Semantically informed methods in structural proof theory

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SEMANTICALLY INFORMED STRUCTURAL PROOF THEORY

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SEMANTICALLY INFORMED METHODS IN STRUCTURAL PROOF THEORY

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Contents

Summary

This thesis is part of a line of research aimed at investigating how insights and results from the algebraic and relational semantics of given families of logics can contribute to the design of ‘good’ proof calculi for these logics. It focuses on the intersection between syntax and semantics in structural proof theory.

The results of the present thesis develop the interface between syntax and semantics for proof-theoretic purposes in very specific ways: on the semantic side, these results build on and further develop the link between relational and algebraic semantics of logics given by discrete dualities and their associated unified correspondence results; on the syntactic side, these results further develop the theory of proper multi-type display calculi.

The results of this thesis include: a systematic connection between the syntactic shape of analytic inductive axioms and the generation of cut-free derivations of these axioms from their associated analytic structural rules; the introduction of proper display calculi for monotone modal logic and conditional logic and a large family of their axiomatic extensions, thanks to a reformulation of their neighbourhood semantics in a suitable multi-type relational environment; extending the semantic analysis for monotone modal logic and conditional logic to graded modal logic and proposing a new definition of graded bisimulation.

The results and methodologies form a base for further investigations at the interface of syntax and semantics for logics.

Samenvatting

Dit proefschrift maakt deel uit van een onderzoekslijn die is gericht op het onderzoeken hoe inzichten en resultaten uit de algebraïsche en relationele semantiek van bepaalde families van logica's kunnen bijdragen aan het ontwerp van 'goede' proof calculi voor deze logica's. Het richt zich op de kruising tussen syntaxis en semantiek in de structurele bewijstheorie.

De resultaten van dit proefschrift ontwikkelen de interface tussen syntaxis en semantiek voor bewijstheoretische doeleinden op zeer specifieke manieren: aan de semantische kant bouwen deze resultaten voort op verbanden tussen relationele en algebraïsche semantiek van logica gegeven door discrete dualiteiten en hun bijbehorende uniforme correspondentieresultaten; aan de syntactische kant ontwikkelen deze resultaten de theorie van de juiste multi-type display-calculi.

De resultaten van dit proefschrift omvatten: een systematisch verband tussen de syntactische vorm van analytische inductieve axioma's en het genereren van cut-free afleidingen van deze axioma's van hun bijbehorende analytische structurele regels; de introductie van correcte display-calculi voor monotone modale logica en conditionele logica en een grote familie van hun axiomatische extensies, dankzij een herformulering van hun semantiek van omgevingsstructuren in een geschikte relationele omgeving met meerdere typen; een uitbreiding van de semantische analyse voor monotone modale logica en conditionele logica tot graduele modale logica en het voorstellen van een nieuwe definitie van graduele bisimulatie.

De resultaten en methodologieen vormen de basis voor verder onderzoek naar de verbanden tussen syntaxis en semantiek in de logica.

Chapter 1

Introduction

1.1 Main focus

This thesis focuses on the intersection between syntax and semantics in structural proof theory; specifically, how insights and results from the algebraic and relational semantics of given families of logics can contribute to the design of 'good' proof calculi for these logics.

Techniques in the interface of syntax and semantics have been used in identity of proofs [60, 66, 41, 61, 1], verification of correctness of computer programs (e.g., model checking [25, 24], theorem provers [63, 3]), and counter-model generation [47, 34].

Using semantic-driven ideas for solving proof-theoretic problems has been a very fruitful modus operandi since the 1970s, giving rise to several successful and mutually intersecting research programs and design frameworks; for instance, semantic tableaux [86, 38], labelled deductive systems [45], labelled sequent calculi [44], deep inference [87], nested sequent calculi [14], and algebraic proof theory [7, 8].

Among these developments, those specifically concerning structural proof theory gave rise to very diverse proof-theoretic frameworks, which nonetheless have a recognizable common conceptual underpinning, namely the notion of *analytic calculi*, which helps to crystallize a set of criteria or desiderata for specifying what it means for a calculus to be 'good' in certain respects and for certain purposes.

The results of the present thesis develop the interface between syntax and semantics for proof-theoretic purposes in very specific ways: on the semantic side, these results build on and further develop the link between relational and algebraic semantics of logics given by discrete dualities and their associated *unified correspondence* results; on the syntactic side, these results further develop the theory of *proper multi-type display calculi*. Before further expanding on the contributions of the present thesis, in what follows we will briefly describe each of the relevant notions and theories mentioned above.

1.2 Analytic calculi

Structural proof theory is the branch of proof theory studying the structural properties of derivation systems. It is pioneered by Gentzen's work towards his celebrated consistency result of Peano Arithmetic. The notion of *analytic calculi* is key to structural proof theory, emerging already in Gentzen's work. Rather than a hard and fast definition, this notion is best captured as a set of properties, conditions and desiderata on the design of a given calculus that ensure that the relevant information for deriving a given formula or sequent in that calculus can be 'extracted', as it were, from the formula or sequent itself. In a slogan, analyticity is Wittgenstein's statement that "there can never be surprises in logic" [96, 6.1261; see also 6.1251] interpreted *normatively*, precisely in order to achieve the equivalence between 'process and result', rather than taking it as a starting point. Concrete instantiations of this idea, starting with the work of Gentzen, have led to the design of calculi in which, for any given derivable formula/sequent, at least a derivation of it exists which only involves subformulas of the formula/sequent to be derived. This typically follows from the theorem of *cut elimination*, stating that the cut rule is redundant, plus the subformula property.

1.3 Cut elimination

The cut elimination theorem is the core result about Gentzen's original sequent calculi as it is the base for establishing various properties, including completeness, subformula property, and decidability. However, many interesting logics cannot be captured by the original framework of Gentzen's calculi with cut elimination. The modal logic $S5$[1] (obtained by extending classical normal modal logic with the axioms $\Box\alpha \to \alpha$, $\Box\alpha \to \Box\Box\alpha$ and $\alpha \to \Box\Diamond\alpha$) and Gödel logic (obtained by extending intuitionistic logic with the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$) are perhaps the best known examples.

For $S5$ and Gödel logic, some approximation results are shown: Takano [90] introduces a Gentzen's calculus for $S5$ and proves that every application of cut can be transformed into one the cut formula of which is a subformula of a formula in the endsequent; Avron [4] proposes an analytic calculus for Gödel logic using hypersequents.

To endow a larger class of logics with cut-eliminable calculi, differents formats of Gentzen-style calculi have been introduced (e.g. labelled sequent calculi [44], hypersequent calculi [5], display calculi [4], deep inference [87], nested sequent calculi [14]), in the context of which both semantic and syntactic methods have been developed to prove cut elimination.

Semantic cut elimination is a core result in algebraic proof theory. Techniques for achieving it have been pioneered by Ono and Komori [78], and Okada and Terui [77], who prove cut elimination and decidability for the full Lambek calculus (and other systems) using monoid semantics and phase spaces. Later, Belardinelli, Jipsen, Ono [7] and Wille [95] give algebraic proofs of cut elimination and decidability for FL-algebras and involutive residuated lattices. Ciabattoni, Galatos and Terui [7] introduce a systematic procedure to transform large classes of axioms into equivalent inference rules in sequent and hypersequent calculi. Cut elimination is preserved under the addition of these rules to the Full Lambek calculus plus exchange (in this sense rules that have this property are called *analytic*). Preservation of cut elimination for these calculi is proved by extending

[1] There are cut-free calculi for $S5$ in which the application of cut is restricted (cf. e.g. [12, 69]). And there are cut-free generalized Gentzen (e.g. labelled [44], hypersequent [80] and display [67]) calculi for $S5$.

Okada's semantic proof of cut elimination for some first-order and higher order logics [76]. More recently, Galatos and Jipsen [46] introduce a type of relational semantics for substructural logics called residuated frames, and illustrate how, based on these frames, a uniform treatment can be introduced for achieving semantic proofs of cut elimination and other properties for axiomatic extensions of the full Lambek calculus. Greco et al. [52] extend these techniques to arbitrary signatures of normal LE-logics (i.e. those logics algebraically captured by varieties of normal lattice expansions) and clarify the role of canonical extensions of normal lattice expansions in establishing semantic cut elimination.

Gentzen's original proof of cut elimination is *syntactic*, and done by induction on the complexity of the cut formula and a subinduction on the sum of the heights of the derivations of the two premises (called *cut-height*). When both cut formulas are *principal*, i.e. introduced by logical rules, transformations are introduced that replace the given cut with one or more cuts on the immediate subformula(s) of the given formula, thereby reducing the complexity of the cut formula; when at least one cut formula is not principal, then transformations are introduced which permute the cut upwards over the other rules, thereby reducing the cut-height. When the cut formula is an atomic formula or a constant and the premises of the cut are axioms and hence leaves of the proof tree, the cut is eliminated and the process is guaranteed to terminate.

Compared to semantic methods for proving cut elimination, the syntactic proof is more informative, since it is algorithmic and constructive. However, it is very lengthy and error-prone, as typically tens of subcases need to be considered, and in addition, it is also not modular, in the sense that a new proof has to be provided if adding axioms or rules to a given calculus.

Belnap's framework of *Display Logic* [4] can be understood as an attempt to gain a more systematic understanding of the underlying mechanism of cut elimination. Belnap is inspired by Curry's approach to the proof of cut elimination for a Gentzen-style calculus for first-order logic [33], which abstracts as much as possible from the specific signature and rules. Belnap's calculus is an abstract proof-theoretic framework in which it is possible to express both the fundamental steps of the proof of cut elimination *à la* Gentzen, and the design principles that a calculus should satisfy in order to make this proof work. These design principles take the form of eight conditions, C_1-C_8, on the behaviour of the calculus and its rules. Condition C_1 requires that the subformula property holds for each rule; conditions C_2-C_3 together require the history[2] of a formula in a derivation to have the shape of a tree; condition C_4 requires any formula introduced in precedent (resp. consequent) position to remain in that position throughout the proof; condition C_5 requires formulas to be never introduced in the scope of any connective; conditions C_6 and C_7 require each rule to be closed under simultaneous substitution of arbitrary structures for certain formulas; condition C_8 requires the existence of a strategy for solving the principal[3] stage of the cut elimination process. Belnap's general cut elimination theorem states that cut elimination holds for any given calculus verifying these eight conditions. Wansing [93] refines Belnap's approach in the context of modal logics. Unlike Belnap, who uses the

[2]The notion of history is formalized by the notions of *parameters* and *congruence*. For definitions, we refer to Section 4.1 in [4] or Section 4.1 in [67].

[3]For the definition of being principal, we refer to Section 4.2 of [67].

structural counterparts of conjunction and the constant truth as complex proxies for box, Wansing associates box operators with dedicated unary structural counterparts which are interpreted as backward-looking modal operators ⧫ when occurring in precedent position, and as modal operators □ when occurring in succedent position. In addition, Wansing also replaces Belnap's conditions C_6 and C_7 by the stronger but more transparent requirement that rules be closed under uniform substitution of all structural variables in any position they occur. Frittella et al. [24] generalize the notion of proper display calculi to multi-type calculi and relax the display property in favour of Sambin's *visibility* requirement [83] for the sake of eliminating side conditions in the formulation of some rules corresponding to axioms of dynamic epistemic logic [22] (I will expand on the multi-type calculi in Section 1.6 below). Greco et al. [60] generalize the proof of cut elimination in sequent calculi admitting heterogeneous turnstiles under the assumption that these calculi verifies the property of visibility in the so-called *focused phases* of a derivation and the property of display in the so called *neutral phases* of a derivation.

From the developments mentioned above, a more refined notion of *analyticity* also emerges, which requires cut elimination to be achieved modularly for classes of logics via a metatheorem. In the light of this stronger understanding of analyticity, the semantic-inspired methods turn out to be especially useful.

1.4 Semantic methods in structural proof theory

As we mentioned in the previous section, semantic methods have been used with great success in achieving core results such as cut elimination. This section, without claiming to be exhaustive, briefly reviews the semantic-inspired methods for analytic calculi as they have been developed in the context of *sequent and labelled calculi* [45, 43, 44], *sequent and hypersequent calculi* [7, 38, 39], and *(proper) display calculi* [37, 10, 34].

Negri and von Plato [45] introduce a methodology, sometimes referred to as *axioms-as-rules*, for transforming *universal axioms* in the language of first-order classical (or intuitionistic) logics into analytic sequent rules. As remarked in the same paper, this methodology has a precursor in Negri [42] for the intuitionistic theories of apartness and order. The rules so generated are then used to expand the sequent calculus **G3c** for first-order classical logic. Negri [43] generalizes the axioms-as-rules methodology so as to capture the so-called *geometric implications* in the language of first-order classical logic, i.e. formulas of the form $\forall \overline{z}(A \rightarrow B)$ where A and B are *geometric formulas* (i.e. first-order formulas not containing $\rightarrow$ or $\forall$). Negri [44] applies the axioms-as-rules methodology to capture various axioms in normal modal logic via equivalent analytic labelled-calculi rules over the basic labelled calculus **G3K** for the modal logic K; moreover, following the standard methods as for the **G3**-style sequent calculi, the admissibility of cut, substitution and contraction is established. Although these calculi do not satisfy the full subformula property, decidability is established thanks to their enjoying the so-called *subterm property* (requiring all the terms in minimal derivations to occur in the endsequent) and height-preserving admissibility of contraction.

Ciabattoni et al. [7] define a *hierarchy* (sometimes referred to as *substructural hierarchy*) of classes of substructural formulas, and shows how to translate substructural axioms up to level $\mathcal{N}_2$ of the hierarchy into equivalent rules of a *Gentzen-style sequent calculus*, and axioms up to a subclass of level $\mathcal{P}_3$ into equivalent rules of a *hypersequent calculus*;

the rules so generated are then transformed into equivalent analytic rules whenever they satisfy an additional condition or the base calculus admits weakening; cut elimination is proved via a semantic argument extending Okada's semantic proof [76] to hypersequent calculi (and Ciabattoni et al. [8] generalize this approach to multi-conclusions hypersequents, and a heuristic is proposed to go beyond $\mathcal{P}_3$ axioms). Lahav [38] identifies n-simple formulas, a particularly well-behaved proper subset of geometric formulas in Negri [44] and introduces a method to transform n-simple formulas into equivalent hypersequent rules for a variety of normal modal logics extending the modal logics **K**, **K4**, or **KB**; cut admissibility is proved for n-simple extensions of **K** and **K4**, and decidability (via standard sub-formula property) is established for n-simple extensions of **KB**. Lellmann [39] introduces the format of hypersequent rules with *context restrictions*, and studies the transformations between rules and modal axioms on a classical or intuitionistic base; decidability and complexity results are proved for a variety of modal logics, as well as uniform cut elimination extending the proof in [7]. Ciabattoni et al. [9] study hypersequent calculi capturing analytic extensions of the full Lambek calculus **FLe**, and introduces a procedure for translating structural rules into equivalent formulas in *disjunction form*. This approach is also applied to some normal modal logics on a classical base. The main goal of [9] is to show that cut-free derivations in hypersequent calculi can be transformed into derivations in sequent calculi satisfying various weaker versions of the subformula property which still guarantee decidability (although not necessarily cut elimination). Specifically, in [9, Theorem 12(i)] it is shown how to construct a derivation in hypersequent calculi of formulas in disjunction form which are equivalent to structural rules.

Kracht [37] characterizes the syntactic shape of *primitive axioms* in the language of tense modal logic on a classical base which can be equivalently captured as analytic structural rules extending the minimal display calculus for tense logic. Ciabattoni and Ramanayake [10] provide an analogous characterization in a more general setting for a given but not fixed display calculus, by introducing a procedure for transforming axioms into analytic structural rules and showing the converse direction whenever the calculus satisfies additional conditions.

Greco et al. [34] obtain a characterization, analogous to the one of [10],[4] of the property of being properly displayable for arbitrary normal (D)LE-logics[5] via a systematic connection between proper display calculi and generalized Sahlqvist correspondence theory (aka unified correspondence [12, 30, 15, 21]). Thanks to this connection, general meta-theoretic results are established for properly displayable (D)LE-logics. In particular, properly displayable (D)LE-logics are syntactically characterized the as the logics axiomatised by *analytic inductive axioms*; moreover, it is shown how the same algorithm ALBA which computes the first-order correspondent of (analytic) inductive (D)LE-axioms can be used to effectively compute their corresponding analytic structural rule(s).

[4] See [34, Section 9] for a comparison between the characterizations of [10] and [34].

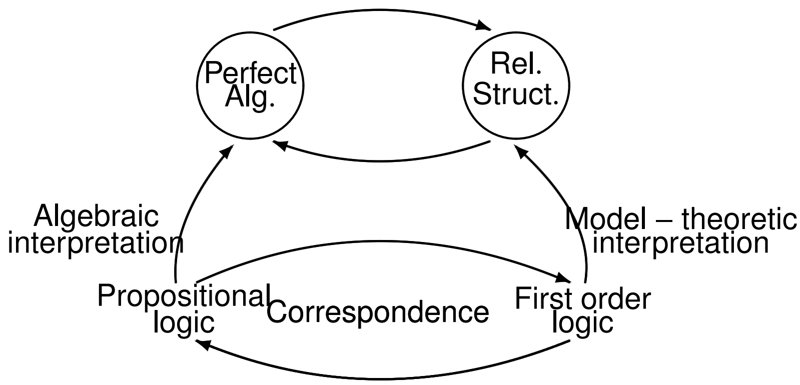
[5] Normal (D)LE-logics are those logics algebraically captured by varieties of normal (distributive) lattice expansions, i.e. (distributive) lattices endowed with additional operations that are finitely join-preserving (resp. meet-reversing) in each positive (resp. negative) coordinate, or are finitely meet-preserving (resp. join-reversing) in each positive (resp. negative) coordinate.

1.5 Unified correspondence theory

Correspondence theory starts as an area of research in the model theory of modal logic which focuses on the relation between modally definable and first-order definable frames. The celebrated *Sahlqvist theory*, originating with Sahlqvist [82] and van Benthem [91], identifies a class of modal formulas, called *Sahlqvist formulas*, each member of which has an effectively computable first-order correspondent (i.e. a first-order formula that defines the class of frames that corresponds to the given modal formula). Sahlqvist-type results have also been achieved in the context of various non-classical logics such as positive modal logic [15], modal relevant logic [85], distributive modal logic [49], modal fixed point logic [9, 92], Lambek calculus [65] and modal substructural logic [89]. Moreover, the original class of Sahlqvist formulas in classical normal modal logic has been properly extended to the class of *inductive formulas* in [51].

The results above have been obtained independently of each other and without an explicit mathematical common ground, and therefore it is very difficult to compare them with each other and with the original result in modal logic. *Unified correspondence* [12] has been introduced precisely in order to provide this common ground, in the form of the recognition that the Sahlqvist mechanism can be analysed in purely algebraic and order-theoretic terms. Accordingly, Sahlqvist and inductive formulas and inequalities have been defined purely in terms of the order-theoretic properties of the algebraic interpretation of the logical connectives, and hence independently of specific signatures. Moreover, an algorithmic and algebraic methodology has been introduced which uses these order-theoretic properties to compute the first-order correspondents of Sahlqvist and inductive axioms across different relational semantic settings [30, 15].

The main insight driving unified correspondence is that the phenomenon of correspondence arises whenever a *discrete duality* exists between some class of *perfect* algebras and some class of set-based (relational) structures. Well-known examples of perfect dualities include those between complete and atomic Boolean algebras and sets [88], perfect distributive lattices and posets [81], perfect lattices and RS-frames [48], Kripke frames and perfect Boolean algebras with operators [43]. Indeed, as shown in the picture below, perfect algebras naturally interpret a suitable propositional language, and the dual set-based relational structures naturally interpret a suitable first-order language. Bridging between the two types of mathematical structures, discrete dualities allow for the translation of validity and satisfaction of formulas from one type to the other. In this way, the phenomenon of correspondence can be regarded as the logical reflection of dualities.



This insight has made it possible to investigate uniformly and systematically the correspondence theory of large families of non-classical logics, which include distributive modal logic [30], normal LE-logics [15], non-normal logics [25, 79], hybrid logic [32], many-valued logics [13] and logics with fixed points [26, 27]. Moreover, stemming from the same methodology, all these results can be compared with one another.

Also, quite surprisingly, the algebraic and algorithmic methodology of unified correspondence theory has facilitated the connection between correspondence phenomena and the theory of proper display calculi. This connection has been seminally observed by Kracht [37] in the classical modal logic setting, and has been developed in [34] in the setting of normal DLE-logics (i.e. those logics that are algebraically captured by varieties of normal distributive lattice expansions, cf. Definition 2.2 in [17]). In particular, in [34], a syntactic characterization is introduced of a proper subclass of inductive axioms, called *analytic inductive axioms* (cf. Definition 2.10 in [17]), which are exactly those that can be equivalently captured by analytic structural rules of a proper display calculus. For each such axiom, these rules can be computed using the same algorithm (referred to as ALBA) that computes the axiom's first-order correspondent.

Let us show by means of an example how ALBA calculates the first-order correspondents and analytic rules of analytic inductive axioms.

The formula $(p \rightarrow (q \vee r)) \rightarrow (p \rightarrow q) \vee (p \rightarrow r)$ is a tautology in classical propositional logic, but is not an intuitionistic tautology. In what follows, we calculate its first-order correspondent on intuitionistic Kripke frames $\mathcal{F} = (W, R)$, where R is a reflexive, transitive and antisymmetric binary relation on W. Seen as an inequality, $p \rightarrow (q \vee r) \leq (p \rightarrow q) \vee (p \rightarrow r)$ is analytic Sahlqvist e.g. for the order type $\varepsilon(p, q, r) = (1, \partial, \partial)$ but it is properly inductive for the order type $\varepsilon'(p, q, r) = (\partial, \partial, \partial)$ and $\Omega = q < p, r < p$. While the order type ε suggests which (positive or negative) occurrences of the propositional variables need to be 'solved for' (we will expand on this below), the dependency order Ω suggests an order in which the variables can be eliminated. In the calculation below, we will solve according to ε and first eliminate q and r, and finally p.

As discussed above, every piece of argument used to prove this correspondence on frames can be translated by duality to complex algebras[6]. We will show how this is done in the case of the example above.

As is well known, complex algebras of intuitionistic Kripke frames (W, R) as above (the elements of which are the R-direct images of subsets of W) can be naturally endowed with the structure of *perfect* (bi-)Heyting algebras, i.e. complete and completely distributive (hence bi-residuated) lattices which are completely join-generated by their completely join-irreducible elements (namely, the R-direct images of singletons, which are also completely join-prime) and also completely meet-generated by their completely meet-irreducible elements (namely, the complements of the R-preimages of singletons, which are also completely meet-prime).

First of all, the condition $\mathcal{F} \Vdash (p \rightarrow (q \vee r)) \rightarrow (p \rightarrow q) \vee (p \rightarrow r)$ translates to the complex algebra $\mathbb{A} = \mathcal{F}^+$ of $\mathcal{F}$ as the inequality $p \rightarrow (q \vee r) \leq (p \rightarrow q) \vee (p \rightarrow r)$ holding in $\mathbb{A}$ for every assignment of p, q, r into $\mathbb{A}$, so this validity clause can be rephrased as follows:

$$\mathbb{A} \models \forall p \forall q \forall r [p \rightarrow (q \vee r) \leq (p \rightarrow q) \vee (p \rightarrow r)], \tag{1.5.1}$$

[6] cf. [11, Definition 5.21].

where the order $\leq$ is interpreted as set inclusion in $\mathbb{A}$. In perfect Heyting algebras, every element is both the join of the completely join-irreducible elements (the set of which is denoted $J^{\infty}(\mathbb{A})$) below it and the meet of the completely meet-irreducible elements (the set of which is denoted $M^{\infty}(\mathbb{A})$) above it[7]. Hence, letting the variables $\mathbf{i}$ and $\mathbf{j}$ range in $J^{\infty}(\mathbb{A})$ and the variables $\mathbf{m}$ and $\mathbf{n}$ range in $M^{\infty}(\mathbb{A})$ (following the literature, we refer to the former variables as *nominals*, and to the latter ones as *co-nominals*), the condition above can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall p \forall q \forall r \Big(\bigvee \{\mathbf{i} \mid \mathbf{i} \leq p \to (q \vee r)\} \leq \bigwedge \{(p \to \mathbf{m}) \vee (p \to \mathbf{n}) \mid q \leq \mathbf{m}, r \leq \mathbf{n}\} \Big).$$

By elementary properties of least upper bounds and greatest lower bounds in posets (cf. [20]), this condition is true if and only if every element in the join is less than or equal to every element in the meet; thus, condition (1.5.1) above can be rewritten as:

$$\mathbb{A} \models \forall pqr \forall \mathbf{imn}[(\mathbf{i} \leq p \to (q \vee r) \ \ \& \ \ q \leq \mathbf{m} \ \ \& \ \ r \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq (p \to \mathbf{m}) \vee (p \to \mathbf{n})], \tag{1.5.2}$$

At this point we are in a position to eliminate q and r simultaneously, and equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall p \forall \mathbf{imn}[\mathbf{i} \leq p \to (\mathbf{m} \vee \mathbf{n}) \Rightarrow \mathbf{i} \leq (p \to \mathbf{m}) \vee (p \to \mathbf{n})], \tag{1.5.3}$$

Let us justify the equivalence between (1.5.2) and (1.5.3): for the direction from top to bottom, fix an interpretation V of the variables $p, \mathbf{i}, \mathbf{m}, \mathbf{n}$ such that the inequality in the antecedent of (1.5.3) is satisfied. To prove that $\mathbf{i} \leq (p \to \mathbf{m}) \vee (p \to \mathbf{n})$ holds under V, it is enough to show that all inequalities in the antecedent of (1.5.2) hold under some $\{q, r\}$-variant V^* of V. It can be easily verified that this condition is met by the $\{q, r\}$–variant V^* of V such that $V^*(q) = \mathbf{m}$ and $V^*(r) = \mathbf{n}$. Conversely, fix an interpretation V of the variables $p, q, r, \mathbf{i}, \mathbf{m}, \mathbf{n}$ such that all inequalities in the antecedent of (1.5.2) are satisfied. Then, because the term function $p \to (q \vee r)$ is monotone in both q and r, the following chain of inequalities holds: $\mathbf{i} \leq p \to (q \vee r) \leq p \to (\mathbf{m} \vee \mathbf{n})$, and hence the inequality in the antecedent of (1.5.3) holds under V, and hence so does $\mathbf{i} \leq (p \to \mathbf{m}) \vee (p \to \mathbf{n})$, as required. This is an instance of the *left Ackermann's lemma* ([2], see also [29]):

> Fix an arbitrary propositional language L. Let $\alpha, \beta(p), \gamma(p)$ be L-formulas such that α is p-free, β is negative and γ is positive in p. For any assignment V on an L-algebra $\mathbb{A}$, the following are equivalent:
>
> 1. $A, V \models \beta(\alpha/p) \leq \gamma(\alpha/p)$;
> 2. $A, V^* \models p \leq \alpha$ and $A, V^* \models \beta(p) \leq \gamma(p)$ for some p-variant V^* of V,
>
> where $\beta(\alpha/p)$ and $\gamma(\alpha/p)$ denote the result of uniformly substituting α for p in β and γ, respectively.

In the lemma above, α is the maximal value (i.e. the 'dually minimal solution') p can take which would satisfy all inequalities in the second clause of the lemma. Left Ackermann's

[7]In perfect BAOs, the completely join-prime elements, the completely join-irreducible elements and the atoms coincide. Moreover, the completely meet-prime elements, the completely meet-irreducible elements and the co-atoms coincide.

lemma is used from bottom to top to eliminate the positive critical occurrences of the variables, i.e. those occurrences of p such that $\varepsilon(p) = 1$. Now we proceed to eliminate the last variable p.

Using again similar arguments as those discussed above, we equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall p \forall \mathbf{imnj}_0\mathbf{j}_1[(\mathbf{i} \leq p \to (\mathbf{m} \vee \mathbf{n}) \ \ \& \ \ \mathbf{j}_0 \leq p \ \ \& \ \ \mathbf{j}_1 \leq p) \Rightarrow \mathbf{i} \leq (\mathbf{j}_0 \to \mathbf{m}) \vee (\mathbf{j}_1 \to \mathbf{n})], \tag{1.5.4}$$

which, by the defining property of the least upper bound (i.e. being the left adjoint to the diagonal map $\Delta : \mathbb{A} \to \mathbb{A} \times \mathbb{A}$ such that $a \mapsto (a, a)$), can be rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{imnj}_0\mathbf{j}_1[(\mathbf{i} \leq p \to (\mathbf{m} \vee \mathbf{n}) \ \ \& \ \ \mathbf{j}_0 \vee \mathbf{j}_1 \leq p) \Rightarrow \mathbf{i} \leq (\mathbf{j}_0 \to \mathbf{m}) \vee (\mathbf{j}_1 \to \mathbf{n})]. \tag{1.5.5}$$

We are again in shape for the application of the following *right Ackermann's lemma*:

> Fix an arbitrary propositional language L. Let $\alpha, \beta(p), \gamma(p)$ be L-formulas such that α is p-free, β is positive and γ is negative in p. For any assignment V on an L-algebra $\mathbb{A}$, the following are equivalent:
>
> 1. $\mathbb{A}, V \models \beta(\alpha/p) \leq \gamma(\alpha/p)$;
> 2. $\mathbb{A}, V^* \models \alpha \leq p$ and $\mathbb{A}, V^* \models \beta(p) \leq \gamma(p)$ for some p-variant V^* of V,
>
> where $\beta(\alpha/p)$ and $\gamma(\alpha/p)$ denote the result of uniformly substituting α for p in β and γ, respectively.

Hence, (1.5.5) is equivalent to the following quasi inequality:

$$\mathbb{A} \models \forall \mathbf{imnj}_0\mathbf{j}_1[\mathbf{i} \leq (\mathbf{j}_0 \vee \mathbf{j}_1) \to (\mathbf{m} \vee \mathbf{n}) \Rightarrow \mathbf{i} \leq (\mathbf{j}_0 \to \mathbf{m}) \vee (\mathbf{j}_1 \to \mathbf{n})], \tag{1.5.6}$$

Taking stock, we have equivalently transformed (1.5.1) into (1.5.6), a condition in which all propositional variables (corresponding to monadic second-order variables) have been eliminated, and all remaining variables range over completely join- and meet-irreducible elements, which, as discussed above, correspond to R-direct images of singletons and complements of R-preimages of singletons of intuitionistic Kripke frames (which we write more concisely as e.g. $x\uparrow$ and $z\downarrow^c$, respectively), which are hence first-order definable. To perform this computation, we have used order-theoretic properties of the complex algebras of intuitionistic frames, namely the fact that these algebras admit special families of join- and meet-generators, and the fact that the interpretations of normal connectives are residuated maps. In fact, to obtain the output (1.5.6) we did not use any property specific to the Heyting algebra setting: this same output could have been obtained in the setting of residuated general lattices. However, we can now translate it into the first-order language of intuitionistic Kripke frames so as to obtain the first-order correspondent of $(p \to (q \vee r)) \to (p \to q) \vee (p \to r)$ on intuitionistic Kripke frames. To facilitate this translation, we first rewrite (1.5.6) as follows:

$$\mathbb{A} \models \forall \mathbf{mnj}_0\mathbf{j}_1[(\mathbf{j}_0 \vee \mathbf{j}_1) \to (\mathbf{m} \vee \mathbf{n}) \leq (\mathbf{j}_0 \to \mathbf{m}) \vee (\mathbf{j}_1 \to \mathbf{n})], \tag{1.5.7}$$

then we replace (co-)nominal variables with their interpretation on complex algebras of intuitionistic Kripke frames:

$$\mathbb{A} \models \forall zz'y_0y_1[(y_0\uparrow \cup y_1\uparrow) \to (z\downarrow^c \cup z'\downarrow^c)) \subseteq (y_0\uparrow \to z\downarrow^c) \cup (y_1\uparrow \to z'\downarrow^c)]. \quad (1.5.8)$$

By set-theoretic manipulation, this is equivalent to

$$\mathbb{A} \models \forall zz'y_0y_1[(y_0\uparrow \cup y_1\uparrow) \to (z\downarrow \cap z'\downarrow)^c \subseteq (y_0\uparrow \to z\downarrow^c) \cup (y_1\uparrow \to z'\downarrow^c)]. \quad (1.5.9)$$

Finally, recalling that, on complex algebras of intuitionistic Kripke frames, $S \to T := (S \cap T^c)\downarrow^c$ for any subsets $S, T \subseteq W$, we can rewrite (1.5.9) as follows:

$$\mathbb{A} \models \forall zz'y_0y_1[((y_0\uparrow \cup y_1\uparrow) \cap (z\downarrow \cap z'\downarrow))\downarrow^c \subseteq (y_0\uparrow \cap z\downarrow)\downarrow^c \cup (y_1\uparrow \cap z'\downarrow)\downarrow^c] \quad (1.5.10)$$

and then, applying set-theoretic manipulations and taking the contrapositive:

$$\mathbb{A} \models \forall zz'y_0y_1[((y_0\uparrow \cap z\downarrow) \cap (y_1\uparrow \cap z'\downarrow))\downarrow \subseteq ((y_0\uparrow \cup y_1\uparrow) \cap (z\downarrow \cap z'\downarrow))\downarrow], \quad (1.5.11)$$

which is equivalent to

$$\mathbb{A} \models \forall zz'y_0y_1[(y_0\uparrow \cap z\downarrow) \cap (y_1\uparrow \cap z'\downarrow) \subseteq ((y_0\uparrow \cup y_1\uparrow) \cap (z\downarrow \cap z'\downarrow))\downarrow]. \quad (1.5.12)$$

Unfolding the abbreviations, the condition above can be rewritten in the first-order frame correspondence language as follows:

$$\begin{aligned}&\forall xzz'y_0y_1\,[\,(y_0Rx \,\&\, xRz \,\&\, y_1Rx \,\&\, xRz')\\ &\qquad\qquad \Rightarrow \exists y(xRy \,\&\, (y_0Ry \;⅋\; y_1Ry) \,\&\, yRz \,\&\, yRz')\,] \quad (1.5.13)\end{aligned}$$

where ⅋ means ‘or’.

Now we show how an analytic rule for $(p \to (q \vee r)) \to (p \to q) \vee (p \to r)$ can also be calculated from the very same ALBA output (1.5.6). Since $\to$ is join-reversing in its first coordinate, and by the defining property of greatest lower bound, (1.5.6) is equivalent to:

$$\begin{aligned}&\mathbb{A} \models \forall \mathbf{imnj}_0\mathbf{j}_1[(\mathbf{i} \leq \mathbf{j}_0 \to (\mathbf{m} \vee \mathbf{n}) \;\&\; \mathbf{i} \leq \mathbf{j}_1 \to (\mathbf{m} \vee \mathbf{n}))\\ &\qquad\qquad \Rightarrow \mathbf{i} \leq (\mathbf{j}_0 \to \mathbf{m}) \vee (\mathbf{j}_1 \to \mathbf{n})]. \quad (1.5.14)\end{aligned}$$

The quasi-inequality above can be read off as the following proper-display-style analytic structural rule:

$$\frac{X \vdash Y_0 \check{\to} (Z \,\check{\vee}\, Z') \qquad X \vdash Y_1 \check{\to} (Z \,\check{\vee}\, Z')}{X \vdash (Y_0 \check{\to} Z) \,\check{\vee}\, (Y_1 \check{\to} Z')}$$

where X, Y_0, Y_1, Z, Z' are structural variables and $\check{\to}$ and $\check{\vee}$ are structural counterparts of $\to$ and $\vee$, respectively.

This rule is analytic for the following reasons: (1) It contains no formulas. Therefore, it satisfies C_1; (2) each of X, Y_0 and Y_1 occurs once both in the upper sequents and the lower sequent. Therefore, it satisfies C_2 and C_3; (3) No formula is introduced in this rule, so it satisfies C_4 and C_5; (4) If we add this rule to the display calculus for modal logic K, the obtained calculus will satisfy C_6 - C_8.

The results discussed above are very powerful and uniform, but they work exactly for logics that are axiomatized by analytic inductive axioms and therefore they are not directly applicable to logics the axiomatization of which violates this specific syntactic shape. This is the case for many well known logics, such as linear logic [30], inquisitive logic [23], semi-De Morgan logic [84], propositional dynamic logic (PDL) [37], dynamic epistemic logic (DEL) [6], bilattice logic [39]. A way to circumvent this limitation is provided by multi-type proper display calculi. These are discussed in the next section.

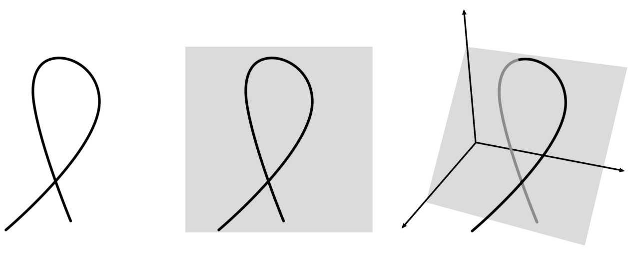
1.6 Proper multi-type display calculi

The systematic connection between unified correspondence theory and proper display calculi has made it possible to endow analytic inductive logics with analytic (specifically: proper display) calculi which are guaranteed by the general theory to satisfy fundamental properties such as soundness, completeness, conservativity, cut-elimination and subformula property. These basic properties form a benchmark of good behaviour that would be desirable to extend also to logics that do not fall into the analytic inductive characterization of [34]. The framework of *proper multi-type display calculi* [70] is specifically designed to achieve this goal, and has been successful with the logics mentioned above, as well as with several others [21, 33, 24, 22, 56, 58, 59, 5, 32, 53].

Although the source of the mathematical difficulties was different for each logic mentioned above, a common core to these difficulties was identified precisely in the encoding of key interactions between entities of different types. For instance, for dynamic epistemic logic the difficulties lay in the interactions between (epistemic) actions, agents' beliefs, and facts of the world; for linear logic, in the interaction between general resources and reusable resources; for propositional dynamic logic, between general and iterative actions; for inquisitive logic, between general formulas (having both a sentential and an inquisitive content) and flat formulas (having only a sentential content). In each case, precisely the formal encoding of these interactions gave rise to non-analytic axioms in the original formulations of the logics.

What sets multi-type calculi apart from other proof-theoretic methodologies is that, in multi-type calculi, entities of different types can coexist and interact on equal ground: each type has its own internal logic (i.e. language and deduction relation), and the interaction between logics of different types is facilitated by special heterogeneous connectives, which are primitive to the language, and which make it possible to express the interactions between entities of different types *within the language* by means of *analytic axioms*. In each case mentioned above, the multi-type approach allowed to redesign the given logic, so as to encode the key interactions into analytic multi-type rules, and define its associated multi-type proper display calculus.

The technical advantage and simplification allowed by the switch from the single-type to the multi-type approach can be described informally with the help of the following pictures, in which the same object (the curve on the left-most picture) is given a two-dimensional representation (the middle picture) in which the curve has a singularity (self-intersection), and a three-dimensional one (the right-most picture) where there is no singularity, since the two arms of the curve lie on different sides of the grey plane. Metaphorically, adding types is analogous to adding dimensions to the analysis of the interactions, thereby making it possible to unravel these interactions, by reformulating them in analytic terms within a richer language.



The multi-type methodology has not only been used to develop analytic reformulations of several important logics, but recently, it has also been used to design novel logical formalisms focusing on agents' abilities and capabilities to manipulate resources, as well as their coordination [5].

More generally and perhaps more fundamentally, multi-type calculi are not just a syntactic bookkeeping device to account for constraints restricting uniform substitution, but they offer a framework which allows to systematically extract proof-theoretic information from the analysis of the relational/algebraic semantics of logical frameworks, and this is exactly how they are used in the present thesis. In particular, all the insights and results of unified correspondence can be applied also in the multi-type setting, and the benchmark results of [34] straightforwardly extend to their multi-type counterparts.

To illustrate how a non analytic axiom can equivalently be encoded in a suitable multi-type language, consider the following axiom of dynamic epistemic logic which encodes the interaction between epistemic actions (denoted by the parameters α and β) and the knowledge of an agent **a** about a statement A.

$$[\alpha][\mathbf{a}]A \;\rightarrow\; \big(Pre(\alpha) \rightarrow \bigwedge\{[\mathbf{a}][\beta]A \mid \alpha\mathbf{a}\beta\}\big)$$

The informal content of this axiom is that, if agent **a** knows fact A whenever epistemic action α has been executed, then, whenever the preconditions for the execution of α (denoted by $Pre(\alpha)$) are verified, agent **a** knows that A is the case after any execution of any epistemic action β which agent **a** cannot distinguish from α (in symbols: $\alpha\mathbf{a}\beta$). This axiom illustrates how logic can illuminate the interactions between entities of different types (in this case between agents and actions in an epistemic setting). However, this axiom is formulated in terms of the extra-logical symbols $Pre(\alpha)$ and $\alpha\mathbf{a}\beta$, which makes it non-analytic. The crucial relations between an action and its preconditions, and between an action and its appearances to an agent, cannot be expressed within the original language of dynamic epistemic logic. However, these relations can be expressed in a suitable multi-type language into which language and axioms of dynamic epistemic logic can be translated. After the translation, the axiom above can be represented by means of the following inequality:

$$\mathsf{a} \triangleright_2 (\alpha \triangleright_0 A) \leq \alpha\triangle\top \rightarrow \mathsf{a} \triangleright ((\mathsf{a}\blacktriangle\alpha) \triangleright A)$$

In this new axiom, all right-pointing triangles are binary *heterogeneous* connectives (i.e., they take arguments in different types) which preserve arbitrary meets in their second coordinate, and reverse arbitrary joins in their first coordinate (i.e., they are all normal). In this language, the information contained in the extra logical symbol $Pre(\alpha)$ can be encoded in the formula-type term $\alpha\triangle\top$, and the set of indices $\alpha\mathbf{a}\beta$ in the action-type term $\mathbf{a}\blacktriangle\alpha$, denoting the weakest action which agent **a** cannot distinguish from α (that is, $\mathbf{a}\blacktriangle\alpha$ has the same interpretation as $\bigvee\{\beta \mid \alpha\mathbf{a}\beta\}$; for full explanation, see [22]). The syntactic shape of this axiom is analytic inductive, in fact Sahlqvist, e.g. for order type $\varepsilon(\mathbf{a}, A, \alpha) = (1, \partial, 1)$, and hence, its corresponding analytic structural rule can be computed using the general ALBA-based methodology.

Summing up, the success of the multi-type methodology in defining analytic calculi for logics as proof-theoretically impervious as dynamic epistemic logic lies in its providing a mathematical environment in which the expressivity problems of the original language can be clearly identified, thus allowing for a suitable re-engineering of the framework itself.

1.7 Contributions

In the light of the results and insights discussed above, we are now in a position to describe and motivate the contributions of the present thesis to the development of semantically informed methods in structural proof theory.

- In Chapter 2, that is a reprint of [17], **we introduce an effective procedure which generates cut-free derivations** in so-called pre-normal form (cf. Definitions 2.24 and 2.25 in Chapter 2) **of any analytic inductive (D)LE-sequent** in the basic proper display calculus for the corresponding language augmented with the analytic structural rule(s) corresponding to the given sequent. This result provides a uniform, constructive, and purely syntactic way of proving the completeness of proper display calculi w.r.t. the logics that they are supposed to capture. So far, in the literature of proper display calculi, this general result has been achieved semantically, and the syntactic route had been implemented only on a case-by-case base. The algorithmic generation of analytic structural rules out of analytic inductive axioms via ALBA has put us in a position to concretely represent the syntactic shape of analytic structural rules, and hence to use this shape in the formulation of the effective procedure for generating the required derivations.

- In Chapter 3, that is a reprint of [16], **we introduce proper multi-type display calculi for basic monotonic modal logic, the conditional logic CK and a number of their axiomatic extensions**. These calculi are sound, complete, conservative, and enjoy cut elimination and subformula property. The design of these calculi is motivated by a semantic analysis, thanks to which neighbourhood frames for monotonic modal logic and conditional logic are represented as multi-type classical Kripke frames. This representation allows to define a syntactic translation from single-type non-normal modal logics to multi-type normal poly-modal logics. The reason why non-normal logics are not amenable to be captured by single-type proper display calculi is that the requirement of the display property would force all connectives to be normal. The semantic analysis precisely allows us to view non-normal connectives as compositions of normal (heterogeneous) connectives, and hence to embed the structural proof theory of non-normal logics into that of multi-type normal logics. Although in this chapter this methodology has been implemented for monotonic modal logic and conditional logic it is applicable more in general to any logics that admit neighbourhood semantics.

- In Chapter 4, that is a reprint of [18], we start exploring the extent to which the techniques developed in Chapter 3 can be applied to graded modal logic. We start from the observation that graded modal logic is a type of monotone modal logic, and accordingly, **we propose a semantic analysis that recasts the standard interpretation of graded modal logic in Kripke frames in the setting of neighbourhood frames**. We show that the ensuing class of neighbourhood frames is not modally definable, and therefore **we propose a new definition of graded bisimulation**, which is an instantiation of monotonic bisimulation.

