda_{S_2}^n(x_i)\right]}{\max\left\langle \begin{array}{l} \sum_{i=1}^{m} w_i\left[\left(M_{S_1}^2(x_i)\right)^n + \left(I_{S_1}^2(x_i)\right)^n + \left(N_{S_1}^2(x_i)\right)^n + \left(\Lambda_{S_1}^2(x_i)\right)^n\right], \\ \sum_{i=1}^{m} w_i\left[\left(M_{S_2}^2(x_i)\right)^n + \left(I_{S_2}^2(x_i)\right)^n + \left(N_{S_2}^2(x_i)\right)^n + \left(\Lambda_{S_2}^2(x_i)\right)^n\right] \end{array} \right\rangle}. \tag{8}$$

Equations (5)–(8) satisfy the following properties:

(a) $K(S_1, S_2) = K(S_2, S_1)$

(b) $0 \leq K(S_1, S_2) \leq 1$

(c) $K(S_1, S_2) = 1$, iff $S_1 = S_2$

3. Proposed New Correlation Coefficient of TSFSs

We propose a new correlation coefficient(s) of TSFSs, including both the weighted and unweighted environments, to overcome the drawbacks of the existing definitions [37, 38]. It has been formulated using a statistical theory of correlation coefficient between two phenomena, that is, the mathematical ratio between the covariance of two TSFSs and the geometric mean of their individual variances. Initially, we introduce the variances and covariance of two TSFSs and use those new definitions to construct our proposed correlation coefficient.

Motivated from [46], we designed the structure of our proposed new correlation coefficient of TSFSs, demonstrated as follows.

Let T_1 and T_2 be two TSFSs $X = \{x_1, x_2, x_3, \cdots, x_m\}$, and we describe two new variables based on the deviations of membership grades as

$$Q_i(T_1) = \left\{ \left\langle M^n_{T_1}(x_i) - \bar{M}^n_{T_1}(x_i) \right\rangle - \left\langle I^n_{T_1}(x_i) - \bar{I}^n_{T_1}(x_i) \right\rangle \right.$$
$$\left. - \left\langle N^n_{T_1}(x_i) - \bar{N}^n_{T_1}(x_i) \right\rangle + \left\langle \Lambda^n_{T_1}(x_i) - \bar{\Lambda}^n_{T_1}(x_i) \right\rangle \right\},$$

$$Q_i(T_2) = \left\{ \left\langle M^n_{T_2}(x_i) - \bar{M}^n_{T_2}(x_i) \right\rangle - \left\langle I^n_{T_2}(x_i) - \bar{I}^n_{T_2}(x_i) \right\rangle \right.$$
$$\left. - \left\langle N^n_{T_2}(x_i) - \bar{N}^n_{T_2}(x_i) \right\rangle + \left\langle \Lambda^n_{T_2}(x_i) - \bar{\Lambda}^n_{T_2}(x_i) \right\rangle \right\}, \forall x_i \in X,$$

$$(9)$$

where

$$\bar{M}^n_{T_1}(x_i) = \frac{1}{m} \sum_{i=1}^m M^n_{T_1}(x_i), \bar{I}^n_{T_1}(x_i) = \frac{1}{m} \sum_{i=1}^m I^n_{T_1}(x_i), \bar{N}^n_{T_1}(x_i)$$
$$= \frac{1}{m} \sum_{i=1}^m N^n_{T_1}(x_i), \bar{\Lambda}^n_{T_1}(x_i) = \frac{1}{m} \sum_{i=1}^m \Lambda^n_{T_1}(x_i),$$

$$\bar{M}^n_{T_2}(x_i) = \frac{1}{m} \sum_{i=1}^m M^n_{T_2}(x_i), \bar{I}^n_{T_2}(x_i) = \frac{1}{m} \sum_{i=1}^m I^n_{T_2}(x_i), \bar{N}^n_{T_2}(x_i)$$
$$= \frac{1}{m} \sum_{i=1}^m N^n_{T_2}(x_i), \text{ and } \bar{\Lambda}^n_{T_2}(x_i) = \frac{1}{m} \sum_{i=1}^m \Lambda^n_{T_2}(x_i).$$

$$(10)$$

Definition 10. For two TSFSs $T_1 = \{\langle x, M^n_{T_1}(x), I^n_{T_1}(x), N^n_{T_1}(x)\rangle / x \in X\}$ and $T_2 = \{\langle x, M^n_{T_2}(x), I^n_{T_2}(x), N^n_{T_2}(x)\rangle / x \in X\}$, variances in terms of new variables can be defined as

$$\mathrm{Var}(T_1) = \frac{1}{m-1} \sum_{i=1}^m Q_i^2(T_1),$$
$$\mathrm{Var}(T_2) = \frac{1}{m-1} \sum_{i=1}^m Q_i^2(T_2),$$

$$(11)$$

while their covariance is mathematically expressed as

$$\mathrm{Cov}(T_1, T_2) = \frac{1}{m-1} \sum_{i=1}^m Q_i(T_1) \times Q_i(T_2). \qquad (12)$$

Proposition 11. *For $Cov(T_1, T_2)$ to be the covariance of two TSFSs T_1 and T_2, it should satisfy the following conditions:*

$$Cov(T_1, T_2) = Cov(T_2, T_1)$$
$$Cov(T_1, T_1) = Var(T_1), \qquad (13)$$
$$|Cov(T_1, T_1)| \leq \langle Var(T_1)\rangle^{1/2} \times \langle Var(T_2)\rangle^{1/2}$$

Proof. (i) and (ii) are straight forward
(iii) Using Cauchy-Schwarz inequality, we have

$$\left\langle \sum_{i=1}^n a_i b_i \right\rangle^2 \leq \left\langle \sum_{i=1}^n a_i^2 \right\rangle \times \left\langle \sum_{i=1}^n b_i^2 \right\rangle \text{ or } \left| \sum_{i=1}^n a_i b_i \right|$$
$$\leq \left\langle \sum_{i=1}^n a_i^2 \right\rangle^{1/2} \times \left\langle \sum_{i=1}^n b_i^2 \right\rangle^{1/2},$$

$$\text{Hence } \langle \mathrm{Cov}(T_1, T_2) \rangle^2 = \left\langle \frac{1}{m-1} \sum_{i=1}^m Q_i(T_1) \times N_i(T_2) \right\rangle^2$$
$$= \frac{1}{(m-1)^2} \left\langle \sum_{i=1}^m Q_i(T_1) \times N_i(T_2) \right\rangle^2,$$
$$\langle \mathrm{Cov}(T_1, T_2)\rangle^2 \leq \frac{1}{(m-1)^2} \left\langle \sum_{i=1}^m Q_i^2(T_1) \times \sum_{i=1}^m Q_i^2(T_2) \right\rangle,$$
$$\langle \mathrm{Cov}(T_1, T_2)\rangle^2 \leq \frac{1}{(m-1)} \left\langle \sum_{i=1}^m Q_i^2(T_1) \right\rangle \times \frac{1}{(m-1)} \left\langle \sum_{i=1}^m Q_i^2(T_2) \right\rangle,$$
$$\langle \mathrm{Cov}(T_1, T_2)\rangle^2 \leq \mathrm{Var}(T_1) \times \mathrm{Var}(T_2),$$
$$|\mathrm{Cov}(T_1, T_2)| \leq \langle \mathrm{Var}(T_1)\rangle^{1/2} \times \langle \mathrm{Var}(T_2)\rangle^{1/2}.$$

$$(14)$$

□

Definition 12. Let T_1 and T_2 be two TSFSs on a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$. The correlation coefficient between T_1 and T_2 is defined as

$$\rho(T_1, T_2) = \frac{\mathrm{Cov}(T_1, T_2)}{\sqrt{\mathrm{Var}(T_1) \times \mathrm{Var}(T_2)}},$$
$$\rho(T_1, T_2) = \frac{\sum_{i=1}^m Q_i(T_1) \times Q_i(T_2)}{\sqrt{\left(\sum_{i=1}^m Q_i^2(T_1)\right) \times \left(\sum_{i=1}^m Q_i^2(T_2)\right)}}.$$

$$(15)$$

Theorem 13. *We have two TSFSs T_1 and T_2, defined over the universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, and then the correlation coefficient between them must fulfill the following conditions:*

(a) $\rho(T_1, T_2) = \rho(T_2, T_1)$

(b) $-1 \le \rho(T_1, T_2) \le +1$

(c) $\rho(T_1, T_2) = 1$, *if* $T_1 = cT_2$ *for any constant* $c > 0$, *where* $T_1 = cT_2$ *means,* $M_{T_1}^n(x) = cM_{T_2}^n(x) \in [0, 1]$, $I_{T_1}^n(x) = cI_{T_2}^n(x) \in [0, 1]$, $N_{T_1}^n(x) = cN_{T_2}^n(x) \in [0, 1]$, *and* $\Lambda_{T_1}^n(x) = c\Lambda_{T_2}^n(x) \in [0, 1]$

(d) $\rho(T_1, T_2) = -1$, *if* $T_1 = cT_2$, *for any constant* $c < 0$

Proof.