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Chapter 2

Syntactic completeness of proper display calculi

2.1 Introduction

In recent years, research in structural proof theory has focused on *analytic calculi* [44, 7, 34, 4, 52, 53], understood as those calculi supporting a *robust* form of cut elimination, i.e. one which is preserved by adding rules of a specific shape (the analytic rules)[1]. Important results on analytic calculi have been obtained in the context of various proof-theoretic formalisms: (classes of) axioms have been identified for which equivalent correspondences with analytic rules have been established algorithmically or semi-algorithmically. Without claiming to be exhaustive, we briefly review this strand of research as it has been developed in the context of *sequent and labelled calculi* [47, 51, 45, 43, 44], *sequent and hypersequent calculi* [7, 38, 39], and *(proper) display calculi* [37, 10, 34].

In [45], a methodology is established, sometimes referred to as *axioms-as-rules*, for transforming *universal axioms* in the language of first order classical (or intuitionistic) logics into analytic sequent rules. As remarked in the same paper, this methodology has a precursor in [42] for the intuitionistic theories of apartness and order. The rules so generated are then used to expand the sequent calculus **G3c** for first order classical logic. In [43], the axioms-as-rules methodology is generalized so to capture the class of *geometric implications*, i.e. first order formulas of the form $\forall\overline{z}(A \to B)$ where A and B are *geometric formulas* (i.e. first-order formulas not containing $\to$ or $\forall$). In [44], the axioms-as-rules methodology is applied for capturing various normal modal logic axioms via equivalent analytic rules of labelled calculi over the basic labelled calculus **G3K** for the modal logic K; moreover, the admissibility of cut and of the substitution and contraction rules is shown with the standard methods for the **G3**-style sequent calculi. Although these calculi do not satisfy the full subformula property, decidability is proved thanks to their enjoying a property referred to as the *subterm property* (requiring all the terms in minimal derivations to occur in the endsequent) and height-preserving admissibility of contraction. The class of geometric formulas was first identified and made relevant to proof theory in [47], where natural deduction calculi (with relational assumptions) and labelled sequent calculi are introduced for uniformly capturing intuitionistic modal logics

[1]In broad terms, a derivation is *analytic* when all information needed to carry it out is already contained in its premises and conclusions. How this goal is concretely carried out within specific calculi depends on the format of each given calculus. Relative to the sequent calculi format, inference rules preserving cut elimination can be understood as *analytic rules*.

defined by geometric theories. The book [51] presents other important instances of the axioms-as-rules methodology, in the form of natural deduction calculi (with relational assumptions) and labelled sequent calculi for a large class of non-classical modal logics, defined by (the more restrictive class of) Horn formulas.

In [7], a hierarchy of classes of substructural formulas is defined (sometimes referred to as *substructural hierarchy*). In the same paper it is shown that substructural axioms up to level $\mathcal{N}_2$ of this hierarchy can be algorithmically translated into equivalent rules of a *Gentzen-style sequent calculus*, and axioms up to a subclass of level $\mathcal{P}_3$ into equivalent rules of a *hypersequent calculus*; the rules so generated are then transformed into equivalent analytic rules whenever they satisfy an additional condition or the base calculus admits weakening; cut-admissibility is proved extending the semantic proof of [46] to hypersequent calculi. In [8], this approach is generalized to multi-conclusions hypersequents, and a heuristic is proposed to go beyond $\mathcal{P}_3$ axioms. In [38], *n*-simple formulas are identified as a particularly well-behaved proper subset of geometric formulas [44], and a method is introduced which transforms *n*-simple formulas into equivalent hypersequent rules for a variety of normal modal logics extending the modal logics **K**, **K4**, or **KB**; cut admissibility is proved for *n*-simple extensions of **K** and **K4**, and decidability (via standard subformula property) is established for *n*-simple extensions of **KB**. In [39], the format of hypersequent rules with *context restrictions* is introduced, and transformations are studied between rules and modal axioms on a classical or intuitionistic base; decidability and complexity results are proved for various modal logics, as well as uniform cut elimination extending the proof in [7]. In [9], hypersequent calculi are studied capturing analytic extensions of the full Lambek calculus **FLe**, and a procedure is introduced for translating structural rules into equivalent formulas in *disjunction form*. This approach is also applied to some normal modal logics on classical propositional base. The main goal of [9] is to show that cut-free derivations in hypersequent calculi can be transformed into derivations in sequent calculi satisfying various weaker versions of the subformula property which still guarantee decidability (although not necessarily cut elimination). Specifically, [9, Theorem 12(i)] shows how to construct derivations in hypersequent calculi of formulas in disjunction form which are equivalent to structural rules.

In [37], the syntactic shape of *primitive axioms* in the language of tense modal logic on classical propositional base is characterized as the one which can be equivalently captured as analytic structural rules extending the minimal display calculus for tense logic. In [10], an analogous characterization is provided in a more general setting for a given but not fixed display calculus, by introducing a procedure for transforming axioms into analytic structural rules and showing the converse direction whenever the calculus satisfies additional conditions.

In [34], which is the contribution in the line of research described above to which the results of the present paper most directly connect, a characterization, analogous to the one of [10],[2] of the property of being properly displayable[3] is obtained for arbitrary normal

[2]For a comparison between the characterizations in [10] and in [34], see [34, Section 9].

[3]The adjective ‘proper’ singles out a subclass of Belnap’s display calculi [4] identified by Wansing in [52, Section 4.1]. A display calculus is *proper* if every structural rule is closed under uniform substitution. This requirement strengthens Belnap’s conditions C_6 and C_7. In [24], this requirement is extended to *multi-type* display calculi. A logic is *(properly) displayable* if it can be captured by some (proper) display calculus (see [34, Section 2.2]).

(D)LE-logics[4] via a systematic connection between proper display calculi and generalized Sahlqvist correspondence theory (aka unified correspondence [12, 13, 14, 21]). Thanks to this connection, general meta-theoretic results are established for properly displayable (D)LE-logics. In particular, in [34], the properly displayable (D)LE-logics are syntactically characterized as the logics axiomatised by *analytic inductive axioms* (cf. Definition 2.2.14); moreover, the same algorithm ALBA which computes the first-order correspondent of (analytic) inductive (D)LE-axioms can be used to effectively compute their corresponding analytic structural rule(s). In [3], following [22], residuated families of unary and binary connectives are studied parametrically in group actions on the coordinates of the relations associated with the connectives.

The *semantic* equivalence between each analytic inductive axiom $\varphi \vdash \psi$ and its corresponding analytic structural rule(s) $R_1, \ldots, R_n$, discussed in [34], is an immediate consequence of the soundness of the rules of ALBA on perfect normal (distributive) lattice expansions (cf. Footnote 12). On the *syntactic* side, a description of the derivation, which relies on the proof-theoretic version of Ackermann's Lemma and therefore involves cuts, is presented in [10]. However, an effective procedure was still missing for building *cut-free* derivations of $\varphi \vdash \psi$ in the proper display calculus obtained by adding $R_1, \ldots, R_n$ to the basic proper display calculus D.LE (resp. D.DLE) of the basic normal (D)LE-logic. Such an effective procedure would establish, via syntactic means, that for any properly displayable (D)LE-logic L, the proper display calculus for L—i.e. the calculus obtained by adding the analytic structural rules corresponding to the axioms of L to the basic calculus D.LE (resp. D.DLE)—derives all the theorems (or derivable sequents) of L. This is what we refer to as the *syntactic completeness* of the proper display calculus for L with respect to any analytic (D)LE-logic L. This syntactic completeness result for all properly displayable logics in arbitrary (D)LE-signatures is the main contribution of the present paper. It is perhaps worth emphasizing that we do not just show that any analytic inductive axiom is derivable in its corresponding proper display calculus, but we also provide an algorithm to generate a *cut-free* derivation of a particular shape that we refer to as being *in pre-normal form* (see Section 2.2.6).

Structure of the paper In Section 2.2, we collect the necessary preliminaries about (D)LE-logics, their language, their basic presentation and notational conventions, algebraic semantics, basic proper display calculi, and analytic inductive LE-inequalities. In Section 3.8, we prove a series of technical properties of the basic proper display calculi which will be needed for achieving our main result, which is then proven in Section 2.4. We conclude in Section 2.5.

2.2 Preliminaries

The present section adapts material from [15, Section 2], [34, Section 2], [28, Section 2], and [18, Section 2].

[4]Normal (D)LE-logics are those logics algebraically captured by varieties of normal (distributive) lattice expansions, i.e. (distributive) lattices endowed with additional operations that are finitely join-preserving or meet-reversing in each coordinate, or are finitely meet-preserving or join-reversing in each coordinate.

2.2.1 Basic normal LE-logics and their algebras

Our base language is an unspecified but fixed language $\mathcal{L}_{\mathrm{LE}}$, to be interpreted over lattice expansions of compatible similarity type. This setting uniformly accounts for many well known logical systems, such as the full Lambek calculus and its axiomatic extensions, the full Lambek-Grishin calculus, and other lattice-based logics.

In our treatment, we make use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$ is an n-tuple $\varepsilon \in \{1, \partial\}^n$. For every order-type ε, we denote its *opposite* order-type by ε^∂, that is, $\varepsilon^\partial_i = \varepsilon^\partial(i) = 1$ iff $\varepsilon_i = \varepsilon(i) = \partial$ for every $1 \leq i \leq n$, and $\varepsilon^\partial_i = \varepsilon^\partial(i) = \partial$ iff $\varepsilon_i = \varepsilon(i) = 1$ for every $1 \leq i \leq n$. For any lattice $\mathbb{A}$, we let $\mathbb{A}^1 := \mathbb{A}$ and $\mathbb{A}^\partial$ be the dual lattice, that is, the lattice associated with the converse partial order of $\mathbb{A}$. For any order-type ε, we let $\mathbb{A}^\varepsilon := \Pi_{i=1}^n \mathbb{A}^{\varepsilon_i}$.

The language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ (from now on abbreviated as $\mathcal{L}_{\mathrm{LE}}$) takes as parameters: a denumerable set of proposition letters AtProp, elements of which are denoted p, q, r, possibly with indexes, and disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$.[5] Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) and is associated with some order-type ε_f over n_f (resp. ε_g over n_g). Unary $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) are sometimes denoted $\Diamond$ (resp. $\Box$) if their order-type is 1, and $\lhd$ (resp. $\rhd$) if their order-type is ∂.[6] The terms (formulas) of $\mathcal{L}_{\mathrm{LE}}$ are defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\varphi_1, \ldots, \varphi_{n_f}) \mid g(\varphi_1, \ldots, \varphi_{n_g})$$

where $p \in \mathsf{AtProp}$. Terms in $\mathcal{L}_{\mathrm{LE}}$ are denoted either by s, t, or by lowercase Greek letters such as φ, ψ, γ.

Definition 2.2.1. For any tuple $(\mathcal{F}, \mathcal{G})$ of disjoint sets of function symbols as above, a *lattice expansion* (abbreviated as LE) is a tuple $\mathbb{A} = (\mathbb{L}, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ such that $\mathbb{L}$ is a bounded lattice, $\mathcal{F}^{\mathbb{A}} = \{f^{\mathbb{A}} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^{\mathbb{A}} = \{g^{\mathbb{A}} \mid g \in \mathcal{G}\}$, such that every $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is an n_f-ary (resp. n_g-ary) operation on $\mathbb{A}$. An LE $\mathbb{A}$ is *normal* if every $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) preserves finite – hence also empty – joins (resp. meets) in each coordinate with $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and reverses finite – hence also empty – meets (resp. joins) in each coordinate with $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$).[7] Let $\mathbb{LE}$ be the class of LEs. Sometimes we will refer to certain LEs as $\mathcal{L}_{\mathrm{LE}}$-algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed.

[5] The connectives in $\mathcal{F}$ (resp. $\mathcal{G}$) correspond to those referred to as *positive* (resp. *negative*) connectives in [7]. This terminology is not adopted in the present paper to avoid confusion with positive and negative nodes in signed generation trees, defined later in this section. Our assumption that the sets $\mathcal{F}$ and $\mathcal{G}$ are disjoint is motivated by the desideratum of generality and modularity. Indeed, for instance, the order theoretic properties of Boolean negation $\neg$ guarantee that this connective belongs both to $\mathcal{F}$ and to $\mathcal{G}$. In such cases we prefer to define two copies $\neg_{\mathcal{F}} \in \mathcal{F}$ and $\neg_{\mathcal{G}} \in \mathcal{G}$, and introduce structural rules which encode the fact that these two copies coincide. Another possibility is to admit a nonempty intersection of the sets $\mathcal{F}$ and $\mathcal{G}$. Notice that only unary connectives can be both left and right adjoints. Whenever a connective belongs both to $\mathcal{F}$ and to $\mathcal{G}$, a completely standard solution in the display calculi literature is also available (cf. Remark 2.2.21 and 2.2.22).

[6] The adjoints of the unary connectives $\Box$, $\Diamond$, $\lhd$ and $\rhd$ are denoted $\blacklozenge$, $\blacksquare$, $\blacktriangleleft$ and $\blacktriangleright$, respectively.

[7] Normal LEs are sometimes referred to as *lattices with operators* (LOs). This terminology comes from the setting of Boolean algebras with operators, in which operators are operations which preserve finite joins in each coordinate. However, this terminology is somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as $\mathbb{A}^\varepsilon \to \mathbb{A}^\eta$ for some order-type ε on n and some order-type $\eta \in \{1, \partial\}$.

In the remainder of the paper, we will often simplify notation and write e.g. f for $f^{\mathbb{A}}$, n for n_f and ε_i for $\varepsilon_f(i)$. We also extend the $\{1, \partial\}$-notation to the symbols $\vee, \wedge, \bot, \top, \leq, \vdash$ by stipulating that the superscript ^{1} denotes the identity map, defining

$$\vee^\partial = \wedge, \qquad \wedge^\partial = \vee, \qquad \bot^\partial = \top, \qquad \top^\partial = \bot, \qquad \leq^\partial = \geq,$$

and stipulating that $\varphi \vdash^\partial \psi$ stands for $\psi \vdash \varphi$.

Henceforth, the adjective 'normal' will typically be dropped. The class of all LEs is equational, and can be axiomatized by the usual lattice identities (cf. [20, Theorem 2.9]) and the following equations for any $f \in \mathcal{F}$, $g \in \mathcal{G}$ and $1 \leq i \leq n$:

$$f(p_1 \ldots, q \vee^{\varepsilon_f(i)} r, \ldots p_{n_f}) = f(p_1 \ldots, q, \ldots p_{n_f}) \vee f(p_1 \ldots, r, \ldots p_{n_f})$$
$$f(p_1 \ldots, \bot^{\varepsilon_f(i)}, \ldots p_{n_f}) = \bot,$$
$$g(p_1 \ldots, q \wedge^{\varepsilon_g(i)} r, \ldots p_{n_g}) = g(p_1 \ldots, q, \ldots p_{n_g}) \wedge g(p_1 \ldots, r, \ldots p_{n_g}),$$
$$g(p_1 \ldots, \top^{\varepsilon_g(i)}, \ldots p_{n_g}) = \top.$$

Each language $\mathcal{L}_{\mathrm{LE}}$ is interpreted in the appropriate class of LEs. In particular, for every LE $\mathbb{A}$, each operation $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \to \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \to \mathbb{A}$).

The generic LE-logic is not equivalent to a sentential logic. Hence the consequence relation of these logics cannot be uniformly captured in terms of theorems, but rather in terms of sequents, which motivates the following definition:

Definition 2.2.2. For any language $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, the *basic*, or *minimal* $\mathcal{L}_{\mathrm{LE}}$*-logic* is a set of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in \mathcal{L}_{\mathrm{LE}}$, which contains as axioms the following sequents for lattice operations and additional connectives:

$$\begin{array}{llll} p \vdash p, & \bot \vdash p, & p \vdash \top, & \\ p \vdash p \vee q & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q, \end{array}$$

$$f(p_1 \ldots, q \vee^{\varepsilon_f(i)} r, \ldots p_{n_f}) \vdash f(p_1 \ldots, q, \ldots p_{n_f}) \vee f(p_1 \ldots, r, \ldots p_{n_f}),$$
$$f(p_1, \ldots, \bot^{\varepsilon_f(i)}, \ldots, p_{n_f}) \vdash \bot,$$
$$g(p_1 \ldots, q, \ldots p_{n_g}) \wedge g(p_1 \ldots, r, \ldots p_{n_g}) \vdash g(p_1 \ldots, q \wedge^{\varepsilon_g(i)} r, \ldots p_{n_g}),$$
$$\top \vdash g(p_1, \ldots, \top^{\varepsilon_g(i)}, \ldots, p_{n_g}),$$

and is closed under the following inference rules (note that $\varphi \vdash^\partial \psi$ means $\psi \vdash \varphi$):

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

$$\frac{\varphi \vdash^{\varepsilon_f(i)} \psi}{f(p_1, \ldots, \varphi, \ldots, p_n) \vdash f(p_1, \ldots, \psi, \ldots, p_n)} \quad \frac{\varphi \vdash^{\varepsilon_g(i)} \psi}{g(p_1, \ldots, \varphi, \ldots, p_n) \vdash g(p_1, \ldots, \psi, \ldots, p_n)}.$$

We let $\mathbf{L}_{\mathrm{LE}}$ denote the minimal $\mathcal{L}_{\mathrm{LE}}$-logic. We typically drop reference to the parameters when they are clear from the context. By an LE-*logic* we understand any axiomatic extension of $\mathbf{L}_{\mathrm{LE}}$ in the language $\mathcal{L}_{\mathrm{LE}}$. If all the axioms in the extension are analytic inductive (cf. Definition 2.2.14) we say that the given LE-logic is *analytic*.

A sequent $\varphi \vdash \psi$ is valid in an LE $\mathbb{A}$ if $v(\varphi) \leq v(\psi)$ for every homomorphism v from the $\mathcal{L}_{\mathrm{LE}}$-algebra of formulas over AtProp to $\mathbb{A}$[8]. The notation $\mathbb{LE} \models \varphi \vdash \psi$ indicates that $\varphi \vdash \psi$ is valid in every LE of the appropriate signature. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal LE-logic $\mathbf{L}_{\mathrm{LE}}$ is sound and complete with respect to its corresponding class of algebras $\mathbb{LE}$, i.e. that any sequent $\varphi \vdash \psi$ is provable in $\mathbf{L}_{\mathrm{LE}}$ iff $\mathbb{LE} \models \varphi \vdash \psi$.

2.2.2 The fully residuated language $\mathcal{L}^*_{\mathrm{LE}}$

Any given language $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ can be associated with the language $\mathcal{L}^*_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}^*, \mathcal{G}^*)$, where $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{G}^* \supseteq \mathcal{G}$ are obtained by expanding $\mathcal{L}_{\mathrm{LE}}$ with the following connectives:

1. an n_f-ary connective $f_i^\sharp$ for $1 \leq i \leq n_f$, the intended interpretation of which is the *right residual* of $f \in \mathcal{F}$ in its ith coordinate if $\varepsilon_f(i) = 1$ (resp. its *Galois-adjoint* if $\varepsilon_f(i) = \partial$)[9];
2. an n_g-ary connective $g_i^\flat$ for $1 \leq i \leq n_g$, the intended interpretation of which is the *left residual* of $g \in \mathcal{G}$ in its ith coordinate if $\varepsilon_g(i) = 1$ (resp. its *Galois-adjoint* if $\varepsilon_g(i) = \partial$)[10].

We stipulate that $f_i^\sharp \in \mathcal{G}^*$ if $\varepsilon_f(i) = 1$, and $f_i^\sharp \in \mathcal{F}^*$ if $\varepsilon_f(i) = \partial$. Dually, $g_i^\flat \in \mathcal{F}^*$ if $\varepsilon_g(i) = 1$, and $g_i^\flat \in \mathcal{G}^*$ if $\varepsilon_g(i) = \partial$. The order-type assigned to the additional connectives is predicated on the order-type of their intended interpretations. That is, for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$,

1. if $\varepsilon_f(i) = 1$, then $\varepsilon_{f_i^\sharp}(i) = 1$ and $\varepsilon_{f_i^\sharp}(j) = \varepsilon_f^\partial(j)$ for any $j \neq i$.
2. if $\varepsilon_f(i) = \partial$, then $\varepsilon_{f_i^\sharp}(i) = \partial$ and $\varepsilon_{f_i^\sharp}(j) = \varepsilon_f(j)$ for any $j \neq i$.
3. if $\varepsilon_g(i) = 1$, then $\varepsilon_{g_i^\flat}(i) = 1$ and $\varepsilon_{g_i^\flat}(j) = \varepsilon_g^\partial(j)$ for any $j \neq i$.

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[8] Of course, the restriction of every such homomorphism v to the set of proposition variables is a *variable assignment*, and conversely, as is well known, every variable assignment $v : \mathsf{AtProp} \to \mathbb{A}$ uniquely extends to a homomorphism from the $\mathcal{L}_{\mathrm{LE}}$-algebra of formulas over AtProp to $\mathbb{A}$. In the remainder of this paper, we will abuse notation and use the same symbol to denote both variable assignments and their homomorphic extensions. Also, sometimes we write $\varphi^{\mathbb{A}}$ for $v(\varphi)$ when the interpretation of v is unambiguous.

[9] That is, the interpretation of $f_i^\sharp$ in any (residuated) LE $\mathbb{A}$ is an n_f-ary operation on $\mathbb{A}$ such that, for any $a_1, \ldots, a_{n_f}, b \in \mathbb{A}$,

$$f(a_1, \ldots, a_i, \ldots, a_{n_f}) \leq b \quad \text{iff} \quad a_i \leq f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f}) \quad \text{if } \varepsilon_f(i) = 1$$

$$f(a_1, \ldots, a_i, \ldots, a_{n_f}) \leq b \quad \text{iff} \quad f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f}) \leq a_i \quad \text{if } \varepsilon_f(i) = \partial$$

[10] That is, the interpretation of $g_i^\flat$ in any (residuated) LE $\mathbb{A}$ is an n_g-ary operation on $\mathbb{A}$ such that, for any $a_1, \ldots, a_{n_f}, b \in \mathbb{A}$,

$$b \leq g(a_1, \ldots, a_i, \ldots, a_{n_f}) \quad \text{iff} \quad g_i^\flat(a_1, \ldots, b, \ldots, a_{n_f}) \leq a_i \quad \text{if } \varepsilon_g(i) = 1$$

$$b \leq g(a_1, \ldots, a_i, \ldots, a_{n_f}) \quad \text{iff} \quad a_i \leq g_i^\flat(a_1, \ldots, b, \ldots, a_{n_f}) \quad \text{if } \varepsilon_g(i) = \partial$$

4. if $\varepsilon_g(i) = \partial$, then $\varepsilon_{g_i^\flat}(i) = \partial$ and $\varepsilon_{g_i^\flat}(j) = \varepsilon_g(j)$ for any $j \neq i$.

For instance, if f and g are binary connectives such that $\varepsilon_f = (1, \partial)$ and $\varepsilon_g = (\partial, 1)$, then $\varepsilon_{f_1^\sharp} = (1, 1)$, $\varepsilon_{f_2^\sharp} = (1, \partial)$, $\varepsilon_{g_1^\flat} = (\partial, 1)$ and $\varepsilon_{g_2^\flat} = (1, 1)$.[11]

Definition 2.2.3. For any language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, its associated basic $\mathcal{L}^*_{\mathrm{LE}}$-*logic* is defined by specializing Definition 2.2.2 to the language $\mathcal{L}^*_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}^*, \mathcal{G}^*)$ and closing under the following additional residuation rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$\frac{f(\varphi_1, \ldots, \varphi, \ldots, \varphi_{n_f}) \vdash \psi}{\varphi \vdash^{\varepsilon_f(i)} f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f})} \qquad \frac{\varphi \vdash g(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}{g_i^\flat(\varphi_1, \ldots, \varphi, \ldots, \varphi_{n_g}) \vdash^{\varepsilon_g(i)} \psi}$$

The double line in each rule above indicates that the rule is invertible (i.e., bidirectional). Let $\mathbf{L}^*_{\mathrm{LE}}$ be the basic $\mathcal{L}^*_{\mathrm{LE}}$-logic.

The algebraic semantics of $\mathbf{L}^*_{\mathrm{LE}}$ is given by the class of fully residuated $\mathcal{L}_{\mathrm{LE}}$-algebras, defined as tuples $\mathbb{A} = (\mathbb{L}, \mathcal{F}^*, \mathcal{G}^*)$ such that $\mathbb{L}$ is a lattice and moreover,

1. for every $f \in \mathcal{F}$ with $n_f \geq 1$, all $a_1, \ldots, a_{n_f}, b \in L$ and $1 \leq i \leq n_f$,
$$f(a_1, \ldots, a_i, \ldots, a_{n_f}) \leq b \quad \text{iff} \quad a_i \leq^{\varepsilon_f(i)} f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f}),$$
2. for every $g \in \mathcal{G}$ with $n_g \geq 1$, all $a_1, \ldots, a_{n_g}, b \in L$ and $1 \leq i \leq n_g$,
$$b \leq g(a_1, \ldots, a_i, \ldots, a_{n_g}) \quad \text{iff} \quad g_i^\flat(a_1, \ldots, b, \ldots, a_{n_g}) \leq^{\varepsilon_g(i)} a_i.$$

It is also routine to prove using the Lindenbaum-Tarski construction that $\mathbb{L}^*_{\mathrm{LE}}$ (as well as any of its canonical axiomatic extensions) is sound and complete with respect to the class of fully residuated $\mathcal{L}_{\mathrm{LE}}$-algebras (or a suitably defined equational subclass, respectively).

Theorem 2.2.4. *The logic* $\mathbf{L}^*_{\mathrm{LE}}$ *is a conservative extension of* $\mathbf{L}_{\mathrm{LE}}$*, i.e. every* $\mathcal{L}_{\mathrm{LE}}$*-sequent* $\varphi \vdash \psi$ *is derivable in* $\mathbf{L}_{\mathrm{LE}}$ *if and only if* $\varphi \vdash \psi$ *is derivable in* $\mathbf{L}^*_{\mathrm{LE}}$.

Proof. We only outline the proof. Clearly, every $\mathcal{L}_{\mathrm{LE}}$-sequent which is $\mathbf{L}_{\mathrm{LE}}$-derivable is also $\mathbf{L}^*_{\mathrm{LE}}$-derivable. Conversely, if an $\mathcal{L}_{\mathrm{LE}}$-sequent $\varphi \vdash \psi$ is not $\mathbf{L}_{\mathrm{LE}}$-derivable, then by the completeness of $\mathbf{L}_{\mathrm{LE}}$ with respect to the class of $\mathcal{L}_{\mathrm{LE}}$-algebras, there exists an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ and a variable assignment v under which $\varphi^{\mathbb{A}} \not\leq \psi^{\mathbb{A}}$. Consider the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$.[12] Since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$, the sequent $\varphi \vdash \psi$ is not satisfied in $\mathbb{A}^\delta$ under

[11] Note that this notation depends on the connective which is taken as primitive, and needs to be carefully adapted to well known cases. For instance, consider the 'fusion' connective $\circ$ (which, when denoted as f, is such that $\varepsilon_f = (1, 1)$). Its residuals $f_1^\sharp$ and $f_2^\sharp$ are commonly denoted $/$ and $\backslash$ respectively. However, if $\backslash$ is taken as the primitive connective g, then $g_2^\flat$ is $\circ = f$, and $g_1^\flat(x_1, x_2) := x_2/x_1 = f_1^\sharp(x_2, x_1)$. This example shows that, when identifying $g_1^\flat$ and $f_1^\sharp$, the conventional order of the coordinates is not preserved, and depends on which connective is taken as primitive.

[12] The *canonical extension* of a bounded lattice L is a complete lattice L^δ with L as a sublattice, satisfying *denseness*: every element of L^δ can be expressed both as a join of meets and as a meet of joins of elements from L, and *compactness*: for all $S, T \subseteq L$, if $\bigwedge S \leq \bigvee T$ in L^δ, then $\bigwedge F \leq \bigvee G$ for some finite sets $F \subseteq S$ and $G \subseteq T$. It is well known that the canonical extension of L is unique up to isomorphism fixing L (cf. e.g. [27, Section 2.2]), and that the canonical extension is a *perfect* bounded lattice, i.e. a complete lattice which is completely join-generated by its completely join-irreducible elements and completely meet-generated by its completely meet-irreducible elements (cf. e.g. [27, Definition 2.14]). The canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the perfect $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}^\delta := (L^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ such that $f^{\mathbb{A}^\delta}$ and $g^{\mathbb{A}^\delta}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$ (cf. [48, 49]).

the variable assignment $\iota \circ v$ (ι denoting the canonical embedding $\mathbb{A} \hookrightarrow \mathbb{A}^\delta$). Moreover, since $\mathbb{A}^\delta$ is a perfect $\mathcal{L}_{\mathrm{LE}}$-algebra, it is naturally endowed with a structure of $\mathcal{L}^*_{\mathrm{LE}}$-algebra. Thus, by the completeness of $\mathbf{L}^*_{\mathrm{LE}}$ with respect to the class of $\mathcal{L}^*_{\mathrm{LE}}$-algebras, the sequent $\varphi \vdash \psi$ is not derivable in $\mathbf{L}^*_{\mathrm{LE}}$, as required. □

The algebraic completeness of the logics $\mathbf{L}_{\mathrm{LE}}$ and $\mathbf{L}^*_{\mathrm{LE}}$ and the canonical embedding of LEs into their canonical extensions immediately entail the completeness of $\mathbf{L}_{\mathrm{LE}}$ and $\mathbf{L}^*_{\mathrm{LE}}$ with respect to the appropriate class of perfect LEs.

2.2.3 Analytic inductive LE-inequalities

In this section we recall the definitions of inductive LE-inequalities introduced in [15] and their corresponding 'analytic' restrictions introduced in [34] in the distributive setting and then generalized to the setting of LEs of arbitrary signatures in [28]. Each inequality in any of these classes is canonical and elementary (cf. [15, Theorems 7.1 and 6.1]).

Definition 2.2.5 (Signed generation tree)**.** The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{LE}}$-term s is defined by labelling the root node of the generation tree (i.e. syntax tree) of s with the sign + (resp. −), and then propagating the labelling on each remaining node as follows:

- For any node labelled with $\vee$ or $\wedge$, assign the same sign to its children nodes.
- For any node labelled with $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$, and for any $1 \leq i \leq n_h$, assign the same (resp. the opposite) sign to its ith child node if $\varepsilon_h(i) = 1$ (resp. if $\varepsilon_h(i) = \partial$).

Nodes in signed generation trees are *positive* (resp. *negative*) if they are signed + (resp. −).

Signed generation trees will mostly be used in the context of term inequalities $s \leq t$. In this context, we will typically consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side.[13] We will also say that a term-inequality $s \leq t$ is *uniform* in a given variable p if all occurrences of p in both $+s$ and $-t$ have the same sign, and that $s \leq t$ is ε*-uniform* in a (sub)array $\overline{r}$ of its variables if each $r \in \overline{r}$ such that $\varepsilon(r) = 1$ (resp. $\varepsilon(r) = \partial$) occurs positively (resp. negatively) in $s \leq t$ [14].

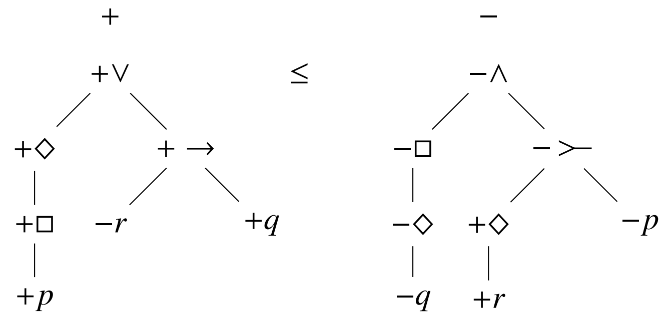
Example 2.2.6. (adapted from [21, Example 2.11]) The language $\mathcal{L}_{\mathrm{LE}}$ of bi-intuitionistic modal logic is obtained by instantiating $\mathcal{F} := \{\Diamond, \succ\!\!-\}$ and $\mathcal{G} := \{\Box, \rightarrow\}$ with $n_\Diamond = n_\Box = 1$, $n_{\succ\!-} = n_\rightarrow = 2$ and $\varepsilon_\Diamond = \varepsilon_\Box = 1$, $\varepsilon_{\succ\!-} = \varepsilon_\rightarrow = (\partial, 1)$. In this language, the signed generation trees associated with the inequality

$$\Diamond\Box p \vee (r \rightarrow q) \leq \Box\Diamond q \wedge (\Diamond r \succ\!\!- p)$$

are represented in the following diagram.

[13] In the context of sequents $s \vdash t$, signed generation trees $+s$ and $-t$ can also be used to specify when subformulas of s (resp. t) occur in precedent or succedent position. Specifically, a given occurrence of formula γ is in *precedent* (resp. *succedent*) position in $s \vdash t$ iff $+\gamma \prec +s$ or $+\gamma \prec -t$ (resp. $-\gamma \prec +s$ or $-\gamma \prec -t$).

[14] If a term inequality $(s \leq t)[\overline{p}/!\overline{x}, \overline{q}]$ is ε-uniform in all variables in $\overline{p}$ (cf. discussion after Definition 3.5.1), then the validity of $s \leq t$ is equivalent to the validity of $(s \leq t)[\overline{\top^{\varepsilon(i)}}/!\overline{x}, \overline{q}]$, where $\top^{\varepsilon(i)} = \top$ if $\varepsilon(i) = 1$ and $\top^{\varepsilon(i)} = \bot$ if $\varepsilon(i) = \partial$.



The inequality above is non-uniform in each variable (since each variable occurs both positively and negatively in it).

For any term $s(p_1, \ldots p_n)$, any order-type ε over n, and any $1 \leq i \leq n$, an *ε-critical node* in a signed generation tree of s is a leaf node $+p_i$ if $\varepsilon(i) = 1$, and a leaf node $-p_i$ if $\varepsilon(i) = \partial$. An *$\varepsilon$-critical branch* in the tree is a branch the leaf of which is an ε-critical node. Variable occurrences corresponding to ε-critical nodes are those used in the runs of the various versions of the algorithm ALBA (cf. [13, 12, 15, 16]) to compute the minimal valuations. For every term $s(p_1, \ldots p_n)$ and every order-type ε, we say that $+s$ (resp. $-s$) *agrees with* ε, and write $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ε-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(\gamma) \prec *s$ (resp. $\varepsilon^\partial(\gamma) \prec *s$) to indicate that the signed subtree γ, with the sign inherited from $*s$, agrees with ε (resp. with ε^∂).

Example 2.2.7. If, for the inequality of the example above, we consider the order-type $\varepsilon(p, q, r) = (1, \partial, \partial)$, the critical nodes in the generation trees pictured above are (from left to right) $+p$, $-r$, and $-q$. Moreover, $+s := +(\Diamond\Box p \vee (r \to q))$ and $-t = -(\Box\Diamond q \wedge (\Diamond r \succ\!\!- p))$ do not agree with either ε^∂ or ε. For the subterms $t' := \Diamond r$ and $t'' := \Diamond r \succ\!\!- p$ of t, we have $+t' \prec -t$ and $-t'' \prec -t$. Finally, $\varepsilon^\partial(t') \prec -t$ and $\varepsilon^\partial(t'') \prec -t$.

Notation 2.2.8. *In what follows, we will often need to use* placeholder variables *to e.g. specify the occurrence of a subformula within a given formula. In these cases, we will write e.g. $\varphi(!z)$ (resp. $\varphi(!\overline{z})$) to indicate that the variable z (resp. each variable z in vector $\overline{z}$) occurs exactly once in φ. Accordingly, we will write $\varphi[\gamma/!z]$ (resp. $\varphi[\overline{\gamma}/!\overline{z}]$ to indicate the formula obtained from φ by substituting γ (resp. each formula γ in $\overline{\gamma}$) for the unique occurrence of (its corresponding variable) z in φ. Also, in what follows, we will find it sometimes useful to group placeholder variables together according to certain assumptions we make about them. So, for instance, we will sometimes write e.g. $\varphi(!\overline{x}, !\overline{y})$ to indicate that $\varepsilon(x) \prec *\varphi$ for all variables x in $\overline{x}$ and $\varepsilon^\partial(y) \prec *\varphi$ for all variables y in $\overline{y}$, or we will write e.g. $f(!\overline{x}, !\overline{y})$ to indicate that f is monotone (resp. antitone) in the coordinates corresponding to every variable x in $\overline{x}$ (resp. y in $\overline{y}$). We will provide further explanations as to the intended meaning of these groupings whenever required. Finally, we will also extend these conventions to inequalities or sequents, and thus write e.g. $(\phi \leq \psi)[\overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$ to indicate the inequality obtained from $\varphi \leq \psi$ by substituting each formula γ in $\overline{\gamma}$ (resp. δ in $\overline{\delta}$) for the unique occurrence of its corresponding variable z (resp. w) in $\varphi \leq \psi$.*

Definition 2.2.9. Non-leaf nodes in signed generation trees are called *Δ-adjoints*, *syntactically left residuals (SLR)*, *syntactically right residuals (SRR)*, and *syntactically right* *adjoints (SRA)*, according to the specification given in Table 3.1. Nodes that are either classified as Δ-adjoints or SLR are collectively referred to as *Skeleton-nodes*, while SRA- and SRR-nodes are referred to as *PIA-nodes*. A branch in a signed generation tree $*s$, with $* \in \{+,-\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes. A good branch is *Skeleton* if the length of P_1 is 0, and is *SLR*, or *definite*, if P_2 only contains SLR nodes.

Skeleton	PIA
Δ-adjoints	Syntactically Right Adjoint (SRA)
$+ \quad \vee$	$+ \quad \wedge \quad g \quad \text{with } n_g = 1$
$- \quad \wedge$	$- \quad \vee \quad f \quad \text{with } n_f = 1$
Syntactically Left Residual (SLR)	Syntactically Right Residual (SRR)
$+ \quad f \quad \text{with } n_f \geq 1$	$+ \quad g \quad \text{with } n_g \geq 2$
$- \quad g \quad \text{with } n_g \geq 1$	$- \quad f \quad \text{with } n_f \geq 2$

Table 2.1: Skeleton and PIA nodes for LE-languages.

Remark 2.2.10. (cf. [15, Remark 3.3]) The classification above follows the general principles of unified correspondence as discussed in [12, Section 1.7.2]. As the names suggest, the subclassification of nodes as SLR, SRR, SRA and Δ-adjoints refers to the inherent order-theoretic properties of the operations interpreting these connectives, whereas the grouping of these classifications into Skeleton and PIA[15] obeys a functional rationale. Indeed, as discussed more in detail in [15], the ALBA reduction strategy involves roughly two tasks, namely approximation (with which the information about the premises of the rules corresponding to a given axiom is isolated from the information about the conclusion of the rule) and display (with which the critical occurrences of propositional variables are brought in display, ready to be eliminated). The order-theoretic properties of Skeleton nodes facilitate approximation, while those of PIA nodes facilitate display. In [13], following [27], the nodes of the signed generation trees were classified according to the *choice* and *universal* terminology. The reader is referred to [12, Section 1.7.2] for an expanded comparison of these two approaches. The convention of considering the positive generation tree of the left-hand side and the negative generation tree of the right-hand side of an inequality comes also from [27].

Definition 2.2.11 (Inductive inequalities)**.** For any order-type ε and any irreflexive and transitive relation (i.e. strict partial order) Ω on $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{-,+\}$) of a term $s(p_1, \ldots p_n)$ is (Ω, ε)*-inductive* if

1. for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good (cf. Definition 3.5.2);

[15]The acronym PIA stands for "Positive Implies Atomic", and was introduced in [50]. The salient property of PIA-formulas is the intersection property, which means that, as term functions, they preserve certain meets.

2. every m-ary SRR-node occurring in the critical branch is of the form

$$\circledast(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1} \ldots, \gamma_m),$$

where for any $\ell \in \{1, \ldots, m\} \setminus \{j\}$,

(a) $\varepsilon^\partial(\gamma_\ell) \prec *s$ (cf. discussion before Definition 3.5.2), and

(b) $p_k <_\Omega p_i$ for every p_k occurring in γ_ℓ and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω, ε)-*inductive* if the signed generation trees $+s$ and $-t$ are (Ω, ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε)-inductive for some Ω and ε.

Figure 2.1 provides a visual representation of the shape of inductive inequalities, where the dashed triangles correspond to non critical subtrees where the branches are not necessarily good.

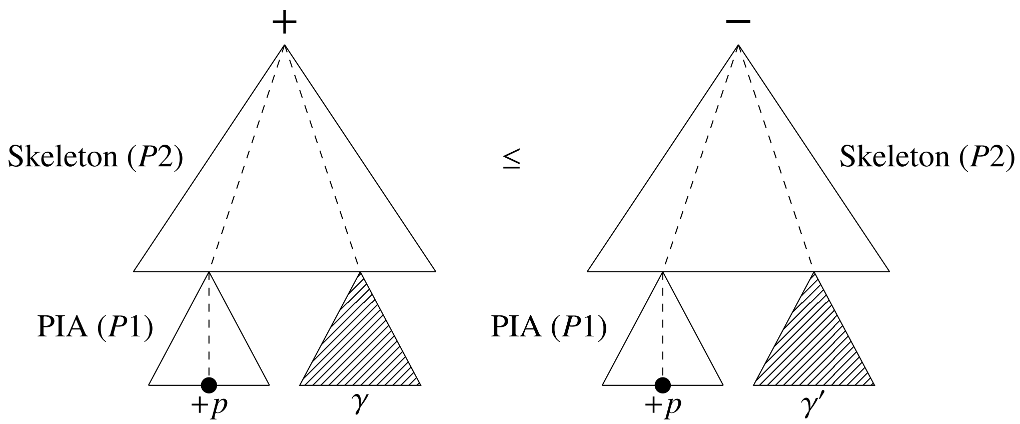
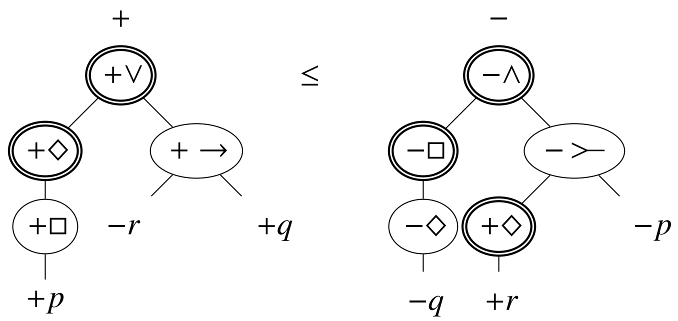


Figure 2.1: The shape of inductive inequalities

Example 2.2.12. Let us concretely illustrate the definitions above by applying them to the inequality of Example 2.2.6. In the following diagram, Skeleton (resp. PIA) nodes occur within a double (resp. single) circle.



The branches ending in $+p$, $-r$, $+q$, $-q$ and $-p$ are good since, traversing each of these branches starting from its leaf, we first encounter PIA nodes, and then only Skeleton nodes. The branch which ends in $+r$ is not good, since (this time traversing it from the root) the Skeleton node $+\Diamond$ occurs in the scope of the PIA node $-\succ\!\!-$. This inequality is (Ω, ε)-inductive for the order-type $\varepsilon(p, q, r) = (1, \partial, \partial)$ and $r <_\Omega q$, and also for the order-type $\varepsilon(p, q, r) = (\partial, \partial, \partial)$ and $q <_\Omega r <_\Omega p$. Notice that, since the positive occurrence of r is the leaf of a branch which is not good, all order-types relative to which the inequality above is inductive must declare the *negative* occurrence of r to be critical. However, $-r$ is the leaf of a branch which traverses the first coordinate of the SRR node $+\rightarrow$, and the subtree corresponding to the second coordinate of that node only consists of the positive occurrence of q. Hence, the minimal valuation of r will depend on that of q, and therefore, for the process of elimination of propositional variables to be well-founded, q needs to be eliminated *before* eliminating r, and independently from r. This poses another restriction on the possible order-types ε and dependency orders Ω; namely, any tuple (Ω, ε) such that the inequality above is (Ω, ε)-inductive must be such that $\varepsilon(r, q) = (\partial, \partial)$ and $q <_\Omega r$. Therefore, the two order-types mentioned above are the only ones for which the inequality above is inductive.

In what follows, we refer to formulas φ such that only PIA nodes occur in $+\varphi$ (resp. $-\varphi$) as *positive* (resp. *negative*) *PIA formulas*, and to formulas ξ such that only Skeleton nodes occur in $+\xi$ (resp. $-\xi$) as *positive* (resp. *negative*) *Skeleton formulas*. It immediately follows from the definitions involved (cf. Table 2.2) that positive (resp. negative) PIA formulas coincide with negative (resp. positive) Skeleton formulas. We also refer to positive and negative PIA (resp. Skeleton) formulas collectively as 'PIA formulas' (resp. 'Skeleton formulas'), for which, the corresponding (positive or negative) generation tree needs to be considered. PIA formulas φ in which no binary SRA-nodes (i.e. $+\wedge$ and $-\vee$) occur in the relevant signed generation tree $*\varphi$ are referred to as *definite*. Skeleton formulas ξ in which no Δ-adjoint nodes (i.e. $-\wedge$ and $+\vee$) occur in $*\xi$ are referred to as *definite*. Hence, ξ (resp. φ) is a definite Skeleton (resp. definite PIA) formula iff all nodes of $*\xi$ (resp. $*\varphi$) are SLR (resp. SRR or unary SRA). The specific order-theoretic properties of definite Skeleton and PIA formulas entail that these are exactly the formulas which can be fully captured at the structural level in display calculi (cf. Remark 2.2.10 and references therein, see also [34]). The following lemma facilitates the connection between Skeleton and PIA formulas and their encoding as structural terms in the language of display calculi.