(a) It is straightforward

(b) From proposition 11 (14), we know that $|\mathrm{Cov}(T_1, T_2)| \le \langle \mathrm{Var}(T_1) \rangle^{1/2} \times \langle \mathrm{Var}(T_2) \rangle^{1/2}$; hence,

$$-\langle \mathrm{Var}(T_1) \rangle^{1/2} \times \langle \mathrm{Var}(T_2) \rangle^{1/2} \le |\mathrm{Cov}(T_1, T_2)|. \tag{16}$$

By using these two inequalities,

$$-\langle \mathrm{Var}(T_1) \rangle^{1/2} \times \langle \mathrm{Var}(T_2) \rangle^{1/2}$$
$$\le |\mathrm{Cov}(T_1, T_2)|$$
$$\le \langle \mathrm{Var}(T_1) \rangle^{1/2} \times \langle \mathrm{Var}(T_2) \rangle^{1/2},$$

$$-1 \le \frac{|\mathrm{Cov}(T_1, T_2)|}{\langle \mathrm{Var}(T_1) \rangle^{1/2} \times \langle \mathrm{Var}(T_2) \rangle^{1/2}} \tag{17}$$
$$\le 1 - 1 \le \rho(T_1, T_2) \le +1.$$

(c) Since for any constant $c > 0$, $M_{T_1}^n(x) = cM_{T_2}^n(x)$, $I_{T_1}^n(x) = cI_{T_2}^n(x)$, $N_{T_1}^n(x) = cN_{T_2}^n(x)$, and $\Lambda_{T_1}^n(x) = c\Lambda_{T_2}^n(x)$, hence

$$Q_i(T_1) = \left\{ \left\langle M_{T_1}^n(x_i) - \bar{M}_{T_1}^n(x_i) \right\rangle - \left\langle I_{T_1}^n(x_i) - \bar{I}_{T_1}^n(x_i) \right\rangle \right.$$
$$\left. - \left\langle N_{T_1}^n(x_i) - \bar{N}_{T_1}^n(x_i) \right\rangle + \left\langle \Lambda_{T_1}^n(x_i) - \bar{\Lambda}_{T_1}^n(x_i) \right\rangle \right\},$$
$$Q_i(T_1) = \left\{ \left\langle cM_{T_2}^n(x_i) - c\bar{M}_{T_2}^n(x_i) \right\rangle - \left\langle cI_{T_2}^n(x_i) - c\bar{I}_{T_2}^n(x_i) \right\rangle \right.$$
$$\left. - \left\langle cN_{T_2}^n(x_i) - c\bar{N}_{T_2}^n(x_i) \right\rangle + \left\langle c\Lambda_{T_2}^n(x_i) - c\bar{\Lambda}_{T_2}^n(x_i) \right\rangle \right\},$$
$$Q_i(T_1) = c\left\{ \left\langle M_{T_2}^n(x_i) - \bar{M}_{T_2}^n(x_i) \right\rangle - \left\langle I_{T_2}^n(x_i) - \bar{I}_{T_2}^n(x_i) \right\rangle \right.$$
$$\left. - \left\langle N_{T_2}^n(x_i) - \bar{N}_{T_2}^n(x_i) \right\rangle + \left\langle \Lambda_{T_2}^n(x_i) - \bar{\Lambda}_{T_2}^n(x_i) \right\rangle \right\},$$
$$Q_i(T_1) = cQ_i(T_2). \tag{18}$$

Therefore, $\mathrm{Cov}(T_1, T_2) = 1/m - 1\sum_{i=1}^m Q_i(T_1) \times Q_i(T_2)$ $= 1/m - 1\sum_{i=1}^m cQ_i(T_2) \times Q_i(T_2)$.

$$\mathrm{Cov}(T_1, T_2) = \frac{1}{m-1} \sum_{i=1}^m cQ_i^2(T_2), \tag{19}$$

$$\mathrm{Cov}(T_1, T_2) = c\mathrm{Var}(T_2),$$

$$\mathrm{Var}(T_1) = \frac{1}{m-1} \sum_{i=1}^m Q_i^2(T_1) = \frac{1}{m-1} \sum_{i=1}^m c^2 Q_i^2(T_2), \tag{20}$$

$$\mathrm{Var}(T_1) = c^2 \mathrm{Var}(T_2).$$

Hence, for any constant $c > 0$,

$$\rho(T_1, T_2) = \frac{\mathrm{Cov}(T_1, T_2)}{\sqrt{\langle \mathrm{Var}(T_1) \rangle \times \langle \mathrm{Var}(T_2) \rangle}}$$
$$= \frac{k\mathrm{Var}(T_2)}{\sqrt{\langle k^2 \mathrm{Var}(T_2) \rangle \times \langle \mathrm{Var}(T_2) \rangle}}, \tag{21}$$
$$\rho(T_1, T_2) = \frac{k\mathrm{Var}(T_2)}{\sqrt{\langle k\mathrm{Var}(T_2) \rangle^2}} = \frac{k\mathrm{Var}(T_2)}{k\mathrm{Var}(T_2)},$$
$$\rho(T_1, T_2) = 1.$$

(d) Since $A = kB$, so for any constant $c < 0$, $M_{T_1}^n(x) = -cM_{T_2}^n(x)$, $I_{T_1}^n(x) = -cI_{T_2}^n(x)$, $N_{T_1}^n(x) = -cN_{T_2}^n(x)$, and $\Lambda_{T_1}^n(x) = -c\Lambda_{T_2}^n(x)$, therefore,

$$Q_i(T_1) = \left\{ \left\langle M_{T_1}^n(x_i) - \bar{M}_{T_1}^n(x_i) \right\rangle - \left\langle I_{T_1}^n(x_i) - \bar{I}_{T_1}^n(x_i) \right\rangle \right.$$
$$\left. - \left\langle N_{T_1}^n(x_i) - \bar{N}_{T_1}^n(x_i) \right\rangle + \left\langle \Lambda_{T_1}^n(x_i) - \bar{\Lambda}_{T_1}^n(x_i) \right\rangle \right\},$$

$$Q_i(T_1) = \left\{ \left\langle -cM_{T_2}^n(x_i) + c\bar{M}_{T_2}^n(x_i) \right\rangle - \left\langle -cI_{T_2}^n(x_i) + c\bar{I}_{T_2}^n(x_i) \right\rangle \right.$$
$$- \left\langle -cN_{T_2}^n(x_i) + \bar{N}_{T_2}^n(x_i) \right\rangle$$
$$\left. + \left\langle -c\Lambda_{T_2}^n(x_i) + c\bar{\Lambda}_{T_2}^n(x_i) \right\rangle \right\}, Q_i(T_1)$$
$$= \left\{ -c\left\langle M_{T_2}^n(x_i) - \bar{M}_{T_2}^n(x_i) \right\rangle + c\left\langle I_{T_2}^n(x_i) - \bar{I}_{T_2}^n(x_i) \right\rangle \right.$$
$$\left. + c\left\langle N_{T_2}^n(x_i) - \bar{N}_{T_2}^n(x_i) \right\rangle - c\left\langle \Lambda_{T_2}^n(x_i) - \bar{\Lambda}_{T_2}^n(x_i) \right\rangle \right\}, Q_i(T_1)$$
$$= -c\left\{ \left\langle M_{T_2}^n(x_i) - \bar{M}_{T_2}^n(x_i) \right\rangle - \left\langle I_{T_2}^n(x_i) - \bar{I}_{T_2}^n(x_i) \right\rangle \right.$$
$$\left. - \left\langle N_{T_2}^n(x_i) - \bar{N}_{T_2}^n(x_i) \right\rangle + \left\langle \Lambda_{T_2}^n(x_i) - \bar{\Lambda}_{T_2}^n(x_i) \right\rangle \right\}, Q_i(T_1)$$
$$= -cQ_i(T_2). \tag{22}$$

Hence,

$$\text{Cov}(T_1, T_2) = \frac{1}{m-1} \sum_{i=1}^{m} Q_i(T_1) \times Q_i(T_2)$$
$$= \frac{1}{m-1} \sum_{i=1}^{m} -cQ_i(T_2) \times Q_i(T_2), \quad (23)$$
$$\text{Cov}(T_1, T_2) = \frac{1}{m-1} \sum_{i=1}^{m} -cQ_i^2(T_2),$$

$$\text{Cov}(T_1, T_2) = -c\text{Var}(T_2), \text{ and}$$

$$\text{Var}(T_1) = \frac{1}{m-1} \sum_{i=1}^{m} Q_i^2(T_1) = \frac{1}{m-1} \sum_{i=1}^{m} c^2 Q_i^2(T_2),$$
$$\text{Var}(T_1) = c^2 \text{Var}(T_2). \quad (24)$$