Lemma 2.2.13. *For every LE-language $\mathcal{L}$,*

1. *if γ is a positive PIA (i.e. negative Skeleton) $\mathcal{L}$-formula, then γ is equivalent to $\bigwedge_{i\in I} \gamma_i$ for some finite set of definite positive PIA (i.e. negative Skeleton) formulas γ_i;*

2. *if δ is a negative PIA (i.e. positive Skeleton) $\mathcal{L}$-formula, then δ is equivalent to $\bigvee_{j\in j} \delta_j$ for some finite set of definite negative PIA (i.e. positive Skeleton) formulas δ_j.*

Proof. By simultaneous induction on γ and δ. The base cases are immediately true. If $\delta := f(\overline{\delta'}, \overline{\gamma'})$, then by the induction hypothesis on each δ' in $\overline{\delta'}$ and each γ' in $\overline{\gamma'}$, the formula δ is equivalent to $f(\overline{\bigvee_{j\in J} \delta'_j}, \overline{\bigwedge_{i\in I} \gamma'_i})$ for some finite sets of definite positive PIA (resp. negative Skeleton) formulas γ'_i and of definite positive Skeleton (resp. negative PIA) formulas δ'_j. By the coordinatewise distribution properties of every $f \in \mathcal{F}$, the term $f(\overline{\bigvee_{j\in J} \delta'_j}, \overline{\bigwedge_{i\in I} \gamma'_i})$ is equivalent to $\bigvee_{j\in J} \bigvee_{i\in I} f(\overline{\delta'_j}, \overline{\gamma'_i})$ with each $f(\overline{\delta'_j}, \overline{\gamma'_i})$ being a definite positive Skeleton (resp. negative PIA) formula, as required. The remaining cases are omitted. □

Definition 2.2.14 (Analytic inductive inequalities)**.** For every order-type ε and every irreflexive and transitive relation Ω on the variables $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{+, -\}$) of a term $s(p_1, \ldots p_n)$ is *analytic* (Ω, ε)*-inductive* if

1. $*s$ is (Ω, ε)-inductive (cf. Definition 3.5.3);

2. every branch of $*s$ is good (cf. Definition 3.5.2).

An inequality $s \leq t$ is *analytic* (Ω, ε)*-inductive* if $+s$ and $-t$ are both analytic (Ω, ε)-inductive. An inequality $s \leq t$ is *analytic inductive* if is analytic (Ω, ε)-inductive for some Ω and ε. An analytic inductive inequality is *definite* if no Δ-adjoint nodes (i.e. $-\wedge$ and $+\vee$) occur in its Skeleton.

Figure 2.2 provides a visual representation of the shape of analytic inductive inequalities, where all branches (even non-critical) have to be good.

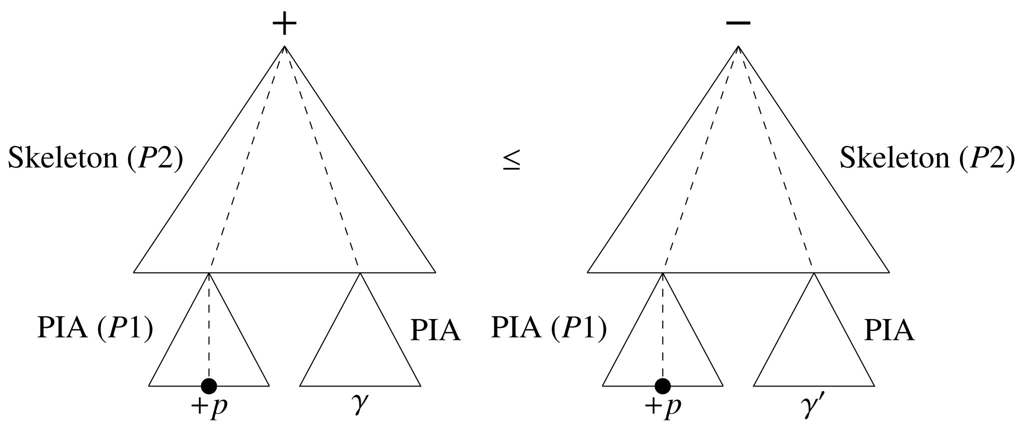


Figure 2.2: The shape of analytic inductive inequalities

Notation 2.2.15. *We will sometimes represent* (Ω, ε)*-analytic inductive inequalities/sequents as follows:*

$$(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}] \qquad (\varphi \vdash \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}],$$

where $(\varphi \leq \psi)[!\overline{x}, !\overline{y}, !\overline{z}, !\overline{w}]$ *is the skeleton of the given inequality,* $\overline{\alpha}$ *(resp.* $\overline{\beta}$*) denotes the vector of positive (resp. negative) maximal PIA subformulas, i.e. each* α *in* $\overline{\alpha}$ *and* β *in* $\overline{\beta}$ *contains at least one* ε*-critical occurrence of some propositional variable, and moreover:*

1. *for each* α *in* $\overline{\alpha}$*, either* $+\alpha \prec +\varphi$ *or* $+\alpha \prec -\psi$*;*

2. *for each* β *in* $\overline{\beta}$*, either* $-\beta \prec +\varphi$ *or* $-\beta \prec -\psi$*,*

and $\overline{\gamma}$ *(resp.* $\overline{\delta}$*) denotes the vector of positive (resp. negative) maximal* ε^∂*-uniform PIA subformulas, and moreover:*

1. *for each γ in $\overline{\gamma}$, either $+\gamma \prec +\varphi$ or $+\gamma \prec -\psi$;*

2. *for each δ in $\overline{\delta}$, either $-\delta \prec +\varphi$ or $-\delta \prec -\psi$.*

For the sake of a more compact notation, in what follows we sometimes write e.g. $(\varphi \leq \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ in place of $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$. The colors are intended to help in identifying which subformula occurrences are in precedent (blue) or succedent (red) position (cf. Footnote 13).[16]

Lemma 2.2.16. *For any LE-language $\mathcal{L}$, any analytic inductive $\mathcal{L}$-sequent*

$$(\varphi \vdash \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$$

is equivalent to the conjunction of definite analytic inductive $\mathcal{L}$-sequents

$$(\varphi_j \vdash \psi_i)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}].$$

Proof. Since by assumption $\varphi(!\overline{x}, !\overline{y}, !\overline{z}, !\overline{w})$ is positive Skeleton and $\psi(!\overline{x}, !\overline{y}, !\overline{z}, !\overline{w})$ is negative Skeleton, by Lemma 2.2.13, the given sequent is equivalent to

$$(\bigvee_{j\in J} \varphi_j \vdash \bigwedge_{i\in I} \psi_i)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}],$$

where every ψ_i is a definite negative Skeleton and every φ_j is a definite positive Skeleton, from which the statement readily follows. $\square$

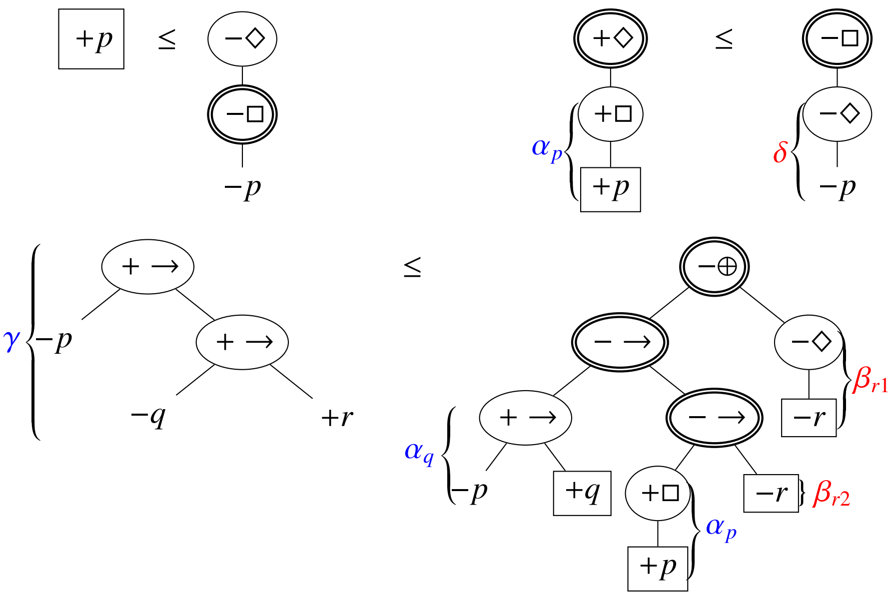
Notation 2.2.17. *We adopt the convention that in graphical representations of signed generation trees the squared variable occurrences are the ε-critical ones, the doubly circled nodes are the Skeleton ones and the single-circle ones are PIA nodes.*

Example 2.2.18. Let $\mathcal{L} := \mathcal{L}(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} := \{\Diamond\}$ and $\mathcal{G} := \{\Box, \oplus, \rightarrow\}$ with the usual arity and order-type. The $\mathcal{L}$-inequality $p \leq \Diamond\Box p$ is ε-inductive for $\varepsilon(p) = 1$, but is not analytic inductive for any order-type, because the negative generation tree of $\Diamond\Box p$, which has only one branch, is not good. The Church-Rosser inequality $\Diamond\Box p \leq \Box\Diamond p$ is analytic ε-inductive for every order-type.

The inequality $p \rightarrow (q \rightarrow r) \leq ((p \rightarrow q) \rightarrow (\Box p \rightarrow r)) \oplus \Diamond r$ is is an analytic (Ω, ε)-inductive inequality, e.g. for $p <_\Omega q <_\Omega r$ and $\varepsilon(p, q, r) = (1, 1, \partial)$.

Below, we represent the signed generation trees pertaining to the inequalities above (see Notation 2.2.17):

[16]The use of colors in this notational convention is inspired by, but different from, the one introduced in [36], where the blue (resp. red) color identifies the logical connectives algebraically interpreted as right (resp. left) adjoints or residuals. However, when restricted to the analytic inductive LE-inequalities, these two conventions coincide, since the main connective of a (non-atomic) positive (resp. negative) maximal PIA-subformula is a right (resp. left) adjoint/residual. Interestingly, the so-called (strong) *focalization* property of the focalized sequent calculi introduced in [36] can be equivalently formulated in terms of maximal PIA-subtrees.



The following auxiliary definition was introduced in [34, Definition 48] as a simplified version of [11, Definition 5.1], and serves to calculate effectively the residuals of definite positive and negative PIA formulas (cf. [34], discussion after Definition 3.5.3) w.r.t. a given variable occurrence x. The intended meaning of symbols such as $\varphi(!x, \overline{z})$ is that the variable x occurs *exactly once* in the formula φ (cf. Notation 2.2.8). In the context of the following definition, the variable x is used (and referred to) as the *pivotal* variable, i.e. the variable that is displayed by effect of the recursive residuation procedure.

Definition 2.2.19. For every definite positive PIA $\mathcal{L}_{\mathrm{LE}}$-formula $\psi = \psi(!x, \overline{z})$, and any definite negative PIA $\mathcal{L}_{\mathrm{LE}}$-formula $\xi = \xi(!x, \overline{z})$ such that x occurs in them exactly once, the $\mathcal{L}^*_{\mathrm{LE}}$-formulas $\mathsf{la}(\psi)(u, \overline{z})$ and $\mathsf{ra}(\xi)(u, \overline{z})$ (for $u \in Var - (x \cup \overline{z})$) are defined by simultaneous recursion as follows:

$$
\begin{array}{rcll}
\mathsf{la}(x) & = & u & (\mathsf{la}1)\\
\mathsf{la}(g(\overline{\psi_{-j}(\overline{z})}, \psi_j(x, \overline{z}), \overline{\xi(\overline{z})})) & = & \mathsf{la}(\psi_j)(g^\flat_j(\overline{\psi_{-j}(\overline{z})}, u, \overline{\xi(\overline{z})}), \overline{z}) & (\mathsf{la}2)\\
\mathsf{la}(g(\overline{\psi(\overline{z})}, \overline{\xi_{-j}(\overline{z})}, \xi_j(x, \overline{z}))) & = & \mathsf{ra}(\xi_j)(g^\flat_j(\overline{\psi(\overline{z})}, \overline{\xi_{-j}(\overline{z})}, u), \overline{z}) & (\mathsf{la}3)\\
\\
\mathsf{ra}(x) & = & u & (\mathsf{ra}1)\\
\mathsf{ra}(f(\overline{\xi_{-j}(\overline{z})}, \xi_j(x, \overline{z}), \overline{\psi(\overline{z})})) & = & \mathsf{ra}(\xi_j)(f^\sharp_j(\overline{\xi_{-j}(\overline{z})}, u, \overline{\psi(\overline{z})}), \overline{z}) & (\mathsf{ra}2)\\
\mathsf{ra}(f(\overline{\xi(\overline{z})}, \overline{\psi_{-j}(\overline{z})}, \psi_j(x, \overline{z}))) & = & \mathsf{la}(\psi_j)(f^\sharp_j(\overline{\xi(\overline{z})}, \overline{\psi_{-j}(\overline{z})}, u), \overline{z}) & (\mathsf{ra}3)
\end{array}
$$

Above, symbols such as $\overline{\psi_{-j}}$ denote the vector obtained by removing the jth coordinate of the vector $\overline{\psi}$.

Example 2.2.20. As mentioned above, $\mathsf{la}(\psi)(u, \overline{z})$ and $\mathsf{ra}(\xi)(u, \overline{z})$ are intended to capture the syntactic shape of the residuals of $\psi(!x, \overline{z})$ and $\xi(!x, \overline{z})$ in their x-coordinates. This means, for instance, that the following equivalence holds (cf. [34, Lemma 49]) if ψ is a definite positive PIA formula which is monotone in x (i.e. $+x \prec +\psi$):

$$\mathsf{la}(\psi)(u, \overline{z}) \leq x \quad \text{iff} \quad u \leq \psi(!x, \overline{z}),$$

while the following equivalence holds if ψ is antitone in x (i.e. $-x \prec +\psi$):

$$x \leq \mathsf{la}(\psi)(u, \overline{z}) \quad \text{iff} \quad u \leq \psi(!x, \overline{z}).$$

For instance, let $\mathcal{L} := \mathcal{L}(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} := \{\succ\!\!-\}$ and $\mathcal{G} := \{\Box, \rightarrow\}$ with the usual arity and order-type. Let $\mathcal{F}^* := \{\succ\!\!-, \succ\!\!-', \blacklozenge, \otimes\}$ and $\mathcal{G}^* := \{\Box, \rightarrow, \rightarrow', \oplus\}$, where $\succ\!\!-$ (resp. $\rightarrow$) has $\succ\!\!-'$ (resp. $\rightarrow'$) as residual in its first coordinate and $\oplus$ (resp. $\otimes$) as residual in its second coordinate. Consider the definite positive PIA $\mathcal{L}$-formula $\psi(!w, !y, !z) := \Box((w \succ\!\!- y) \rightarrow z)$, which is monotone in w and z, and antitone in y. If $x := w$, then $\mathsf{la}(\psi)(u, !y, !z) = (\blacklozenge u \rightarrow' z) \succ\!\!-' y$. Indeed, ψ can be represented as $g(\psi'(!w, !y, !z))$ where $g := \Box$ and $\psi'(!w, !y, !z) := (w \succ\!\!- y) \rightarrow z$. So $\mathsf{la}(\psi)(u, !y, !z) = \mathsf{la}(\psi')(v, !y, !z)[\blacklozenge u/v]$, where $g_1^\flat := \blacklozenge$. Next, $\psi'(!w, !y, !z)$ can be represented as $g'(\varphi'(!w, !y, !z), \psi''(!y, !z))$, where $g' := \rightarrow$, $\varphi'(!w, !y, !z) := w \succ\!\!- y$, and $\psi''(!y, !z) := z$. Then, $\mathsf{la}(\psi')(v, !y, !z) = \mathsf{ra}(\varphi')(t, !y, !z)[\blacklozenge u \rightarrow' z/t]$, where $g'^\flat_1 := \rightarrow'$. Finally, $\varphi'(!w, !y, !z)$ can be represented as $f(\psi'''(!w, !y, !z), \varphi''(!y, !z))$, where $f := \succ\!\!-$, $\psi'''(!w, !y, !z) := w$, and $\varphi''(!y, !z) := y$. Hence, $\mathsf{ra}(\varphi')(t, !y, !z) := t \succ\!\!-' y$, where $f_1^\sharp := \succ\!\!-'$.

Let us verify that $(\blacklozenge u \rightarrow' z) \succ\!\!-' y \leq w$ iff $u \leq \Box((w \succ\!\!- y) \rightarrow z)$:

$$\begin{array}{rll} u \leq \Box((w \succ\!\!- y) \rightarrow z) & \text{iff} & \blacklozenge u \leq (w \succ\!\!- y) \rightarrow z \\ & \text{iff} & w \succ\!\!- y \leq \blacklozenge u \rightarrow' z \\ & \text{iff} & (\blacklozenge u \rightarrow' z) \succ\!\!-' y \leq w. \end{array}$$

Likewise, if $x := y$, then $\mathsf{la}(\psi')(u, !w, !z) = (\blacklozenge u \rightarrow' z) \oplus w$, and the following chain of equivalences holds:

$$\begin{array}{rll} u \leq \Box((w \succ\!\!- y) \rightarrow z) & \text{iff} & \blacklozenge u \leq (w \succ\!\!- y) \rightarrow z \\ & \text{iff} & w \succ\!\!- y \leq \blacklozenge u \rightarrow' z \\ & \text{iff} & y \leq (\blacklozenge u \rightarrow' z) \oplus w. \end{array}$$

If $x := z$, then $\mathsf{la}(\psi)(u, !w, !y) = (w \succ\!\!- y) \otimes \blacklozenge u$, and the following chain of equivalences holds:

$$\begin{array}{rll} u \leq \Box((w \succ\!\!- y) \rightarrow z) & \text{iff} & \blacklozenge u \leq (w \succ\!\!- y) \rightarrow z \\ & \text{iff} & (w \succ\!\!- y) \otimes \blacklozenge u \leq z. \end{array}$$

2.2.4 Display calculi for basic normal LE-logics

In this section we define the proper display calculus D.LE for the basic normal $\mathcal{L}_{\mathrm{LE}}$-logic in a fixed but arbitrary LE-signature $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$ (cf. Section 2.2.1). Let $S_{\mathcal{F}} := \{\hat{f} \mid f \in \mathcal{F}^*\}$ and $S_{\mathcal{G}} := \{\check{g} \mid g \in \mathcal{G}^*\}$ be the sets of structural connectives associated with $\mathcal{F}^*$ and $\mathcal{G}^*$ respectively (cf. Section 2.2.2). Each such structural connective comes with an arity and an order-type which coincide with those of its associated operational connective in $\mathcal{F}^*$ and $\mathcal{G}^*$.

Remark 2.2.21. If $f \in \mathcal{F}$ and $g \in \mathcal{G}$ form a dual pair,[17] then $n_f = n_g$ and $\varepsilon_f = \varepsilon_g$. Then f and g can be assigned one and the same structural operator H, which is interpreted as f when occurring in precedent position and as g when occurring in succedent position (cf. Footnote 13):

[17] Examples of dual pairs are $(\top, \bot)$, $(\wedge, \vee)$, $(\succ\!\!-, \rightarrow)$, $(-\!\!\prec, \leftarrow)$, and $(\Diamond, \Box)$ where $\Diamond$ is defined as $\neg\Box\neg$.

Structural symbols	H	
Operational symbols	f	g

Moreover, for any $1 \leq i \leq n_f = n_g$, the residuals $f_i^\sharp$ and $g_i^\flat$ are dual to one another. Hence they can also be assigned one and the same structural connective as follows:

Order-type	$\varepsilon_f(i) = \varepsilon_g(i) = 1$		$\varepsilon_f(i) = \varepsilon_g(i) = \partial$	
Structural symbols	H_i		H_i	
Operational symbols	$(g_i^\flat)$	$(f_i^\sharp)$	$(f_i^\sharp)$	$(g_i^\flat)$

This observation has made it possible to associate one structural connective with two logical connectives, which has become common in the display calculi literature. In this paper, we prefer to maintain a strict one-to-one correspondence between operational and structural symbols.

If we admit that the sets $\mathcal{F}$ and $\mathcal{G}$ have a non empty intersection (cf. Footnote 5), then a unary connective $h \in \mathcal{F} \cap \mathcal{G}$ can be assigned one and the same structural operator $\tilde{h}$, which is interpreted as h when occurring in precedent position and in succedent position:

Structural symbols	$\tilde{h}$	
Operational symbols	h	h

For notational convenience, we let $\mathcal{F}^\partial := \mathcal{G}$ and $\mathcal{G}^\partial := \mathcal{F}$. Moreover, given the sets $\mathsf{Str}_\mathcal{F}$, $\mathsf{Str}_\mathcal{G}$ defined below and any order-type ε on n, we let $\mathsf{Str}_\mathcal{F}^\varepsilon := \prod_{i=1}^n \mathsf{Str}_{\mathcal{F}^{\varepsilon(i)}}$ and $\mathsf{Str}_\mathcal{G}^\varepsilon := \prod_{i=1}^n \mathsf{Str}_{\mathcal{G}^{\varepsilon(i)}}$.

The calculus D.LE manipulates sequents $\Pi \vdash \Sigma$ where the structures Π (for precedent) and Σ (for succedent) are defined by the following simultaneous recursion:

$$\mathsf{Str}_\mathcal{F} \ni \Pi ::= \ \varphi \mid \hat{\top} \mid \hat{f}\,(\overline{\Pi}^{(\varepsilon_f)})$$
$$\mathsf{Str}_\mathcal{G} \ni \Sigma ::= \ \varphi \mid \check{\bot} \mid \check{g}\,(\overline{\Sigma}^{(\varepsilon_g)})$$

with $\varphi \in \mathcal{L}_{\mathrm{LE}}$, and $\hat{f} \in S_\mathcal{F}$, $\check{g} \in S_\mathcal{G}$, $\overline{\Pi}^{(\varepsilon_f)} \in \mathsf{Str}_\mathcal{F}^{\varepsilon_f}$ and $\overline{\Sigma}^{(\varepsilon_g)} \in \mathsf{Str}_\mathcal{G}^{\varepsilon_g}$. Notice that for any connective h of arity $n \geq 1$ the notational convention $\hat{h}$ conveys also the information that h is a left-adjoint/residual and the notational convention $\check{h}$ conveys the information that h is a right-adjoint/residual.

In what follows, we use $\Upsilon_1, \ldots, \Upsilon_n$ as structure metavariables in $\mathsf{Str}_\mathcal{F} \cup \mathsf{Str}_\mathcal{G}$. The introduction rules of the calculus below will guarantee that $\Upsilon \in \mathsf{Str}_\mathcal{F}$ whenever it occurs in precedent position, and $\Upsilon \in \mathsf{Str}_\mathcal{G}$ whenever it occurs in succedent position. The calculus D.LE = D.LE$_\mathcal{L}$ consists of the following rules:[18]

- Identity and cut rules:[19]

$$\mathrm{Id}\ \frac{}{p \vdash p} \qquad \frac{\Pi \vdash \varphi \qquad \varphi \vdash \Sigma}{\Pi \vdash \Sigma}\ \mathrm{Cut}$$

[18]For any LE-language $\mathcal{L}$, we will sometimes let D.LE* := D.LE$_{\mathcal{L}^*}$, i.e. we will let D.LE* denote the calculus obtained by instantiating the general definition of the basic calculus D.LE$_\mathcal{L}$ to $\mathcal{L} := \mathcal{L}^*$.

[19]In the display calculi literature, the identity rule is sometimes defined as $\varphi \vdash \varphi$, where φ is an arbitrary, possibly complex, formula. The difference is inessential, given that, in any display calculus, $p \vdash p$ is an instance of $\varphi \vdash \varphi$, and $\varphi \vdash \varphi$ is derivable for any formula φ whenever $p \vdash p$ is the Identity rule.

- Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$: for any $1 \leq i, j \leq n_f$ and $1 \leq h, k \leq n_g$,

If $\varepsilon_f(i) = 1$ and $\varepsilon_g(h) = 1$,

$$\hat{f} \dashv \check{f}_i^\sharp \; \frac{\hat{f}(\Upsilon_1, \ldots, \Pi_i, \ldots, \Upsilon_{n_f}) \vdash \Sigma}{\Pi_i \vdash \check{f}_i^\sharp(\Upsilon_1, \ldots, \Sigma, \ldots, \Upsilon_{n_f})} \qquad \frac{\Pi \vdash \check{g}(\Upsilon_1 \ldots, \Sigma_h, \ldots \Upsilon_{n_g})}{\hat{g}_h^\flat(\Upsilon_1, \ldots, \Pi, \ldots, \Upsilon_{n_g}) \vdash \Sigma_h} \; \hat{g}_h^\flat \dashv \check{g}$$

If $\varepsilon_f(j) = \partial$ and $\varepsilon_g(k) = \partial$,

$$(\hat{f}, \hat{f}_j^\sharp) \; \frac{\hat{f}(\Upsilon_1, \ldots, \Sigma_j, \ldots, \Upsilon_{n_f}) \vdash \Sigma}{\hat{f}_j^\sharp(\Upsilon_1, \ldots, \Sigma, \ldots, \Upsilon_{n_f}) \vdash \Sigma_j} \qquad \frac{\Pi \vdash \check{g}(\Upsilon_1, \ldots, \Pi_k, \ldots, \Upsilon_{n_g})}{\Pi_k \vdash \check{g}_k^\flat(\Upsilon_1, \ldots, \Pi, \ldots, \Upsilon_{n_g})} \; (\check{g}, \check{g}_k^\flat)$$

- Structural rules for lattice connectives:

$$\top_W \; \frac{\hat{\top} \vdash \Sigma}{\Pi \vdash \Sigma} \qquad \frac{\Pi \vdash \check{\bot}}{\Pi \vdash \Sigma} \; \bot_W$$

- Logical introduction rules for lattice connectives:

$$\top_L \; \frac{\hat{\top} \vdash \Sigma}{\top \vdash \Sigma} \qquad \frac{}{\hat{\top} \vdash \top} \; \top_R \qquad \bot_L \; \frac{}{\bot \vdash \check{\bot}} \qquad \frac{\Pi \vdash \check{\bot}}{\Pi \vdash \bot} \; \bot_R$$

$$\wedge_{L2} \; \frac{\psi \vdash \Sigma}{\varphi \wedge \psi \vdash \Sigma} \qquad \wedge_{L1} \; \frac{\varphi \vdash \Sigma}{\varphi \wedge \psi \vdash \Sigma} \qquad \frac{\Pi \vdash \varphi \quad \Pi \vdash \psi}{\Pi \vdash \varphi \wedge \psi} \; \wedge_R$$

$$\vee_L \; \frac{\varphi \vdash \Sigma \quad \psi \vdash \Sigma}{\varphi \vee \psi \vdash \Sigma} \qquad \frac{\Pi \vdash \varphi}{\Pi \vdash \varphi \vee \psi} \; \vee_{R1} \qquad \frac{\Pi \vdash \psi}{\Pi \vdash \varphi \vee \psi} \; \vee_{R2}$$

- Logical introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$\frac{\Big(\Upsilon_i \vdash \varphi_i \quad \varphi_j \vdash \Upsilon_j \mid 1 \leq i, j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\Big)}{\hat{f}(\Upsilon_1, \ldots, \Upsilon_{n_f}) \vdash f(\varphi_1, \ldots, \varphi_{n_f})} \; f_R$$

$$g_L \; \frac{\Big(\varphi_i \vdash \Upsilon_i \quad \Upsilon_j \vdash \varphi_j \mid 1 \leq i, j \leq n_g, \varepsilon_g(i) = 1 \text{ and } \varepsilon_g(j) = \partial\Big)}{g(\varphi_1, \ldots, \varphi_{n_g}) \vdash \check{g}(\Upsilon_1, \ldots, \Upsilon_{n_g})}$$

$$f_L \; \frac{\hat{f}(\varphi_1, \ldots, \varphi_{n_f}) \vdash \Sigma}{f(\varphi_1, \ldots, \varphi_{n_f}) \vdash \Sigma} \qquad \frac{\Pi \vdash \check{g}(\varphi_1, \ldots, \varphi_{n_g})}{\Pi \vdash g(\varphi_1, \ldots, \varphi_{n_g})} \; g_R$$

If f and g are 0-ary (i.e. they are constants), the rules f_R and g_L above reduce to the axioms (aka 0-ary rules) $\hat{f} \vdash f$ and $g \vdash \check{g}$.

Remark 2.2.22. If we let $\mathcal{F}$ and $\mathcal{G}$ have a nonempty intersection (cf. Footnote 5), then the rules capturing a generic connective $h \in (\mathcal{F} \cap \mathcal{G})$ of arity $n = 1$ are as follows (notice that the notational convention $\tilde{h}$ conveys also the information that h is both a left adjoint and a right adjoint):

- Display postulates for $h \in (\mathcal{F} \cap \mathcal{G})$ occurring in precedent and in succedent position:

If $\varepsilon_h(1) = 1$,

$$\tilde{h} \dashv \check{h}^{\sharp}\ \begin{array}{c} \tilde{h}\,\Pi \vdash \Sigma \\ \hline\hline \Pi \vdash \check{h}^{\sharp}\Sigma \end{array} \qquad \begin{array}{c} \Pi \vdash \tilde{h}\,\Sigma \\ \hline\hline \hat{h}^{\flat}\,\Pi \vdash \Sigma \end{array}\ \hat{h}^{\flat} \dashv \tilde{h}$$

If $\varepsilon_h(1) = \partial$,

$$(\hat{h}^{\sharp}, \tilde{h})\ \begin{array}{c} \tilde{h}\,\Sigma_1 \vdash \Sigma_2 \\ \hline\hline \hat{h}^{\sharp}\,\Sigma_2 \vdash \Sigma_1 \end{array} \qquad \begin{array}{c} \Pi_1 \vdash \tilde{h}\,\Pi_2 \\ \hline\hline \Pi_2 \vdash \check{h}^{\flat}\,\Pi_1 \end{array}\ (\check{h}^{\flat}, \tilde{h})$$

- Structural rules for $h \in (\mathcal{F} \cap \mathcal{G})$:

If $\varepsilon_h(1) = 1$,

$$\tilde{h}\ \frac{\Pi \vdash \Sigma}{\tilde{h}\,\Pi \vdash \tilde{h}\,\Sigma} \qquad (\tilde{h}, \hat{h}^{\flat})\ \frac{\tilde{h}\,\hat{h}^{\flat}\,\Pi \vdash \Sigma}{\Pi \vdash \Sigma}$$

If $\varepsilon_h(1) = \partial$,

$$\frac{\Pi \vdash \Sigma}{\tilde{h}\,\Sigma \vdash \tilde{h}\,\Pi}\ \tilde{h} \qquad \frac{\Pi \vdash \tilde{h}\,\check{h}^{\sharp}\,\Sigma}{\Pi \vdash \Sigma}\ (\tilde{h}, \check{h}^{\sharp})$$

- Logical introduction rules for $h \in (\mathcal{F} \cap \mathcal{G})$ occurring in precedent and in succedent position:

$$h_L\ \frac{\tilde{h}\,(\varphi_1, \ldots, \varphi_{n_h}) \vdash \Sigma}{h(\varphi_1, \ldots, \varphi_{n_h}) \vdash \Sigma} \qquad \frac{\Pi \vdash \tilde{h}\,(\varphi_1, \ldots, \varphi_{n_h})}{\Pi \vdash h(\varphi_1, \ldots, \varphi_{n_h})}\ h_R$$

Let $\underline{\text{D.LE}}$ denote the calculus obtained by removing Cut in D.LE. In what follows, we indicate that the sequent $\varphi \vdash \psi$ is derivable in D.LE (resp. in $\underline{\text{D.LE}}$) by $\vdash_{\text{D.LE}} \varphi \vdash \psi$ (resp. $\vdash_{\underline{\text{D.LE}}} \varphi \vdash \psi$).

Proposition 2.2.23 (Soundness)**.** *The calculus* D.LE *(hence also* $\underline{\text{D.LE}}$*) is sound w.r.t. the class of complete $\mathcal{L}$-algebras.*

Proof. The soundness of the basic lattice rules is clear. The soundness of the remaining rules is due to the monotonicity (resp. antitonicity) of the algebraic connectives interpreting each $f \in \mathcal{F}$ and $g \in \mathcal{G}$, and their adjunction/residuation properties, which hold since any complete $\mathcal{L}$-algebra is an $\mathcal{L}^*$-algebra. □

Proposition 2.2.24. *The calculus* D.LE *is a proper display calculus (cf. [34, Theorem 26]), and hence cut elimination holds for it as a consequence of a Belnap-style cut elimination meta-theorem (cf. [34, Section 2.2 and Appendix A] and [32, Theorem 2]).*

2.2.5 The setting of distributive LE-logics

In this section we discuss how the general setting presented above can account for the assumption that the given LE-logic is distributive, i.e. that the *distributive laws* $(p \vee r) \wedge (p \vee q) \vdash p \vee (r \wedge q)$ and $p \wedge (r \vee q) \vdash (p \wedge r) \vee (p \wedge q)$ are valid. Such logics will be referred to as DLE-logics, since they are algebraically captured by varieties of *normal distributive lattice expansions* (DLEs), i.e. LE-algebras as in Definition 2.2.1 such that $\mathbb{L}$ is assumed to be a *bounded distributive lattice*. For any (D)LE-language, the basic $\mathcal{L}_{\mathrm{DLE}}$-logic is defined as in Definition 2.2.2 augmented with the distributive laws above.

Since $\wedge$ and $\vee$ distribute over each other, besides being Δ-adjoints, they can also be treated as elements of $\mathcal{F}$ and $\mathcal{G}$ respectively. In particular, the binary connectives $\leftarrow$ and $\rightarrow$ occur in the fully residuated language $\mathcal{L}^*_{\mathrm{DLE}}$, the intended interpretations of which are the right residuals of $\wedge$ in the first and second coordinate respectively, as well as the binary connectives $-\!\!<$ and $>\!\!-$, the intended interpretations of which are the left residuals of $\vee$ in the first and second coordinate, respectively. Following the general convention discussed in Section 2.2.2, we stipulate that $>\!\!-, -\!\!< \in \mathcal{F}^*$ and $\rightarrow, \leftarrow \in \mathcal{G}^*$. The basic fully residuated $\mathcal{L}^*_{\mathrm{DLE}}$*-logic*, which will sometimes be referred to as the *basic bi-intuitionistic 'tense'* DLE-logic, is given as per Definition 2.2.3. In particular, the residuation rules for the lattice connectives are specified as follows:[20]

$$\frac{\varphi \wedge \psi \vdash \chi}{\psi \vdash \varphi \rightarrow \chi} \qquad \frac{\psi \wedge \varphi \vdash \chi}{\psi \vdash \chi \leftarrow \varphi} \qquad \frac{\varphi \vdash \psi \vee \chi}{\psi >\!\!- \varphi \vdash \chi} \qquad \frac{\varphi \vdash \chi \vee \psi}{\varphi -\!\!< \psi \vdash \chi}$$

When interpreting LE-languages on perfect distributive lattice expansions (perfect DLEs, cf. Footnote 12), the logical disjunction is interpreted by means of the coordinatewise completely $\wedge$-preserving join operation of the lattice, and the logical conjunction with the coordinatewise completely $\vee$-preserving meet operation of the lattice. Hence we are justified in listing $+\wedge$ and $-\vee$ among the SLRs, and $+\vee$ and $-\wedge$ among the SRRs, as is done in Table 2.2. Consequently, the classes of (*analytic*) *inductive* $\mathcal{L}_{\mathrm{DLE}}$*-inequalities* are obtained by simply applying Definitions 3.5.2 and 3.5.3 with respect to Table 2.2 below.

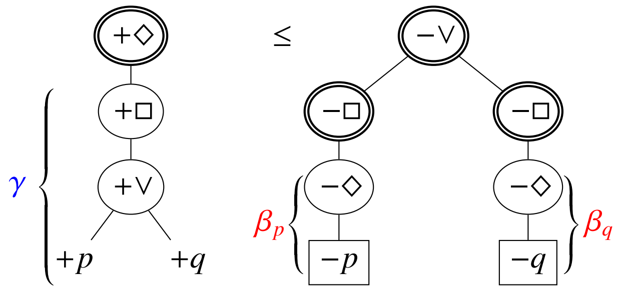
Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee$	$+ \quad \wedge \quad g \quad$ with $n_g = 1$
$- \quad \wedge$	$- \quad \vee \quad f \quad$ with $n_f = 1$
SLR	SRR
$+ \quad \wedge \quad f \quad$ with $n_f \geq 1$	$+ \quad \vee \quad g \quad$ with $n_g \geq 2$
$- \quad \vee \quad g \quad$ with $n_g \geq 1$	$- \quad \wedge \quad f \quad$ with $n_f \geq 2$

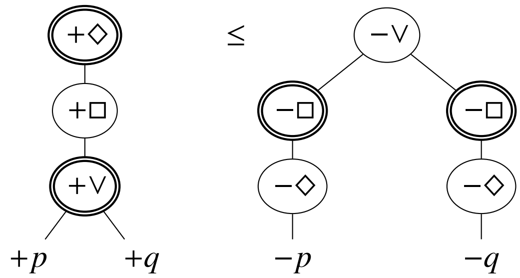
Table 2.2: Skeleton and PIA nodes for $\mathcal{L}_{\mathrm{DLE}}$.

Precisely because, as reported in Table 2.2, the nodes $+\wedge$ and $-\vee$ are now also SLR nodes, and $+\vee$ and $-\wedge$ are also SRR nodes (see also Remark 2.2.27), the classes of (analytic) inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities are strictly larger than the classes of (analytic) inductive $\mathcal{L}_{\mathrm{LE}}$-inequalities in the same signature, as shown in the next example.

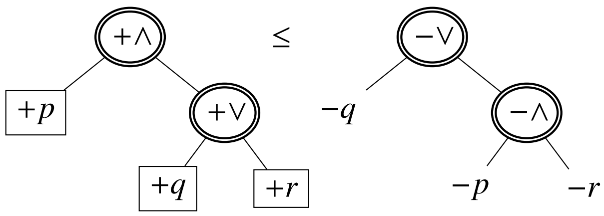
[20]Notice that $\varphi \rightarrow \chi$ and $\chi \leftarrow \varphi$ are interderivable for any φ and ψ, since $\wedge$ is commutative; similarly, $\psi >\!\!- \varphi$ and $\varphi -\!\!< \psi$ are interderivable, since $\vee$ is commutative. Hence in what follows we consider explicitly only $\rightarrow$ and $-\!\!<$.

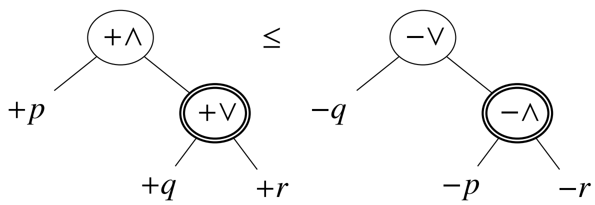
Example 2.2.25. The inequality $\Diamond\Box(p \vee q) \leq \Box\Diamond p \vee \Box\Diamond q$ is not an inductive $\mathcal{L}_{\mathrm{LE}}$-inequality for any order-type, but it is an ε-Sahlqvist $\mathcal{L}_{\mathrm{DLE}}$-inequality e.g. for $\varepsilon(p, q) = (\partial, \partial)$. The classification of nodes in the signed generation trees of $\Diamond\Box(p\vee q) \leq \Box\Diamond p\vee\Box\Diamond q$ as an $\mathcal{L}_{\mathrm{DLE}}$-inequality is on the left-hand side of the picture below, and the one as an $\mathcal{L}_{\mathrm{LE}}$-inequality is on the right (see Notation 2.2.17). In the classification on the right, no branch is good, therefore $\Diamond\Box(p \vee q) \leq \Box\Diamond p \vee \Box\Diamond q$ is not an inductive $\mathcal{L}_{\mathrm{LE}}$-inequality for any order-type.





The inequality $p \wedge (q \vee r) \leq q \vee (p \wedge r)$ is an ε-Sahlqvist $\mathcal{L}_{\mathrm{DLE}}$-inequality e.g. for $\varepsilon(p, q, r) = (1, 1, 1)$, but is not an inductive $\mathcal{L}_{\mathrm{LE}}$-inequality for any order-type. The classification of nodes in the signed generation trees of $p \wedge (q \vee r) \leq q \vee (p \wedge r)$ as an $\mathcal{L}_{\mathrm{DLE}}$-inequality is represented on the left-hand side of the picture below (where the squared variable occurrences are ε-critical; recall that Skeleton nodes are doubly circled, while PIA nodes are circled, cf. Notation 2.2.17), and the one as an $\mathcal{L}_{\mathrm{LE}}$-inequality is on the right. In the classification on the right, no branch is good leading to occurrences of r, therefore $p \wedge (q \vee r) \leq q \vee (p \wedge r)$ is not an inductive $\mathcal{L}_{\mathrm{LE}}$-inequality for any order-type.





Also, definite Skeleton and definite PIA $\mathcal{L}_{\mathrm{DLE}}$-formulas are defined verbatim in the same way as in the setting of $\mathcal{L}_{\mathrm{LE}}$-formulas. Namely, $*\xi$ (resp. $*\varphi$) is a definite Skeleton (resp. definite PIA) iff all nodes of $*\xi$ (resp. $*\varphi$) are SLR (resp. SRR). However, the classification of nodes we need to consider is now the one of Table 2.2, where $+\wedge$ and $-\vee$ are also SLR-nodes, and $+\vee$ and $-\wedge$ are also SRR-nodes. Definition 2.2.19 is specified for $\wedge$, $\vee$, $\rightarrow$ and $-\!\!\!<$ as follows:

$$
\begin{array}{rcl}
\mathsf{la}(\xi(\overline{z}) \rightarrow \psi(x,\overline{z})) & = & \mathsf{la}(\psi)(u \wedge \xi(\overline{z}), \overline{z}); \\
\mathsf{la}(\psi_1(\overline{z}) \vee \psi_2(x,\overline{z})) & = & \mathsf{la}(\psi_2)(u -\!\!\!< \psi_1(\overline{z}), \overline{z}); \\
\mathsf{la}(\xi(x,\overline{z}) \rightarrow \psi(\overline{z})) & = & \mathsf{ra}(\xi)(u \rightarrow \psi(\overline{z}), \overline{z}); \\
 & & \\
\mathsf{ra}(\xi(x,\overline{z}) -\!\!\!< \psi(\overline{z})) & = & \mathsf{ra}(\xi)(\psi(\overline{z}) \vee u, \overline{z}); \\
\mathsf{ra}(\xi_1(\overline{z}) \wedge \xi_2(x,\overline{z})) & = & \mathsf{ra}(\xi_2)(\xi_1(\overline{z}) \rightarrow u, \overline{z}); \\
\mathsf{ra}(\xi(\overline{z}) -\!\!\!< \psi(x,\overline{z})) & = & \mathsf{la}(\psi)(\xi(\overline{z}) -\!\!\!< u, \overline{z});
\end{array}
$$

Finally, as to the display calculus D.DLE for the basic $\mathcal{L}_{\mathrm{DLE}}$-logic, its language is obtained by augmenting the language of D.LE with the following structural symbols for the lattice operators and their residuals:[21]

Structural symbols	$\hat{\top}$	$\check{\bot}$	$\hat{\wedge}$	$\check{\vee}$	$\hat{\succ\!\!-}$	$\check{\rightarrow}$	$\hat{-\!\!\!<}$	$\check{\leftarrow}$
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$(\succ\!\!-)$	$(\rightarrow)$	$(-\!\!\!<)$	$(\leftarrow)$

Display postulates for lattice connectives and their residuals are specified as follows:

$$
\hat{\wedge} \dashv \check{\rightarrow}\ \frac{\Pi_1 \mathbin{\hat{\wedge}} \Pi_2 \vdash \Sigma}{\Pi_2 \vdash \Pi_1 \mathbin{\check{\rightarrow}} \Sigma}
\qquad
\hat{\wedge} \dashv \check{\leftarrow}\ \frac{\Pi_1 \mathbin{\hat{\wedge}} \Pi_2 \vdash \Sigma}{\Pi_1 \vdash \Sigma \mathbin{\check{\leftarrow}} \Pi_2}
$$

$$
\frac{\Pi \vdash \Sigma_1 \mathbin{\check{\vee}} \Sigma_2}{\Sigma_1 \mathbin{\hat{\succ\!\!-}} \Pi \vdash \Sigma_2}\ \hat{\succ\!\!-} \dashv \check{\vee}
\qquad
\frac{\Pi \vdash \Sigma_1 \mathbin{\check{\vee}} \Sigma_2}{\Pi \mathbin{\hat{-\!\!\!<}} \Sigma_2 \vdash \Sigma_1}\ \hat{-\!\!\!<} \dashv \check{\vee}
$$

Moreover, D.DLE is augmented with the following structural rules encoding the characterizing properties of the lattice connectives:

$$
\hat{\top}_L\ \frac{\Pi \vdash \Sigma}{\hat{\top} \mathbin{\hat{\wedge}} \Pi \vdash Y}
\qquad
\frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma \mathbin{\check{\vee}} \check{\bot}}\ \check{\bot}_R
\qquad
E_L\ \frac{\Pi_1 \mathbin{\hat{\wedge}} \Pi_2 \vdash \Sigma}{\Pi_2 \mathbin{\hat{\wedge}} \Pi_1 \vdash \Sigma}
\qquad
\frac{\Pi \vdash \Sigma_1 \mathbin{\check{\vee}} \Sigma_2}{\Pi \vdash \Sigma_2 \mathbin{\check{\vee}} \Sigma_1}\ E_R
$$

$$
W_L\ \frac{\Pi_2 \vdash \Sigma}{\Pi_1 \mathbin{\hat{\wedge}} \Pi_2 \vdash \Sigma}
\qquad
\frac{\Pi \vdash \Sigma_1}{\Pi \vdash \Sigma_1 \mathbin{\check{\vee}} \Sigma_2}\ W_R
\qquad
C_L\ \frac{\Pi \mathbin{\hat{\wedge}} \Pi \vdash \Sigma}{\Pi \vdash \Sigma}
\qquad
\frac{\Pi \vdash \Sigma \mathbin{\check{\vee}} \Sigma}{\Pi \vdash \Sigma}\ C_R
$$

$$
A_L\ \frac{\Pi_1 \mathbin{\hat{\wedge}} (\Pi_2 \mathbin{\hat{\wedge}} \Pi_3) \vdash \Sigma}{(\Pi_1 \mathbin{\hat{\wedge}} \Pi_2) \mathbin{\hat{\wedge}} \Pi_3 \vdash \Sigma}
\qquad
\frac{\Pi \vdash (\Sigma_1 \mathbin{\check{\vee}} \Sigma_2) \mathbin{\check{\vee}} \Sigma_3}{\Pi \vdash \Sigma_1 \mathbin{\check{\vee}} (\Sigma_2 \mathbin{\check{\vee}} \Sigma_3)}\ A_R
$$

and the introduction rules for the lattice connectives (and their residuals) follow the same pattern as the introduction rules of any $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$
\bot_L\ \frac{}{\bot \vdash \check{\bot}}
\qquad
\frac{\Pi \vdash \check{\bot}}{\Pi \vdash \bot}\ \bot_R
\qquad\Bigg|\qquad
\top_L\ \frac{\hat{\top} \vdash \Sigma}{\top \vdash \Sigma}
\qquad
\frac{}{\hat{\top} \vdash \top}\ \top_R
$$

$$
\wedge_L\ \frac{\varphi \mathbin{\hat{\wedge}} \psi \vdash \Sigma}{\varphi \wedge \psi \vdash \Sigma}
\qquad
\frac{\Pi_1 \vdash \varphi \qquad \Pi_2 \vdash \psi}{\Pi_1 \mathbin{\hat{\wedge}} \Pi_2 \vdash \varphi \wedge \psi}\ \wedge_R
\qquad\Bigg|\qquad
\vee_L\ \frac{\varphi \vdash \Sigma_1 \qquad \psi \vdash \Sigma_2}{\varphi \vee \psi \vdash \Sigma_1 \mathbin{\check{\vee}} \Sigma_2}
\qquad
\frac{\Pi \vdash \varphi \mathbin{\check{\vee}} \psi}{\Pi \vdash \varphi \vee \psi}\ \vee_R
$$

[21] In the presence of the exchange rules E_L and E_R, the structural connectives $\hat{\succ\!\!-}$, $\check{\leftarrow}$ and the corresponding operational connectives $\succ\!\!-$, $\leftarrow$ are redundant. For simplicity, we consider languages and calculi where the operational connectives $\succ\!\!-$ and $\leftarrow$ and their introduction rules are not included.