So, for $c < 0$,

$$\rho(T_1, T_2) = \frac{\text{Cov}(T_1, T_2)}{\sqrt{\langle \text{Var}(T_1) \rangle \times \langle \text{Var}(T_2) \rangle}} = \frac{-c\text{Var}(T_2)}{\sqrt{\langle c^2\text{Var}(T_2) \rangle \times \langle \text{Var}(T_2) \rangle}},$$
$$\rho(T_1, T_2) = \frac{-c\text{Var}(T_2)}{\sqrt{\langle c\text{Var}(T_2) \rangle^2}} = \frac{-c\text{Var}(T_2)}{c\text{Var}(T_2)},$$
$$\rho(T_1, T_2) = -1. \quad (25)$$

□

3.1. Special Cases

(a) When $n = 2$, equation (15) reduces to the correlation coefficient of SFSs

(b) When $n = 1$, equation (15) reduces to the correlation coefficient of PFSs

(c) When $n = q$ and $I_A(x_i) = I_B(x_i) = 0$, equation (15) reduces to the correlation coefficient of Q-ROFSs

(d) When $n = 2$ and $\Lambda_{T_1}(x_i) = \Lambda_{T_2}(x_i) = 0$, equation (15) reduces to the correlation coefficient of PyFSs

(e) When $n = 1$ and $\Lambda_{T_1}(x_i) = \Lambda_{T_2}(x_i) = 0$, equation (15) reduces to the correlation coefficient of IFSs

We also propose a weighted correlation coefficient between two TSFSs in the following, to deal with the practical problems and situations, with different objects having unequal importance, in which we consider their weights $-\sqrt{Var_w(A)} \times \sqrt{Var_w(B)} \leq |Cov_w(A, B)| \leq \sqrt{Var_w(A)} \times \sqrt{Var_w(B)}$ keeping in view their relative importance.

Definition 14. Suppose we have two TSFSs T_1 and T_2 defined over a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, having a weight vector $w = \{w_1, w_2, w_3, \cdots, w_m\}$ for its elements $x_i \in X$, where $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$, then their weighted correlation coefficient can be defined as

$$\rho_w(T_1, T_2) = \frac{\text{Cov}_w(T_1, T_2)}{\sqrt{\text{Var}_w(T_1) \times \text{Var}_w(T_2)}},$$
$$\rho_w(T_1, T_2) = \frac{\sum_{i=1}^{m} w_i Q_i(T_1) \times Q_i(T_2)}{\sqrt{\left(\sum_{i=1}^{m} w_i Q_i^2(T_1)\right) \times \left(\sum_{i=1}^{m} w_i Q_i^2(T_2)\right)}}. \quad (26)$$

If all of the elements $x_i \in X$ have equal weights (i.e. $w = \{(1/m), (1/m), (1/m), \cdots, (1/m)\}$), then equation (26) can be reduced to equation (15).

From the above definitions, we derive the following proposition and theorem.

Proposition 15. *If we have two SFSs $\rho_w(A, B) = Cov_w(A, B)/\sqrt{Var(A)Var_w(B)} = aVar_w(B)/\sqrt{a^2Var_w(B)Var_w(B)}$ and $\rho_w(A, B) = aVar_w(B)/\sqrt{a^2Var_w(B)^2} = aVar_w(B)/aVar_w(B)$ in a universe of discourse $\rho_w(A, B) = 1$, then their weighted covariance satisfies the following conditions:*

(i) $Cov_w(A, B) = Cov_w(B, A)$

(ii) $Cov_w(A, A) = Var_w(A)$

(iii) $|Cov_w(A, B)| \leq \langle Var_w(A) \rangle^{1/2} \times \langle Var_w(B) \rangle^{1/2}$

Proof. It can be proved like Proposition 11. □

Theorem 16. *For two TSFSs T_1 and T_2 defined over $X = \{x_1, x_2, x_3, \cdots, x_m\}$, then the weighted correlation coefficient between them must satisfy the following properties:*

(a) $\rho_w(T_1, T_2) = \rho_w(T_2, T_1)$

(b) $-1 \leq \rho_w(T_1, T_2) \leq +1$

(c) $\rho_w(T_1, T_2) = 1$, if $T_1 = cT_2$ for any constant $c > 0$

(d) $\rho_w(T_1, T_2) = -1$, if $T_1 = cT_2$, for any constant $c < 0$

Proof. Theorem 16 can also be proved like Theorem 13. □

Definition 17. Let T be a TSFS defined over a universe of discourse $X = \{x_1, x_2, x_3, \cdots, x_m\}$, and then standard negation set for it be characterized as $M_{N(T)}(x_i) = N_T(x_i)$, $I_{N(T)}(x_i) = \Lambda_T(x_i)$ and $N_{N(T)}(x_i) = M_T(x_i)$ for all $\delta F_{\min} = \min_i |F_A(x_i) - F(x_i)|$ and can be mathematically expressed as $N(T) = \{\langle x, N_T(x), \Lambda_T(x), M_T(x) \rangle / x \in X\}$, where $0 \leq M_T^n(x) + I_T^n(x) + N_B^n(x) \leq 1$ and where $\Lambda_T(x) = \sqrt{1 - \{M_T^n(x) + I_T^n(x) + N_T^n(x)\}}$ is the degree of refusal for an element $x \in X$ to be in SFSs T.

4. Comparison with Existing Methods

This section has been dedicated to establishing a full comparative study of the results acquired from our suggested scheme and other current approaches [42, 43] using numerical examples to demonstrate the advantage and novelty of

our concept. Using these examples and numerical results, we also intuitively showed the advantages of our proposed method to address the shortcomings of the existing methods. From a statistical as well as intuitive point of view, if two phenomena or datasets are moving in the same direction, then they have a positive relationship, while if their movement is in the opposite direction, then their relationship would be inverse. The correlation coefficient is an important instrument in statistical theory for determining the degree and nature of a link between two processes or variables.

So, our definitions of correlation coefficient should accurately measure both the degree and the nature of relationship between two TSFSs.

We present some numerical examples in the following to show those characteristics of our proposed scheme and its comparison with existing methods.

Example 18. Let T_1 and T_2 be two TSFSs in a universe of discourse $X = \{x_1, x_2, x_3\}$, characterized as

$$T_1 = \{\langle x_1, 0.9, 0.5, 0.7\rangle, \langle x_2, 0.7, 0.6, 0.8\rangle, \langle x_3, 0.5, 0.8, 0.9\rangle\},$$
$$T_2 = \{\langle x_1, 0.7, 0.6, 0.9\rangle, \langle x_2, 0.8, 0.5, 0.7\rangle, \langle x_3, 0.9, 0.4, 0.6\rangle\}. \tag{27}$$

By using our new definition, the standard negation set of a TSFS T_1 is $SN(T_1) = \{\langle x_1, 0.7, 0., 0.9\rangle, \langle x_2, 0.8, 0., 0.7\rangle, \langle x_3, 0.9, 0., 0.5\rangle\}$.

The following Table 1 represents the comparative results of our proposed method and the existing definitions [37, 38] of the correlation coefficient when applied to the above TSFSs.

It is intuitively justifiable that the relationship between a TSFS and its negation set should have an inverse relation, with a negative correlation coefficient. It can be observed from the above Table 1 that none of the definitions [42, 43] can accurately measure the negative relationship between TSFS T_1 and its standard negation set $SN(T)_1$, rather reflecting a positive relation (0.3798, 0.3505) which cannot be possible, while our proposed correlation coefficient shows that there is a negatively strong correlation (-0.9865) between TSFS T_1 and its standard negation set $SN(T)_1$. Similarly, it can be seen that the corresponding membership grades of TSFSs T_1 and T_2 are moving in an opposite direction over the various point of the universe of discourse $X = \{x_1, x_2, x_3\}$. So, they should have an inverse relationship between them, with a negative correlation coefficient. It can be clearly seen from the above Table 1 that the value of our proposed correlation coefficient (-0.9859) portrays that there exists a strongly negative correlation between TSFSs T_1 and T_2, as it should be statistically as well as intuitively. On the other and the existing methods show a positive correlation (0.4304., 0.4051) between them and overlook their inverse relationship. Hence, it shows the superiority and upper hand of our proposed method over the existing definitions of correlation coefficient for TSFSs.

Example 19. Let $w = \{0.22, 0.33, 0.45\}$ be the weight vector for the elements of TSFSs in Example 18. After applying our proposed weighted correlation coefficient and the exist-

TABLE 1: Coefficient of correlation.

$\rho(T_i, T_j)$	Existing definitions [37, 38]	[37, 38]	Our proposed idea
$\rho(T_1, SN(T_1))$	0.3798	0.3505	-0.9865
$\rho(T_1, T_2)$	0.4304	0.4051	-0.9859

TABLE 2: Weighted correlation coefficient.

$\rho_w(T_i, T_j)$	Existing definitions [42, 43]	[42, 43]	Our proposed definition
$\rho_w(T_1, SN(T_1))$	0.3153	0.2850	-0.9862
$\rho_w(T_1, T_2)$	0.3774	0.3473	-0.9859

TABLE 3: T-spherical fuzzy information of alternatives $A_j (j = 1, 2, 3)$.