Remark 2.2.26. Rules $\wedge_{L1}$, $\wedge_{L2}$, $\wedge_L$, $\vee_{R1}$, $\vee_{R2}$ and $\vee_R$ in D.LE* are derivable in D.DLE as follows:

$$\wedge_{L2}:\quad \wedge_L\,\dfrac{W_L\,\dfrac{\psi \vdash \Sigma}{\varphi \,\hat{\wedge}\, \psi \vdash \Sigma}}{\varphi \wedge \psi \vdash \Sigma} \qquad\qquad \wedge_{L1}:\quad \wedge_L\,\dfrac{E_L\,\dfrac{W_L\,\dfrac{\varphi \vdash \Sigma}{\psi \,\hat{\wedge}\, \varphi \vdash \Sigma}}{\varphi \,\hat{\wedge}\, \psi \vdash \Sigma}}{\varphi \wedge \psi \vdash \Sigma}$$

$$\wedge_R:\quad C_L\,\dfrac{\dfrac{\Pi \vdash \varphi \qquad \Pi \vdash \psi}{\Pi \,\hat{\wedge}\, \Pi \vdash \varphi \wedge \psi}\,\wedge_R}{\Pi \vdash \varphi \wedge \psi} \qquad\qquad \vee_L:\quad \dfrac{\vee_L\,\dfrac{\varphi \vdash \Sigma \qquad \psi \vdash \Sigma}{\varphi \vee \psi \vdash \Sigma \,\check{\vee}\, \Sigma}}{\varphi \vee \psi \vdash \Sigma}\,C_R$$

$$\vee_{R1}:\quad \dfrac{\dfrac{\Pi \vdash \varphi}{\Pi \vdash \varphi \,\check{\vee}\, \psi}\,W_R}{\Pi \vdash \varphi \vee \psi}\,\vee_R \qquad\qquad \vee_{R_2}:\quad \dfrac{\dfrac{\dfrac{\Pi \vdash \psi}{\Pi \vdash \psi \,\check{\vee}\, \varphi}\,W_R}{\Pi \vdash \varphi \,\check{\vee}\, \psi}\,E_R}{\Pi \vdash \varphi \vee \psi}\,\vee_R$$

Remark 2.2.27. In what follows, we will work in the non-distributive setting with the calculus D.LE and its extensions. However, all the results we obtain about derivations in D.LE straightforwardly transfer to D.DLE using the following procedure: all applications of $\wedge_{L1}$, $\wedge_{L2}$, $\wedge_R$, $\vee_{R1}$, $\vee_{R2}$ and $\vee_L$ will be replaced by their derivations in D.DLE (cf. Remark 2.2.26).

All occurrences of $\wedge$ (resp. $\vee$) in an inductive $\mathcal{L}_{\mathrm{DLE}}$-inequality which are classified as SLR (resp. SRR) will be treated as connectives in $\mathcal{F}$ (resp. $\mathcal{G}$).

2.2.6 Derivations in pre-normal form

In Section 2.4, we will show that any analytic inductive LE-axiom $\varphi \vdash \psi$ can be effectively derived in the corresponding basic cut-free calculus $\underline{\text{D.LE}}$ enriched with the structural analytic rules $R_1, \ldots, R_n$ corresponding to $\varphi \vdash \psi$. In fact, the cut-free derivation we produce has a particular shape, referred to as *pre-normal form*, which we define in the present section. Informally, in a derivation in pre-normal form, a division of labor is effected on the applications of rules:[22] some rules are applied only before the application of R_i and some rules are applied only after the application of R_i.

Before moving on to the definitions, we highlight the following fact: when using ALBA to compute the analytic structural rule(s) corresponding to a given analytic inductive LE-axiom $\varphi \vdash \psi$, if $+\wedge$ and $-\vee$ occur as SRA nodes in a non-critical maximal PIA subtree (cf. Notation 2.2.15) of $\varphi \vdash \psi$, then this subtree will generate two or more premises of one of the corresponding rules (depending on the number of occurrences of $+\wedge$ and $-\vee$). If $-\wedge$ and $+\vee$ occur as Δ-adjoints in the Skeleton of $\varphi \vdash \psi$, then the axiom is non-definite, and by exhaustively permuting those occurrences upwards, i.e. towards the roots of the signed generation trees, and then applying the ALBA splitting rules, the given axiom can be equivalently transformed into a set of definite axioms, each of which will correspond to one analytic structural rule.

[22]The name 'pre-normal' is intended to remind of a similar division of labor, among rules applied in derivations in normal form of the well known natural deduction systems for classical and intuitionistic logic.

Definition 2.2.28. A derivation π in D.LE of the analytic inductive axiom $\varphi \vdash \psi$ (also indicated as Ax) is in *pre-normal form* if the unique application of each rule in its corresponding set of analytic structural rules $R_1(Ax), \ldots, R_m(Ax)$ computed by ALBA splits π into the following components:

$$
\begin{array}{l}
\text{PIA}(\pi)\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\text{Skeleton}(\pi)\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right.
\end{array}
\begin{array}{c}
\dfrac{\begin{array}{ccc} p_{1.1}\vdash p_{1.1}\cdots p_{1.k}\vdash p_{1.k} & \cdots & p_{n.1}\vdash p_{n.1}\cdots p_{n.\ell}\vdash p_{n.\ell} \\ \vdots & & \vdots \\ \Pi^1_1\vdash\Sigma^1_1 & \cdots & \Pi^1_n\vdash\Sigma^1_n \end{array}}{\Pi^1\vdash\Sigma^1}\,R_1(Ax)
\quad \cdots \quad
\dfrac{\begin{array}{ccc} p_{1.1}\vdash p_{1.1}\cdots p_{1.k'}\vdash p_{1.k'} & \cdots & p_{n'.1}\vdash p_{n'.1}\cdots p_{n'.\ell'}\vdash p_{n'.\ell'} \\ \vdots & & \vdots \\ \Pi^m_1\vdash\Sigma^m_1 & \cdots & \Pi^m_{n'}\vdash\Sigma^m_{n'} \end{array}}{\Pi^m\vdash\Sigma^m}\,R_m(Ax) \\
\vdots \\
\varphi\vdash\psi
\end{array}
$$

where:

(i) Skeleton(π) is the proof-subtree of π containing the root of π and applications of *invertible rules* for the introduction of all connectives occurring in the Skeleton of $\varphi \vdash \psi$ (possibly modulo applications of display rules);

(ii) PIA(π) is a collection of proof-subtrees of π containing the initial axioms of π and all the applications of *non-invertible rules* for the introduction of connectives occurring in the maximal PIA-subtrees (cf. Notation 2.2.15) in the signed generation trees of $\varphi \vdash \psi$ (possibly modulo applications of display rules) and such that

(iii) the root of each proof-subtree in PIA(π) coincides with a premise of the application of $R(Ax)$ in π, where the atomic structural variables are suitably instantiated with maximal PIA-subformulas of $\varphi \vdash \psi$.

Definition 2.2.29. A derivation π in D.DLE of the analytic inductive axiom $\varphi \vdash \psi$ (also indicated as Ax) is in *pre-normal form* if the unique application of each rule in its corresponding set of analytic structural rules $R_1(Ax), \ldots, R_m(Ax)$ computed by ALBA splits π into the following components:

$$
\begin{array}{l}
\text{PIA}(\pi)\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\text{Skeleton}(\pi)\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right.
\end{array}
\begin{array}{c}
\dfrac{\begin{array}{ccc} p_{1.1}\vdash p_{1.1}\cdots p_{1.k}\vdash p_{1.k} & \cdots & p_{n.1}\vdash p_{n.1}\cdots p_{n.\ell}\vdash p_{n.\ell} \\ \vdots & & \vdots \\ \Pi^1_1\vdash\Sigma^1_1 & \cdots & \Pi^1_n\vdash\Sigma^1_n \end{array}}{\Pi^1\vdash\Sigma^1}\,R_1(Ax)
\quad \cdots \quad
\dfrac{\begin{array}{ccc} p_{1.1}\vdash p_{1.1}\cdots p_{1.k'}\vdash p_{1.k'} & \cdots & p_{n'.1}\vdash p_{n'.1}\cdots p_{n'.\ell'}\vdash p_{n'.\ell'} \\ \vdots & & \vdots \\ \Pi^m_1\vdash\Sigma^m_1 & \cdots & \Pi^m_{n'}\vdash\Sigma^m_{n'} \end{array}}{\Pi^m\vdash\Sigma^m}\,R_m(Ax) \\
\vdots \\
\varphi\vdash\psi
\end{array}
$$

where:

(i) Skeleton(π) is the proof-subtree of π containing, possibly modulo applications of display rules, the root of π and applications of

(a) *invertible rules* for the introduction of all connectives occurring as SLR nodes in the Skeleton of $\varphi \vdash \psi$;

(b) *non-invertible rules* and Contraction for the introduction of all connectives occurring as Δ-adjoint nodes in the Skeleton of $\varphi \vdash \psi$;

(ii) PIA(π) is a collection of proof-subtrees of π containing, possibly modulo applications of display rules, the initial axioms of π and applications of

(a) *non-invertible rules* for the introduction of all connectives occurring as unary SRA nodes or as SRR nodes in the maximal PIA-subtrees in the signed generation trees of $\varphi \vdash \psi$;

(b) *invertible rules* and Weakening for the introduction of all lattice connectives occurring as SRA nodes in the maximal PIA-subtrees in the signed generation trees of $\varphi \vdash \psi$;

and such that

(iii) the root of each proof-subtree in PIA(π) coincides with a premise of the application of $R(ax)$ in π, where the atomic structural variables are suitably instantiated with operational maximal PIA-subtrees of $\varphi \vdash \psi$.

The key tools for obtaining the sub-derivations in PIA(π) introducing the connectives occurring as unary SRA nodes or as SRR nodes are given in Proposition 2.3.3 and Corollary 2.3.9. An inspection on the proofs of these results reveals that indeed only non-invertible logical rules and display rules are applied. The key tools involving the introduction of the lattice connectives occurring as SRA nodes in PIA(π) (resp. as Δ-adjoint nodes in Skeleton(π)) are given in Proposition 2.3.6 (resp. Proposition 2.3.12). Again, inspecting the proofs of these results reveals that only introduction rules of *one type* are applied in each component.

Remark 2.2.30. The binary introduction rules of $\underline{\text{D.LE}}$ for lattice connectives are invertible, while the corresponding rules of $\underline{\text{D.DLE}}$ are not, and Contraction is needed to derive these rules of $\underline{\text{D.LE}}$ in $\underline{\text{D.DLE}}$. Likewise, the unary introduction rules of $\underline{\text{D.LE}}$ for lattice connectives are not invertible, while the corresponding rules of $\underline{\text{D.DLE}}$ are, and so Weakening is needed to derive these rules of $\underline{\text{D.LE}}$ in $\underline{\text{D.DLE}}$. This is why derivations in pre-normal form of analytic inductive axioms in the general lattice setting of Definition 2.2.28 can be described purely in terms of invertible and non-invertible introduction rules, while in the distributive lattice setting of Definition 2.2.29, the occurrences of lattice connectives in 'Δ-adjoint/SRA-position' in the signed generation trees of a given analytic inductive axiom need to be accounted for separately (cf. clauses (b) of Definition 2.2.29). However, if $\varphi \vdash \psi$ is an analytic inductive axiom in the general lattice setting, applying the process described in Remark 2.2.27 to a derivation of $\varphi \vdash \psi$ in $\underline{\text{D.LE}}$ in pre-normal form according to Definition 2.2.28 results in a derivation of $\varphi \vdash \psi$ in $\underline{\text{D.DLE}}$ which is in pre-normal form according to Definition 2.2.29.

Remark 2.2.31. If $\varphi \vdash \psi$ is a *definite* analytic inductive axiom, then ALBA yields a *single* analytic structural rule corresponding to it. So, both in the general lattice and in the distributive settings, the Skeleton part of the derivation of $\varphi \vdash \psi$ in pre-normal form will only have one branch, yielding the following simpler shape of π:

$$
\begin{array}{rc}
\mathrm{PIA}(\pi)\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. &
\begin{array}{ccc}
p_{1.1}\vdash p_{1.1} \quad \cdots \quad p_{1.k}\vdash p_{1.k} & \cdots & p_{n.1}\vdash p_{n.1} \quad \cdots \quad p_{n.\ell}\vdash p_{n.\ell} \\
\ddots\vdots\iddots & & \ddots\vdots\iddots \\
\Pi_1\vdash\Sigma_1 & \cdots & \Pi_n\vdash\Sigma_n \\ \hline
\end{array}\; R(Ax) \\
\mathrm{Skeleton}(\pi)\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. &
\begin{array}{c}
\Pi\vdash\Sigma \\
\vdots \\
\varphi\vdash\psi
\end{array}
\end{array}
$$

All derivations in Examples 2.4.7 and 2.4.11 are derivations of definite analytic inductive axioms in pre-normal form.

2.3 Properties of the basic display calculi D.LE

In this section, we will state and prove the key lemmas needed for the proof of the syntactic completeness. Throughout this section, we let $\mathcal{L}_{\mathrm{LE}}$ (resp. $\mathcal{L}_{\mathrm{DLE}}$) be an arbitrary but fixed (D)LE-language, and D.LE (resp. D.DLE) denote the proper display calculi for the basic $\mathcal{L}_{\mathrm{LE}}$-logic (resp. $\mathcal{L}_{\mathrm{DLE}}$-logic).

Notation 2.3.1. *For any definite Skeleton (resp. definite PIA) formula φ (resp. $\psi, \gamma, \delta, \xi, \ldots$), we let its corresponding capital Greek letter Φ (resp. $\Psi, \Gamma, \Delta, \Xi, \ldots$) denote its structural counterpart, defined by induction as follows (cf. Notation 2.2.8):*

1. *if $\varphi := p \in \mathsf{AtProp}$, then $\Phi := p$;*
2. *if $\varphi := f(\overline{\xi}, \overline{\psi})$, then $\Phi := \hat{f}(\overline{\Xi}, \overline{\Psi})$;*
3. *if $\varphi := g(\overline{\psi}, \overline{\xi})$, then $\Phi := \check{g}(\overline{\Psi}, \overline{\Xi})$.*

Notice that items 2 and 3 above cover also the case of zero-ary connectives (and of $\wedge$ and $\vee$ in the setting of D.DLE*).*

Also, notice that the introduction rules of D.LE (resp. D.DLE) are such that structural counterparts of connectives in $\mathcal{F}$ (resp. in $\mathcal{F}\cup\{\wedge\}$) can only occur in precedent position, and structural counterparts of connectives in $\mathcal{G}$ (resp. in $\mathcal{G}\cup\{\vee\}$) can only occur in succedent position, which is why Notation 2.3.1 only applies to definite Skeleton and definite PIA formulas.

Notation 2.3.2. *In what follows, we let $\overline{\sigma}$, $\overline{S}$ and $\overline{\sigma\vdash S}$ (resp. $\overline{\tau}$, $\overline{U}$ and $\overline{U\vdash\tau}$) denote finite vectors of formulas, of structures in $\mathsf{Str}_{\mathcal{G}}$ (resp. $\mathsf{Str}_{\mathcal{F}}$) and of* D.(D)LE*-sequents.*

Proposition 2.3.3. *For every definite positive PIA (i.e. definite negative Skeleton) formula $\gamma(!\overline{x}, !\overline{y})$ and every definite negative PIA (i.e. a definite positive Skeleton) formula $\delta(!\overline{y}, !\overline{x})$,*

1. *if $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$ are derivable in* D.LE *(resp.* D.DLE*), then so is $\gamma[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Gamma[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]$;*
2. *if $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$ are derivable in* D.LE *(resp.* D.DLE*), then so is $\Delta[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \delta[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$.*

with derivations such that, if any rules are applied other than right-introduction rules for negative SRR-connectives and negative unary SRA-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), and left-introduction rules for positive SRR-connectives and positive unary SRA-connectives, then they are applied only in the derivations of $\overline{\sigma \vdash S}$ *and* $\overline{U \vdash \tau}$.

Proof. By simultaneous induction on γ and δ. If $\gamma := x$, then $\Gamma := x$. Hence, $\gamma[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Gamma[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]$ reduces to $\sigma \vdash S$, which is derivable by assumption. The case of $\delta := y$ is shown similarly. As to the inductive steps, let $\gamma(!\overline{x}, !\overline{y}) := g(\overline{\psi}(!\overline{x}, !\overline{y}), \overline{\xi}(!\overline{x}, !\overline{y}))$ with $\overline{\psi}$ definite positive PIA-formulas and $\overline{\xi}$ definite negative PIA-formulas. Then $\gamma[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] = g(\overline{\psi}[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}], \overline{\xi}[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}])$ and $\Gamma[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}] = \check{g}(\overline{\Psi}[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}], \overline{\Xi}[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}])$.

By the induction hypothesis, all sequents in the following vectors are derivable in D.LE (resp. D.DLE):

$$\overline{\psi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Psi[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]} \quad \text{and} \quad \overline{\Xi[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}.$$

Then we can derive the required sequent $\gamma[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Gamma[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]$ by prolonging all these derivations with an application of g_L as follows:

$$g_L\,\frac{\overline{\psi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Psi[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]} \qquad \overline{\Xi[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}}{g(\overline{\psi}[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}], \overline{\xi}[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]) \vdash \check{g}(\overline{\Psi}[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}], \overline{\Xi}[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}])}$$

Let $\delta(!\overline{x}, !\overline{y}) := f(\overline{\xi}(!\overline{x}, !\overline{y}), \overline{\psi}(!\overline{x}, !\overline{y}))$ with $\overline{\xi}$ definite negative PIA-formulas (i.e. positive Skeleton-formulas) and $\overline{\psi}$ definite positive PIA-formulas (i.e. negative Skeleton-formulas). Then $\delta[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] = f(\overline{\xi}[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}], \overline{\psi}[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}])$ and $\Delta[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] = \hat{f}(\overline{\Xi}[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}], \overline{\Psi}[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}])$. By the induction hypothesis, all sequents in the following vectors are derivable in D.LE (resp. D.DLE):

$$\overline{\psi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Psi[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]} \quad \text{and} \quad \overline{\Xi[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}]) \vdash \xi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}.$$

Then we can derive the required sequent $\Delta[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \delta[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$ by prolonging all these derivations with an application of f_R as follows:

$$\frac{\overline{\Xi[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]} \qquad \overline{\psi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Psi[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]}}{\hat{f}(\overline{\Xi}[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}], \overline{\Psi}[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]) \vdash f(\overline{\xi}[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}], \overline{\psi}[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}])}\, f_R$$

The proof, specific to the setting of D.DLE, of the case in which $\gamma := \gamma_1 \vee \gamma_2$ (resp. $\delta := \delta_1 \wedge \delta_2$) goes like the case of arbitrary $g \in \mathcal{G}$ (resp. $f \in \mathcal{F}$) discussed above, using the D.DLE-rule $\vee_L$ (resp. $\wedge_R$). □

By instantiating $\overline{\sigma \vdash S}$ and $\overline{U \vdash \tau}$ in the proposition above to identity axioms, we immediately get the following

Corollary 2.3.4. *Any calculus* D.LE *(resp.* D.DLE*) derives the following sequents (cf. Notation 2.3.1):*

1. $\gamma \vdash \Gamma$ *for every definite positive PIA (i.e. a definite negative Skeleton) formula* γ*;*
2. $\Delta \vdash \delta$ *for every definite negative PIA (i.e. a definite positive Skeleton) formula* δ*,*

with derivations which only consist of identity axioms, and applications of right-introduction rules for negative SRR-connectives and negative unary SRA-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), and left-introduction rules for positive SRR-connectives and positive unary SRA-connectives.

Example 2.3.5. The formula $\Diamond(p \otimes q) \to (q \oplus p)$ is definite positive PIA in any (D)LE-language such that $\Diamond, \otimes \in \mathcal{F}$ and $\oplus, \to \in \mathcal{G}$ with $n_\Diamond = 1$ and $\varepsilon_\Diamond(1) = 1$, and $n_\otimes = n_\oplus = n_\to = 2$ and $\varepsilon_\circ(i) = 1$ for every $\circ \in \{\otimes, \oplus, \to\}$ and every $1 \leq i \leq 2$ except $\varepsilon_\to(1) = \partial$. Then, instantiating the argument above, we can derive the sequent $\Diamond(p \otimes q) \to (q \oplus p) \vdash \hat{\Diamond}(p \hat{\otimes} q) \check{\to} (q \check{\oplus} p)$ in D.LE (resp. D.DLE) as follows:

$$
\to_L \dfrac{\dfrac{\dfrac{p \vdash p \qquad q \vdash q}{p \hat{\otimes} q \vdash p \otimes q}\,\otimes_R}{\hat{\Diamond}(p \hat{\otimes} q) \vdash \Diamond(p \otimes q)}\,\Diamond_R \qquad \oplus_L\,\dfrac{p \vdash p \qquad q \vdash q}{p \oplus q \vdash p \check{\oplus} q}}{\Diamond(p \otimes q) \to (p \oplus q) \vdash \hat{\Diamond}(p \hat{\otimes} q) \check{\to} p \check{\oplus} q}
$$

The formula $\Diamond p \otimes q$ in the same language is definite negative PIA. Then, instantiating the argument above, we can derive the sequent $\hat{\Diamond} p \hat{\otimes} q \vdash \Diamond p \otimes q$ in D.LE (resp. D.DLE) as follows:

$$
\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R \qquad q \vdash q}{\hat{\Diamond} p \hat{\otimes} q \vdash \Diamond p \otimes q}\,\otimes_R
$$

Proposition 2.3.6. *Let $\gamma = \gamma(!\overline{x}, !\overline{y})$ and $\delta = \delta(!\overline{y}, !\overline{x})$ be a positive and a negative PIA formula, respectively, and let $\bigwedge_{i \in I} \gamma_i$ and $\bigvee_{j \in J} \delta_j$ be their equivalent rewritings as per Lemma 2.2.13, so that each γ_i (resp. each δ_j) is definite positive (resp. negative) PIA.*

1. *If $\overline{\sigma \vdash S}$ and $\overline{U \vdash \tau}$ are derivable in D.LE (resp. D.DLE), then so is $\gamma[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Gamma_i[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}]$ for each $i \in I$;*

2. *if $\overline{\sigma \vdash S}$ and $\overline{U \vdash \tau}$ are derivable in D.LE (resp. D.DLE), then so is $\Delta_j[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \delta[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$ for each $j \in J$,*

with derivations such that, if any rules are applied other than right-introduction rules for negative PIA-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE), then they are applied only in the derivations of $\overline{\sigma \vdash S}$ and $\overline{U \vdash \tau}$.

Proof. Let $n_\gamma(+\wedge)$ (resp. $n_\delta(+\wedge)$) be the number of occurrences of $+\wedge$ in $+\gamma$ (resp. $-\delta$), and let $n_\gamma(-\vee)$ (resp. $n_\delta(-\vee)$) be the number of occurrences of $-\vee$ in $+\gamma$ (resp. $-\delta$). The proof is by simultaneous induction on $n_\gamma = n_\gamma(+\wedge) + n_\gamma(-\vee)$ and $n_\delta = n_\delta(+\wedge) + n_\delta(-\vee)$.

If $n_\gamma = n_\delta = 0$, then γ (resp. δ) is definite positive (resp. negative) PIA. Then the claims follow from Proposition 2.3.3.

If $n_\gamma \geq 1$, then let us consider one occurrence of $+\wedge$ or $-\vee$ in $+\gamma$, which we will refer to as 'the focal occurrence'. Let us assume that the focal occurrence of $+\wedge$ or $-\vee$ in $+\gamma$ is an occurrence of $-\vee$ (the case in which it is an occurrence of $+\wedge$ is argued similarly).

Let $-\xi'$ and $-\xi''$ be the two subtrees under the focal occurrence of $-\vee$. Then $\xi' \vee \xi''$ is a subformula of γ such that ξ' and ξ'' are negative PIA formulas, and $n_{\xi'}$ and $n_{\xi''}$ are strictly smaller than n_γ. Let u be a fresh variable which does not occur in γ, and let γ' be the formula obtained by substituting the occurrence of $\xi' \vee \xi''$ in γ with u. Then γ' is a positive PIA formula such that $n_{\gamma'}$ is strictly smaller than n_γ, and $\gamma = \gamma'[(\xi' \vee \xi'')/!u]$. Let $\bigwedge_{i\in I}\gamma_i$, $\bigwedge_{j\in J}\gamma'_j$, $\bigvee_{h\in H}\xi'_h$ and $\bigvee_{k\in K}\xi''_k$ be the equivalent rewritings of γ, γ', ξ' and ξ'', respectively, resulting from distributing exhaustively $+\wedge$ and $-\vee$ over each connective in γ, γ', ξ' and ξ'', respectively. Then,

$$\{\gamma_i \mid i \in I\} = \{\gamma'_j[\xi'_h/!u] \mid j \in J \text{ and } h \in H\} \cup \{\gamma'_j[\xi''_k/!u] \mid j \in J \text{ and } k \in K\}.$$

By the induction hypothesis, the following sequents are derivable in D.LE (resp. D.DLE) for every $h \in H$ and $k \in K$:

$$\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \quad \text{and} \quad \Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}].$$

Then, by prolonging the derivations of the two sequents above with suitable applications of $(\vee_{R1})$ and $(\vee_{R2})$, we obtain derivations in D.LE (resp. D.DLE)[23] of the following sequents for every $h \in H$ and $k \in K$:

$$\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash (\xi' \vee \xi'')[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \quad \text{and} \quad \Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash (\xi' \vee \xi'')[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]. \tag{2.3.1}$$

$$\frac{\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}{\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \vee \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ \vee_{R1}$$

$$\frac{\Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}{\Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \vee \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ \vee_{R2}$$

By the induction hypothesis on γ', the following sequents are also derivable in D.LE (resp. D.DLE) for every $j \in J$, $h \in H$ and $k \in K$:

$$\gamma'[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}, (\xi' \vee \xi'')[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]/!u] \vdash \Gamma'_j[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}, \Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}]/!u] \quad \text{and}$$
$$\gamma'[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}, (\xi' \vee \xi'')[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]/!u] \vdash \Gamma'_j[\overline{S}/!\overline{x}, \overline{U}/!\overline{y}, \Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}]/!u].$$

which is enough to prove the statement, since $\gamma = \gamma'[(\xi' \vee \xi'')/!u]$, and for every $i \in I$, either $\gamma_i = \gamma'_j[\xi'_h/!u]$ for some $j \in J$ and $h \in H$, or $\gamma_i = \gamma'_j[\xi'_k/!u]$ for some $j \in J$ and $k \in K$. The induction step for $n_\delta \geq 1$ is similar to the induction step above. ◻

[23] In the calculus D.DLE, we obtain the derivations of the sequents in (2.3.1) by replacing the applications of the derivable rules $(\vee_{R1})$ and $(\vee_{R2})$ with the derivations of those applications as shown in Remark 2.2.26, thereby implementing the general strategy outlined in Remark 2.2.27. That is:

$$\dfrac{\dfrac{\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}{\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \,\check{\vee}\, \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ W_R}{\Xi_h[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \vee \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ \vee_R$$

$$\dfrac{\dfrac{\dfrac{\Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}{\Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \,\check{\vee}\, \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ W_R}{\Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \,\check{\vee}\, \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ E_R}{\Xi_k[\overline{U}/!\overline{y}, \overline{S}/!\overline{x}] \vdash \xi'[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \vee \xi''[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}\ \vee_R$$

By instantiating $\overline{\sigma \vdash S}$ and $\overline{U \vdash \tau}$ in the proposition above to identity axioms, we immediately get the following

Corollary 2.3.7. *For any positive (resp. negative) PIA formula γ (resp. δ), let $\bigwedge_{i\in I}\gamma_i$ (resp. $\bigvee_{j\in J}\delta_j$) be its equivalent rewriting as per Lemma 2.2.13, so that each γ_i (resp. δ_j) is definite positive (resp. negative) PIA. Then the following sequents are derivable in* $\underline{\text{D.LE}}$ *(resp.* $\underline{\text{D.DLE}}$*, cf. Notation 2.3.1):*

1. $\gamma \vdash \Gamma_i$ *for every $i \in I$;*
2. $\Delta_j \vdash \delta$ *for every $j \in J$,*

with derivations which only consist of identity axioms, and applications of right- introduction rules for negative PIA-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of $\underline{\text{D.DLE}}$*).*

Example 2.3.8. The formula $\Box(p \wedge q)$ is a positive PIA in any (D)LE-language such that $\Box \in \mathcal{G}$, and is equivalent to $\Box p \wedge \Box q$. Since $p \vdash \check{\blacksquare}\Diamond p$ and $q \vdash \check{\blacksquare}\Diamond q$ are derivable sequents

$$\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R}{p \vdash \check{\blacksquare}\Diamond p} \qquad \dfrac{\dfrac{q \vdash q}{\hat{\Diamond} q \vdash \Diamond q}\,\Diamond_R}{q \vdash \check{\blacksquare}\Diamond q}$$

instantiating the argument in Proposition 2.3.6, we can derive the sequents $\Box(p\wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p$ and $\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond q$ in D.LE and in D.DLE as follows:

$\underline{\text{D.LE}}$-derivation of $\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p$:

$$\Box_L\,\dfrac{\wedge_L\,\dfrac{\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R}{p \vdash \check{\blacksquare}\Diamond p}}{p \wedge q \vdash \check{\blacksquare}\Diamond p}}{\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p}$$

$\underline{\text{D.DLE}}$-derivation of $\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p$:

$$\Box_L\,\dfrac{\wedge_L\,\dfrac{W_L\,\dfrac{\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R}{p \vdash \check{\blacksquare}\Diamond p}}{p \mathbin{\hat{\wedge}} q \vdash \check{\blacksquare}\Diamond p}}{p \wedge q \vdash \check{\blacksquare}\Diamond p}}{\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p}$$

$\underline{\text{D.LE}}$-derivation of $\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond q$:

$$\Box_L\,\dfrac{\wedge_L\,\dfrac{\dfrac{\dfrac{q \vdash q}{\hat{\Diamond} q \vdash \Diamond q}\,\Diamond_R}{q \vdash \check{\blacksquare}\Diamond q}}{p \wedge q \vdash \check{\blacksquare}\Diamond q}}{\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond q}$$

$\underline{\text{D.DLE}}$-derivation of $\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond q$:

$$\Box_L\,\dfrac{\wedge_L\,\dfrac{E_L\,\dfrac{W_L\,\dfrac{\dfrac{\dfrac{q \vdash q}{\hat{\Diamond} q \vdash \Diamond q}\,\Diamond_R}{q \vdash \check{\blacksquare}\Diamond q}}{q \mathbin{\hat{\wedge}} p \vdash \check{\blacksquare}\Diamond q}}{p \mathbin{\hat{\wedge}} q \vdash \check{\blacksquare}\Diamond q}}{p \wedge q \vdash \check{\blacksquare}\Diamond q}}{\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond q}$$

In the remainder of the present section, if $\varphi(!\overline{x}, !\overline{y})$ (resp. $\psi(!\overline{y}, !\overline{x})$) is a definite positive (resp. negative) PIA formula, we will need to fix one variable in $\overline{x}$ or in $\overline{y}$ and make it the pivotal variable for the computation of the corresponding $\mathsf{la}(\varphi)(u, \overline{z})$ (resp. $\mathsf{ra}(\psi)(u, \overline{z})$), where the vector $\overline{z}$ of parametric variables exactly includes all the placeholder variables in $\overline{x}$ and in $\overline{y}$ different from the pivotal one. So we write e.g. φ_x (resp. φ_y) to indicate that we are choosing the pivotal variable among the variables in $\overline{x}$ (resp. $\overline{y}$). In order to simplify the notation, we leave it to be understood that the set of parametric variables does not contain the pivotal one, although we do not make this fact explicit in the notation. In the remainder of the paper, we will let e.g. $\mathsf{LA}(\varphi)(u,\overline{z})$ denote the structural counterpart of $\mathsf{la}(\varphi)(u,\overline{z})$ (cf. Definition 2.2.19).

Corollary 2.3.9. *Let $\psi(!\overline{x},!\overline{y})$ and $\xi(!\overline{y},!\overline{x})$ be a positive and a negative PIA formula respectively, and let $\bigwedge_{i\in I}\psi_i$ and $\bigvee_{j\in J}\xi_j$ be their equivalent rewritings as per Lemma 2.2.13, so that each φ_i (resp. ξ_j) is a definite positive (resp. negative) PIA formula. Then:*

1. *if $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$ are derivable in* <u>D.LE</u> *(resp.* <u>D.DLE</u>*), then so is*

$$\mathsf{LA}(\psi_i)[\psi_x[\overline{\sigma}/!\overline{x},\overline{\tau}/!\overline{y}]/!u,\overline{S}/!\overline{x},\overline{U}/!\overline{y}]\vdash S_x,$$

where ψ_i is the definite positive PIA formula in which the pivotal variable x occurs;

2. *if $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$ are derivable in* <u>D.LE</u> *(resp.* <u>D.DLE</u>*), then so is*

$$U_y\vdash\mathsf{LA}(\psi_i)[\psi_y[\overline{\sigma}/!\overline{x},\overline{\tau}/!\overline{y}]/!u,\overline{S}/!\overline{x},\overline{U}/!\overline{y}],$$

where ψ_i is the definite positive PIA formula in which the pivotal variable y occurs;

3. *if $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$ are derivable in* <u>D.LE</u> *(resp.* <u>D.DLE</u>*), then so is*

$$\mathsf{RA}(\xi_j)[\xi_x[\overline{\tau}/!\overline{y},\overline{\sigma}/!\overline{x}]/!u,\overline{U}/!\overline{y},\overline{S}/!\overline{x}]\vdash S_x,$$

where ξ_j is the definite negative PIA formula in which the pivotal variable x occurs;

4. *if $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$ are derivable in* <u>D.LE</u> *(resp.* <u>D.DLE</u>*), then so is*

$$U_y\vdash\mathsf{RA}(\xi_j)[\xi_y[\overline{\tau}/!\overline{y},\overline{\sigma}/!\overline{x}]/!u,\overline{U}/!\overline{y},\overline{S}/!\overline{x}],$$

where ξ_j is the definite negative PIA formula in which the pivotal variable y occurs,

with derivations such that, if any rules are applied other than display rules, right- introduction rules for negative PIA-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of <u>D.DLE</u>*), then they are applied only in the derivations of $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$.*

Proof. 1. Let Ψ_x denote the structural counterpart of ψ_x (cf. Notation 2.3.1). The assumptions imply, by Proposition 2.3.6, that the sequent $\psi_x[\overline{\sigma}/!\overline{x},\overline{\tau}/!\overline{y}]\vdash\Psi_i[\overline{S}/!\overline{x},\overline{U}/!\overline{y}]$ is derivable in <u>D.LE</u> (resp. <u>D.DLE</u>) with a derivation such that, if any rules are applied other than right-introduction rules for negative PIA-connectives and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of <u>D.DLE</u>), then they are applied only in the derivations of $\overline{\sigma\vdash S}$ and $\overline{U\vdash\tau}$. Then, we can prolong this derivation by applying display rules to each node of the branch of Ψ_i leading to the pivotal variable x, so as to obtain a derivation of the required sequent

$$\mathsf{LA}(\psi_i)[\psi_x[\overline{\sigma}/!\overline{x},\overline{\tau}/!\overline{y}]/u,\overline{S}/!\overline{x},\overline{U}/!\overline{y}]\vdash S_x.$$

The remaining items are proved similarly. ◻

By instantiating $\overline{\sigma \vdash S}$ and $\overline{U \vdash \tau}$ in the corollary above to identity axioms, we immediately get the following

Corollary 2.3.10. *The following sequents are derivable in* D.LE *(resp.* D.DLE*) for any positive PIA (i.e. negative Skeleton) formula $\psi(!\overline{x}, !\overline{y})$ and any negative PIA (i.e. positive Skeleton) formula $\xi(!\overline{y}, !\overline{x})$, such that $\bigwedge_{i\in I}\psi_i$ and $\bigvee_{j\in J}\xi_j$ are their equivalent rewritings as per Lemma 2.2.13, so that each ψ_i (resp. ξ_j) is a definite positive (resp. negative) PIA formula.*

1. $\mathsf{LA}(\psi_i)[\psi_x/u] \vdash x$*, where ψ_i is the definite positive PIA formula in which the pivotal variable x occurs;*

2. $y \vdash \mathsf{LA}(\psi_i)[\psi_y/u]$*, where ψ_i is the definite positive PIA formula in which the pivotal variable y occurs;*

3. $\mathsf{RA}(\xi_j)[\xi_x/u] \vdash x$*, where ξ_j is the definite negative PIA formula in which the pivotal variable x occurs;*

4. $y \vdash \mathsf{RA}(\xi_j)[\xi_y/u]$*, where ξ_j is the definite negative PIA formula in which the pivotal variable y occurs,*

with derivations which only consist of identity axioms, and applications of display rules, right-introduction rules for negative PIA-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE*).*

Example 2.3.11. The formula $\Box((\Box(p \wedge q)) \oplus r)$ is a positive PIA in any (D)LE-language such that $\Box, \oplus \in \mathcal{G}$, and is equivalent to $\Box((\Box p) \oplus r) \wedge \Box((\Box q) \oplus r)$. Let $x = p$, then $\mathsf{LA}(\Box((\Box p) \oplus r))[\Box((\Box(p \wedge q)) \oplus r)/u] = \hat{\blacklozenge}(\hat{\blacklozenge}\Box((\Box(p \wedge q)) \oplus r) \mathbin{\hat{\prec}} r)$, where $\hat{\prec}$ is the left residual of $\oplus$ on the first coordinate. As shown in Example 2.3.8 $p \vdash \check{\blacksquare}\Diamond p$ and $r \vdash \check{\blacksquare}\Diamond r$ are derivable sequents. Instantiating the argument in Corollary 2.3.9, we can derive the sequent $\hat{\blacklozenge}(\hat{\blacklozenge}\Box((\Box(p \wedge q)) \oplus r) \mathbin{\hat{\prec}} \check{\blacksquare}\Diamond r) \vdash \check{\blacksquare}\Diamond p$ as follows:

D.LE-derivation of $\hat{\blacklozenge}(\hat{\blacklozenge}\Box((\Box(p \wedge q)) \oplus r) \mathrel{\hat{-\!\!\!<}} \check{\blacksquare}\Diamond r) \vdash \check{\blacksquare}\Diamond p$:

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond} p \vdash \Diamond p$}
\UnaryInfC{$p \vdash \check{\blacksquare}\Diamond p$}
\LeftLabel{$\wedge_L$}
\UnaryInfC{$p \wedge q \vdash \check{\blacksquare}\Diamond p$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p$}
\AxiomC{$r \vdash r$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond} r \vdash \Diamond r$}
\UnaryInfC{$r \vdash \check{\blacksquare}\Diamond r$}
\LeftLabel{$\oplus_L$}
\BinaryInfC{$\Box(p \wedge q) \oplus r \vdash \check{\Box}\check{\blacksquare}\Diamond p \,\check{\oplus}\, \check{\blacksquare}\Diamond r$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box(\Box(p \wedge q) \oplus r) \vdash \check{\Box}(\check{\Box}\check{\blacksquare}\Diamond p \,\check{\oplus}\, \check{\blacksquare}\Diamond r)$}
\UnaryInfC{$\hat{\blacklozenge}\Box(\Box(p \wedge q) \oplus r) \vdash \check{\Box}\check{\blacksquare}\Diamond p \,\check{\oplus}\, \check{\blacksquare}\Diamond r$}
\UnaryInfC{$\hat{\blacklozenge}\Box(\Box(p \wedge q) \oplus r) \mathrel{\hat{-\!\!\!<}} \check{\blacksquare}\Diamond r \vdash \check{\Box}\check{\blacksquare}\Diamond p$}
\UnaryInfC{$\hat{\blacklozenge}(\hat{\blacklozenge}\Box(\Box(p \wedge q) \oplus r) \mathrel{\hat{-\!\!\!<}} \check{\blacksquare}\Diamond r) \vdash \check{\blacksquare}\Diamond p$}
\end{prooftree}
$$

D.DLE-derivation of $\hat{\blacklozenge}(\hat{\blacklozenge}\Box((\Box(p \wedge q)) \oplus r) \mathrel{\hat{-\!\!\!<}} \check{\blacksquare}\Diamond r) \vdash \check{\blacksquare}\Diamond p$:

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond} p \vdash \Diamond p$}
\UnaryInfC{$p \vdash \check{\blacksquare}\Diamond p$}
\LeftLabel{W_L}
\UnaryInfC{$p \,\hat{\wedge}\, q \vdash \check{\blacksquare}\Diamond p$}
\LeftLabel{$\wedge_L$}
\UnaryInfC{$p \wedge q \vdash \check{\blacksquare}\Diamond p$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p$}
\AxiomC{$r \vdash r$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond} r \vdash \Diamond r$}
\UnaryInfC{$r \vdash \check{\blacksquare}\Diamond r$}
\LeftLabel{$\oplus_L$}
\BinaryInfC{$\Box(p \wedge q) \oplus r \vdash \check{\Box}\check{\blacksquare}\Diamond p \,\check{\oplus}\, \check{\blacksquare}\Diamond r$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box(\Box(p \wedge q) \oplus r) \vdash \check{\Box}(\check{\Box}\check{\blacksquare}\Diamond p \,\check{\oplus}\, \check{\blacksquare}\Diamond r)$}
\UnaryInfC{$\hat{\blacklozenge}\Box(\Box(p \wedge q) \oplus r) \vdash \check{\Box}\check{\blacksquare}\Diamond p \,\check{\oplus}\, \check{\blacksquare}\Diamond r$}
\UnaryInfC{$\hat{\blacklozenge}\Box(\Box(p \wedge q) \oplus r) \mathrel{\hat{-\!\!\!<}} \check{\blacksquare}\Diamond r \vdash \check{\Box}\check{\blacksquare}\Diamond p$}
\UnaryInfC{$\hat{\blacklozenge}(\hat{\blacklozenge}\Box(\Box(p \wedge q) \oplus r) \mathrel{\hat{-\!\!\!<}} \check{\blacksquare}\Diamond r) \vdash \check{\blacksquare}\Diamond p$}
\end{prooftree}
$$

Proposition 2.3.12. *Let $\varphi = \varphi(!\overline{x}, !\overline{y})$ and $\psi = \psi(!\overline{y}, !\overline{x})$ be a positive and a negative Skeleton formula, respectively, and let $\bigvee_{j\in J} \varphi_j$ and $\bigwedge_{i\in I} \psi_i$ be their equivalent rewritings as per Lemma 2.2.13, so that each φ_j (resp. each ψ_i) is definite positive (resp. negative) Skeleton. Then:*

1. *if $\Phi_j[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Sigma$ is derivable in D.LE (resp. D.DLE) for every $j \in J$, then so is $\varphi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Sigma$;*

2. *if $\Pi \vdash \Psi_i[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$ is derivable in D.LE (resp. D.DLE) for every $i \in I$, then so is $\Pi \vdash \psi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$,*

with derivations such that, if any rules are applied other than display rules, left- introduction rules for positive Skeleton-connectives (cf. Tables 3.1 and 2.2, and Definition 3.5.1), right-introduction rules for negative Skeleton-connectives, (and contraction in the case of D.DLE), then they are applied only in the derivations of $\Phi_j[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Sigma$ and $\Pi \vdash \Psi_i[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$.