Alternatives	F_1	F_2	F_3
A_1	<0.8, 0.5, 0.4>	<0.7, 0.4, 0.4>	<0.3, 0.5, 0.4>
A_2	<0.9, 0.2, 0.4>	<0.6, 0.3, 0.2>	<0.4, 0.1, 0.7>
A_3	<0.5, 0.5, 0.5>	<0.8, 0.2, 0.3>	<0.6, 0.4, 0.3>

TABLE 4: Association between A^* and $A_j (j = 1, 2, 3)$.

$\rho_w(A^*, A_i)$	A_1	A_2	A_3
$\rho_w(A^*, A_i)$	0.9703	0.8307	-0.2530

ing methods for weighted TSFSs [42, 43], we present the comparative performance results in the following Table 2.

It can also be clearly seen from Table 2 that our proposed method accurately measures the negative relationship between TSFSs T_1 and its standard negation set $SN(T_1)$ having a weighted correlation coefficient -0.9862, rationally endorsing the strongly negative correlation between them. Contrarily, the existing definitions [42, 43] show a positive correlation (0.3153, 0.2850) between TSFSs T_1 and its standard negation set $SN(T_1)$, which cannot be logically possible. Moreover, there should be a negative correlation between TSFSs T_1 and T_1 (as explained in example 18). Table 2 shows that our method demonstrates a negative correlation (-0.9859) between T_1 and T_1 while the existing methods cannot measure it accurately, as they show a positive correlation coefficient (0.3774, 0.3473), which cannot be intuitively possible, and these shortcomings reduce the reliability and accuracy of the existing definitions. Furthermore, the competency of our proposed method to address these drawbacks shows its effectiveness and advantages.

5. Application in Multicriteria Decision-Making and Pattern Recognition

This section has been designated to illustrate the application of our proposed idea of the correlation coefficient for TSFSs in

TABLE 5: T-spherical fuzzy pattern information matrix (TSFIM).

Patterns	x_1	x_2	x_3
T_1	<0.42, 0.42, 0.42>	<0.42, 0.42, 0.62>	<0.63, 0.63, 0.72>
T_2	<0.42, 0.42, 0.62>	<0.52, 0.63, 0.82>	<0.72, 0.72, 0.82>
T_3	<0.42, 0.52, 0.63>	<0.52, 0.63, 0.72>	<0.63, 0.72, 0.82>
T	<0.42, 0.42, 0.42>	<0.42, 0.42, 0.52>	<0.63, 0.63, 0.72>

the practical domain of MCDM and pattern recognition to show its suitability and usefulness. Pattern recognition and machine learning are versatile practices that have widely discussed in different disciplines such as [19, 20, 24–26, 47, 48]: these are some examples.

5.1. Application in MCDM. This subsection has been planned to show the application of our proposed scheme in the field of multicriteria decision-making by using a real MCDM problem, taken from [28]. To do so, we present here the implementation procedure of our proposed scheme for dealing with an MCDM problem. Suppose there is a set of "k" alternatives defined as TSFSs $A = \{A_1, A_2, \cdots, A_k\}$ and a set of "m" evaluation features $F = \{F_1, F_2, \cdots, F_m\}$, having weight vector $w = \{w_1, w_2, \cdots, w_m\}^T$, $0 \leq w \leq 1$, and $\sum_{i=1}^{m} w_i = 1$, we define the ideal alternative A^* as a TSFS by taking maximum membership value, minimum nonmembership, and minimum indeterminacy value for each evaluation criteria across all of the given alternatives, like an ideal positive solution in TOPSIS. $A^* = \{(F_i, \max_{1 \leq j \leq k} \langle M_{A_j}(F_i) \rangle, \min_{1 \leq j \leq k} \langle I_{A_j}(F_i) \rangle, \min_{1 \leq j \leq k} \langle N_{A_j}(F_i) \rangle)$ $/F_i \in F\}$. By applying our proposed correlation coefficient, we calculate the relationship between ideal choice A^* and the given alternatives. Furthermore, by using the principle of the maximum correlation coefficient, we select an alternative as the best option, which has the highest correlation coefficient with an ideal alternative A^*.

Example 20. In order to strengthen the sales and purchase department, a firm had advertised the position of manager and selected 3 candidates after conducting an initial screening of the prospective candidates, who had applied. For the selection of the best candidate, the top management of the firm had considered the following three features to evaluate the aforementioned initially short-listed candidates. (i) F_1: communication skills, (ii) F_2: sense of responsibility, and (iii) F_3: creativity. The candidates, evaluation criteria, and their weights have been presented in the form of sets as a set of alternatives, set of evaluation criteria, and weights vector, respectively, as follows.