Proof. Let $n_\varphi(+\vee)$ (resp. $n_\psi(+\vee)$) be the number of occurrences of $+\vee$ in $+\varphi$ (resp. $-\psi$), and let $n_\varphi(-\wedge)$ (resp. $n_\psi(-\wedge)$) be the number of occurrences of $-\wedge$ in $+\varphi$ (resp. $-\psi$). The proof is by simultaneous induction on $n_\varphi = n_\varphi(+\vee) + n_\varphi(-\wedge)$ and $n_\psi = n_\psi(+\vee) + n_\psi(-\wedge)$.

If $n_\psi = n_\varphi = 0$, then φ (resp. ψ) is definite positive (resp. negative) Skeleton. Then from a derivation of $\Phi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Sigma$ (resp. $\Pi \vdash \Psi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$) we obtain a derivation of $\varphi[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \Sigma$ (resp. $\Pi \vdash \psi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$) by applications of left-introduction rules for positive SLR-connectives, and right-introduction rules for negative SLR-connectives, interleaved with applications of display rules.

If $n_\psi \geq 1$, then let us consider one occurrence of $-\wedge$ or $+\vee$ in $-\psi$, which we will refer to as 'the focal occurrence'. Let us assume that the focal occurrence of $-\wedge$ or $+\vee$ in $-\psi$ is an occurrence of $+\vee$ (the case in which it is an occurrence of $-\wedge$ is argued similarly). Let $+\xi'$ and $+\xi''$ be the two subtrees under the focal occurrence of $+\vee$. Then $\xi' \vee \xi''$ is a subformula of ψ such that ξ' and ξ'' are positive Skeleton formulas, and $n_{\xi'}$ and $n_{\xi''}$ are strictly smaller than n_ψ. Let u be a fresh variable which does not occur in ψ, and let ψ' be the formula obtained by substituting the occurrence of $\xi' \vee \xi''$ in ψ with u. Then ψ' is a negative Skeleton formula such that $n_{\psi'}$ is strictly smaller than n_ψ, and $\psi = \psi'[(\xi' \vee \xi'')/!u]$.

Let $\bigwedge_{i\in I} \psi_i$, $\bigwedge_{j\in J} \psi'_j$, $\bigvee_{h\in H} \xi'_h$ and $\bigvee_{k\in K} \xi''_k$ be the equivalent rewritings of ψ, ψ', ξ' and ξ'', respectively, resulting from applying Lemma 2.2.13 to ψ, ψ', ξ' and ξ'', respectively. Then,

$$\{\psi_i \mid i \in I\} = \{\psi'_j[\xi'_h/!u] \mid j \in J \text{ and } h \in H\} \cup \{\psi'_j[\xi''_k/!u] \mid j \in J \text{ and } k \in K\}.$$

Hence, the assumptions can be equivalently reformulated as the following sequents being derivable in D.LE (resp. D.DLE) for every $j \in J$, $h \in H$, and $k \in K$:

$$\Pi \vdash \Psi'_j[\Xi'_h/!u] \qquad \Pi \vdash \Psi'_j[\Xi''_k/!u].$$

By prolonging those derivations with consecutive applications of display rules, we obtain derivations in D.LE (resp. D.DLE) of the following sequents, for every $j \in J$, $h \in H$, and $k \in K$:

$$\Xi'_h \vdash \mathsf{LA}(\psi'_j)[\Pi/!v] \qquad \Xi''_k \vdash \mathsf{LA}(\psi'_j)[\Pi/!v].$$

Hence, by the induction hypothesis on ξ' and ξ'', the following sequents are derivable in D.LE (resp. D.DLE) for every $j \in J$:

$$\xi' \vdash \mathsf{LA}(\psi'_j)[\Pi/!v] \qquad \xi'' \vdash \mathsf{LA}(\psi'_j)[\Pi/!v].$$

Then, by prolonging the derivations of the two sequents above with suitable applications of $(\vee_L)$, we obtain derivations in D.LE (resp. D.DLE)[24] of the following sequents for every $j \in J$:

$$\frac{\xi'[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \mathsf{LA}(\psi'_j)[\Pi/!v, \overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}] \qquad \xi''[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \mathsf{LA}(\psi'_j)[\Pi/!v, \overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]}{(\xi' \vee \xi'')[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}] \vdash \mathsf{LA}(\psi'_j)[\Pi/!v, \overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]} \vee_L$$

By prolonging the derivations above with consecutive applications of display rules we obtain derivations in D.LE (resp. D.DLE) of the following sequents for every $j \in J$:

$$\Pi \vdash \Psi'_j[(\xi_1 \vee \xi_2)[\overline{\sigma}/!\overline{x}, \overline{\tau}/!\overline{y}]/!v, \overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}].$$

By the induction hypothesis on ψ', and recalling that $\psi = \psi'[(\xi' \vee \xi'')/!u]$, we can conclude that $\Pi \vdash \psi[\overline{\tau}/!\overline{y}, \overline{\sigma}/!\overline{x}]$ is derivable, as required. □

[24] In the calculus D.DLE, we obtain the derivations of the sequents in (2.3.1) by replacing the applications of the derivable rules $(\vee_L)$ with the derivations of those applications as shown in Remark 2.2.26, thereby implementing the general strategy outlined in Remark 2.2.27, which in this specific case involves the application of structural contraction rule.

Example 2.3.13. The formula $\Diamond(p \vee \Diamond p)$ is a negative PIA in any (D)LE-language such that $\Diamond \in \mathcal{F}$, and is equivalent to $\Diamond p \vee \Diamond\Diamond p$. Assuming that $\hat{\Diamond} p \vdash \check{\Box}\Diamond p$ and $\hat{\Diamond}\hat{\Diamond} p \vdash \check{\Box}\Diamond p$ are derivable sequents, instantiating the argument in Proposition 2.3.12, we can derive the sequent $\Diamond(p \vee \Diamond p) \vdash \check{\Box}\Diamond p$ in D.LE as follows:

$$
\begin{prooftree}
\AxiomC{$\hat{\Diamond} p \vdash \check{\Box}\Diamond p$}
\LeftLabel{$\hat{\Diamond} \dashv \check{\blacksquare}$}
\UnaryInfC{$p \vdash \check{\blacksquare}\check{\Box}\Diamond p$}
\AxiomC{$\hat{\Diamond}\hat{\Diamond} p \vdash \check{\Box}\Diamond p$}
\LeftLabel{$\hat{\Diamond} \dashv \check{\blacksquare}$}
\UnaryInfC{$\hat{\Diamond} p \vdash \check{\blacksquare}\check{\Box}\Diamond p$}
\LeftLabel{$\Diamond_L$}
\UnaryInfC{$\Diamond p \vdash \check{\blacksquare}\check{\Box}\Diamond p$}
\LeftLabel{$\vee_L$}
\BinaryInfC{$p \vee \Diamond p \vdash \check{\blacksquare}\check{\Box}\Diamond p$}
\LeftLabel{$\hat{\Diamond} \dashv \check{\blacksquare}$}
\UnaryInfC{$\hat{\Diamond}(p \vee \Diamond p) \vdash \check{\Box}\Diamond p$}
\LeftLabel{$\Diamond_L$}
\UnaryInfC{$\Diamond(p \vee \Diamond p) \vdash \check{\Box}\Diamond p$}
\end{prooftree}
$$

2.4 Syntactic completeness

In the present section, we fix an arbitrary LE-language $\mathcal{L}_{\mathrm{LE}}$, for which we prove our main result (cf. Theorem 2.4.10), via an effective procedure which generates cut-free derivations in pre-normal form (cf. Definitions 2.2.28 and 2.2.29) of any analytic inductive $\mathcal{L}_{\mathrm{LE}}$-sequent in D.LE (resp. D.DLE) augmented with the analytic structural rule(s) corresponding to the given sequent. In Section 2.4.1, we will first illustrate some of the main ideas of the proof in the context of a proper subclass of analytic inductive sequents, which we refer to as *quasi-special inductive*. Then in Section 2.4.2, we state and prove this result for arbitrary analytic inductive sequents.

Notation 2.4.1. *In this section, we will often deal with vectors of formulas $\overline{\gamma}$ and $\overline{\delta}$ such that each γ in $\overline{\gamma}$ (resp. δ in $\overline{\delta}$) is a positive (resp. negative) PIA formula, and hence, by Lemma 2.2.13, is equivalent to $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$). To avoid overloading notation, we will slightly abuse it and write $\overline{\gamma^\lambda}$ (resp. $\overline{\delta^\mu}$), understanding that, for each element of these vectors, each λ and μ range over different sets.*

2.4.1 Syntactic completeness for quasi-special inductive sequents

Definition 2.4.2. For every analytic (Ω, ε)-inductive inequality $s \leq t$, if every ε-critical branch of the signed generation trees $+s$ and $-t$ consists solely of Skeleton nodes, then $s \leq t$ is a *quasi-special inductive inequality*. Such an inequality is *definite* if none of its Skeleton nodes is $+\vee$ or $-\wedge$.

In terms of the convention introduced in Notation 2.2.15, quasi special inductive sequents can be represented as $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$, i.e. as those $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ such that each α in $\overline{\alpha}$ and β in $\overline{\beta}$ is an atomic proposition.

Figure 2.3 provides a visual representation of the shape of quasi special inductive inequalities, where the critical branches are all skeleton.

Example 2.4.3. Let $\mathcal{L} := \mathcal{L}(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} := \{\wedge, \otimes, \Diamond\}$ and $\mathcal{G} := \{\vee, \oplus, \Box\}$. The $\mathcal{L}_{\mathrm{LE}}$-inequality $\Diamond p \leq \Box\Diamond p$, known in the modal logic literature as axiom 5, is a definite quasi-special inductive inequality, e.g. for $<_\Omega = \emptyset$ and $\varepsilon(p) = 1$, as can be seen from the signed generation tree below (see Notation 2.2.17):

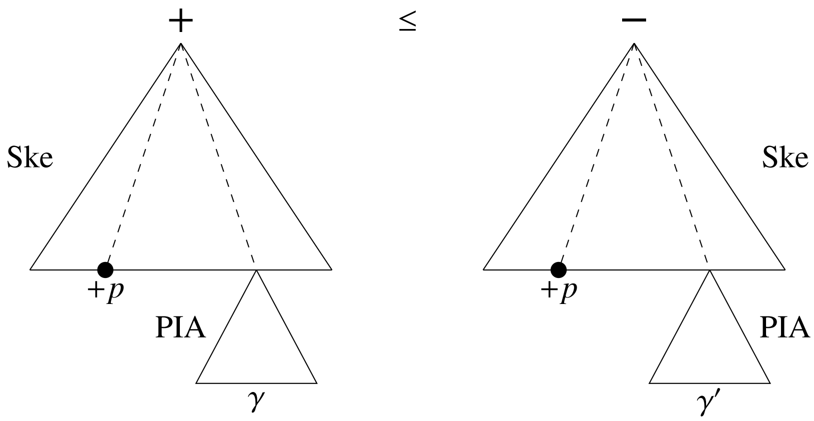
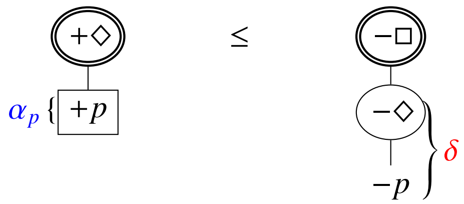
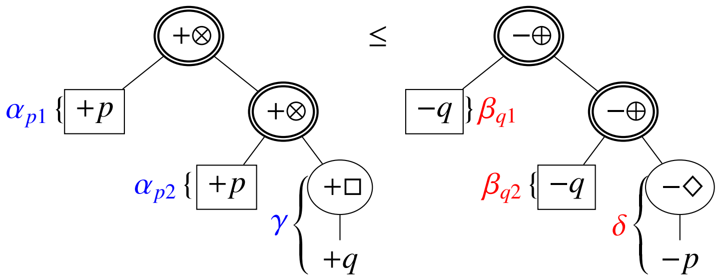


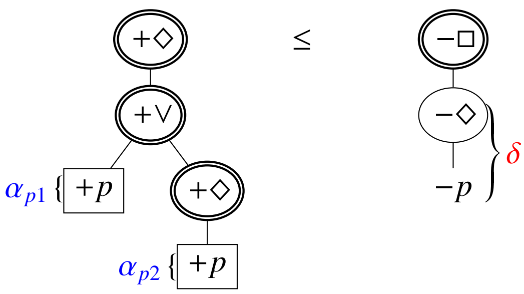
Figure 2.3: The shape of quasi-special inductive inequalities



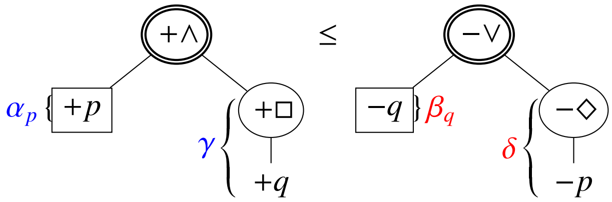
The $\mathcal{L}_{\mathrm{LE}}$-inequality $p \otimes (p \otimes \Box q) \leq q \oplus (q \oplus \Diamond p)$ is a definite quasi-special inductive inequality, e.g. for $p <_\Omega q$ and $\varepsilon(p, q) = (1, \partial)$, as can be seen from the signed generation tree below (cf. Notation 2.2.17):



The $\mathcal{L}_{\mathrm{LE}}$-inequality $\Diamond(p \vee \Diamond p) \leq \Box\Diamond p$ is a *non-definite* quasi-special inductive inequality, e.g. for $<_\Omega = \emptyset$ and $\varepsilon(p) = 1$, as can be seen from the signed generation tree below (see Notation 2.2.17):



Finally, in the distributive case, the $\mathcal{L}_{\mathrm{DLE}}$-inequality $p \wedge \Box q \leq q \vee \Diamond p$, is a definite quasi-special inductive inequality, e.g. for $p <_\Omega q$ and $\varepsilon(p, q) = (1, \partial)$, as can be seen from the signed generation tree below (cf. Notation 2.2.17):



Lemma 2.4.4. *If $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ is a quasi-special inductive inequality (cf. Notation 2.2.15), then each of its corresponding rules has the following shape:*

$$\frac{\overline{(Z \vdash \Gamma^\lambda)_\lambda}[\overline{X}/\overline{p}, \overline{Y}/\overline{q}] \qquad \overline{(\Delta^\mu \vdash W)_\mu}[\overline{X}/\overline{p}, \overline{Y}/\overline{q}]}{(\Phi_j \vdash \Psi_i)[\overline{X}, \overline{Y}, \overline{Z}, \overline{W}]}$$

where for each γ in $\overline{\gamma}$ (resp. each δ in $\overline{\delta}$), each Γ^λ (resp. Δ^μ) is the structural counterpart of some conjunct (resp. disjunct) γ^λ (resp. δ^μ) of the equivalent rewriting of γ (resp. δ) as $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$), as per Lemma 2.2.13, with each γ^λ (resp. δ^μ) being a definite positive (resp. negative) PIA formula, and each Φ_j (resp. Ψ_i) is the structural counterpart of some disjunct (resp. conjunct) φ_j (resp. ψ_i) of the equivalent rewriting of φ (resp. ψ) as $\bigvee_{j\in J} \varphi_j$ (resp. $\bigwedge_{i\in I} \psi_i$), as per Lemma 2.2.13, with each φ_j (resp. ψ_i) being a definite positive (resp. negative) Skeleton formula.

Proof. Let us apply the algorithm ALBA to $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ to compute its corresponding analytic rules. Modulo pre-processing, we can assume w.l.o.g. that $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ is definite,[25] and hence we can proceed with first approximation:

$$\forall \overline{xyzw}[(\overline{x \vdash p} \ \& \ \overline{q \vdash y} \ \& \ \overline{z \vdash \gamma} \ \& \ \overline{\delta \vdash w}) \Rightarrow (\varphi \vdash \psi)[\overline{x}, \overline{y}, \overline{z}, \overline{w}]] \tag{2.4.1}$$

Since γ (resp. δ) can be equivalently rewritten as $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$), as per Lemma 2.2.13, the quasi-inequality above can be equivalently rewritten as follows:

$$\forall \overline{xyzw}\, [(\overline{x \vdash p} \ \& \ \overline{q \vdash y} \ \& \ \overline{(z \vdash \gamma^\lambda)_\lambda} \ \& \ \overline{(\delta^\mu \vdash w)_\mu}) \Rightarrow (\varphi \vdash \psi)[\overline{x}, \overline{y}, \overline{z}, \overline{w}]]. \tag{2.4.2}$$

If every p in $\overline{p}$ and q in $\overline{q}$ has one critical occurrence, then we are in Ackermann-shape and hence we can eliminate the variables $\overline{p}$ and $\overline{q}$ as follows (since by assumption $\overline{\gamma}$ and $\overline{\delta}$ agree with ε^∂):

$$\forall \overline{xyzw}\, [(\overline{(z \vdash \gamma^\lambda)_\lambda}[\overline{x}/\overline{p}, \overline{y}/\overline{q}] \ \& \ \overline{(\delta^\mu \vdash w)_\mu}[\overline{x}/\overline{p}, \overline{y}/\overline{q}]) \Rightarrow (\varphi \vdash \psi)[\overline{x}, \overline{y}, \overline{z}, \overline{w}]] \tag{2.4.3}$$

which yields a rule of the desired shape. If there are multiple critical occurrences of some p in $\overline{p}$ or q in $\overline{q}$, then the Ackermann-shape looks as in (2.4.2), but with $\bigvee_{k=1}^{n_i} x_k \vdash p_i$ and $q_j \vdash \bigwedge_{h=1}^{m_j} y_h$, where n_i (resp. m_j) is the number of critical occurrences of p_i (resp. q_j). Hence, by applying the Ackermann rule we obtain a quasi-inequality similar to (2.4.3), except that the sequents in the antecedent have the following shape:

$$\overline{(z \vdash \gamma^\lambda)_\lambda}\left[\overline{\bigvee_{k=1}^{n_i} x_k / p_i}, \overline{\bigwedge_{h=1}^{m_j} y_h / q_j}\right] \qquad \overline{(\delta^\mu \vdash w)_\mu}\left[\overline{\bigvee_{k=1}^{n_i} x_k / p_i}, \overline{\bigwedge_{h=1}^{m_j} y_h / q_j}\right]. \tag{2.4.4}$$

[25] If $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ is not definite, then by Lemma 2.2.16 it can equivalently be transformed into the conjunction of *definite* quasi-special inductive sequents which we can treat separately as shown in the proof.

Since by assumption $\varepsilon(p) = 1$ for every p in $\overline{p}$ and $\varepsilon(q) = \partial$ for every q in $\overline{q}$, recalling that $+\gamma^\lambda$ and $-\delta^\mu$ agree with ε^∂ for each γ^λ in $\overline{\gamma^\lambda}$ and δ^μ in $\overline{\delta^\mu}$, and moreover every γ^λ in $\overline{\gamma^\lambda}$ (resp. δ^μ in $\overline{\delta^\mu}$) is positive (resp. negative) PIA, the following semantic equivalences hold for each γ^λ in $\overline{\gamma^\lambda}$ and δ^μ in $\overline{\delta^\mu}$:

$$\gamma^\lambda\left[\overline{\bigvee_{k=1}^{n_i} x_k/p_i}, \overline{\bigwedge_{h=1}^{m_j} y_h/q_j}\right] = \bigwedge_{h=1}^{m_j}\bigwedge_{k=1}^{n_i}\gamma^\lambda\left[\overline{x_k/p_i}, \overline{y_h/q_j}\right],$$

$$\delta^\mu\left[\overline{\bigvee_{k=1}^{n_i} x_k/p_i}, \overline{\bigwedge_{h=1}^{m_j} y_h/q_j}\right] = \bigvee_{h=1}^{m_j}\bigvee_{k=1}^{n_i}\delta^\mu\left[\overline{x_k/p_i}, \overline{y_h/q_j}\right]. \tag{2.4.5}$$

Hence, for every γ^λ in $\overline{\gamma^\lambda}$ and δ^μ in $\overline{\delta^\mu}$, the corresponding sequents in (2.4.4) can be equivalently replaced by (at most) $\Sigma_{n,m}(n_i m_j)$ sequents of the form

$$z \vdash \gamma^\lambda\left[\overline{x_k/p_i}, \overline{y_h/q_j}\right] \qquad \delta^\mu\left[\overline{x_k/p_i}, \overline{y_h/q_j}\right] \vdash w, \tag{2.4.6}$$

yielding again a rule of the desired shape. □

As discussed, the Lemma above applies for both the non-distributive and distributive setting, following Remark 2.2.27.

Example 2.4.5. Let us illustrate the procedure described in the lemma above by applying it to the sequents discussed in Example 2.4.3.

	ALBA-run computing the structural rule for $\Diamond p \vdash \Box\Diamond p$:	
	$\Diamond p \vdash \Box\Diamond p$	
iff	$\forall p \forall x \forall w[x \vdash p \;\&\; \Diamond p \vdash w \Rightarrow \Diamond x \vdash \Box w]$	Instance of (2.4.1)
iff	$\forall x \forall w[\Diamond x \vdash w \Rightarrow \Diamond x \vdash \Box w]$	Instance of (2.4.3)

Hence, the analytic rule corresponding to $\Diamond p \vdash \Box\Diamond p$ is

$$\frac{\hat{\Diamond} X \vdash W}{\hat{\Diamond} X \vdash \check{\Box} W}\, R_1$$

ALBA-run computing the structural rule for $p \wedge \Box q \vdash q \vee \Diamond p$:

$p \wedge \Box q \vdash q \vee \Diamond p$

iff $\forall p \forall q \forall x \forall y \forall z \forall w[(x \vdash p \;\&\; q \vdash y \;\&\; z \vdash \Box q \;\&\; \Diamond p \vdash w) \Rightarrow x \wedge z \vdash y \vee w]$

Instance of (2.4.1)

iff $\forall x \forall y \forall z \forall w[(z \vdash \Box y \;\&\; \Diamond x \vdash w) \Rightarrow x \wedge z \vdash y \vee w]$

Instance of (2.4.3)

Hence, the analytic rule corresponding to $p \wedge \Box q \vdash q \vee \Diamond p$ is

$$\frac{Z \vdash \check{\Box} Y \qquad \hat{\Diamond} X \vdash W}{X \,\hat{\wedge}\, Z \vdash Y \,\check{\vee}\, W}\, R_2$$

ALBA-run computing the structural rule for $p \otimes (p \otimes \Box q) \vdash q \oplus (q \oplus \Diamond p)$:

$$
\begin{array}{lll}
& p \otimes (p \otimes \Box q) \vdash q \oplus (q \oplus \Diamond p) & \\
\text{iff} & \forall p \forall q \forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z \forall w & \\
& [(x_1 \vdash p \;\&\; x_2 \vdash p \;\&\; q \vdash y_1 \;\&\; q \vdash y_2 \;\&\; z \vdash \Box q \;\&\; \Diamond p \vdash w) & \\
& \qquad \Rightarrow x_1 \otimes (x_2 \otimes z) \vdash y_1 \oplus (y_2 \oplus w)] & (2.4.1) \\
\text{iff} & \forall p \forall q \forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z \forall w & \\
& [(x_1 \vee x_2 \vdash p \;\&\; q \vdash y_1 \wedge y_2 \;\&\; z \vdash \Box q \;\&\; \Diamond p \vdash w) & \\
& \qquad \Rightarrow x_1 \otimes (x_2 \otimes z) \vdash y_1 \oplus (y_2 \oplus w)] & (2.4.4) \\
\text{iff} & \forall p \forall q \forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z \forall w & \\
& [(z \vdash \Box(y_1 \wedge y_2) \;\&\; \Diamond(x_1 \vee x_2) \vdash w) \Rightarrow x_1 \otimes (x_2 \otimes z) \vdash y_1 \oplus (y_2 \oplus w)] & (2.4.5) \\
\text{iff} & \forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z \forall w & \\
& [(z \vdash \Box y_1 \;\&\; z \vdash \Box y_2 \;\&\; \Diamond x_1 \vdash w \;\&\; \Diamond x_2 \vdash w) \Rightarrow x_1 \otimes (x_2 \otimes z) \vdash y_1 \oplus (y_2 \oplus w)] & (2.4.6)
\end{array}
$$

Hence, the analytic rule corresponding to $p \otimes (p \otimes \Box q) \vdash q \oplus (q \oplus \Diamond p)$ is

$$
\frac{Z \vdash \check{\Box} Y_1 \qquad Z \vdash \check{\Box} Y_2 \qquad \hat{\Diamond} X_1 \vdash W \qquad \hat{\Diamond} X_2 \vdash W}{X_1 \hat{\otimes} (X_2 \hat{\otimes} Z) \vdash Y_1 \check{\oplus} (Y_2 \oplus W)} \; R_3
$$

For the last inequality $\Diamond(p \vee \Diamond p) \leq \Box\Diamond p$, we first need to preprocess the inequality and obtain two definite inequalities $\Diamond p \leq \Box\Diamond p$ and $\Diamond\Diamond p \leq \Box\Diamond p$. We now need to compute the rule for the second one (the first was already computed above):

ALBA-run computing the structural rule for $\Diamond\Diamond p \vdash \Box\Diamond p$:

$$
\begin{array}{lll}
& \Diamond\Diamond p \vdash \Box\Diamond p & \\
\text{iff} & \forall p \forall x \forall w [x \vdash p \;\&\; \Diamond p \vdash w \Rightarrow \Diamond\Diamond x \vdash \Box w] & \text{Instance of (2.4.1)} \\
\text{iff} & \forall x \forall w [\Diamond x \vdash w \Rightarrow \Diamond\Diamond x \vdash \Box w] & \text{Instance of (2.4.3)}
\end{array}
$$

Hence, the analytic rule corresponding to $\Diamond\Diamond p \vdash \Box\Diamond p$ is

$$
\frac{\hat{\Diamond} X \vdash W}{\hat{\Diamond}\hat{\Diamond} X \vdash \check{\Box} W} \; R_4
$$

Theorem 2.4.6. *If $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ is a quasi-special inductive inequality, then a cut-free derivation in pre-normal form exists of $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ in $\underline{\mathrm{D.LE}} + \mathcal{R}$ (resp. $\underline{\mathrm{D.DLE}} + \mathcal{R}$), where $\mathcal{R}$ denotes the finite set of analytic structural rules corresponding to $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ as in Lemma 2.4.4.*

Proof. Recall that each γ in $\overline{\gamma}$ (resp. δ in $\overline{\delta}$) is a positive (resp. negative) PIA formula. Hence, let $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$) denote the equivalent rewriting of γ (resp. δ) as conjunction (resp. disjunction) of definite positive (resp. negative) PIA formulas, as per Lemma 2.2.13. Let us assume that $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ is definite,[26] and hence $\mathcal{R}$ has only one element R, which has the following shape (cf. Lemma 2.4.4):

[26]If $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ is not definite, then, by Lemma 2.2.16, it can equivalently be transformed into the conjunction of *definite* quasi-special inductive sequents $(\varphi_i \vdash \psi_j)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$, where φ is equivalent to $\bigvee_i \varphi_i$ and ψ is equivalent to $\bigwedge_j \psi_j$, which we can treat separately as shown in the proof. Then, a derivation of the original sequent can be obtained by applying the procedure indicated in the proof of Proposition 2.3.12 twice: by applying the procedure once, from derivations of $(\Phi_i \vdash \Psi_j)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ for every i and j we obtain derivations of $(\varphi \vdash \Psi_j)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$ for every j. Then, by applying the procedure again on these sequents, we obtain the required derivation of $(\varphi \vdash \psi)[\overline{p}, \overline{q}, \overline{\gamma}, \overline{\delta}]$.

$$\frac{\overline{(Z \vdash \Gamma^{\lambda})_{\lambda}}[\overline{X}/\overline{p},\overline{Y}/\overline{q}] \qquad \overline{(\Delta^{\mu} \vdash W)_{\mu}}[\overline{X}/\overline{p},\overline{Y}/\overline{q}]}{(\Phi \vdash \Psi)[\overline{X},\overline{Y},\overline{Z},\overline{W}]}$$

Then, modulo application of display rules, we can apply left-introduction (resp. right-introduction) rules to positive (resp. negative) SLR-connectives bottom-up, so as to transform all Skeleton connectives into structural connectives:

$$\begin{array}{c} (\Phi \vdash \Psi)[\overline{p},\overline{q},\overline{\gamma},\overline{\delta}] \\ \vdots \\ (\varphi \vdash \psi)[\overline{p},\overline{q},\overline{\gamma},\overline{\delta}] \end{array} \tag{2.4.7}$$

Notice that $(\Phi \vdash \Psi)[\overline{p},\overline{q},\overline{\gamma},\overline{\delta}]$ is an instance of the conclusion of R with $\overline{p}/\overline{X}$, $\overline{q}/\overline{Y}$, $\overline{\gamma}/\overline{Z}$ and $\overline{\delta}/\overline{W}$. Hence, we can apply R bottom-up and obtain:

$$\frac{\overline{(\gamma \vdash \Gamma^{\lambda})_{\lambda}}[\overline{p}/\overline{p},\overline{q}/\overline{q}] \qquad \overline{(\Delta^{\mu} \vdash \delta)_{\mu}}[\overline{p}/\overline{p},\overline{q}/\overline{q}]}{(\Phi \vdash \Psi)[\overline{p},\overline{q},\overline{\gamma},\overline{\delta}]} \tag{2.4.8}$$

By Corollary 2.3.7, the sequents $(\gamma \vdash \Gamma^{\lambda})[\overline{p}/\overline{p},\overline{q}/\overline{q}]$ and $(\Delta^{\mu} \vdash \delta)[\overline{p}/\overline{p},\overline{q}/\overline{q}]$ are cut-free derivable in D.LE (resp. D.DLE), with derivations which only contain identity axioms, and applications of right-introduction rules for negative PIA-connectives, and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of D.DLE). Moreover, as discussed above (cf. also Proposition 2.3.12), the rules applied after applying the rules in $\mathcal{R}$ are only display rules, left-introduction rules for positive Skeleton-connectives and right-introduction rules for negative Skeleton-connectives (plus contraction in the case of D.DLE). This completes the proof that the cut-free derivation in D.LE + $\mathcal{R}$ (resp. D.DLE + $\mathcal{R}$) of $(\varphi \vdash \psi)[\overline{p},\overline{q},\overline{\gamma},\overline{\delta}]$ is in pre-normal form. □

Example 2.4.7. Let us illustrate the procedure described in the proposition above by deriving the sequents in Example 2.4.3.

D.LE-derivation of $\Diamond p \vdash \Box\Diamond p$:

$$\left.\begin{array}{c}\\ \text{(2.4.7)}\end{array}\right\{ \dfrac{\dfrac{\left.\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,{\scriptstyle \Diamond_R}}{\hat{\Diamond} p \vdash \check{\Box}\Diamond p}\,{\scriptstyle R_1}\right\}\text{(2.4.8)}}{\Diamond p \vdash \check{\Box}\Diamond p}\,{\scriptstyle \Diamond_L}}{\Diamond p \vdash \Box\Diamond p}\,{\scriptstyle \Box_R}$$

D.DLE-derivation of $p \wedge \Box q \vdash q \vee \Diamond p$:

$$\left.\begin{array}{c}\\ \text{(2.4.7)}\end{array}\right\{ \dfrac{\dfrac{\left.\dfrac{{\scriptstyle \Box_L}\dfrac{q \vdash q}{\Box q \vdash \check{\Box} q} \qquad \dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,{\scriptstyle \Diamond_R}}{p \,\hat{\wedge}\, \Box q \vdash q \,\check{\vee}\, \Diamond p}\,{\scriptstyle R_2}\right\}\text{(2.4.8)}}{p \wedge \Box q \vdash q \,\check{\vee}\, \Diamond p}\,{\scriptstyle \wedge_L}}{p \wedge \Box q \vdash q \vee \Diamond p}\,{\scriptstyle \vee_R}$$

D.LE-derivation of $p \otimes (p \otimes \Box q) \vdash q \oplus (q \oplus \Diamond p)$:

$$(2.4.7)\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box_L\,\dfrac{q \vdash q}{\Box q \vdash \check{\Box} q} \qquad \Box_L\,\dfrac{q \vdash q}{\Box q \vdash \check{\Box} q} \qquad \dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R \qquad \dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R}{p \,\hat{\otimes}\, (p \,\hat{\otimes}\, \Box q) \vdash q \,\check{\oplus}\, (q \,\check{\oplus}\, \Diamond p)}\, R_3 \;(2.4.8)}{p \,\hat{\otimes}\, \Box q \vdash p \;\check{\backslash}\; (q \,\check{\oplus}\, (q \,\check{\oplus}\, \Diamond p))}}{p \otimes \Box q \vdash p \;\check{\backslash}\; (q \,\check{\oplus}\, (q \,\check{\oplus}\, \Diamond p))}\,\otimes_L}{p \,\hat{\otimes}\, (p \otimes \Box q) \vdash q \,\check{\oplus}\, (q \,\check{\oplus}\, \Diamond p)}}{p \otimes (p \otimes \Box q) \vdash q \,\check{\oplus}\, (q \,\check{\oplus}\, \Diamond p)}\,\otimes_L}{q \,\hat{\obslash}\, (p \otimes (p \otimes \Box q)) \vdash q \,\check{\oplus}\, \Diamond p}}{q \,\hat{\obslash}\, (p \otimes (p \otimes \Box q)) \vdash q \oplus \Diamond p}\,\oplus_R}{p \otimes (p \otimes \Box q) \vdash q \,\check{\oplus}\, (q \oplus \Diamond p)}}{p \otimes (p \otimes \Box q) \vdash q \oplus (q \oplus \Diamond p)}\,\oplus_R$$

D.LE-derivation of $\Diamond(p \vee \Diamond p) \vdash \Box\Diamond p$:

$$\text{Footnote (26)}\quad \dfrac{\dfrac{\dfrac{\dfrac{(2.4.8)\;\dfrac{\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R}{\hat{\Diamond} p \vdash \check{\Box}\Diamond p}\, R_1}{p \vdash \check{\blacksquare}\check{\Box}\Diamond p}\,\hat{\Diamond} \dashv \check{\blacksquare} \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\hat{\Diamond} p \vdash \Diamond p}\,\Diamond_R}{\hat{\Diamond}\hat{\Diamond} p \vdash \check{\Box}\Diamond p}\, R_4\;(2.4.8)}{\hat{\Diamond} p \vdash \check{\blacksquare}\check{\Box}\Diamond p}\,\hat{\Diamond} \dashv \check{\blacksquare}}{\Diamond p \vdash \check{\blacksquare}\check{\Box}\Diamond p}\,\Diamond_L}{p \vee \Diamond p \vdash \check{\blacksquare}\check{\Box}\Diamond p}\,\vee_L}{\hat{\Diamond}(p \vee \Diamond p) \vdash \check{\Box}\Diamond p}\,\hat{\Diamond} \dashv \check{\blacksquare}}{\Diamond(p \vee \Diamond p) \vdash \check{\Box}\Diamond p}\,\Diamond_L}{\Diamond(p \vee \Diamond p) \vdash \Box\Diamond p}\,\Box_R \quad \text{Footnote (26)}$$

2.4.2 Syntactic completeness for analytic inductive sequents

Lemma 2.4.8. *If* $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ *is an analytic* (Ω, ε)*-inductive sequent, then each of its corresponding rules has the following shape:*

$$\frac{\overline{(Z \vdash \Gamma^{\lambda})}_{\lambda}[\overline{\mathsf{MV}(p)/\overline{p}}, \overline{\mathsf{MV}(q)/\overline{q}}] \qquad \overline{(\Delta^{\mu} \vdash W)}_{\mu}[\overline{\mathsf{MV}(p)/\overline{p}}, \overline{\mathsf{MV}(q)/\overline{q}}]}{(\Phi_j \vdash \Psi_i)[\overline{X}, \overline{Y}, \overline{Z}, \overline{W}]} \tag{2.4.9}$$

where Γ^{λ}*,* Δ^{μ}*,* Φ_j *and* Ψ_i *are as in Lemma 2.4.4, and* $\mathsf{MV}(p)$ *and* $\mathsf{MV}(q)$ *denote the structural counterparts of the components of the minimal and maximal valuations* $\mathsf{mv}(p) \in \mathsf{Mv}(p)$ *and* $\mathsf{mv}(q) \in \mathsf{Mv}(q)$ *defined in the proof below.*

Proof. Let us apply the algorithm ALBA to $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ to compute its corresponding analytic rules. Modulo pre-processing, we can assume w.l.o.g. that $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is definite,[27] and hence we can proceed with first approximation, which yields the following quasi-inequality:

$$\forall \overline{xyzw}[(\overline{x \vdash \alpha} \;\&\; \overline{\beta \vdash y} \;\&\; \overline{z \vdash \gamma} \;\&\; \overline{\delta \vdash w}) \Rightarrow (\varphi \vdash \psi)[\overline{x}, \overline{y}, \overline{z}, \overline{w}]], \tag{2.4.10}$$

[27] If $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is not definite, then by Lemma 2.2.16 it can equivalently be transformed into the conjunction of *definite* analytic inductive sequents which we can treat separately as shown in the proof.

Modulo distribution and splitting (cf. Lemma 2.2.13), the quasi-inequality above can be equivalently rewritten as follows :

$$\forall \overline{xyzw}[(\overline{x \vdash \alpha_p} \ \& \ \overline{x \vdash \alpha_q} \ \& \ \overline{\beta_p \vdash y} \ \& \ \overline{\beta_q \vdash y} \ \& \ \overline{z \vdash \gamma} \ \& \ \overline{\delta \vdash w}) \Rightarrow (\varphi \vdash \psi)[\overline{x}, \overline{y}, \overline{z}, \overline{w}]], \tag{2.4.11}$$

where each α_p and α_q (resp. β_p and β_q) is *definite* positive (resp. negative) PIA and contains a unique ε-critical propositional variable occurrence, which we indicate in its subscript. By applying adjunction and residuation ALBA-rules on all definite PIA-formulas α_p, α_q, β_p and β_q using each ε-critical propositional variable occurrence as the pivotal variable, the antecedent of the quasi-inequality above can be equivalently written as follows:

$$\begin{gathered} \overline{\mathsf{la}(\alpha_p)[x/u, \overline{p}, \overline{q}] \vdash p} \ \& \ \overline{\mathsf{ra}(\beta_p)[y/u, \overline{p}, \overline{q}] \vdash p} \ \& \\ \overline{q \vdash \mathsf{la}(\alpha_q)[x/u, \overline{p}, \overline{q}]} \ \& \ \overline{q \vdash \mathsf{ra}(\beta_q)[y/u, \overline{p}, \overline{q}]} \ \& \\ \overline{z \vdash \gamma} \ \& \ \overline{\delta \vdash w} \end{gathered} \tag{2.4.12}$$

Since each γ (resp. δ) is a positive (resp. negative) PIA formula, by Lemma 2.2.13 it is equivalent to $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$), where each γ^λ (resp. δ^μ) is *definite* positive (resp. negative) PIA. Therefore (2.4.12) can be equivalently rewritten as follows:

$$\begin{gathered} \overline{\mathsf{la}(\alpha_p)[x/u, \overline{p}, \overline{q}] \vdash p} \ \& \ \overline{\mathsf{ra}(\beta_p)[y/u, \overline{p}, \overline{q}] \vdash p} \ \& \\ \overline{q \vdash \mathsf{la}(\alpha_q)[x/u, \overline{p}, \overline{q}]} \ \& \ \overline{q \vdash \mathsf{ra}(\beta_q)[y/u, \overline{p}, \overline{q}]} \ \& \\ \overline{(z \vdash \gamma^\lambda)_\lambda} \ \& \ \overline{(\delta^\mu \vdash w)_\mu} \end{gathered} \tag{2.4.13}$$

Notice that the 'parametric' (i.e. non-critical) variables in $\overline{p}$ and $\overline{q}$ actually occurring in each formula $\mathsf{la}(\alpha_p)[x/u, \overline{p}, \overline{q}]$, $\mathsf{ra}(\beta_p)[y/u, \overline{p}, \overline{q}]$, $\mathsf{la}(\alpha_q)[x/u, \overline{p}, \overline{q}]$, and $\mathsf{ra}(\beta_q)[y/u, \overline{p}, \overline{q}]$ are those that are strictly $<_\Omega$-smaller than the (critical and pivotal) variable indicated in the subscript of the given PIA-formula. After applying adjunction and residuation as indicated above, the quasi-inequality (2.4.11) is in Ackermann shape relative to the $<_\Omega$-minimal variables.

For every $p \in \overline{p}$ and $q \in \overline{q}$, let us define the sets $\mathsf{Mv}(p)$ and $\mathsf{Mv}(q)$ by recursion on $<_\Omega$ as follows:

- $\mathsf{Mv}(p) := \{\mathsf{la}(\alpha_p)[x_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}], \mathsf{ra}(\beta_p)[y_h/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}] \mid 1 \leq k \leq n_{i_1}, 1 \leq h \leq n_{i_2}, \overline{\mathsf{mv}(p)} \in \overline{\mathsf{Mv}(p)}, \overline{\mathsf{mv}(q)} \in \overline{\mathsf{Mv}(q)}\}$
- $\mathsf{Mv}(q) := \{\mathsf{la}(\alpha_q)[x_h/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}], \mathsf{ra}(\beta_q)[y_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}) \mid 1 \leq h \leq m_{j_1}, 1 \leq k \leq m_{j_2}, \overline{\mathsf{mv}(p)} \in \overline{\mathsf{Mv}(p)}, \overline{\mathsf{mv}(q)} \in \overline{\mathsf{Mv}(q)}\}$

where, n_{i_1} (resp. n_{i_2}) is the number of occurrences of p in αs (resp. in βs) for every $p \in \overline{p}$, and m_{j_1} (resp. m_{j_2}) is the number of occurrences of q in αs (resp. in βs) for every $q \in \overline{q}$. By induction on $<_\Omega$, we can apply the Ackermann rule exhaustively so as to eliminate all variables p and q. Then the antecedent of the quasi-inequality has the following form:

$$\overline{(z \vdash \gamma^\lambda)_\lambda}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] \qquad \overline{(\delta^\mu \vdash w)_\mu}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]. \tag{2.4.14}$$

Since by assumption $\varepsilon(p) = 1$ for every p in $\overline{p}$ and $\varepsilon(q) = \partial$ for every q in $\overline{q}$, recalling that $\overline{\gamma^\lambda}$ and $\overline{\delta^\mu}$ agree with ε^∂, and moreover every γ^λ in $\overline{\gamma^\lambda}$ (resp. δ^μ in $\overline{\delta^\mu}$) is positive (resp. negative) PIA, the following semantic equivalences hold for each γ^λ in $\overline{\gamma^\lambda}$ and δ^μ in $\overline{\delta^\mu}$:

$$\gamma^\lambda\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] = \bigwedge \gamma^\lambda\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right]$$

$$\delta^\mu\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] = \bigvee \delta^\mu\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right].$$

Hence for every γ^λ in $\overline{\gamma^\lambda}$ and δ^μ in $\overline{\delta^\mu}$, the corresponding sequents in (2.4.14) can be equivalently replaced by (at most) $\Sigma_{n,m}(n_i m_j)$ sequents of the form

$$z \vdash \gamma^\lambda\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right] \qquad \delta^\mu\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right] \vdash w, \tag{2.4.15}$$

yielding a rule of the desired shape. □

Example 2.4.9. Let us illustrate the procedure described in the lemma above by applying it to the sequents discussed in Example 2.2.18.

	ALBA-run computing the structural rule for $\Diamond\Box p \vdash \Box\Diamond p$:	
	$\Diamond\Box p \vdash \Box\Diamond p$	
iff	$\forall p\forall x\forall w[x \vdash \Box p \;\&\; \Diamond p \vdash w \Rightarrow \Diamond x \vdash \Box w]$	Instance of (2.4.10)
iff	$\forall p\forall x\forall w[\blacklozenge x \vdash p \;\&\; \Diamond p \vdash w \Rightarrow \Diamond x \vdash \Box w]$	Instance of (2.4.12)
iff	$\forall x\forall w[\Diamond\blacklozenge x \vdash w \Rightarrow \Diamond x \vdash \Box w]$	Instance of (2.4.15)

Then the analytic rule corresponding to $\Diamond\Box p \vdash \Box\Diamond p$ is:

$$\frac{\hat{\Diamond}\hat{\blacklozenge} X \vdash W}{\hat{\Diamond} X \vdash \check{\Box} W}\, R_4$$

ALBA-run computing the structural rule for $p \to (q \to r) \leq ((p \to q) \to (\Box p \to r)) \oplus \Diamond r$:

$p \to (q \to r) \leq ((p \to q) \to (\Box p \to r)) \oplus \Diamond r$

iff $\forall p\forall q\forall r\forall x_1\forall x_2\forall y_1\forall y_2\forall z$
$[x_1 \vdash \Box p \;\&\; x_2 \vdash p \to q \;\&\; \Diamond r \vdash y_1 \;\&\; r \vdash y_2 \;\&\; z \vdash p \to (q \to r)$
$\Rightarrow z \vdash (x_2 \to (x_1 \to y_2)) \oplus y_1]$ (2.4.10)

iff $\forall p\forall q\forall r\forall x_1\forall x_2\forall y_1\forall y_2\forall z$
$[\blacklozenge x_1 \vdash p \;\& p\; \wedge x_2 \vdash q \;\&\; r \vdash \blacksquare y_1 \;\&\; r \vdash y_2 \;\&\; z \vdash p \to (q \to r)$
$\Rightarrow z \vdash (x_2 \to (x_1 \to y_2)) \oplus y_1]$ (2.4.12)

iff $\forall q\forall r\forall x_1\forall x_2\forall y_1\forall y_2\forall z$
$[\blacklozenge x_1 \wedge x_2 \vdash q \;\&\; r \vdash \blacksquare y_1 \;\&\; r \vdash y_2 \;\&\; z \vdash \blacklozenge x_1 \to (q \to r)$
$\Rightarrow z \vdash (x_2 \to (x_1 \to y_2)) \oplus y_1]$

iff $\forall r\forall x_1\forall x_2\forall y_1\forall y_2\forall z$
$[r \vdash \blacksquare y_1 \;\&\; r \vdash y_2 \;\&\; z \vdash \blacklozenge x_1 \to (\blacklozenge x_1 \wedge x_2 \to r)$
$\Rightarrow z \vdash x_2 \to (x_1 \to y_2)) \oplus y_1]$

iff $\forall x_1\forall x_2\forall y_1\forall y_2\forall z$
$[z \vdash \blacklozenge x_1 \to (\blacklozenge x_1 \wedge x_2 \to \blacksquare y_1 \vee y_2) \Rightarrow z \vdash (x_2 \to (x_1 \to y_2)) \oplus y_1]$ (2.4.14)

iff $\forall x_1\forall x_2\forall y_1\forall y_2\forall z$
$[z \vdash \blacklozenge x_1 \to (\blacklozenge x_1 \wedge x_2 \to \blacksquare y_1) \;\&\; z \vdash \blacklozenge x_1 \to (\blacklozenge x_1 \wedge x_2 \to y_2)$
$\Rightarrow z \vdash (x_2 \to (x_1 \to y_2)) \oplus y_1]$ (2.4.15)

Then the analytic rule corresponding to $p \to (q \to r) \leq ((p \to q) \to (\Box p \to r)) \oplus \Diamond r$ is:

$$\frac{Z \vdash \hat{\blacklozenge} X_1 \check{\to} (\hat{\blacklozenge} X_1 \hat{\wedge} X_2 \check{\to} \check{\blacksquare} Y_1) \qquad Z \vdash \hat{\blacklozenge} X_1 \check{\to} (\hat{\blacklozenge} X_1 \hat{\wedge} X_2 \check{\to} Y_2)}{Z \vdash (X_2 \check{\to} (X_1 \check{\to} Y_2)) \check{\oplus} Y_1} R_5$$

ALBA-run computing the structural rule for $\Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q$:

$$\begin{array}{lll}
 & \Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q & \\
\text{iff} & \forall p \forall x \forall w [x \vdash \Box(p \wedge q)\ \&\ \Diamond p \vdash y\ \&\ \Diamond q \vdash z \Rightarrow \Diamond x \vdash \Box y \vee \Box z] & \text{Instance of (2.4.10)} \\
\text{iff} & \forall p \forall x \forall w [x \vdash \Box(p \wedge q)\ \&\ p \vdash \blacksquare y\ \&\ q \vdash \blacksquare z \Rightarrow \Diamond x \vdash \Box y \vee \Box z] & \text{Instance of (2.4.12)} \\
\text{iff} & \forall p \forall x \forall w [x \vdash \Box(\blacksquare y \wedge \blacksquare z) \Rightarrow \Diamond x \vdash \Box y \vee \Box z] & \text{Instance of (2.4.14)} \\
\text{iff} & \forall p \forall x \forall w [x \vdash \Box\blacksquare y\ \&\ x \vdash \Box\blacksquare z \Rightarrow \Diamond x \vdash \Box y \vee \Box z] & \text{Instance of (2.4.15)}
\end{array}$$

Then the analytic rule corresponding to $\Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q$ is:

$$\frac{X \vdash \check{\Box}\check{\blacksquare} Y \qquad X \vdash \check{\Box}\check{\blacksquare} Z}{\hat{\Diamond} X \vdash \check{\Box} Y \check{\vee} \check{\Box} Z} R_6$$

Theorem 2.4.10. *If* $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ *is an analytic inductive sequent, then a cut-free derivation in pre-normal form exists of* $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ *in* $\underline{\text{D.LE}} + \mathcal{R}$ *(resp.* $\underline{\text{D.DLE}} + \mathcal{R}$*), where* $\mathcal{R}$ *denotes the finite set of analytic structural rules corresponding to* $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ *as in Lemma 2.4.8.*

Proof. Recall that each γ in $\overline{\gamma}$ (resp. δ in $\overline{\delta}$) is a positive (resp. negative) PIA formula. Let $\bigwedge_\lambda \gamma^\lambda$ (resp. $\bigvee_\mu \delta^\mu$) denote the equivalent rewriting of γ (resp. δ) as conjunction (resp. disjunction) of definite positive (resp. negative) PIA formulas as per Lemma 2.2.13. Let us assume that $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is definite,[28] and hence $\mathcal{R}$ has only one element R, with the following shape (cf. Lemma 2.4.8):

$$\frac{\overline{(Z \vdash \Gamma^\lambda)_\lambda}[\overline{\mathsf{MV}(p)}/\overline{p}, \overline{\mathsf{MV}(q)}/\overline{q}] \qquad \overline{(\Delta^\mu \vdash W)_\mu}[\overline{\mathsf{MV}(p)}/\overline{p}, \overline{\mathsf{MV}(q)}/\overline{q}]}{(\Phi \vdash \Psi)[\overline{X}, \overline{Y}, \overline{Z}, \overline{W}]} \tag{2.4.16}$$

Then, modulo application of display rules, we can apply left-introduction (resp. right-introduction) rules to positive (resp. negative) SLR-connectives bottom-up, so as to transform all Skeleton connectives into structural connectives:

$$\begin{array}{c}
(\Phi \vdash \Psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}] \\
\vdots \\
(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]
\end{array} \tag{2.4.17}$$

Notice that $(\Phi \vdash \Psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is an instance of the conclusion of R. Hence we can apply R bottom-up and obtain:

[28] If $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is not definite, then, by Lemma 2.2.16, it can equivalently be transformed into the conjunction of *definite* analytic inductive sequents $(\varphi_i \vdash \psi_j)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$, where φ is equivalent to $\bigvee_i \varphi_i$ and ψ is equivalent to $\bigwedge_j \psi_j$, which we can treat separately as shown in the proof. Then, a derivation of the original sequent can be obtained by applying the procedure indicated in the proof of Proposition 2.3.12 twice: by applying the procedure once, from derivations of $(\Phi_i \vdash \Psi_j)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ for every i and j we obtain derivations of $(\varphi \vdash \Psi_j)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ for every j. Then, by applying the procedure again on these sequents, we obtain the required derivation of $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$.

$$\frac{\overline{(\gamma \vdash \Gamma^{\lambda})_{\lambda}}[\overline{\mathsf{MV}(p)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{p}, \overline{\mathsf{MV}(q)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{q}] \qquad \overline{(\Delta^{\mu} \vdash \delta)_{\mu}}[\overline{\mathsf{MV}(p)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{p}, \overline{\mathsf{MV}(q)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{q}]}{(\Phi \vdash \Psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]}$$

To finish the proof that $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is derivable in in $\underline{\text{D.LE}}$+R (resp. $\underline{\text{D.DLE}}$+R), it is enough to show that the sequents

$$\overline{(\gamma \vdash \Gamma^{\lambda})_{\lambda}}[\overline{\mathsf{MV}(p)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{p}, \overline{\mathsf{MV}(q)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{q}]$$

and

$$\overline{(\Delta^{\mu} \vdash \delta)_{\mu}}[\overline{\mathsf{MV}(p)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{p}, \overline{\mathsf{MV}(q)}[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]/\overline{q}]$$

are derivable in $\underline{\text{D.LE}}$ (resp. $\underline{\text{D.DLE}}$). Recalling that each γ in $\overline{\gamma}$ (resp. δ in $\overline{\delta}$) is a positive (resp. negative) PIA formula, by Proposition 2.3.6, it is enough to show that for every p and q, the sequents $\mathsf{MV}(p)[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}] \vdash p$ and $q \vdash \mathsf{MV}(q)[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]$ are derivable in $\underline{\text{D.LE}}$ (resp. $\underline{\text{D.DLE}}$) for all formulas $\mathsf{mv}(p) \in \mathsf{Mv}(p)$ and $\mathsf{mv}(q) \in \mathsf{Mv}(q)$. Let us show this latter statement. Each sequent $\mathsf{MV}(p)[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}] \vdash p$ is of either of the following forms:

$$\mathsf{LA}(\alpha_p)[\alpha[\overline{p}/\overline{p}, \overline{q}/\overline{q}]/!u, \overline{\mathsf{MV}(p)}/\overline{p}, \overline{\mathsf{MV}(q)}/\overline{q}] \vdash p$$
$$\mathsf{RA}(\beta_p)[\beta[\overline{p}/\overline{p}, \overline{q}/\overline{q}]/!u, \overline{\mathsf{MV}(p)}/\overline{p}, \overline{\mathsf{MV}(q)}/\overline{q}] \vdash p$$

where α_p (resp. β_p) denotes the definite positive (resp. negative) PIA formula which occurs as a conjunct (resp. disjunct) of α (resp. β) as per Lemma 2.2.13, which contains the ε-critical occurrence of p as a subformula (cf. discussion around (2.4.11)). By Corollary 2.3.9, it is enough to show that $\mathsf{MV}(p') \vdash p'$ and $q' \vdash \mathsf{MV}(q')$ are derivable in $\underline{\text{D.LE}}$ (resp. $\underline{\text{D.DLE}}$) for each $p', q' <_{\Omega} p$ (which is true by the induction hypothesis, while the basis of the induction holds by Corollary 2.3.10), and $p \vdash p$ is derivable in $\underline{\text{D.LE}}$ (resp. $\underline{\text{D.DLE}}$), which is of course the case. Likewise, one shows that the sequents $q \vdash \mathsf{MV}(q)[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}]$ are derivable in $\underline{\text{D.LE}}$ (resp. $\underline{\text{D.DLE}}$), which completes the proof that $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is derivable in in $\underline{\text{D.LE}}$ + R (resp. $\underline{\text{D.DLE}}$ + R). Finally, the derivation so generated only consists of identity axioms, and applications of display rules, right-introduction rules for negative PIA-connectives, and left-introduction rules for positive PIA-connectives (and weakening and exchange rules in the case of $\underline{\text{D.DLE}}$) before the application of a rule in $\mathcal{R}$ (cf. Proposition 2.3.6 and Corollaries 2.3.9 and 2.3.10); moreover, after applying a rule in $\mathcal{R}$, the only rules applied are display rules, left-introduction rules for positive Skeleton-connectives and right-introduction rules for negative Skeleton-connectives (plus contraction in the case of $\underline{\text{D.DLE}}$), cf. Proposition 2.3.12 and Footnote 28). This completes the proof that the cut-free derivation in $\underline{\text{D.LE}} + \mathcal{R}$ (resp. $\underline{\text{D.DLE}} + \mathcal{R}$) of $(\varphi \vdash \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ is in pre-normal form. □

Example 2.4.11. Let us illustrate the procedure described in the proposition above by deriving the sequents in Example 2.2.18 using the rules computed in Example 2.4.9. In the last derivation below, the symbol $\hat{/}_{\oplus}$ denotes the left residual of $\check{\oplus}$ in its first coordinate.

D.DLE-derivation of $\Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q$:

$$
\begin{array}{rc}
 & p \vdash p \\ \hline
\Diamond_R & \hat{\Diamond} p \vdash \Diamond p \\ \hline
 & p \vdash \check{\blacksquare}\Diamond p \\ \hline
W_L & p \mathbin{\hat{\wedge}} q \vdash \check{\blacksquare}\Diamond p \\ \hline
\wedge_L & p \wedge q \vdash \check{\blacksquare}\Diamond p \\ \hline
\Box_L & \Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond p
\end{array}
\qquad
\begin{array}{rc}
 & q \vdash q \\ \hline
\Diamond_R & \hat{\Diamond} q \vdash \Diamond q \\ \hline
 & q \vdash \check{\blacksquare}\Diamond q \\ \hline
W_L & q \mathbin{\hat{\wedge}} p \vdash \check{\blacksquare}\Diamond q \\ \hline
E_L & p \mathbin{\hat{\wedge}} q \vdash \check{\blacksquare}\Diamond q \\ \hline
\wedge_L & p \wedge q \vdash \check{\blacksquare}\Diamond q \\ \hline
\Box_L & \Box(p \wedge q) \vdash \check{\Box}\check{\blacksquare}\Diamond q
\end{array}
$$

$$
\begin{array}{rc}
R_4 \; (2.4.9) & \hat{\Diamond}\Box(p \wedge q) \vdash \check{\Box}\Diamond p \mathbin{\check{\vee}} \check{\Box}\Diamond q \\ \hline
(2.4.17) & \hat{\Diamond}\Box(p \wedge q) \mathbin{\hat{-\!\!<}} \check{\Box}\Diamond q \vdash \check{\Box}\Diamond p \\ \hline
\Box_R & \hat{\Diamond}\Box(p \wedge q) \mathbin{\hat{-\!\!<}} \check{\Box}\Diamond q \vdash \Box\Diamond p \\ \hline
 & \hat{\Diamond}\Box(p \wedge q) \vdash \Box\Diamond p \mathbin{\check{\vee}} \check{\Box}\Diamond q \\ \hline
 & \Box\Diamond p \mathbin{\hat{>\!\!-}} \hat{\Diamond}\Box(p \wedge q) \vdash \check{\Box}\Diamond q \\ \hline
\Box_R & \Diamond p \mathbin{\hat{>\!\!-}} \hat{\Diamond}\Box(p \wedge q) \vdash \Box\Diamond q \\ \hline
 & \hat{\Diamond}\Box(p \wedge q) \vdash \Box\Diamond p \mathbin{\check{\vee}} \Box\Diamond q \\ \hline
\vee_R & \hat{\Diamond}\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q \\ \hline
\Diamond_L & \Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q
\end{array}
$$

D.DLE-derivation of $p \rightarrow (q \rightarrow r) \vdash ((p \rightarrow q) \rightarrow (\Box p \rightarrow r)) \oplus \Diamond r$:

$$
\begin{array}{rc}
 & q \vdash q \qquad r \vdash r \\ \hline
\rightarrow_L & q \rightarrow r \vdash q \mathbin{\check{\rightarrow}} r \quad (\text{with } p \vdash p) \\ \hline
\rightarrow_L & p \rightarrow (q \rightarrow r) \vdash p \mathbin{\check{\rightarrow}} (q \mathbin{\check{\rightarrow}} r) \\ \hline
 & p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)) \vdash q \mathbin{\check{\rightarrow}} r \\ \hline
 & p \vdash (q \mathbin{\check{\rightarrow}} r) \mathbin{\check{\leftarrow}} (p \rightarrow (q \rightarrow r)) \\ \hline
\Box_L & \Box p \vdash \check{\Box}((q \mathbin{\check{\rightarrow}} r) \mathbin{\check{\leftarrow}} (p \rightarrow (q \rightarrow r))) \\ \hline
 & \hat{\blacklozenge}\Box p \vdash ((q \mathbin{\check{\rightarrow}} r) \mathbin{\check{\leftarrow}} (p \rightarrow (q \rightarrow r))) \\ \hline
 & \hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)) \vdash q \mathbin{\check{\rightarrow}} r \\ \hline
 & q \mathbin{\hat{\wedge}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \vdash r \\ \hline
 & q \vdash r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \quad (\text{with } p \vdash p) \\ \hline
\rightarrow_L & p \rightarrow q \vdash p \mathbin{\check{\rightarrow}} (r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \\ \hline
 & p \mathbin{\hat{\wedge}} (p \rightarrow q) \vdash r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \\ \hline
 & p \vdash (r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \mathbin{\check{\leftarrow}} (p \rightarrow q) \\ \hline
\Box_L & \Box p \vdash \check{\Box}(r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \mathbin{\check{\leftarrow}} (p \rightarrow q) \\ \hline
 & \hat{\blacklozenge}\Box p \vdash r \mathbin{\check{\leftarrow}} ((\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \mathbin{\check{\leftarrow}} (p \rightarrow q) \\ \hline
 & \hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q) \vdash r \mathbin{\check{\leftarrow}} ((\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \\ \hline
 & (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q)) \mathbin{\hat{\wedge}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \vdash r \\ \hline
 & \hat{\Diamond}((\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q)) \mathbin{\hat{\wedge}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \vdash \Diamond r \\ \hline
 & (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q)) \mathbin{\hat{\wedge}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \vdash \check{\blacksquare}\Diamond r \\ \hline
 & p \rightarrow (q \rightarrow r) \vdash \hat{\blacklozenge}\Box p \mathbin{\check{\rightarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q) \mathbin{\check{\rightarrow}} \check{\blacksquare}\Diamond r)
\end{array}
$$

$$
\begin{array}{rc}
 & q \vdash q \qquad r \vdash r \\ \hline
\rightarrow_L & q \rightarrow r \vdash q \mathbin{\check{\rightarrow}} r \quad (\text{with } p \vdash p) \\ \hline
\rightarrow_L & p \rightarrow (q \rightarrow r) \vdash p \mathbin{\check{\rightarrow}} (q \mathbin{\check{\rightarrow}} r) \\ \hline
 & p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)) \vdash q \mathbin{\check{\rightarrow}} r \\ \hline
 & p \vdash (q \mathbin{\check{\rightarrow}} r) \mathbin{\check{\leftarrow}} (p \rightarrow (q \rightarrow r)) \\ \hline
\Box_L & \Box p \vdash \check{\Box}((q \mathbin{\check{\rightarrow}} r) \mathbin{\check{\leftarrow}} (p \rightarrow (q \rightarrow r))) \\ \hline
 & \hat{\blacklozenge}\Box p \vdash ((q \mathbin{\check{\rightarrow}} r) \mathbin{\check{\leftarrow}} (p \rightarrow (q \rightarrow r))) \\ \hline
 & \hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)) \vdash q \mathbin{\check{\rightarrow}} r \\ \hline
 & q \mathbin{\hat{\wedge}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \vdash r \\ \hline
 & q \vdash r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \quad (\text{with } p \vdash p) \\ \hline
\rightarrow_L & p \rightarrow q \vdash p \mathbin{\check{\rightarrow}} (r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \\ \hline
 & p \mathbin{\hat{\wedge}} (p \rightarrow q) \vdash r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r))) \\ \hline
 & p \vdash (r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \mathbin{\check{\leftarrow}} (p \rightarrow q) \\ \hline
\Box_L & \Box p \vdash \check{\Box}(r \mathbin{\check{\leftarrow}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \mathbin{\check{\leftarrow}} (p \rightarrow q) \\ \hline
 & \hat{\blacklozenge}\Box p \vdash r \mathbin{\check{\leftarrow}} ((\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \mathbin{\check{\leftarrow}} (p \rightarrow q) \\ \hline
 & \hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q) \vdash r \mathbin{\check{\leftarrow}} ((\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \\ \hline
 & (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q)) \mathbin{\hat{\wedge}} (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)))) \vdash r \\ \hline
 & \hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow (q \rightarrow r)) \vdash (\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q)) \mathbin{\check{\rightarrow}} r \\ \hline
 & p \rightarrow (q \rightarrow r) \vdash \hat{\blacklozenge}\Box p \mathbin{\check{\rightarrow}} ((\hat{\blacklozenge}\Box p \mathbin{\hat{\wedge}} (p \rightarrow q)) \mathbin{\check{\rightarrow}} r)
\end{array}
$$

$$
\begin{array}{rc}
R_5 & p \rightarrow (q \rightarrow r) \vdash ((p \rightarrow q) \mathbin{\check{\rightarrow}} (\Box p \mathbin{\check{\rightarrow}} r)) \mathbin{\check{\oplus}} \Diamond r \\ \hline
 & (p \rightarrow (q \rightarrow r)) \mathbin{\hat{/}_{\oplus}} \Diamond r \vdash (p \rightarrow q) \mathbin{\check{\rightarrow}} (\Box p \mathbin{\check{\rightarrow}} r) \\ \hline
 & (p \rightarrow q) \mathbin{\hat{\wedge}} ((p \rightarrow (q \rightarrow r)) \mathbin{\hat{/}_{\oplus}} \Diamond r) \vdash \Box p \mathbin{\check{\rightarrow}} r \\ \hline
\rightarrow_R & (p \rightarrow q) \mathbin{\hat{\wedge}} ((p \rightarrow (q \rightarrow r)) \mathbin{\hat{/}_{\oplus}} \Diamond r) \vdash \Box p \rightarrow r \\ \hline
 & (p \rightarrow (q \rightarrow r)) \mathbin{\hat{/}_{\oplus}} \Diamond r \vdash (p \rightarrow q) \mathbin{\check{\rightarrow}} (\Box p \rightarrow r) \\ \hline
\rightarrow_R & (p \rightarrow (q \rightarrow r)) \mathbin{\hat{/}_{\oplus}} \Diamond r \vdash (p \rightarrow q) \rightarrow (\Box p \rightarrow r) \\ \hline
 & p \rightarrow (q \rightarrow r) \vdash ((p \rightarrow q) \rightarrow (\Box p \rightarrow r)) \mathbin{\check{\oplus}} \Diamond r \\ \hline
\oplus_R & p \rightarrow (q \rightarrow r) \vdash ((p \rightarrow q) \rightarrow (\Box p \rightarrow r)) \oplus \Diamond r
\end{array}
$$

<u>D.LE</u>-derivation of $\Diamond\Box p \vdash \Box\Diamond p$:

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box p \vdash \check{\Box} p$}
\UnaryInfC{$\hat{\blacklozenge}\Box p \vdash p$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond}\hat{\blacklozenge}\Box p \vdash \Diamond p$}
\UnaryInfC{$\hat{\Diamond}\Box p \vdash \check{\Box}\Diamond p$}
\LeftLabel{$\Diamond_L$}
\UnaryInfC{$\Diamond\Box p \vdash \check{\Box}\Diamond p$}
\RightLabel{$\Box_R$}
\UnaryInfC{$\Diamond\Box p \vdash \Box\Diamond p$}
\end{prooftree}
$$

(2.4.9) (2.4.17)

We finish the present section with an example which, strictly speaking, does not fall into the scope of the present paper, since it concerns a *non-normal* logical framework. The example below intends to illustrate how the present results naturally extend beyond normal LE-logics. We expand on this topic also in a dedicated paragraph in Section 2.5.

Example 2.4.12. (cf. [6, Section 7.2]) In [6], a proper (multi-type) display calculus is introduced for basic monotone modal logic via a semantic analysis which allows to equivalently represent the non-normal (monotone) modal operator ∇ as the composition of *normal* (multi-type) modal operators in two ways, namely both as a 'box-diamond' and as a 'diamond-box' composition. This justifies the introduction of a syntactic translation from the (single-type) language of monotone modal logic to a (multi-type) *normal* distributive modal language, under which, the monotone normal axiom C translates as follows:

$$\nabla p \wedge \nabla q \to \nabla(p \wedge q) \quad \rightsquigarrow \quad \langle \nu \rangle [\ni] p \wedge \langle \nu \rangle [\ni] q \vdash [\nu^c] \langle \not\ni \rangle (p \wedge q).$$

The translation of axiom C can be straightforwardly recognized as an *analytic inductive* axiom/inequality, since the operations interpreting the heterogeneous connectives $\langle \nu \rangle$, $[\ni]$, $[\nu^c]$, $\langle \not\ni \rangle$ enjoy all the relevant order-theoretic properties required by the theory of algorithmic correspondence and proper display calculi. Applying the tools for the algorithmic generation of analytic structural rules to the translated axiom yields the following rule:

$$\frac{\langle \hat{\not\ni} \rangle (\langle \hat{\in} \rangle \Gamma \mathbin{\hat{\wedge}} \langle \hat{\in} \rangle \Delta) \vdash \Theta}{\langle \hat{\nu} \rangle \Gamma \mathbin{\hat{\wedge}} \langle \hat{\nu} \rangle \Delta \vdash [\check{\nu^c}] \Theta}\ C$$

The rule above can be then added to the (multi-type) proper display calculus for the basic heterogeneous normal modal logic, so to obtain a calculus in which the following derivation in pre-normal form can be generated for the translated axiom:

<u>D.DLE</u>-derivation of $\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{[\ni]p \vdash [\check{\ni}]p}}{\langle\hat{\in}\rangle[\ni]p \vdash p} \qquad \dfrac{\dfrac{q \vdash q}{[\ni]q \vdash [\check{\ni}]q}}{\langle\hat{\in}\rangle[\ni]q \vdash q}}{\langle\hat{\in}\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle[\ni]q \vdash p \wedge q}}{\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle[\ni]q) \vdash \langle\not\ni\rangle(p \wedge q)}}{\langle\hat{\nu}\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\hat{\nu}\rangle[\ni]q \vdash [\check{\nu^c}]\langle\not\ni\rangle(p \wedge q)}\ (2.4.9)
$$

$$
(2.4.17)\left\{
\begin{array}{l}
\langle\hat{\nu}\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\hat{\nu}\rangle[\ni]q \vdash [\check{\nu^c}]\langle\not\ni\rangle(p \wedge q) \\ \hline
\langle\hat{\nu}\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\hat{\nu}\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q) \\ \hline
\qquad \langle\hat{\nu}\rangle[\ni]q \vdash \langle\hat{\nu}\rangle[\ni]p \mathbin{\check{\rightarrow}} [\nu^c]\langle\not\ni\rangle(p \wedge q) \\ \hline
\qquad \langle\nu\rangle[\ni]q \vdash \langle\hat{\nu}\rangle[\ni]p \mathbin{\check{\rightarrow}} [\nu^c]\langle\not\ni\rangle(p \wedge q) \\ \hline
\langle\hat{\nu}\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\nu\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q) \\ \hline
\qquad \langle\hat{\nu}\rangle[\ni]p \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q) \mathbin{\check{\leftarrow}} \langle\nu\rangle[\ni]q \\ \hline
\qquad \langle\nu\rangle[\ni]p \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q) \mathbin{\check{\leftarrow}} \langle\nu\rangle[\ni]q \\ \hline
\langle\nu\rangle[\ni]p \mathbin{\hat{\wedge}} \langle\nu\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q)
\end{array}
\right.
$$

$$
\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q)
$$

2.5 Conclusions

Main contribution In this chapter we showed that, for any *properly displayable* (D)LE-logic L (i.e. a (D)LE-logic axiomatized by *analytic inductive* axioms, cf. Definition 2.2.14), the proper display calculus for L—i.e. the calculus obtained by adding the analytic structural rules corresponding to the axioms of L to the basic calculus D.LE (resp. D.DLE)—derives all the theorems of L. This is what we refer to as the *syntactic completeness* of the proper display calculus for L with respect to L. In [10], this is achieved by showing the syntactic equivalence between axioms and their corresponding rules (cf. [10, Propositions 3.14 and 3.28]). This equivalence relies on the use of the syntactic version of Ackermann's Lemma ([10, Lemma 3.6]), and involves the cut rule. Using semantic means, paralleling this process, in [34] the equivalence between analytic inductive axioms and their corresponding analytic structural rules is achieved by transforming the algebraic inequalities corresponding to each given axiom to a set of quasi-inequalities via ALBA. Then completeness follows from the soundness of the rules of ALBA on perfect normal (distributive) lattice expansions and the completeness of the display calculus augmented with the analytic structural rules with respect to perfect normal (distributive) lattice expansions satisfying these quasi-inequalities. So, the completeness results described in both [10] and [34] have to rely on the cut-elimination algorithm to provide cut-free derivations of the analytic axioms from the rules. In the present chapter, we achieve completeness syntactically, as in [10], and moreover we do so by providing an *effective procedure* for generating a derivation which is not only *cut-free* but also in *pre-normal form* (cf. Definitions 2.2.28 and 2.2.29).

Scope Since (D)LE-logics encompass a wide family of well known logics (modal, intuitionistic, substructural), and since analytic inductive axioms provide a formulation of the notion of analyticity based on the syntactic shape of formulas/sequents, the results of the present paper directly apply to all logical settings for which analytic (proper display) calculi have been defined, such as those of [34, 4, 52, 53, 37, 10]. Moreover, in the present paper we have worked in a single-type environment, mainly for ease of exposition. However, all the results mentioned above straightforwardly apply also to properly displayable *multi-type calculi*, which have been recently introduced to extend the scope and benefits of proper display calculi also to a wide range of logics that for various reasons do not fall into the scope of the analytic inductive definition. These logics crop up in various areas of the literature and include well known logics such as linear logic [35], dynamic epistemic logic [25], semi De Morgan logic [31, 32], bilattice logic [33], inquisitive logic [26], non normal modal logics [6], the logics of classes of rough algebras [30, 29]. Interestingly, the multi-type framework can be also usefully deployed to introduce logics specifically designed to describe and reason about the interaction of entities of different types, as done e.g. in [5, 23].

The syntax-semantics interface on analytic calculi The main insight developed in the research line to which the present paper pertains is that there is a close connection between *semantic* results pertaining to correspondence theory and the *syntactic* theory of analytic calculi. This close connection, which has been observed and also exploited by several authors in various proof-theoretic settings (cf. e.g. [37, 44, 7]), gave rise in [34] to the notion of analytic inductive inequalities as the uniform and independent identification, across signatures, of the syntactic shape (semantically motivated by the order-theoretic properties of the algebraic interpretation of the logical connectives) which guarantees the desiderata of analyticity. In this context, the core of the "syntax-semantic interface" is the algorithm ALBA, which serves to compute *both* the first-order correspondent of analytic inductive axioms *and* their corresponding analytic structural rules. In this paper, we saw the analytic structural rules computed by ALBA at work as the key cogs of the machinery of proper display calculi to derive the axioms that had generated them. This result can be understood as the purely syntactic counterpart of the proof that ALBA preserves semantic equivalence on complete algebras (cf. [13, 19, 17]), which has been used in [34] to motivate the semantic equivalence of any given analytic inductive axiom with its corresponding ALBA-generated structural rules. This observation paves the way to various questions, among which, whether information about the derivation of a given analytic inductive axiom can be extracted directly from its successful ALBA-run, or conversely, whether information about (optimal) ALBA-runs of analytic inductive axioms can be extracted from its derivation in pre-normal form, or whether the recent independent topological characterization of analytic inductive inequalities established in [21] can be exploited for proof-theoretic purposes.

Focused derivations *Focused sequent calculi* [1, 2, 40, 41] make use of syntactic restrictions on the applicability of inference rules forcing a special normal form of cut-free derivations. Strategies for generating focused derivations can be taken as starting points for the development of efficient theorem provers. For a detailed discussion on the applicability and the structure of focused derivations, we refer the reader to the introduction of [36] both from an algebraic and a proof-theoretic perspective. Here we recall that the distinction between so-called *positive* versus *negative* formulas (cf. Footnote 5 and 16) discussed in [36]) is a key ingredient for organising derivations as focused derivations. In future work, we will investigate the relationship between derivations in prenormal form and focalized derivations.

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Chapter 3

Non-normal modal logics and conditional logics: semantic analysis and proof theory

3.1 Introduction

By *non-normal logics* we understand those propositional logics algebraically captured by varieties of *(general, distributive or Boolean) lattice expansions*, i.e. algebras $\mathbb{A} = (\mathbb{B}, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ such that $\mathbb{B}$ is a (general, distributive, or Boolean) lattice, and $\mathcal{F}^{\mathbb{A}}$ and $\mathcal{G}^{\mathbb{A}}$ are finite, possibly empty families of operations on $\mathbb{B}$ in which, in contrast to the corresponding connectives of logics such as normal modal logic and the Lambek calculus, the requirement that each operation in $\mathcal{F}^{\mathbb{A}}$ be finitely join-preserving or meet-reversing in each coordinate and each operation in $\mathcal{G}^{\mathbb{A}}$ be finitely meet-preserving or join-reversing in each coordinate is omitted. Well known examples of non-normal logics are *monotonic modal logic* [4] and *conditional logic* [57, 8], which have been extensively investigated, since they capture key aspects of agents' reasoning, such as the epistemic [64], strategic [3, 2], and hypothetical [26, 51].

Non-normal logics have been investigated both with model-theoretic tools [10, 4] and with proof-theoretic tools [56, 58, 29]. Specific to proof theory, the main challenge is to endow non-normal logics with analytic calculi which can be modularly expanded with additional rules so as to uniformly capture wide classes of axiomatic extensions of the basic frameworks, while preserving key properties such as cut elimination. In this chapter, which builds and expands on [5], we propose to achieve this goal by applying a method which proved successful in very diverse logical contexts, each of which presented its own specific challenges [21, 22, 24, 32, 34, 37, 33, 63]. We illustrate this method by specializing it to the case studies of monotonic modal logic and conditional logic.

Our approach is based on (semantically motivated) translations of the languages of monotonic modal logic and conditional logic into suitable poly-modal signatures in which all connectives are *normal*. Both validity (cf. Propositions 3.3.11 and 3.3.14 for algebraic and relational semantics, respectively) and derivability (cf. Section 3.8.2) are preserved by these translations. Thanks to these translations, non-normal connectives can be captured as compositions of normal connectives.

Via these translations, monotonic modal logic and conditional logic can be endowed with *proper (multi-type) display calculi (see Proposition 3.4.1 and Theorem 3.6.4),*[1] *a*

[1] It is perhaps worth stressing that, by their definition, the (interpretation of) non-normal connectives lack

general format of analytic calculi characterized by a "division of labour" between introduction rules and structural rules (cf. [23, Subsection 2.2] and in particular the so-called Došen's principle *in [67, Subsection1.5]). Specifically, in proper display calculi, the rules introducing logical connectives encode the minimal properties of each connective (namely, its arity and tonicity), while the (analytic) structural rules capture the additional properties of the connectives, including their relations with each other. Together with the defining features of analytic structural rules, this division of labour makes it possible to endow large classes of axiomatic extensions of a given base logic with analytic calculi uniformly and in full generality, simply by adding analytic structural rules to the calculus for the base logic, while preserving cut elimination. Finally, if (the translation of) the axioms defining a given axiomatic extension of a logic which is captured by a proper display calculus are of a certain syntactic shape (namely, the* analytic inductive *shape [35, Definition 55]), the analytic structural rules corresponding to these axioms can be algorithmically generated (cf. [35, Proposition 59], [11, Lemma 4.8]).*

Our starting point for defining the translations mentioned above is the observation, very well-known e.g. from [4, 27, 10], that, under the interpretation of the modal connective of monotonic modal logic in monotone neighbourhood frames $\mathbb{F} = (W, \nu)$, the monotonic 'box' operation can be understood as the composition of a *normal* (i.e. finitely join-preserving) semantic diamond $\langle \nu \rangle$ and a *normal* (i.e. finitely meet-preserving) semantic box $[\ni]$. The binary relations R_ν and $R_\ni$ corresponding to these normal operators are not defined on one and the same domain, but span over two domains. Namely, $R_\nu \subseteq W \times \mathcal{P}(W)$ is such that $wR_\nu X$ iff $X \in \nu(w)$, and $R_\ni \subseteq \mathcal{P}(W) \times W$ is such that $XR_\ni w$ iff $w \in X$ (cf. [10, Definition 5.7] and [44, 27]).

In the present chapter, these relations and their associated *heterogeneous* semantic normal modal operators $\langle \nu \rangle : \mathcal{P}\mathcal{P}(W) \to \mathcal{P}(W)$ and $[\ni] : \mathcal{P}(W) \to \mathcal{P}\mathcal{P}(W)$ become core elements in the definition of the *multi-type*, in this case, two-sorted, (algebraic and relational) semantics interpreting the target languages of the translations, and the observations above are further refined and expanded so as to: (a) introduce a semantic environment of two-sorted Kripke frames (cf. Definition 3.3.1) and their heterogeneous algebras (cf. Definition 3.3.2) for monotonic modal logic and conditional logic; (b) outline a network of discrete dualities and correspondences among these semantic structures and the algebras and frames for monotonic modal logic and conditional logic (cf. Propositions 3.2.2, 3.3.7, 3.3.11, 3.3.14); (c) based on these semantic relationships, introduce multi-type *normal* logics into which the original non-normal logics can be embedded via the semantically motivated translations discussed above (cf. Section 3.4); (d) retrieve well-known dual characterization results for axiomatic extensions of monotonic modal logic and conditional logics as instances of general algorithmic correspondence theory for normal (multi-type) LE-logics applied to the translated axioms (cf. Section 3.9); (e) extract analytic structural rules from the computations of the first-order correspondents of the translated axioms, so that, again by general results on proper display calculi [35] (which, as discussed in [5], can be applied also to multi-type logical frameworks) the resulting

the minimum order-theoretic properties necessary for any non-normal logic to be *properly displayable*, i.e. amenable to be equivalently captured by a proper display calculus, in its original presentation. A characterization of properly displayable logics is given in [35]. This explains why translations are necessary components of our proposed method. This situation is common to all the logical frameworks to which this methodology was successfully applied, none of which is properly displayable in its original presentation for different reasons.

calculi are sound, complete, conservative, and enjoy cut elimination and subformula property.

Besides allowing for the principled design of proper display calculi for the two non-normal logics considered in the present chapter and an infinite class of their axiomatic extensions (cf. Definition 3.6.2), the equivalent multi-type presentations of monotonic modal logic and conditional logic and their semantics are interesting both per se, and also because they introduce new possibilities to the conceptual understanding of non-normal logics. Indeed, firstly, by making it possible to consider states and neighbourhoods as entities of different types, non-normal logics can be regarded as logics describing and reasoning about the specific behaviour of each type as well as their interaction. For instance, if states are interpreted as 'states of affairs' and neighbourhoods as 'pieces of evidence', then by translating the formula $\nabla\varphi$ as $\langle\nu\rangle[\ni]\varphi$ we access a formal language in which it is possible to unpack the meaning of e.g. φ being 'definitely true' at a given state, and reformulate it in terms of the availability of a piece of evidence accessible at that state and supporting the truth of φ. Secondly, the translation we have just discussed is not the only one suggested by the multi-type reformulation of neighbourhood semantics; in fact, another is possible which translates $\nabla\varphi$ as $[\nu^c]\langle\not\ni\rangle\varphi$, and which, when used in combination with the first, makes it possible to obtain analytic translations for a large class of axioms (cf. Remark 3.6.1). Besides being technically useful, this translation has the potential to support different interpretations of the two types and their interaction. Thirdly, the multi-type reformulation of non-normal logics facilitates establishing connections with other areas of investigation in logic and neighbouring fields in which logical frameworks connecting entities of different types have already been studied and exploited. One such area is structural control, which has given rise to a rich literature both in substructural logic [30, 36, 42, 16, 65] and in formal linguistics [53, 45, 54, 55, 39, 3, 66].

Related work. In what follows, without claiming to be exhaustive, we briefly review the literature on proof systems for non-normal logics developed in the context of labelled sequent calculi [29, 15, 56], sequent calculi [48, 40, 41, 60, 49], and nested sequent calculi [1, 50].

In [29], labelled sequent systems are introduced for the classical cube of non-normal modal logics, i.e. the basic non-monotonic modal logic E (also called congruential logic) and its axiomatic extensions with the axioms N, C, or the rule M and their combinations. The approach captures all logics in the cube by extending the basic system via so-called *systems of rules* still preserving cut elimination. The approach of [29] is similar to the approach followed in the present chapter both because it makes use of a poly-modal translation of the original signature, and because a preliminary analysis (i.e. a syntactic characterization) of the first order correspondents of axiomatic extensions is key to the generation of equivalent rules. In [15], analytic and modular labelled sequent systems are introduced for the same non-normal modal logics treated in [29]. While no syntactic translation of the formulas of the original language intervenes in these calculi, the distinction between worlds and neighbourhoods is encoded in the label language. The approach of [15] relies on the methodology introduced in [56], the distinctive feature of which is the introduction of so-called *bi-neighbourhood semantics*, i.e. each world is associated with (sets of) pairs of neighbourhoods rather than single neighbourhoods (this is also reflected in the richer label language). As soon as the logic satisfies the rule M, the bi-neighbourhood semantics collapses onto the standard one.

In [48], complete sequent systems with cut elimination are introduced for the classical cube of non-normal modal logics, which are used to prove finite model property and provide bounds on the cardinality of countermodels. In [40], (resp. [41]), sequent calculi are introduced for monotonic modal logic (resp. congruential logic E with or without axiom N) and its axiomatic extensions with all combinations of axioms D, T, 4, B and 5. In order to capture the axiomatic extensions, the rules introducing the modal connectives differ from one calculus to another. As a consequence, cut elimination, which is proved for most of these calculi by a standard Gentzen argument, is not shown uniformly for all calculi, but has to be proved separately for each of them.

In [60], cut-free sequent calculi are introduced for the minimal conditional logic extended with axioms CEM, MP and their combination, and for the minimal conditional logic extended with axiom ID. Cut elimination is shown via a meta-theorem called *generic cut-elimination*, i.e. an argument that holds for an entire class of sequent calculi satisfying certain local conditions. In [49], the generic cut elimination method, also referred to as *cut elimination by saturation*, is extended to logics over an intuitionistic propositional base and to logics introduced by axioms of arbitrary modal rank, and in particular, to a cut-free sequent calculus introduced for Lewis' conditional logic VA.

In [1], cut-free nested sequent calculi are introduced for the basic conditional logic CK and its axiomatic extensions with ID, MP and CEM and their combinations, with the exception of CK + MP + CEM (+ID). These calculi are all *internal*, i.e. every sequent can be translated into a formula of the original language of CK. In [50], (linear) nested sequent calculi are introduced for a large class of logics which includes the classical non-normal modal cube and various extensions with axioms from P, D, T, 4, and 5.

Structure of the chapter. In Section 3.2, we collect well-known definitions and facts about monotonic modal logic and conditional logic, their algebraic and state-based semantics, and the connection between the two. In Section 3.3, we introduce the multi-type environment (both in the form of heterogeneous algebras and of multi-type Kripke frames) which will provide the semantic justification for the two-sorted modal logics introduced in Section 3.4, as well as for the syntactic translation of the original languages of monotonic modal logic and conditional logic into suitable (multi-type) normal modal languages. In Section 3.5 we specify the definitions of inductive and analytic inductive inequalities/sequents for the multi-type language of monotonic modal logic and conditional logic. In Section 3.6, the theory of unified correspondence is applied to this two-sorted environment to establish a Sahlqvist-type correspondence framework for monotonic modal logic and conditional logic which encompasses and extends the extant correspondence-theoretic results for these logics. In Section 3.7, proper (multi-type) display calculi are introduced for the basic two sorted normal modal languages and for some of their best known extensions. The main properties of these calculi are discussed in Section 3.8. Conclusions and further directions are discussed in Section 3.9.

3.2 Preliminaries

Notation. Throughout the chapter, the superscript $(\cdot)^c$ denotes the relative complement of the subset of a given set. When the given set is a singleton $\{x\}$, we will write x^c instead of $\{x\}^c$. For any binary relation $R \subseteq S \times T$, let $R^{-1} \subseteq T \times S$ be the *converse relation* of R, i.e. $tR^{-1}s$ iff sRt. For any $S' \subseteq S$ and $T' \subseteq T$, we let $R[S'] := \{t \in T \mid (s,t) \in R \text{ for some } s \in S'\}$ and $R^{-1}[T'] := \{s \in S \mid (s,t) \in R \text{ for some } t \in T'\}$. As usual, we write $R[s]$ and $R^{-1}[t]$ in place of $R[\{s\}]$ and $R^{-1}[\{t\}]$, respectively. For any ternary relation $R \subseteq S \times T \times U$ and subsets $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$, we also let

- $R^{(0)}[T', U'] = \{s \in S \mid \exists t \exists u (R(s,t,u) \ \& \ t \in T' \ \& \ u \in U')\}$,
- $R^{(1)}[S', U'] = \{t \in T \mid \exists s \exists u (R(s,t,u) \ \& \ s \in S' \ \& \ u \in U')\}$,
- $R^{(2)}[S', T'] = \{u \in U \mid \exists s \exists t (R(s,t,u) \ \& \ s \in S' \ \& \ t \in T')\}$.