$A = \{A_1, A_2, A_3\}, F = \{F_1, F_2, F_3\}$, and $w = \{0.20, 0.35, 0.45\}^T$. After a comprehensive evaluation of the candidates, their decision information has been presented in the following Table 3 as a T-spherical fuzzy decision matrix (TSFDM).

It can be clearly seen that the decision information in the above Table 3 is TSFI $n = 3$. So, we apply our proposed scheme to find a correlation coefficient between the ideal candidate $A^* = \{\langle 0.9, 0.2, 0.4 \rangle, \langle 0.8, 0.2, 0.2 \rangle, \langle 0.6, 0.1, 0.3 \rangle\}$

TABLE 6: Relationship between T and $T_j (j = 1, 2, 3)$.

$\rho_w(T, T_j)$	T_1	T_2	T_3
$\rho_w(T, T_j)$	0.9566	0.5185	0.9548

and the initially short-listed three candidates. The results have been furnished in the following Table 4.

It can be clearly seen that the candidate A_1 has the highest degree of relationship with the ideal choice A^* as compared to that of the other alternatives. Hence, using the principle of maximum correlation, we can say that the candidate A_1 is the most suitable choice for the position of manager. This is in complete agreement with the decision, from where this example has been taken.

5.2. Application in Pattern Recognition. In this section, we show how we used our proposed theory to solve a pattern recognition problem and recognize an unknown pattern while taking into account the properties of several well-known patterns. We develop a relationship between the unknown sample pattern P^* and the set of known patterns $P = \{P_1, P_2, \cdots, P_k\}$ using our proposed correlation coefficient on the basis of a set of evaluation criteria. Taking motivation from the idea of the recognition principle [49], we developed an analogy as a degree of belongingness of the unknown pattern to any one of the known patterns to recognize its actual class as $j^* = \arg \max_j \{\rho(P^*, P_j)\}$, where $\rho(P^*, P_j)$ is the degree of relationship between P^* and P_j that can be calculated using our proposed correlation coefficient. More will be the value of j^*, and more will be the closeness P^* to the jth known pattern. A synthetic/fictitious numerical example has been given in the following to demonstrate the application of our method in pattern recognition problems.

Example 21. Let T_1, T_2, and T_3 be three patterns which are known, and T be an unknown sample pattern, defined as follows in the form of TSFSs. These TSFSs have been defined over a space of points $X = \{x_1, x_2, x_3\}$, and weights have been expressed in the form of a weight vector as $w = \{0.31, 0.36, 0.33\}^T$. The characteristic values of the above-mentioned patterns have been furnished in the following Table 5.

By applying our proposed scheme $n = 5$, we measure the relationship between the unknown sample pattern T and the other three known patterns (T_1, T_2, T_3) to identify its

belongingness. The obtained results have been presented in the following Table 4.

Table 6 clearly shows that the unknown sample pattern T has the highest correlation coefficient with the known pattern T_1 as compared to the other known patterns T_2 and T_3. Hence, using the principle of maximum belongingness $j^* = \arg \max_j \{\rho(P^*, P_j)\}$, we can conclude that the unknown sample pattern T belongs to the known pattern T_1.

6. Conclusions

The TSFS is a useful tool for displaying the degree of positive, neutral, and negative membership, as well as the information's dependability owing to rejection membership. Major correlation coefficient approaches based on the unit interval [0, 1] have been defined in the literature. In accordance with the conventional correlation coefficient in statistics, our newly proposed technique correlation coefficient for T-spherical fuzzy sets is the best choice for dealing with positive, negative, and no correlation [-1, 1]. The mathematical derivations for the upper and lower bound of correlation have been given. To show the novelty, we compare our method with existing methods, and the results demonstrate the good aspect of the proposed idea. Furthermore, to demonstrate the feasibility, usefulness, and practical application, we applied our proposed scheme to solve technical and scientific problems of multicriteria decision-making and pattern recognition.

Data Availability

The data used in this research can be obtained from the corresponding authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest to disclose.

Authors' Contributions

Conceptualization was contributed by WA, MA, SH, SSU, and IH. Methodology was contributed by WA, MA, AB, and RA. Validation was contributed by WA, MA, SH, SSU, AB, RA, and IH. Investigation and funding were contributed by RA and AB. Writing the original draft preparation was contributed by WA, MA, SH, SSU, AB, RA, and IH. Writing-review and editing was contributed by FU, MA, IH, SSU, and SH. Visualization was contributed by WA, AB, RA, SSU, and FU. Project administration was contributed by WA, SH, IH, SSU, FU, AB, and RA.

Acknowledgments

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