Any binary relation $R \subseteq S \times T$ gives rise to the *modal operators* $\langle R\rangle, [R], [R\rangle, \langle R] : \mathcal{P}(T) \to \mathcal{P}(S)$ s.t. for any $T' \subseteq T$

- $\langle R\rangle T' := R^{-1}[T'] = \{s \in S \mid \exists t(sRt \ \& \ t \in T')\}$;
- $[R]T' := (R^{-1}[T'^c])^c = \{s \in S \mid \forall t(sRt \ \Rightarrow \ t \in T')\}$;
- $[R\rangle T' := (R^{-1}[T'])^c = \{s \in S \mid \forall t(sRt \ \Rightarrow \ t \notin T')\}$;
- $\langle R]T' := R^{-1}[T'^c] = \{s \in S \mid \exists t(sRt \ \& \ t \notin T')\}$.

By construction, these modal operators are normal. Specifically, $\langle R\rangle$ is completely join-preserving, $[R]$ is completely meet-preserving, $[R\rangle$ is completely join-reversing and $\langle R]$ is completely meet-reversing.[2] Hence, their adjoint maps exist and coincide with

$$[R^{-1}], \langle R^{-1}\rangle, [R^{-1}\rangle, \langle R^{-1}] : \mathcal{P}(S) \to \mathcal{P}(T),$$

respectively. That is, for any $T' \subseteq T$ and $S' \subseteq S$,

$$\begin{array}{lcl} \langle R\rangle T' \subseteq S' & \text{iff} & T' \subseteq [R^{-1}]S', \\ S' \subseteq [R]T' & \text{iff} & \langle R^{-1}\rangle S' \subseteq T', \\ S' \subseteq [R\rangle T' & \text{iff} & T' \subseteq [R^{-1}\rangle S' \\ \langle R]T' \subseteq S' & \text{iff} & \langle R^{-1}]S' \subseteq T'. \end{array}$$

Any ternary relation $R \subseteq S \times T \times U$ gives rise to binary modal operators

$$\rhd_R : \mathcal{P}(T) \times \mathcal{P}(U) \to \mathcal{P}(S) \quad \blacktriangle_R : \mathcal{P}(T) \times \mathcal{P}(S) \to \mathcal{P}(U) \quad \blacktriangleright_R : \mathcal{P}(S) \times \mathcal{P}(U) \to \mathcal{P}(T)$$

s.t. for any $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$,

- $T' \rhd_R U' := (R^{(0)}[T', U'^c])^c = \{s \in S \mid \forall t \forall u (R(s,t,u) \ \& \ t \in T' \Rightarrow u \in U')\}$;
- $T' \blacktriangle_R S' := R^{(2)}[T', S'] = \{u \in U \mid \exists t \exists s (R(s,t,u) \ \& \ t \in T' \ \& \ s \in S')\}$;
- $S' \blacktriangleright_R U' := (R^{(1)}[S', U'^c])^c = \{t \in T \mid \forall s \forall u (R(s,t,u) \ \& \ s \in S' \Rightarrow u \in U')\}$.

The stipulations above guarantee that these modal operators are normal. In particular, $\rhd_R$ and $\blacktriangleright_R$ are completely join-reversing in their first coordinate and completely meet-preserving in their second coordinate, and $\blacktriangle_R$ is completely join-preserving in both coordinates. These three maps are residual to each other, i.e. for any $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$,

$$S' \subseteq T' \rhd_R U' \quad \text{iff} \quad T' \blacktriangle_R S' \subseteq U' \quad \text{iff} \quad T' \subseteq S' \blacktriangleright_R U'.$$

[2] That is, $\langle R\rangle \bigcup_{i\in I} T_i = \bigcup_{i\in I} \langle R\rangle T_i$, $[R] \bigcap_{i\in I} T_i = \bigcap_{i\in I} [R] T_i$, $[R\rangle \bigcup_{i\in I} T_i = \bigcap_{i\in I} [R\rangle T_i$ and $\langle R] \bigcap_{i\in I} T_i = \bigcup_{i\in I} \langle R] T_i$. For a general overview of normal logics and the order-theoretic properties characterizing their algebraic semantics, see e.g. [13].

3.2.1 Basic monotonic modal logic and conditional logic

In this section, we collect the necessary preliminaries on the logical frameworks considered in this chapter, and introduce the notation that will be used throughout the chapter. An overview of monotonic modal logic and conditional logic can be found in [4].

Syntax. For a countable set of propositional variables Prop, the languages $\mathcal{L}_\nabla$ and $\mathcal{L}_>$ of monotonic modal logic and conditional logic over Prop are defined as follows:

$$\mathcal{L}_\nabla \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \nabla\varphi \qquad\qquad \mathcal{L}_> \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi > \varphi.$$

The connectives $\top, \wedge, \vee, \rightarrow$ and $\leftrightarrow$ are defined as usual.

The *basic monotonic modal logic* $\mathbf{L}_\nabla$ (resp. *basic conditional logic* $\mathbf{L}_>$) is a set of $\mathcal{L}_\nabla$-formulas (resp. $\mathcal{L}_>$-formulas) containing the axioms of classical propositional logic and closed under modus ponens, uniform substitution and the following rule(s) M (resp. $RCEA$ and RCK_n for all $n \geq 0$):

$$\text{M}\ \frac{\varphi \rightarrow \psi}{\nabla\varphi \rightarrow \nabla\psi} \qquad \text{RCEA}\ \frac{\varphi \leftrightarrow \psi}{(\varphi > \chi) \leftrightarrow (\psi > \chi)} \qquad \text{RCK}_n\ \frac{\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \psi}{(\chi > \varphi_1) \wedge \ldots \wedge (\chi > \varphi_n) \rightarrow (\chi > \psi)}$$

Algebraic semantics. A *monotone Boolean algebra expansion*, abbreviated as *m-algebra* (resp. *conditional algebra*, abbreviated as *c-algebra*) is a pair $\mathbb{A} = (\mathbb{B}, \nabla^{\mathbb{A}})$ (resp. $\mathbb{A} = (\mathbb{B}, >^{\mathbb{A}})$) s.t. $\mathbb{B}$ is a Boolean algebra and $\nabla^{\mathbb{A}}$ is a unary monotone operation on $\mathbb{B}$ (resp. $>^{\mathbb{A}}$ is a binary operation on $\mathbb{B}$ which is finitely meet-preserving in its second coordinate). Such an m-algebra (resp. c-algebra) is *perfect* if $\mathbb{B}$ is a complete and atomic Boolean algebra (and, for c-algebras, $>^{\mathbb{A}}$ is completely meet-preserving in its second coordinate). Hence, the underlying Boolean algebra of any perfect m-algebra (resp. c-algebra) can be identified with the powerset algebra $\mathcal{P}(W)$ for some set W.

Interpretation of formulas in algebras under assignments $h : \mathcal{L}_\nabla \rightarrow \mathbb{A}$ (resp. $h : \mathcal{L}_> \rightarrow \mathbb{A}$) and validity of formulas in algebras (in symbols: $\mathbb{A} \models \varphi$) are defined as usual. By a routine Lindenbaum-Tarski construction one can show that $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) is sound and complete w.r.t. the class of m-algebras V_m (resp. c-algebras V_c).

Canonical extensions. The *canonical extension* of an m-algebra (resp. c-algebra) $\mathbb{A}$ is $\mathbb{A}^\delta := (\mathbb{B}^\delta, \nabla^\sigma)$ (resp. $\mathbb{A}^\delta := (\mathbb{B}^\delta, >^\pi)$), where $\mathbb{B}^\delta \cong \mathcal{P}(Ult(\mathbb{B}))$, with $Ult(\mathbb{B})$ denoting the set of the ultrafilters of $\mathbb{B}$, is the canonical extension of $\mathbb{B}$ [43], and ∇^σ (resp. $>^\pi$) is the σ-extension of $\nabla^{\mathbb{A}}$ (resp. the π-extension of $>^{\mathbb{A}}$). Let us recall that for all $u, u_1, u_2 \in \mathbb{B}^\delta$,

$$\nabla^\sigma u := \bigvee\{\bigwedge\{\nabla a \mid a \in \mathbb{B} \text{ and } k \leq a\} \mid k \in K(\mathbb{B}^\delta) \text{ and } k \leq u\},$$

$$u_1 >^\pi u_2 := \bigwedge\{\bigvee\{a_1 > a_2 \mid a_i \in \mathbb{B} \text{ and } o_i \leq a_i \leq k_i\} \mid k_i \in K(\mathbb{B}^\delta), o_i \in O(\mathbb{B}^\delta) \text{ and } k_i \leq u_i \leq o_i\}, \quad (3.2.1)$$

where $K(\mathbb{B}^\delta)$ and $O(\mathbb{B}^\delta)$ respectively denote the join-closure and the meet-closure of $\mathbb{B}$ in $\mathbb{B}^\delta$ under the canonical embedding, mapping each $a \in \mathbb{B}$ to $\{U \in Ult(\mathbb{B}) \mid a \in U\}$.

By definition and general results on canonical extensions of maps (cf. [28]), the canonical extension of an m-algebra (resp. c-algebra) as above is a perfect m-algebra (resp. c-algebra).

Frames and models. A *neighbourhood frame*, abbreviated as *n-frame* (resp. *conditional frame*, abbreviated as *c-frame*) is a pair $\mathbb{F} = (W, \nu)$ (resp. $\mathbb{F} = (W, f)$) s.t. W is a non-empty set and $\nu : W \to \mathcal{P}(\mathcal{P}(W))$ is a *neighbourhood function* ($f : W \times \mathcal{P}(W) \to \mathcal{P}(W)$ is a *selection function*). In the remainder of the chapter, even if it is not explicitly indicated, we will assume that n-frames are *monotone*, i.e. s.t. for every $w \in W$, if $X \in \nu(w)$ and $X \subseteq Y$, then $Y \in \nu(w)$. For any n-frame (resp. c-frame) $\mathbb{F}$, the *complex algebra* of $\mathbb{F}$ is $\mathbb{F}^* := (\mathcal{P}(W), \nabla^{\mathbb{F}^*})$ (resp. $\mathbb{F}^* := (\mathcal{P}(W), >^{\mathbb{F}^*})$) s.t. for all $X, Y \in \mathcal{P}(W)$,

$$\nabla^{\mathbb{F}^*} X := \{w \mid X \in \nu(w)\} \qquad\qquad X >^{\mathbb{F}^*} Y := \{w \mid f(w, X) \subseteq Y\}.$$

Proposition 3.2.1. *If $\mathbb{F}$ is an n-frame (resp. a c-frame), then $\mathbb{F}^*$ is a perfect m-algebra (resp. c-algebra).*

Proof. Let $\mathbb{F} = (W, \nu)$ be an n-frame. Recall that, by definition, $\nu(w)$ is an upward-closed collection of subsets of W. To show that $\mathbb{F}^*$ is a perfect m-algebra, it is enough to show that $\nabla^{\mathbb{F}^*}$ is monotone. Let $w \in W$ and $X \subseteq Y \subseteq W$. Since $\nu(w)$ is upward-closed, $X \in \nu(w)$ implies that $Y \in \nu(w)$. Hence, $\nabla^{\mathbb{F}^*} X = \{w \mid X \in \nu(w)\} \subseteq \{w \mid Y \in \nu(w)\} = \nabla^{\mathbb{F}^*} Y$.

Let $\mathbb{F} = (W, f)$ be a c-frame. To show that $\mathbb{F}^*$ is a perfect c-algebra, it is enough to show that $>^{\mathbb{F}^*}$ is completely meet-preserving in its second coordinate. For any $X \subseteq W$,

$$X >^{\mathbb{F}^*} \top^{\mathbb{F}^*} = X >^{\mathbb{F}^*} W = \{w \mid f(w, X) \subseteq W\} = W = \top^{\mathbb{F}^*},$$

and for every $\mathcal{X} \subseteq \mathcal{P}(W)$,

$$\begin{aligned} X >^{\mathbb{F}^*} \bigcap \mathcal{X} &= \{w \in W \mid f(w, X) \subseteq \bigcap \mathcal{X}\} \\ &= \{w \in W \mid f(w, X) \subseteq Y \text{ for any } Y \in \mathcal{X}\} \\ &= \bigcap \{(X >^{\mathbb{F}^*} Y) \mid Y \in \mathcal{X}\}. \end{aligned}$$

□

Models are pairs $\mathbb{M} = (\mathbb{F}, V)$ such that $\mathbb{F}$ is a frame and $V : \mathcal{L} \to \mathbb{F}^*$ is a homomorphism of the appropriate type. Hence, the truth of formulas at states in models is defined as $\mathbb{M}, w \Vdash \varphi$ iff $w \in V(\varphi)$, and unravelling this stipulation for ∇- and $>$-formulas, we get:

$$\mathbb{M}, w \Vdash \nabla\varphi \quad \text{iff} \quad V(\varphi) \in \nu(w) \qquad\qquad \mathbb{M}, w \Vdash \varphi > \psi \quad \text{iff} \quad f(w, V(\varphi)) \subseteq V(\psi).$$

Local validity (notation: $\mathbb{F}, w \Vdash \varphi$) is defined as local satisfaction for every valuation V. Global satisfaction (notation: $\mathbb{M} \Vdash \varphi$) and frame validity (notation: $\mathbb{F} \Vdash \varphi$) are defined in the usual way as local satisfaction/validity at every state. Thus, by definition, $\mathbb{F} \Vdash \varphi$ iff $\mathbb{F}^* \models \varphi$, from which the soundness of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) w.r.t. the corresponding class of frames immediately follows from the algebraic soundness. Completeness follows from algebraic completeness, by observing that (a) the canonical extension of any algebra refuting φ will also refute φ; (b) canonical extensions are perfect algebras; (c) perfect m-algebras (resp. c-algebras) can be associated with n-frames (resp. c-frames) as follows: for any $\mathbb{A} = (\mathcal{P}(W), \nabla^{\mathbb{A}})$ (resp. $\mathbb{A} = (\mathcal{P}(W), >^{\mathbb{A}})$) let $\mathbb{A}_* := (W, \nu_{\nabla^{\mathbb{A}}})$ (resp. $\mathbb{A}_* := (W, f_{>^{\mathbb{A}}})$) s.t. for all $w \in W$ and $X \subseteq W$,

$$\nu_{\nabla^{\mathbb{A}}}(w) := \{X \subseteq W \mid w \in \nabla^{\mathbb{A}} X\} \qquad\qquad f_{>^{\mathbb{A}}}(w, X) := \bigcap \{Y \subseteq W \mid w \in X >^{\mathbb{A}} Y\}.$$

That $\mathbb{A}_*$ is a monotone n-frame can be proved as follows: if $X \in \nu_\nabla(w)$ and $X \subseteq Y$, then the monotonicity of $\nabla^{\mathbb{A}}$ implies that $\nabla^{\mathbb{A}} X \subseteq \nabla^{\mathbb{A}} Y$ and hence $Y \in \nu_{\nabla^{\mathbb{A}}}(w)$, as required.

Let $\varphi \in \mathcal{L}_\nabla$ (resp. $\varphi \in \mathcal{L}_>$). It can be shown by a straightforward induction on φ that $w \in V(\varphi)$ iff $(\mathbb{A}_*, V), w \Vdash \varphi$ for any perfect algebra $\mathbb{A}$ and assignment V. Then, $\mathbb{A} \models \varphi$ iff $\mathbb{A}_* \Vdash \varphi$. This completes the argument deriving the frame completeness of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) from its algebraic completeness.

Proposition 3.2.2. *If $\mathbb{A}$ is a perfect m-algebra (resp. c-algebra) and $\mathbb{F}$ is an n-frame (resp. c-frame), then $(\mathbb{F}^*)_* \cong \mathbb{F}$ and $(\mathbb{A}_*)^* \cong \mathbb{A}$.*

Proof. Let $\mathbb{F} = (W, \nu)$ be an n-frame. By definition, $(\mathbb{F}^*)_* = (W, \nu_{\nabla^{\mathbb{F}^*}})$, where, for every $w \in W$,

$$\begin{aligned} \nu_{\nabla^{\mathbb{F}^*}}(w) &= \{X \subseteq W \mid w \in \nabla^{\mathbb{F}^*} X\} \\ &= \{X \subseteq W \mid w \in \{u \mid X \in \nu(u)\}\} \\ &= \{X \subseteq W \mid X \in \nu(w)\} \\ &= \nu(w), \end{aligned}$$

which shows that $(\mathbb{F}^*)_* = \mathbb{F}$, as required. Let $\mathbb{F} = (W, f)$ be a c-frame. By definition, $(\mathbb{F}^*)_* = (W, f_{>^{\mathbb{F}^*}})$, where, for every $w \in W$ and $X \subseteq W$,

$$\begin{aligned} f_{>^{\mathbb{F}^*}}(w, X) &= \bigcap\{Y \subseteq W \mid w \in X >^{\mathbb{F}^*} Y\} \\ &= \bigcap\{Y \subseteq W \mid w \in \{u \in W \mid f(u, X) \subseteq Y\}\} \\ &= \bigcap\{Y \subseteq W \mid f(w, X) \subseteq Y\} \\ &= f(w, X), \end{aligned}$$

which shows that $(\mathbb{F}^*)_* = \mathbb{F}$, as required. Let $\mathbb{A} = (\mathcal{P}(W), \nabla^{\mathbb{A}})$ be a perfect m-algebra (up to isomorphism). Then $(\mathbb{A}_*)^* = (\mathcal{P}(W), \nabla^{(\mathbb{A}_*)^*})$, where for every $X \subseteq W$,

$$\begin{aligned} \nabla^{(\mathbb{A}_*)^*} X &= \{w \in W \mid X \in \nu_{\nabla^{\mathbb{A}}}(w)\} \\ &= \{w \in W \mid X \in \{Y \subseteq W \mid w \in \nabla^{\mathbb{A}} Y\}\} \\ &= \{w \in W \mid w \in \nabla^{\mathbb{A}} X\} \\ &= \nabla^{\mathbb{A}} X, \end{aligned}$$

which shows that $(\mathbb{A}_*)^* \cong \mathbb{A}$, as required. Let $\mathbb{A} = (\mathcal{P}(W), >^{\mathbb{A}})$ be a perfect c-algebra (up to isomorphism). Then $(\mathbb{A}_*)^* = (\mathcal{P}(W), >^{(\mathbb{A}_*)^*})$, where for all $X, Y \subseteq W$,

$$\begin{aligned} X >^{(\mathbb{A}_*)^*} Y &= \{w \in W \mid f_{>^{\mathbb{A}}}(w, X) \subseteq Y\} \\ &= \{w \in W \mid \bigcap\{Z \subseteq W \mid w \in X >^{\mathbb{A}} Z\} \subseteq Y\} \\ &= X >^{\mathbb{A}} Y. \end{aligned}$$

Let us show the last equality. If $w \in X >^{\mathbb{A}} Y$, then $Y \in \{Z \subseteq W \mid w \in X >^{\mathbb{A}} Z\}$, and hence $\bigcap\{Z \subseteq W \mid w \in X >^{\mathbb{A}} Z\} \subseteq Y$. Conversely, let $w \in W$ be s.t. $\bigcap\{Z \subseteq W \mid w \in X >^{\mathbb{A}} Z\} \subseteq Y$. Since $>^{\mathbb{A}}$ is completely meet-preserving in the second coordinate, this implies that

$$w \in \bigcap\{X >^{\mathbb{A}} Z \mid Z \subseteq W \text{ and } w \in X >^{\mathbb{A}} Z\} = X >^{\mathbb{A}} \bigcap\{Z \subseteq W \mid w \in X >^{\mathbb{A}} Z\} \subseteq X >^{\mathbb{A}} Y,$$

as required. This completes the proof that $(\mathbb{A}_*)^* \cong \mathbb{A}$. $\square$

Axiomatic extensions. A *monotonic modal logic* (resp. a *conditional logic*) is any extension of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) with $\mathcal{L}_\nabla$-axioms (resp. $\mathcal{L}_>$-axioms). The correspondence results collected in the theorem below mostly concern well known axioms and are well known from the literature (cf. [10, Theorem 5.1] [58]).[3] The axiom *CN* below is inspired by the Connex axiom of *V*-logics presented in [51].

Theorem 3.2.3. *For every n-frame (resp. c-frame)* $\mathbb{F}$*,*

N	$\mathbb{F} \Vdash \nabla\top$	*iff*	$\mathbb{F} \models \forall w[W \in \nu(w)]$
P	$\mathbb{F} \Vdash \neg\nabla\bot$	*iff*	$\mathbb{F} \models \forall w[\varnothing \notin \nu(w)]$
C	$\mathbb{F} \Vdash \nabla p \wedge \nabla q \to \nabla(p \wedge q)$	*iff*	$\mathbb{F} \models \forall w \forall X \forall Y[(X \in \nu(w)\ \&\ Y \in \nu(w)) \Rightarrow X \cap Y \in \nu(w)]$
T	$\mathbb{F} \Vdash \nabla p \to p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \in \nu(w) \Rightarrow w \in X]$
4	$\mathbb{F} \Vdash \nabla\nabla p \to \nabla p$	*iff*	$\mathbb{F} \models \forall w \forall YX[(X \in \nu(w)\ \&\ \forall x(x \in X \Rightarrow Y \in \nu(x))) \Rightarrow Y \in \nu(w)]$
4'	$\mathbb{F} \Vdash \nabla p \to \nabla\nabla p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \in \nu(w) \Rightarrow \{y \mid X \in \nu(y)\} \in \nu(w)]$
5	$\mathbb{F} \Vdash \neg\nabla\neg p \to \nabla\neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \notin \nu(w) \Rightarrow \{y \mid X \in \nu(y)\}^c \in \nu(w)]$
B	$\mathbb{F} \Vdash p \to \nabla\neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w \forall X[w \in X \Rightarrow \{y \mid X^c \in \nu(y)\}^c \in \nu(w)]$
D	$\mathbb{F} \Vdash \nabla p \to \neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \in \nu(w) \Rightarrow X^c \notin \nu(w)]$
CS	$\mathbb{F} \Vdash (p \wedge q) \to (p > q)$	*iff*	$\mathbb{F} \models \forall x \forall Z[x \in Z \Rightarrow f(x, Z) \subseteq \{x\}]$
CEM	$\mathbb{F} \Vdash (p > q) \vee (p > \neg q)$	*iff*	$\mathbb{F} \models \forall X \forall y[\lvert f(y, X)\rvert \leq 1]$
ID	$\mathbb{F} \Vdash p > p$	*iff*	$\mathbb{F} \models \forall x \forall Z[f(x, Z) \subseteq Z]$.
CN	$\mathbb{F} \Vdash (p > q) \vee (q > p)$	*iff*	$\mathbb{F} \models \forall X \forall Y \forall z[(f(z, X) \subseteq Y)$ *or* $(f(z, Y) \subseteq X)]$.

In the following section we introduce a semantic environment thanks to which the correspondence results above can be obtained as instances of a suitable multi-type version of unified correspondence theory [12, 13]. This environment also motivates the introduction of proper display calculi for the logics axiomatised by those axioms—among the ones listed above—the translation of which is analytic inductive (cf. Section 3.4).

3.3 Semantic analysis

3.3.1 Two-sorted Kripke frames and their discrete duality

Structures similar to those below are considered implicitly in [10], and explicitly in [25].

Definition 3.3.1. A *two-sorted n-frame* (resp. *c-frame*) is a structure $\mathbb{K} := (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{K} := (X, Y, R_\ni, R_{\not\ni}, T_f)$) such that X and Y are nonempty sets, $R_\ni, R_{\not\ni} \subseteq Y \times X$ and $R_\nu, R_{\nu^c} \subseteq X \times Y$ and $T_f \subseteq X \times Y \times X$. Such an n-frame is *supported* if for every $D \subseteq X$,

$$R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c. \tag{3.3.1}$$

For any two-sorted n-frame (resp. c-frame) $\mathbb{K}$, the *complex algebra* of $\mathbb{K}$ is

$$\mathbb{K}^+ := (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{K}^+}, \langle\not\ni\rangle^{\mathbb{K}^+}, \langle\nu\rangle^{\mathbb{K}^+}, [\nu^c]^{\mathbb{K}^+})$$
$$\text{(resp. } \mathbb{K}^+ := (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{K}^+}, [\not\ni\rangle^{\mathbb{K}^+}, \rhd^{\mathbb{K}^+})\text{), s.t.}$$

$$\begin{array}{rlrlrl}
\langle\nu\rangle^{\mathbb{K}^+}: & \mathcal{P}(Y) \to \mathcal{P}(X) & [\ni]^{\mathbb{K}^+}: & \mathcal{P}(X) \to \mathcal{P}(Y) & \langle\not\ni\rangle^{\mathbb{K}^+}: & \mathcal{P}(X) \to \mathcal{P}(Y) \\
U \mapsto & R_\nu^{-1}[U] & D \mapsto & (R_\ni^{-1}[D^c])^c & D \mapsto & R_{\not\ni}^{-1}[D] \\
[\nu^c]^{\mathbb{K}^+}: & \mathcal{P}(Y) \to \mathcal{P}(X) & [\not\ni\rangle^{\mathbb{K}^+}: & \mathcal{P}(X) \to \mathcal{P}(Y) & \rhd^{\mathbb{K}^+}: & \mathcal{P}(Y) \times \mathcal{P}(X) \to \mathcal{P}(X) \\
U \mapsto & (R_{\nu^c}^{-1}[U^c])^c & D \mapsto & (R_{\not\ni}^{-1}[D])^c & (U, D) \mapsto & (T_f^{(0)}[U, D^c])^c
\end{array}$$

[3] The scope of applicability of the methodology presented in this chapter is specified in Definition 3.6.2, Example 3.6.3 and Theorem 3.6.4.

The adjoints and residuals of the maps above (cf. Section 3.2) are defined as follows:

$$
\begin{array}{rcl}
[\lambda]^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) & \langle\in\rangle^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) & [\notin]^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) \\
D \mapsto (R_\nu[D^c])^c & U \mapsto R_\ni[U] & U \mapsto (R_{\not\ni}[U^c])^c \\
\\
\langle\lambda^c\rangle^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) & [\notin\rangle^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) & \blacktriangleright^{\mathbb{K}^+} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(Y) \\
D \mapsto R_{\nu^c}[D] & U \mapsto (R_{\not\ni}[U])^c & (C, D) \mapsto (T_f^{(1)}[C, D^c])^c \\
 & \blacktriangle^{\mathbb{K}^+} : \mathcal{P}(Y) \times \mathcal{P}(X) \to \mathcal{P}(X) & \\
 & (U, D) \mapsto T_f^{(2)}[U, D] &
\end{array}
$$

Complex algebras of two-sorted frames can be recognized as perfect heterogeneous algebras (cf. [6]) of the following kind:

Definition 3.3.2. A *heterogeneous m-algebra* (resp. *c-algebra*) is a structure

$$\mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}}) \qquad (\text{resp. } \mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}}))$$

such that $\mathbb{A}$ and $\mathbb{B}$ are Boolean algebras, $\langle\nu\rangle^{\mathbb{H}}, [\nu^c] : \mathbb{B} \to \mathbb{A}$ are finitely join-preserving and finitely meet-preserving respectively, $[\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}} : \mathbb{A} \to \mathbb{B}$ are finitely meet-preserving, finitely join-reversing, and finitely join-preserving respectively, and $\rhd^{\mathbb{H}}: \mathbb{B} \times \mathbb{A} \to \mathbb{A}$ is finitely join-reversing in its first coordinate and finitely meet-preserving in its second coordinate. Such an $\mathbb{H}$ is *complete* if $\mathbb{A}$ and $\mathbb{B}$ are complete Boolean algebras and the operations above enjoy the complete versions of the finite preservation properties indicated above, and is *perfect* if it is complete and $\mathbb{A}$ and $\mathbb{B}$ are perfect. The *canonical extension* of a heterogeneous m-algebra (resp. c-algebra) $\mathbb{H}$ is $\mathbb{H}^\delta := (\mathbb{A}^\delta, \mathbb{B}^\delta, [\ni]^{\mathbb{H}^\delta}, \langle\not\ni\rangle^{\mathbb{H}^\delta}, \langle\nu\rangle^{\mathbb{H}^\delta}, [\nu^c]^{\mathbb{H}^\delta})$ (resp. $\mathbb{H}^\delta := (\mathbb{A}^\delta, \mathbb{B}^\delta, [\ni]^{\mathbb{H}^\delta}, [\not\ni\rangle^{\mathbb{H}^\delta}, \rhd^{\mathbb{H}^\delta})$), where $\mathbb{A}^\delta$ and $\mathbb{B}^\delta$ are the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ respectively [43], moreover $[\ni]^{\mathbb{H}^\delta}$, $[\not\ni\rangle^{\mathbb{H}^\delta}$, $[\nu^c]^{\mathbb{H}^\delta}$, $\rhd^{\mathbb{H}^\delta}$ are the π-extensions of $[\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}}, \rhd^{\mathbb{H}}$ respectively, and $\langle\nu\rangle^{\mathbb{H}^\delta}$, $\langle\not\ni\rangle^{\mathbb{H}^\delta}$ are the σ-extensions of $\langle\nu\rangle^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}$ respectively.

Definition 3.3.3. A heterogeneous m-algebra $\mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ is *supported* if $\langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}a = [\nu^c]^{\mathbb{H}} \langle\not\ni\rangle^{\mathbb{H}}a$ for every $a \in \mathbb{A}$.

It immediately follows from the definitions that

Lemma 3.3.4. *The complex algebra of a supported two-sorted n-frame is a perfect heterogeneous supported m-algebra.*

Proof. Let $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ be a supported two-sorted n-frame. Then its complex algebra is $\mathbb{K}^+ = (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{K}^+}, \langle\not\ni\rangle^{\mathbb{K}^+}, \langle\nu\rangle^{\mathbb{K}^+}, [\nu^c]^{\mathbb{K}^+})$, which is clearly perfect. Since $\mathbb{K}$ is also supported, $R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c$ for any $D \subseteq \mathbb{K}$. Hence,

$$\langle\nu\rangle^{\mathbb{K}^+}[\ni]^{\mathbb{K}^+}D = R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c = [\nu^c]^{\mathbb{K}^+}\langle\not\ni\rangle^{\mathbb{K}^+}D.$$

□

Definition 3.3.5. If $\mathbb{H} = (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ is a perfect heterogeneous m-algebra (resp. $\mathbb{H} = (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})$ is a perfect heterogeneous c-algebra), its associated two-sorted n-frame (resp. c-frame) is

$$\mathbb{H}_+ := (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c}) \qquad (\text{resp. } \mathbb{H}_+ := (X, Y, R_\ni, R_{\not\ni}, T_f)), \text{ s.t.}$$

- $R_\ni \subseteq Y \times X$ is defined by $yR_\ni x$ iff $y \notin [\ni]^{\mathbb{H}} x^c$,
- $R_{\not\ni} \subseteq Y \times X$ is defined by $xR_{\not\ni}y$ iff $y \in \langle\not\ni\rangle^{\mathbb{H}}\{x\}$ (resp. $y \notin [\not\ni\rangle^{\mathbb{H}}\{x\}$),
- $R_\nu \subseteq X \times Y$ is defined by $xR_\nu y$ iff $x \in \langle\nu\rangle^{\mathbb{H}}\{y\}$,
- $R_{\nu^c} \subseteq X \times Y$ is defined by $xR_{\nu^c}y$ iff $x \notin [\nu^c]^{\mathbb{H}} y^c$,
- $T_f \subseteq X \times Y \times X$ is defined by $(x', y, x) \in T_f$ iff $x' \notin \{y\} \rhd^{\mathbb{H}} x^c$.

Lemma 3.3.6. *If $\mathbb{H}$ is a perfect supported heterogeneous m-algebra, then $\mathbb{H}_+$ is a supported two-sorted n-frame.*

Proof. To show that $\mathbb{H}_+$ is supported, for every $D \subseteq X$,

$$R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = \langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}D = [\nu^c]^{\mathbb{H}}\langle\not\ni\rangle^{\mathbb{H}}D = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c.$$

□

The duality between perfect BAOs and Kripke frames can be readily extended to the present two-sorted case. The following proposition collects these well-known facts, the proofs of which are analogous to those of the single-sorted case, hence are omitted.

Proposition 3.3.7. *For every heterogeneous m-algebra (resp. c-algebra) $\mathbb{H}$ and every two-sorted n-frame (resp. c-frame) $\mathbb{K}$,*

1. *$\mathbb{K}^+$ is a perfect heterogeneous m-algebra (resp. c-algebra);*
2. *$(\mathbb{K}^+)_+ \cong \mathbb{K}$, and if $\mathbb{H}$ is perfect, then $(\mathbb{H}_+)^+ \cong \mathbb{H}$.*

3.3.2 Equivalent representation of m-algebras and c-algebras

Every supported heterogeneous m-algebra (resp. c-algebra) can be associated with an m-algebra (resp. a c-algebra) as follows:

Definition 3.3.8. For every supported heterogeneous m-algebra $\mathbb{H} = (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ (resp. c-algebra $\mathbb{H} = (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})$), let $\mathbb{H}_\bullet := (\mathbb{A}, \nabla^{\mathbb{H}_\bullet})$ (resp. $\mathbb{H}_\bullet := (\mathbb{A}, >^{\mathbb{H}_\bullet})$), where for every $a \in \mathbb{A}$ (resp. $a, b \in \mathbb{A}$),

$$\nabla^{\mathbb{H}_\bullet}a = \langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}a = [\nu^c]^{\mathbb{H}}\langle\not\ni\rangle^{\mathbb{H}}a \qquad (\text{resp. } a >^{\mathbb{H}_\bullet} b := ([\ni]^{\mathbb{H}}a \wedge [\not\ni\rangle^{\mathbb{H}}a) \rhd^{\mathbb{H}} b).$$

It immediately follows from the stipulations above that $\nabla^{\mathbb{H}_\bullet}$ is a monotone map (resp. $>^{\mathbb{H}_\bullet}$ is finitely meet-preserving in its second coordinate), and hence $\mathbb{H}_\bullet$ is an m-algebra (resp. a c-algebra). Conversely, every complete m-algebra (resp. c-algebra) can be associated with a complete supported heterogeneous m-algebra (resp. a c-algebra) as follows:

Definition 3.3.9. For every complete m-algebra $\mathbb{C} = (\mathbb{A}, \nabla^{\mathbb{C}})$ (resp. complete c-algebra $\mathbb{C} = (\mathbb{A}, >^{\mathbb{C}})$), let $\mathbb{C}^\bullet := (\mathbb{A}, \mathcal{P}(\mathbb{A}), [\ni]^{\mathbb{C}^\bullet}, \langle\not\ni\rangle^{\mathbb{C}^\bullet}, \langle\nu\rangle^{\mathbb{C}^\bullet}, [\nu^c]^{\mathbb{C}^\bullet})$ (resp. $\mathbb{C}^\bullet := (\mathbb{A}, \mathcal{P}(\mathbb{A}), [\ni]^{\mathbb{C}^\bullet}, [\not\ni\rangle^{\mathbb{C}^\bullet}, \rhd^{\mathbb{C}^\bullet})$), where for every $a \in \mathbb{A}$ and $B \in \mathcal{P}(\mathbb{A})$,

$$[\ni]^{\mathbb{C}^\bullet}a := \{b \in \mathbb{A} \mid b \leq a\} \qquad \langle\nu\rangle^{\mathbb{C}^\bullet}B := \bigvee\{\nabla^{\mathbb{C}}b \mid b \in B\} \qquad [\not\ni\rangle^{\mathbb{C}^\bullet}a := \{b \in \mathbb{A} \mid a \leq b\}$$

$$[\nu^c]^{\mathbb{C}^\bullet}B := \bigwedge\{\nabla^{\mathbb{C}}b \mid b \notin B\} \quad B \rhd^{\mathbb{C}^\bullet} a := \bigwedge\{b >^{\mathbb{C}} a \mid b \in B\} \quad \langle\not\ni\rangle^{\mathbb{C}^\bullet}a := \{b \in \mathbb{A} \mid a \not\leq b\}.$$

Lemma 3.3.10. *If $\mathbb{C}$ is a complete m-algebra (resp. complete c-algebra), then $\mathbb{C}^\bullet$ is a complete supported heterogeneous m-algebra (resp. c-algebra).*

Proof. Let $\mathbb{C} = (\mathbb{A}, \nabla^{\mathbb{C}})$ be a complete m-algebra. First we show that $\mathbb{C}^\bullet$ is a complete heterogeneous m-algebra. For $X \subseteq \mathbb{A}$ and $\Gamma \subseteq \mathcal{P}(\mathbb{A})$,

$$
\begin{aligned}
[\ni]^{\mathbb{C}^\bullet} \bigwedge X &= \{b \in \mathbb{A} \mid b \leq \bigwedge X\} = \bigcap_{x \in X} \{b \in \mathbb{A} \mid b \leq x\} = \bigcap_{x \in X} [\ni]^{\mathbb{C}^\bullet} x \\
\langle \not\ni \rangle^{\mathbb{C}^\bullet} \bigvee X &= \{b \in \mathbb{A} \mid \bigvee X \not\leq b\} = \bigcup_{x \in X} \{b \in \mathbb{A} \mid x \not\leq b\} = \bigcup_{x \in X} \langle \not\ni \rangle^{\mathbb{C}^\bullet} x \\
\langle \nu \rangle^{\mathbb{C}^\bullet} \bigcup \Gamma &= \bigvee \{\nabla^{\mathbb{C}} b \mid b \in \bigcup \Gamma\} = \bigvee_{Y \in \Gamma} \bigvee \{\nabla^{\mathbb{C}} b \mid b \in Y\} = \bigvee_{Y \in \Gamma} \langle \nu \rangle^{\mathbb{C}^\bullet} Y \\
[\nu^c]^{\mathbb{C}^\bullet} \bigcap \Gamma &= \bigwedge \{\nabla^{\mathbb{C}} b \mid b \notin \bigcap \Gamma\} = \bigcap_{Y \in \Gamma} \bigwedge \{\nabla^{\mathbb{C}} b \mid b \notin Y\} = \bigcap_{Y \in \Gamma} [\nu^c]^{\mathbb{C}^\bullet} Y.
\end{aligned}
$$

Let us show that $\mathbb{C}^\bullet$ is supported. For every $a \in \mathbb{A}$,

$$
\begin{aligned}
\langle \nu \rangle^{\mathbb{C}^\bullet} [\ni]^{\mathbb{C}^\bullet} a &= \langle \nu \rangle^{\mathbb{C}^\bullet} \{b \in \mathbb{A} \mid b \leq a\} = \bigvee \{\nabla^{\mathbb{C}} b \mid b \leq a\} = \nabla^{\mathbb{C}} a, \\
[\nu^c]^{\mathbb{C}^\bullet} \langle \not\ni \rangle^{\mathbb{C}^\bullet} a &= [\nu^c]^{\mathbb{C}^\bullet} \{b \in \mathbb{A} \mid a \nleq b\} = \bigwedge \{\nabla^{\mathbb{C}} b \mid a \leq b\} = \nabla^{\mathbb{C}} a.
\end{aligned}
$$

Hence, $\langle \nu \rangle^{\mathbb{C}^\bullet} [\ni]^{\mathbb{C}^\bullet} a = [\nu^c]^{\mathbb{C}^\bullet} \langle \not\ni \rangle^{\mathbb{C}^\bullet} a$.

Let $\mathbb{C} = (\mathbb{A}, >^{\mathbb{C}})$ be a complete c-algebra. That $[\ni]^{\mathbb{C}^\bullet}$ is completely join preserving can be proved as shown above. As to the remaining connectives, for any $X \subseteq \mathbb{A}$ and $\Gamma \subseteq \mathcal{P}$,

$$
\begin{aligned}
[\not\ni\rangle^{\mathbb{C}^\bullet} \bigvee X &= \{b \in \mathbb{A} \mid \bigvee X \leq b\} = \bigcap_{x \in X} \{b \in \mathbb{A} \mid x \leq b\} = \bigcap_{x \in X} [\not\ni\rangle^{\mathbb{C}^\bullet} x \\
\bigcup \Gamma \rhd^{\mathbb{C}^\bullet} a &= \bigwedge \{b >^{\mathbb{C}} a \mid b \in \bigcup \Gamma\} = \bigwedge_{Y \in \Gamma} \bigwedge \{b >^{\mathbb{C}} a \mid b \in Y\} = \bigwedge_{Y \in \Gamma} (Y \rhd^{\mathbb{C}^\bullet} a) \\
B \rhd^{\mathbb{C}^\bullet} \bigwedge X &= \bigwedge \{b >^{\mathbb{C}} \bigwedge X \mid b \in B\} = \bigwedge_{x \in X} \bigwedge \{b >^{\mathbb{C}} x \mid b \in B\} = \bigwedge_{x \in X} (B \rhd^{\mathbb{C}^\bullet} x).
\end{aligned}
$$

□

Proposition 3.3.11. *If $\mathbb{C}$ is a complete m-algebra (resp. c-algebra), then $\mathbb{C} \cong (\mathbb{C}^\bullet)_\bullet$. Moreover, if $\mathbb{H}$ is a complete supported heterogeneous m-algebra (resp. c-algebra), then $\mathbb{H} \cong \mathbb{C}^\bullet$ for some complete m-algebra (resp. c-algebra) $\mathbb{C}$ iff $\mathbb{H} \cong (\mathbb{H}_\bullet)^\bullet$.*

Proof. For the first part of the statement, by definition, $\mathbb{C}$ and $(\mathbb{C}^\bullet)_\bullet$ have the same underlying Boolean algebra. Moreover, $\nabla^{(\mathbb{C}^\bullet)_\bullet} a = \langle \nu \rangle^{\mathbb{C}^\bullet} [\ni]^{\mathbb{C}^\bullet} a = \nabla^{\mathbb{C}} a$ for every $a \in \mathbb{C}$, the first identity holding by definition, the second one being shown in the proof of Lemma 3.3.10.

As to the second part, for the left to right direction, assume that $\mathbb{H} \cong \mathbb{C}^\bullet$ for some complete m-algebra (resp. c-algebra) $\mathbb{C}$. From the first part of the proposition we know that $\mathbb{C} \cong (\mathbb{C}^\bullet)_\bullet$. Then $\mathbb{H} \cong \mathbb{C}^\bullet \cong ((\mathbb{C}^\bullet)_\bullet)^\bullet \cong (\mathbb{H}_\bullet)^\bullet$. For the right to left direction, $\mathbb{H}_\bullet$ is the required complete m-algebra (resp. c-algebra). □

The proposition above characterizes up to isomorphism the supported heterogeneous m-algebras (resp. c-algebras) which arise from single-type m-algebras (resp. c-algebras).

3.3.3 Representing n-frames and c-frames as two-sorted Kripke frames

Thanks to the discrete dualities discussed in Sections 3.2.1 and 3.3.1, we can transfer the algebraic characterization of Proposition 3.3.11 to the side of frames, as detailed in this subsection.

Definition 3.3.12. For any n-frame (resp. c-frame) $\mathbb{F}$, we let $\mathbb{F}^\star := ((\mathbb{F}^\circledast)^\bullet)_+$, and for every supported two-sorted n-frame (resp. c-frame) $\mathbb{K}$, we let $\mathbb{K}_\star := ((\mathbb{K}^+)_\bullet)_\circledast$.

Spelling out the definition above, if $\mathbb{F} = (W, \nu)$ (resp. $\mathbb{F} = (W, f)$) then $\mathbb{F}^\star = (W, \mathcal{P}(W), R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{F}^\star = (W, \mathcal{P}(W), R_{\not\ni}, R_\ni, T_f)$) where:

- $R_\nu \subseteq W \times \mathcal{P}(W)$ is defined as $xR_\nu Z$ iff $Z \in \nu(x)$;
- $R_{\nu^c} \subseteq W \times \mathcal{P}(W)$ is defined as $xR_{\nu^c}Z$ iff $Z \notin \nu(x)$;
- $R_\ni \subseteq \mathcal{P}(W) \times W$ is defined as $ZR_\ni x$ iff $x \in Z$;
- $R_{\not\ni} \subseteq \mathcal{P}(W) \times W$ is defined as $ZR_{\not\ni}x$ iff $x \notin Z$;
- $T_f \subseteq W \times \mathcal{P}(W) \times W$ is defined as $T_f(x, Z, x')$ iff $x' \in f(x, Z)$.

Moreover, if $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, T_f)$), then $\mathbb{K}_\star = (X, \nu_\star)$ (resp. $\mathbb{K}_\star = (X, f_\star)$) where:

- $\nu_\star(x) = \{D \subseteq X \mid x \in R_\nu^{-1}[(R_\ni^{-1}[D^c])^c]\} = \{D \subseteq X \mid x \in (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c\}$;
- $f_\star(x, D) = \bigcap\{C \subseteq X \mid x \in T_f^{(0)}[\{C\}, D^c]\}$.

Lemma 3.3.13. *If $\mathbb{F} = (W, \nu)$ is an n-frame, then $\mathbb{F}^\star$ is a supported two-sorted n-frame.*

Proof. By definition, $\mathbb{F}^\star$ is a two-sorted n-frame. Moreover, for any $D \subseteq W$,

$$\begin{array}{rcll}(R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c & = & \{w \mid \forall X(X \notin \nu(w) \Rightarrow \exists u(X \not\ni u \;\&\; u \in D))\} & \\ & = & \{w \mid \forall X(X \notin \nu(w) \Rightarrow D \not\subseteq X)\} & \\ & = & \{w \mid \forall X(D \subseteq X \Rightarrow X \in \nu(w))\} & \\ & = & \{w \mid \exists X(X \in \nu(w) \;\&\; X \subseteq D)\} & (*) \\ & = & R_\nu^{-1}[(R_\ni^{-1}[D^c])^c]. & \end{array}$$

To show the identity marked with $(*)$, from top to bottom, take $X := D$; conversely, if $D \subseteq Z$ then $X \subseteq Z$, and since by assumption $X \in \nu(w)$ and $\nu(w)$ is upward closed, we conclude that $Z \in \nu(w)$, as required. □

The next proposition is the frame-theoretic counterpart of Proposition 3.3.11.

Proposition 3.3.14. *If $\mathbb{F}$ is an n-frame (resp. c-frame), then $\mathbb{F} \cong (\mathbb{F}^\star)_\star$. Moreover, if $\mathbb{K}$ is a supported two-sorted n-frame (resp. c-frame), then $\mathbb{K} \cong \mathbb{F}^\star$ for some n-frame (resp. c-frame) $\mathbb{F}$ iff $\mathbb{K} \cong (\mathbb{K}_\star)^\star$.*

Proof. For the first part of the statement,

$$\begin{array}{rcll}(\mathbb{F}^\star)_\star & = & (((((\mathbb{F}^\circledast)^\bullet)_+)^+)_\bullet)_\circledast & \text{definition of } (-)^\star \text{ and } (-)_\star \\ & \cong & (((\mathbb{F}^\circledast)^\bullet)_\bullet)_\circledast & \text{Proposition 3.3.7.2, } (\mathbb{F}^\circledast)^\bullet \text{ perfect heterogeneous algebra} \\ & = & (\mathbb{F}^\circledast)_\circledast & \text{Proposition 3.3.11, since } \mathbb{F}^\circledast \text{ is complete} \\ & = & \mathbb{F}. & \text{Proposition 3.2.2}\end{array}$$

As to the second part, for the left to right direction, assume that $\mathbb{K} \cong \mathbb{F}^\star$ for some m-frame (resp. c-frame) $\mathbb{F}$. From the first part of the statement we know that $\mathbb{F} \cong (\mathbb{F}^\star)_\star$. Then $\mathbb{K} \cong \mathbb{F}^\star \cong ((\mathbb{F}^\star)_\star)^\star \cong (\mathbb{K}_\star)^\star$. For the right to left direction, $\mathbb{K}_\star$ is the required m-frame (resp. c-frame).

□

3.4 Embedding non-normal logics into two-sorted normal logics

The two-sorted frames and heterogeneous algebras discussed in the previous section serve as semantic environment for the multi-type languages defined below.

Multi-type languages. For a denumerable set Prop of atomic propositions, the language $\mathcal{L}_{MT\nabla}$ for monotonic modal logic, in types S (sets) and N (neighbourhoods) over Prop, is defined as:

$$\mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle \nu \rangle \alpha \mid [\nu^c]\alpha$$
$$\mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid \langle \not\ni \rangle \alpha$$

and the language $\mathcal{L}_{MT>}$ for conditional logic, in types S (sets) and N (neighbourhoods) over Prop, is defined as:

$$\mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A$$
$$\mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid [\not\ni\rangle A.$$

Algebraic semantics. Interpretation of $\mathcal{L}_{MT\nabla}$-formulas (resp. $\mathcal{L}_{MT>}$formulas) in heterogeneous m-algebras (resp. c-algebras) under homomorphic assignments $h : \mathcal{L}_{MT\nabla} \to \mathbb{H}$ (resp. $h : \mathcal{L}_{MT>} \to \mathbb{H}$) and validity of formulas in heterogeneous algebras ($\mathbb{H} \models \Theta$) are defined as usual.

Frames and models. $\mathcal{L}_{MT\nabla}$*-models* (resp. $\mathcal{L}_{MT>}$*-models*) are pairs $\mathbb{N} = (\mathbb{K}, V)$ s.t. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ is a supported two-sorted n-frame (resp. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, T_f)$ is a two-sorted c-frame) and $V : \mathcal{L}_{MT} \to \mathbb{K}^+$ is a heterogeneous algebra homomorphism of the appropriate signature. Hence, truth of formulas at states in models is defined as $\mathbb{N}, z \Vdash \Theta$ iff $z \in V(\Theta)$ for every $z \in X \cup Y$ and $\Theta \in \mathsf{S} \cup \mathsf{N}$, and unravelling this stipulation for formulas with a modal operator as main connective, we get:

- $\mathbb{N}, x \Vdash \langle \nu \rangle \alpha$ iff $\mathbb{N}, y \Vdash \alpha$ for some y s.t. $xR_\nu y$;
- $\mathbb{N}, x \Vdash [\nu^c]\alpha$ iff $\mathbb{N}, y \Vdash \alpha$ for all y s.t. $xR_{\nu^c} y$;
- $\mathbb{N}, y \Vdash [\ni]A$ iff $\mathbb{N}, x \Vdash A$ for all x s.t. $yR_\ni x$;
- $\mathbb{N}, y \Vdash \langle \not\ni \rangle A$ iff $\mathbb{N}, x \Vdash A$ for some x s.t. $yR_{\not\ni} x$;
- $\mathbb{N}, y \Vdash [\not\ni\rangle A$ iff $\mathbb{N}, x \nVdash A$ for all x s.t. $yR_{\not\ni} x$;
- $\mathbb{N}, x \Vdash \alpha \rhd A$ iff for all y and all x', if $T_f(x, y, x')$ and $\mathbb{N}, y \Vdash \alpha$ then $\mathbb{N}, x' \Vdash A$.

Global satisfaction (notation: $\mathbb{N} \Vdash \Theta$) is defined relative to the domain of the appropriate type, and frame validity (notation: $\mathbb{K} \Vdash \Theta$) is defined as usual. Thus, by definition, $\mathbb{K} \Vdash \Theta$ iff $\mathbb{K}^+ \models \Theta$, and if $\mathbb{H}$ is a perfect heterogeneous algebra, then $\mathbb{H} \models \Theta$ iff $\mathbb{H}_+ \Vdash \Theta$.

Correspondence theory for multi-type normal logics. The semantic environment introduced above supports a straightforward extension of unified correspondence theory for multi-type normal logics, which includes the definition of inductive and analytic inductive formulas and inequalities in $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ (cf. Section 3.5), and a corresponding version of the algorithm ALBA [13] for computing their first-order correspondents and analytic structural rules.

Translation. Correspondence theory and analytic calculi for the non-normal logics $\mathbf{L}_\nabla$ and $\mathbf{L}_>$ and their analytic extensions can be then obtained 'via translation', i.e. by recursively defining translations $\tau_1, \tau_2 : \mathcal{L}_\nabla \to \mathcal{L}_{MT\nabla}$ and $(\cdot)^\tau : \mathcal{L}_> \to \mathcal{L}_{MT>}$ as follows:

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\tau_1(p) & = & p & \tau_2(p) & = & p & p^\tau & = & p\\
\tau_1(\varphi\wedge\psi) & = & \tau_1(\varphi)\wedge\tau_1(\psi) & \tau_2(\varphi\wedge\psi) & = & \tau_2(\varphi)\wedge\tau_2(\psi) & (\varphi\wedge\psi)^\tau & = & \varphi^\tau\wedge\psi^\tau\\
\tau_1(\neg\varphi) & = & \neg\tau_2(\varphi) & \tau_2(\neg\varphi) & = & \neg\tau_1(\varphi) & (\neg\varphi)^\tau & = & \neg\varphi^\tau\\
\tau_1(\nabla\varphi) & = & \langle\nu\rangle[\ni]\tau_1(\varphi) & \tau_2(\nabla\varphi) & = & [\nu^c]\langle\not\ni\rangle\tau_2(\varphi) & & &
\end{array}
$$

and

$$(\varphi > \psi)^\tau = ([\ni]\varphi^\tau \cap [\not\ni\rangle\varphi^\tau) \rhd \psi^\tau$$

Let $\tau(\varphi \vdash \psi) := \varphi^\tau \vdash \psi^\tau$ if $\varphi \vdash \psi$ is an $\mathcal{L}_>$-sequent, and $\tau(\varphi \vdash \psi) := \tau_1(\varphi) \vdash \tau_2(\psi)$ if $\varphi \vdash \psi$ is an $\mathcal{L}_\nabla$-sequent.

Proposition 3.4.1. *If $\mathbb{F}$ is an n-frame (resp. c-frame) and $\varphi \vdash \psi$ is an $\mathcal{L}_\nabla$-sequent (resp. an $\mathcal{L}_>$-sequent), then $\mathbb{F} \Vdash \varphi \vdash \psi$ iff $\mathbb{F}^\star \Vdash \tau(\varphi \vdash \psi)$.*

Proof. When $\mathbb{F}$ is an n-frame, the proposition is an immediate consequence of the following claim:

$$(\mathbb{F}, V), w \Vdash \varphi \quad \text{iff} \quad (\mathbb{F}^\star, V), w \Vdash \tau_1(\varphi) \quad \text{iff} \quad (\mathbb{F}^\star, V), w \Vdash \tau_2(\varphi),$$

which can be proved by induction on φ. We only sketch the case in which $\varphi := \nabla\psi$. In this case, $\tau_1(\nabla\psi) = \langle\nu\rangle[\ni]\tau_1(\psi)$ and $\tau_2(\nabla\psi) = [\nu^c]\langle\not\ni\rangle\tau_2(\psi)$.

$$
\begin{array}{rll}
\mathbb{F}, V, w \Vdash \nabla\psi & \text{iff} & \exists D(D \in \nu(w) \ \&\ D \subseteq V(\psi))\\
 & \text{iff} & \exists D(wR_\nu D \ \&\ \forall d(DR_\ni d \Rightarrow d \in V(\psi)))\\
 & \text{iff} & \exists D(wR_\nu D \ \&\ \forall d(DR_\ni d \Rightarrow d \in V(\tau_1(\psi)))) \qquad \text{Induction hypothesis}\\
 & \text{iff} & \mathbb{F}^\star, V, w \Vdash \langle\nu\rangle[\ni]\tau_1(\psi)
\end{array}
$$

$$
\begin{array}{rll}
\mathbb{F}, V, w \Vdash \nabla\psi & \text{iff} & \exists D(D \in \nu(w) \ \&\ D \subseteq V(\psi))\\
 & \text{iff} & \exists D(wR_\nu D \ \&\ \forall d(DR_\ni d \Rightarrow d \in V(\psi)))\\
(*) & \text{iff} & \forall D(wR_{\nu^c} D \Rightarrow \exists d(DR_{\not\ni} d \ \&\ d \in V(\psi)))\\
 & \text{iff} & \forall D(wR_{\nu^c} D \Rightarrow \exists d(DR_{\not\ni} d \ \&\ d \in V(\tau_2(\psi)))) \qquad \text{Induction hypothesis}\\
 & \text{iff} & \mathbb{F}^\star, V, w \Vdash [\nu^c]\langle\not\ni\rangle\tau_2(\psi).
\end{array}
$$

The equivalence marked by $(*)$ follows from Lemma 3.3.13.

When $\mathbb{F}$ is a c-frame, the proposition is an immediate consequence of the following claim, which can be shown by induction on φ.

$$(\mathbb{F}, V), w \Vdash \varphi \quad \text{iff} \quad (\mathbb{F}^\star, V), w \Vdash \varphi^\tau.$$

We only sketch the case in which $\varphi := \varphi > \psi$. In this case, $(\varphi > \psi)^\tau = ([\ni]\varphi^\tau \cap [\not\ni\rangle\varphi^\tau) \rhd \psi^\tau$.

$$
\begin{aligned}
(\mathbb{F}, V), w \Vdash \varphi > \psi \text{ iff } & f(w, V(\varphi)) \subseteq V(\psi) \\
\text{iff } & \forall x(x \in f(w, V(\varphi)) \Rightarrow x \in V(\psi)) \\
\text{iff } & \forall x \forall Y(x \in f(w, Y) \ \&\ Y = V(\varphi) \Rightarrow x \in V(\psi)) \\
\text{iff } & \forall x \forall Y(x \in f(w, Y) \ \&\ Y = V(\varphi^\tau) \Rightarrow x \in V(\psi^\tau)) \qquad \text{I.H.} \\
\text{iff } & \forall x \forall Y(T_f(w, Y, x) \ \&\ (\forall y(Y R_\ni y \Rightarrow y \in V(\varphi^\tau))) \ \& \\
& \qquad (\forall y(Y R_{\not\ni} y \Rightarrow y \notin V(\varphi^\tau))) \Rightarrow x \in V(\psi^\tau)) \\
\text{iff } & (\mathbb{F}^\star, V), w \Vdash ([\ni]\varphi^\tau \cap [\not\ni\rangle\varphi^\tau) \rhd \psi^\tau.
\end{aligned}
$$

□

With this framework in place, we are in a position to (a) retrieve correspondence results in the setting of *non-normal* logics, such as those collected in Theorem 3.2.3, as instances of the general Sahlqvist theory for multi-type *normal* logics, and (b) recognize whether the translation of a non-normal axiom is analytic inductive, and compute its corresponding analytic structural rules (cf. Section 3.9).

3.5 Analytic inductive inequalities

In the present section, we specialize the definitions of *inductive inequalities* (cf. [13, Definition 3.4]) and *analytic inductive inequalities* (cf. [35, Definition 55]) to the multi-type languages $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ provided in Section 3.4.

An *order-type* over $n \in \mathbb{N}$ is an n-tuple $\varepsilon \in \{1, \partial\}^n$. If ε is an order type, ε^∂ is its *opposite* order type; i.e. $\varepsilon^\partial(i) = 1$ iff $\varepsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_\mathsf{S} \cup \mathcal{F}_\mathsf{N} \cup \mathcal{F}_\mathrm{MT}$ and $\mathcal{G} := \mathcal{G}_\mathsf{S} \cup \mathcal{G}_\mathsf{N} \cup \mathcal{G}_\mathrm{MT}$, defined as follows:

$$
\begin{array}{ll}
\mathcal{F}_\mathsf{S} := \{\neg\} & \mathcal{G}_\mathsf{S} = \{\neg\} \\
\mathcal{F}_\mathsf{N} := \{\sim\} & \mathcal{G}_\mathsf{N} := \{\sim\} \\
\mathcal{F}_\mathrm{MT} := \{\langle\nu\rangle, \langle\not\ni\rangle\} & \mathcal{G}_\mathrm{MT} := \{[\ni], [\nu^c], \rhd, [\not\ni\rangle\}
\end{array}
$$

For any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), we let $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) denote the arity of f (resp. g), and the order-type ε_f (resp. ε_g) on n_f (resp. n_g) indicate whether the ith coordinate of f (resp. g) is positive ($\varepsilon_f(i) = 1$, $\varepsilon_g(i) = 1$) or negative ($\varepsilon_f(i) = \partial$, $\varepsilon_g(i) = \partial$).

Definition 3.5.1 (Signed Generation Tree). The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_\mathrm{MT}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: For any node labelled with $\ell \in \mathcal{F} \cup \mathcal{G}$ of arity n_ℓ, and for any $1 \leq i \leq n_\ell$, assign the same (resp. the opposite) sign to its ith child node if $\varepsilon_\ell(i) = 1$ (resp. if $\varepsilon_\ell(i) = \partial$). Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).

For any term $s(p_1, \ldots p_n)$, any order type ε over n, and any $1 \leq i \leq n$, an *ε-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon(i) = 1$ or $-p_i$ with $\varepsilon(i) = \partial$. An *$\varepsilon$-critical branch* in the tree is a branch ending in an ε-critical node. For any term $s(p_1, \ldots p_n)$ and any order type ε over n, we say that $+s$ (resp. $-s$) *agrees with ε*, and

Skeleton	PIA
Δ-adjoints	SRA
$+$ $\vee$ $\cup$	$+$ $\wedge$ $\cap$ $[\ni]$ $[\nu^c]$ $\rhd$ $[\not\ni\rangle$ $\neg$ $\sim$
$-$ $\wedge$ $\cap$	$-$ $\vee$ $\cup$ $\langle\nu\rangle$ $\langle\not\ni\rangle$ $\neg$ $\sim$
SLR	SRR
$+$ $\wedge$ $\cap$ $\langle\nu\rangle$ $\langle\not\ni\rangle$ $\neg$ $\sim$	$+$ $\vee$ $\cup$
$-$ $\vee$ $\cup$ $[\ni]$ $[\nu^c]$ $\rhd$ $[\not\ni\rangle$ $\neg$ $\sim$	$-$ $\wedge$ $\cap$

Table 3.1: Skeleton and PIA nodes.

write $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ε-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(s') \prec *s$ (resp. $\varepsilon^\partial(s') \prec *s$) to indicate that the signed subtree s', with the sign inherited from $*s$, agrees with ε (resp. with ε^∂).

Definition 3.5.2 (Good branch). Nodes in signed generation trees are called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 3.1. A branch in a signed generation tree $*s$, with $* \in \{+,-\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

Definition 3.5.3 (Inductive $\mathcal{L}_{MT\nabla}$- and $\mathcal{L}_{MT>}$-inequalities). For any order type ε and any irreflexive and transitive relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{-,+\}$) of an $\mathcal{L}_{MT\nabla}$-term (resp. $\mathcal{L}_{MT>}$-term) $s(p_1, \ldots p_n)$ is (Ω, ε)-*inductive* if

1. for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good (cf. Definition 3.5.2);
2. for all $1 \leq i \leq n$, every SRR-node occurring in any ε-critical branch with leaf p_i is of the form $\circledast(s, \beta)$ or $\circledast(\beta, s)$, where the critical branch goes through β and
   (a) $\varepsilon^\partial(s) \prec *s$ (cf. discussion before Definition 3.5.2), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in s and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An $\mathcal{L}_{MT\nabla}$-inequality (resp. $\mathcal{L}_{MT>}$-inequality) $s \leq t$ is (Ω, ε)-*inductive* if the signed generation trees $+s$ and $-t$ are (Ω, ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε)-inductive for some Ω and ε.

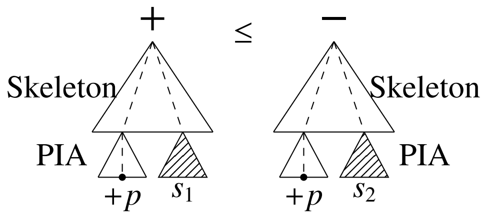
Definition 3.5.4 (Analytic inductive $\mathcal{L}_{MT\nabla}$- and $\mathcal{L}_{MT>}$-inequalities). For any order type ε and any irreflexive and transitive relation Ω on the variables $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{+,-\}$) of an $\mathcal{L}_{MT\nabla}$-term (resp. $\mathcal{L}_{MT>}$-term) $s(p_1, \ldots p_n)$ is *analytic* (Ω, ε)-*inductive* if

1. $*s$ is (Ω, ε)-inductive (cf. Definition 3.5.3);

2. every branch of $*s$ is good (cf. Definition 3.5.2).

an inequality $s \leq t$ is *analytic* (Ω, ε)*-inductive* if $+s$ and $-t$ are both (Ω, ε)-analytic inductive. An inequality $s \leq t$ is *analytic inductive* if is (Ω, ε)-analytic inductive for some Ω and ε.

The syntactic shape of analytic inductive inequalities is illustrated by the following picture:



Remark 3.5.5. In what follows, in order to be consistent with proof-theoretic notation, we also refer to *inductive sequents* and *analytic inductive sequents*. Since inductive and analytic inductive inequalities are syntactic objects, inductive and analytic inductive sequents are obtained from the former by substituting the inequality symbol with the symbol $\vdash$.

3.6 Algorithmic correspondence for non-normal logics

In this section, we detail how the two-sorted environment introduced and discussed in the previous sections can be used to establish a Sahlqvist-type correspondence framework for *classes* of non-normal logics (see Theorem 3.6.4 and the discussion in Section 3.9) which can be specialized to the signatures of monotonic modal logic and conditional logic, encompasses and extends the well-known correspondence-theoretic results for these logics collected in Theorem 3.2.3, and brings them into the fold of unified correspondence theory [12, 13]. The unified correspondence approach pivots on the order theoretic properties of the algebraic interpretation of logical connectives. As pointed out in [5], when the relevant order theoretic properties hold in a given multi-type setting such as the one introduced in Section 3.3, the insights, tools and results of unified correspondence theory can be straightforwardly transferred to it. As the first step of this process, specifically for the present cases of monotonic modal logic and conditional logic, we have specialized the definition of inductive and analytic inductive inequalities/sequents to the languages $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ (cf. Definitions 3.5.3 and 3.5.4); in the following table, we list the translations of the axioms of Theorem 3.2.3, for each of which, the last column of the table specifies whether its translation is analytic inductive.

	Axiom	Translation	Inductive	Analytic
N	$\nabla\top$	$\top \leq [\nu^c]\langle\not\ni\rangle\top$	✓	✓
P	$\neg\nabla\bot$	$\top \leq \neg\langle\nu\rangle[\ni]\bot$	✓	✓
C	$\nabla p \wedge \nabla q \to \nabla(p \wedge q)$	$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq [\nu^c]\langle\not\ni\rangle(p \wedge q)$	✓	✓
T	$\nabla p \to p$	$\langle\nu\rangle[\ni]p \leq p$	✓	✓
4	$\nabla\nabla p \to \nabla p$	$\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \leq [\nu^c]\langle\not\ni\rangle p$	✓	×
4’	$\nabla p \to \nabla\nabla p$	$\langle\nu\rangle[\ni]p \leq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$	✓	×
5	$\neg\nabla\neg p \to \nabla\neg\nabla\neg p$	$\neg[\nu^c]\langle\not\ni\rangle\neg p \leq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	✓	×
B	$p \to \nabla\neg\nabla\neg p$	$p \leq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	✓	×
D	$\nabla p \to \neg\nabla\neg p$	$\langle\nu\rangle[\ni]p \leq \neg\langle\nu\rangle[\ni]\neg p$	✓	✓
CS	$(p \wedge q) \to (p > q)$	$p \wedge q \leq ([\ni]p \cap [\not\ni\rangle p) \rhd q$	✓	✓
CEM	$(p > q) \vee (p > \neg q)$	$\top \leq (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]p \cap [\not\ni\rangle p) \rhd \neg q)$	✓	✓
ID	$p > p$	$\top \leq ([\ni]p \cap [\not\ni\rangle p) \rhd p$	✓	✓
CN	$(p > q) \vee (q > p)$	$\top \leq (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]q \cap [\not\ni\rangle q) \rhd p)$	✓	✓

Remark 3.6.1. The positional translation of $\mathcal{L}_\nabla$-axioms/sequents alllows for a larger set of translated axioms to be (analytic) inductive, compared to e.g. using only τ_1. To illustrate this point, consider axioms 4 and C above; translating them using τ_1 respectively yields $\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \leq \langle\nu\rangle[\ni]p$, which is not inductive since no branch is good, and $\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq \langle\nu\rangle[\ni](p \wedge q)$, which is inductive but not analytic, since in $-\langle\nu\rangle[\ni](p \wedge q)$ some branches (in fact all) are not good (cf. Definition 3.5.2). This trick is not a panacea: also under the positional translation, nested occurrences of ∇ connectives, as in axioms 4, 4’, 5 and B, give rise to nestings of modal operators in which Skeleton nodes occur in the scope of PIA nodes, violating the ‘good branch’ requirement (cf. Definition 3.5.2). We return to this point in the discussion after Definition 3.6.2.
An analogous positional translation trick is not applicable to $\mathcal{L}_>$. The reason is that $>$ is (finitely/completely) meet-preserving in its second coordinate and arbitrary in its first coordinate, thereby forcing any *normal* connective $*$ at the root of the translated term to be (at least) *binary*, and (finitely/completely) meet-preserving in one of its coordinates. However, to occur as a positive Skeleton node, the connective $*$ would also need to be (finitely/completely) join-preserving (resp. meet-reversing) in its positive (resp. negative) coordinates and, as observed in [11, Footnote 4], all normal operations endowed with both sets of properties need to be *unary*.

The algorithm ALBA defined in [13] can straightforwardly be adapted to $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ and their algebraic and relational semantics; since the translations of all the axioms listed above are inductive, by the general theory, ALBA succeeds in eliminating the propositional variables occurring in them and in equivalently transforming their validity on frames into suitable conditions expressible in the predicate languages canonically associated with n-frames (resp. c-frames). The ALBA runs on these axioms are reported in Section 3.9.

To further expand on how the correspondence results of Theorem 3.2.3 can be obtained as instances of algorithmic correspondence on two-sorted frames and their complex algebras, let $\mathbb{F}$ be an n-frame (resp. a c-frame) and $\varphi \vdash \psi$ an $\mathcal{L}_\nabla$-sequent (resp. $\mathcal{L}_\nabla$-sequent). Let $\tau(\varphi \vdash \psi)$ denote $\tau_1(\varphi) \vdash \tau_2(\psi)$ or $\varphi^\tau \vdash \psi^\tau$ as appropriate. Let $ALBA(\tau(\varphi \vdash \psi))$ denote an output of ALBA when run on $\tau(\varphi \vdash \psi)$, and $\mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$ be its standard translation in the appropriate predicate language of n-frames (resp. c-frames). Then the following chain of equivalences holds[4]:

[4]In the last equivalence the relations are interpreted according to Definition 3.3.12.

	$\mathbb{F} \Vdash \varphi \vdash \psi$	
iff	$\mathbb{F}^\star \Vdash \tau(\varphi \vdash \psi)$	Proposition 3.4.1
iff	$(\mathbb{F}^\star)^+ \models \tau(\varphi \vdash \psi)$	def. of validity on two sorted-frames
iff	$(\mathbb{F}^\star)^+ \models ALBA(\tau(\varphi \vdash \psi))$	two-sorted correspondence
iff	$\mathbb{F}^\star \models \mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$	
iff	$\mathbb{F} \models \mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$	

Let us concretely illustrate this proof pattern by applying it to the following axiom:

$$\nabla p \wedge \nabla q \vdash \nabla(p \wedge q). \tag{3.6.1}$$

Let $\mathbb{F} = (W, \nu)$ be a n-frame, and $\mathbb{F}^\star = (W, \mathcal{P}(W), R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ be its associated two-sorted n-frame, where e.g. $wR_\nu Z$ iff $Z \in \nu(w)$ and so on (full details are in Definition 3.3.12). By Proposition 3.4.1, the validity of axiom (3.6.1) on $\mathbb{F}$ is equivalent to its translation

$$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q) \tag{3.6.2}$$

being valid on $\mathbb{F}^\star$, which, by definition of satisfaction and validity in the two-sorted environment, is equivalent to the validity of axiom (3.6.2) on the complex algebra $(\mathbb{F}^\star)^+ = (\mathcal{P}(W), \mathcal{PP}(W), [\ni], \langle\not\ni\rangle, \langle\nu\rangle, [\nu^c])$.

According to Definition 3.5.4, axiom (3.6.2) is a (Ω, ε)-analytic inductive inequality for $p <_\Omega q$ and $\varepsilon(p) = \varepsilon(q) = 1$. Let us now run ALBA on axiom (3.6.2). In what follows we let $\mathbf{i}_1$ and $\mathbf{i}_2$ be nominal variables of type N and $\mathbf{m}$ be a co-nominal variable of type N. This means that $\mathbf{i}_1$ and $\mathbf{i}_2$ are interpreted as — and hence range in the set of — atoms of the second domain $\mathcal{PP}(W)$ of the perfect heterogeneous c-algebra $(\mathbb{F}^\star)^+$ (i.e. singleton subsets $\{Z\}$ for $Z \subseteq W$), while $\mathbf{m}$ ranges over the set of coatoms of $\mathcal{PP}(W)$, and hence is interpreted as the collection of subsets $\{Z\}^c := \{Y \subseteq W \mid Y \neq Z\}$ for an arbitrary $Z \subseteq W$.

As no preprocessing is needed, ALBA performs first approximation, which equivalently transforms

$$\forall p \forall q[\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq [\nu^c]\langle\not\ni\rangle(p \wedge q)]$$

into the following quasi-inequality:

$$\forall p \forall q \forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[(\mathbf{i}_1 \leq [\ni]p \ \&\ \mathbf{i}_2 \leq [\ni]q \ \&\ \langle\not\ni\rangle(p \wedge q) \leq \mathbf{m}) \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}].$$

Recall that $\langle\in\rangle$ and $[\ni]$ form a residuation pair. Hence, $\mathbf{i}_1 \leq [\ni]p$ is equivalent to $\langle\in\rangle\mathbf{i}_1 \leq p$ and $\mathbf{i}_2 \leq [\ni]q$ is equivalent to $\langle\in\rangle\mathbf{i}_2 \leq q$. Then the quasi inequality above is equivalent to the following quasi-inequality:

$$\forall p \forall q \forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[(\langle\in\rangle\mathbf{i}_1 \leq p \ \&\ \langle\in\rangle\mathbf{i}_2 \leq q \ \&\ \langle\not\ni\rangle(p \wedge q) \leq \mathbf{m}) \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}].$$

The quasi inequality above is in Ackermann shape, hence the Ackermann rule can be applied (cf. [13, Lemma 4.2]) to eliminate all occurrences of p and q, yielding the following (pure) quasi inequality in output

$$\forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[\langle\not\ni\rangle(\langle\in\rangle\mathbf{i}_1 \wedge \langle\in\rangle\mathbf{i}_2) \leq \mathbf{m} \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}],$$

which, for the sake of convenience, applying adjunction, we equivalently rewrite as

$$\forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[\langle\in\rangle\mathbf{i}_1 \wedge \langle\in\rangle\mathbf{i}_2 \leq [\not\in]\mathbf{m} \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}]. \tag{3.6.3}$$

Let $ALBA(\tau(\nabla p \wedge \nabla q \vdash \nabla(p \wedge q)))$ denote the quasi inequality above. The soundness of ALBA on perfect heterogeneous m-algebras and the validity of (3.6.2) on $(\mathbb{F}^\star)^+$ imply that $ALBA(\tau(\nabla p \wedge \nabla q \vdash \nabla(p \wedge q)))$ holds in $(\mathbb{F}^\star)^+$. The next step is to translate this quasi-inequality into a condition on $\mathbb{F}^\star$ expressible in its appropriate correspondence language.

As discussed above, nominal and conominal variables correspond to subsets of W. Moreover, recall that the heterogeneous connectives $[\ni], \langle\not\ni\rangle, \langle\nu\rangle, [\nu^c]$ are interpreted in $(\mathbb{F}^\star)^+$ as heterogeneous operations defined by the following assignments: for any $D \in \mathcal{P}(W)$ and $U \in \mathcal{P}\mathcal{P}(W)$ (cf. Definition 3.3.1),

$$[\notin]U = (R_{\not\ni}[U^c])^c \quad \langle\in\rangle U = R_{\ni}[U] \quad \langle\nu\rangle U = R_\nu^{-1}[U] \quad [\nu^c]U = (R_{\nu^c}^{-1}[U^c])^c.$$

Let $Z_1, Z_2, Z_3 \subseteq W$ and $\{Z_1\}, \{Z_2\}, \{Z_3\}^c$ be the interpretations of $\mathbf{i}_1, \mathbf{i}_2, \mathbf{m}$, respectively. Then, writing $R_\circ[Z]$ for $R_\circ[\{Z\}]$ for any $\circ \in \{\ni, \not\ni, \nu, \nu^c\}$, we can translate (3.6.3) as follows:

$$\begin{aligned}
&\forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[\langle\in\rangle\mathbf{i}_1 \wedge \langle\in\rangle\mathbf{i}_2 \leq [\notin]\mathbf{m} \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}]\\
=&\forall Z_1 \forall Z_2 \forall Z_3[\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\} \leq [\notin]\{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \leq [\nu^c]\{Z_3\}^c]\\
=&\forall Z_1 \forall Z_2 \forall Z_3[R_\ni[Z_1] \cap R_\ni[Z_2] \subseteq (R_{\not\ni}[\{Z_3\}^{cc}])^c \Rightarrow R_\nu^{-1}[Z_1] \cap R_\nu^{-1}[Z_2] \subseteq (R_{\nu^c}^{-1}[\{Z_3\}^{cc}])^c]\\
=&\forall Z_1 \forall Z_2 \forall Z_3[R_\ni[Z_1] \cap R_\ni[Z_2] \subseteq (R_{\not\ni}[Z_3])^c \Rightarrow R_\nu^{-1}[Z_1] \cap R_\nu^{-1}[Z_2] \subseteq (R_{\nu^c}^{-1}[Z_3])^c].
\end{aligned}$$

Thus, we have obtained

$$\mathbb{F}^\star \models \forall Z_1 \forall Z_2 \forall Z_3[R_\ni[Z_1] \cap R_\ni[Z_2] \subseteq (R_{\not\ni}[Z_3])^c \Rightarrow R_\nu^{-1}[Z_1] \cap R_\nu^{-1}[Z_2] \subseteq (R_{\nu^c}^{-1}[Z_3])^c].$$

The final step is to translate this condition into a condition on $\mathbb{F}$. Recalling the definitions of $R_{\not\ni}, R_\ni, R_\nu, R_{\nu^c}$ in Definition 3.3.12, it is easy to see that for any $Z \subseteq W$,

$$R_\ni[Z] = Z = (R_{\not\ni}[Z])^c \quad \text{ and } \quad R_\nu^{-1}[Z] = \{w \in W \mid Z \in \nu(w)\} = (R_{\nu^c}^{-1}[Z])^c.$$

Hence, we get:

$$\mathbb{F} \models \forall Z_1 \forall Z_2 \forall Z_3[Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow \forall x[(Z_1 \in \nu(x) \,\&\, Z_2 \in \nu(x)) \Rightarrow Z_3 \in \nu(x)]],$$

which, by uncurrying and then currying again, and suitably distributing quantifiers, is equivalent to

$$\mathbb{F} \models \forall Z_1 \forall Z_2 \forall x[(Z_1 \in \nu(x) \,\&\, Z_2 \in \nu(x)) \Rightarrow \forall Z_3[Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow Z_3 \in \nu(x)]],$$

which is equivalent to

$$\mathbb{F} \models \forall Z_1 \forall Z_2 \forall x[(Z_1 \in \nu(x) \,\&\, Z_2 \in \nu(x)) \Rightarrow Z_1 \cap Z_2 \in \nu(x)]] :$$

Indeed, for the top-to-bottom direction, take $Z_3 = Z_1 \cap Z_2$. Conversely, assume that $Z_1 \cap Z_2 \subseteq Z_3$, and that $Z_1 \in \nu(x)$ and $Z_2 \in \nu(x)$. Then, the assumption implies that $Z_1 \cap Z_2 \in \nu(x)$. Since $\nu(x)$ is upward-closed, $Z_1 \cap Z_2 \subseteq Z_3$ implies that $Z_3 \in \nu(x)$. This completes the algorithmic proof of item C of Theorem 3.2.3. The remaining items can be obtained by similar arguments. In Appendix 3.9 we collect the relevant ALBA runs and translations of their output.

The discussion above also motivates the following definition, aimed at identifying those $\mathcal{L}_\nabla$- and $\mathcal{L}_>$-inequalities $\varphi \vdash \psi$ such that $\tau(\varphi \vdash \psi)$ is (analytic) inductive.

Skeleton	PIA	Non-Normal
Δ-adjoints	SRA	
$+\quad\vee$	$+\quad\wedge\quad\neg$	$+\quad\nabla$
$-\quad\wedge$	$-\quad\vee\quad\neg$	
SLR	SRR	$-\quad\nabla\quad>$
$+\quad\wedge\quad\neg$	$+\quad\vee$	
$-\quad\vee\quad\neg$	$-\quad\wedge$	

Table 3.2: Skeleton, PIA and Non-normal nodes.

Definition 3.6.2 ((Analytic) inductive $\mathcal{L}_\nabla$- and $\mathcal{L}_>$-inequalities). Nodes in signed generation trees will be called Δ*-adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, *syntactically right adjoint (SRA)*, and *Non-Normal* according to the specification given in Table 3.2. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, with at most one Non-Normal connective in between P_1 and P_2, and such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

Inductive and analytic inductive $\mathcal{L}_\nabla$- and $\mathcal{L}_>$-inequalities are defined verbatim as in Definitions 3.5.3 and 3.5.4, relative to the definition of good branch given above, with the additional restriction that a (sub)formula $\varphi > \psi$ is allowed to occur positively in signed generation trees of inductive $\mathcal{L}_>$-inequalities only if every leaf of φ is a constant.

The translations of Section 3.4 map good branches defined above to good branches as in Definition 3.5.2; indeed, occurrences of $+\nabla$ (resp. $-\nabla$) are translated as $+\langle\nu\rangle[\ni]$ (resp. $-[\nu^c]\langle\not\ni\rangle$). Also, occurrences of $>$ in good branches can only be negative, and conditional formulas $-\varphi > \psi$ are translated as $-([\ni]\varphi \cap [\not\ni\rangle\varphi) \rhd \psi$. In all these cases, PIA nodes occur in the scope of a Skeleton node, as required by the 'good branch' shape of the target language. Moreover, Non-Normal connectives are allowed to occur at most once in a good branch, since, as discussed in Remark 3.6.1, independently of their sign, the nested occurrence of Non-Normal connectives would create, when translated, at least one branch in which a Skeleton node occurs in the scope of a PIA node, which violates the 'good branch' requirement. Finally, the requirement that positive occurrences of $\varphi > \psi$ are allowed in signed generation trees only if φ does not contain occurrences of atomic propositions is motivated by the fact that, as discussed in Remark 3.6.1, no positional translation is available which would map branches with occurrences of $+ >$ nodes to good branches of the target language. Hence, applying the translation $(\cdot)^\tau$ to $+(\varphi > \psi)$ generates Skeleton nodes in the scope of a PIA node. Therefore, the translation $+([\ni]\varphi\cap[\not\ni\rangle\varphi) \rhd \psi$ of this formula would be allowed to occur as a subformula in an (Ω, ε)-inductive inequality only if it was ε^∂-uniform. However, if φ contains atomic propositions, then by construction, the formula $([\ni]\varphi \cap [\not\ni\rangle\varphi) \rhd \psi$ is not ε^∂-uniform for any order-type ε.

Example 3.6.3. Axioms N and P give rise to the $\mathcal{L}_\nabla$-inequalities $\top \leq \nabla\top$ and $\top \leq \neg\nabla\bot$ which are trivially analytic inductive. Axioms C, T and D give rise to the $\mathcal{L}_\nabla$-inequalities $\nabla p \wedge \nabla q \leq \nabla(p \wedge q)$, $\nabla p \leq p$ and $\nabla p \leq \neg\nabla\neg p$ which are analytic (Ω, ε)-inductive for any order type ε and the empty Ω. Axioms 4', 5, and B give rise to $\nabla p \leq \nabla\nabla p$,

$\neg\nabla\neg p \leq \nabla\neg\nabla\neg p$ and $p \leq \nabla\neg\nabla\neg p$ which are (Ω, ε)-inductive for the order type $\varepsilon(p) = 1$ and the empty Ω, but not for $\varepsilon(p) = \partial$, and hence are not analytic. Axiom 4 gives rise to $\nabla\nabla p \leq \nabla p$ which is (Ω, ε)-inductive for the order type $\varepsilon(p) = \partial$ and the empty Ω, but not for $\varepsilon(p) = 1$, and hence is not analytic. The $\mathcal{L}_>$-inequality $(p > q) \leq (p \to q)$, arising from Axiom MP from [58], is not inductive, since $>$ occurs positively and the atomic proposition p occurs in the scope of its first coordinate. Axioms CS, CEM, ID, CN give rise to the $\mathcal{L}_>$-inequalities $(p \wedge q) \leq (p > q)$, $\top \leq (p > q) \vee (p > \neg q)$, $\top \leq p > p$, and $\top \leq (p > q) \vee (q > p)$, which are analytic (Ω, ε)-inductive for any order type ε and the empty Ω.
The class of inductive $\mathcal{L}_\nabla$-inequalities properly includes those arising from *KW-formulas* (cf. [10, Definition 5.13]). Indeed, it is not difficult to see that if $\varphi \to \psi$ is a KW-formula, then $\tau(\varphi \vdash \psi)$ is a Sahlqvist (and hence inductive) inequality.[5] To see that the inclusion is proper, consider e.g. $\nabla(p \vee q) \leq (\nabla\nabla p) \vee q$, which is (Ω, ε)-inductive for $\varepsilon(p, q) = (1, \partial)$ and $q <_\Omega p$, but $\nabla(p \vee q) \to ((\nabla\nabla p) \vee q)$ is not a KW-formula.

The tools of unified correspondence can be used also for computing analytic rules corresponding to analytic inductive axioms in the given two-sorted languages, so to obtain analytic calculi for some axiomatic extensions of the basic monotonic modal logic and basic conditional logic as an application of the theory developed in [35]. This treatment yields the analytic calculi defined in the next section. In particular, in the light of the general results [13, Theorem 6.1, Theorem 8.8] and [35, Proposition 59], the discussion so far yields the following

Theorem 3.6.4. *For any inductive $\mathcal{L}_\nabla$-sequent (resp. $\mathcal{L}_>$-sequent) $\varphi \vdash \psi$, ALBA successfully terminates on $\tau(\varphi \vdash \psi)$, and if $\mathbb{F}$ is an n-frame (resp. c-frame), then $\mathbb{F} \Vdash \varphi \vdash \psi$ if and only if $\mathbb{F} \models \mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$. Furthermore, if $\varphi \vdash \psi$ is analytic inductive, then one or more analytic structural rules in the language of the display calculus D.MT∇ (resp. D.MT$>$)—cf. Section 3.7—can be read off from the same ALBA run of $\tau(\varphi \vdash \psi)$, and hence will be semantically equivalent to $\varphi \vdash \psi$.*

3.7 Proper display calculi for non-normal logics

In this section we introduce proper multi-type display calculi for $\mathbf{L}_\nabla$ and $\mathbf{L}_>$ and their axiomatic extensions generated by the analytic axioms considered in Section 3.6. For an introduction to (proper) display calculi we refer the reader to [4, 67, 31]. For a generalization to multi-type (proper, display) calculi we refer the reader to [20, 21, 22, 24, 32, 34, 37, 33, 63].
Languages. The language $\mathcal{L}_{DMT\nabla}$ of the calculus D.MT∇ for $\mathbf{L}_\nabla$ is defined as follows:

$$\mathsf{S}\left\{\begin{array}{l} A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle \nu \rangle \alpha \mid [\nu^c]\alpha \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid \langle \hat{\nu} \rangle \Gamma \mid [\check{\nu}^c]\Gamma \mid \langle \hat{\in} \rangle \Gamma \mid [\check{\notin}]\Gamma \end{array}\right.$$

$$\mathsf{N}\left\{\begin{array}{l} \alpha ::= [\ni]A \mid \langle \not\ni \rangle A \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid [\check{\ni}]X \mid \langle \hat{\not\ni} \rangle X \mid [\check{\lambda}]X \mid \langle \hat{\lambda}^c \rangle X \end{array}\right.$$

[5] In fact, it is already pointed out in the proof of [10, Theorem 5.14] that the definition of KW-formulas is designed so as to target Sahlqvist formulas under a translation that is slightly different, but not in essential ways, from the one adopted in the present chapter.

The language $\mathcal{L}_{DMT>}$ of the calculus D.MT> for $\mathbf{L}_{>}$ is defined as follows:

$$
\mathsf{S}\begin{cases} A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid \langle\hat{\in}\rangle\Gamma \mid \Gamma \mathbin{\check{\rhd}} X \mid \Gamma \mathbin{\hat{\blacktriangle}} X \mid [\check{\notin}\rangle\Gamma \end{cases}
$$

$$
\mathsf{N}\begin{cases} \alpha ::= [\ni]A \mid [\not\ni\rangle A \mid \alpha \cap \alpha \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid [\check{\ni}]X \mid [\check{\not\ni}\rangle X \mid X \mathbin{\check{\blacktriangleright}} X \end{cases}
$$

Multi-type display calculi. In what follows, we use X, Y, W, Z as structural S-variables, and $\Gamma, \Delta, \Sigma, \Pi$ as structural N-variables.

Propositional base. The calculi D.MT∇ and D.MT> share the rules listed below.

- Identity and Cut:

$$
Id_{\mathsf{S}}\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\ Cut_{\mathsf{S}} \qquad \frac{\Gamma \vdash \alpha \quad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\ Cut_{\mathsf{N}}
$$

- Pure S-type display rules:

$$
\bot\ \frac{}{\bot \vdash \check{\bot}} \qquad \frac{}{\hat{\top} \vdash \top}\ \top
$$

$$
gal_{\mathsf{S}}\ \frac{\tilde{\neg} X \vdash Y}{\tilde{\neg} Y \vdash X} \qquad \frac{X \vdash \tilde{\neg} Y}{Y \vdash \tilde{\neg} X}\ gal_{\mathsf{S}} \qquad res_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \vdash \tilde{\neg} X \mathbin{\check{\vee}} Z} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{\tilde{\neg} Y \mathbin{\hat{\wedge}} X \vdash Z}\ res_{\mathsf{S}}
$$

- Pure N-type display rules:

$$
gal_{\mathsf{N}}\ \frac{\tilde{\sim}\Gamma \vdash \Delta}{\tilde{\sim}\Delta \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{\sim}\Delta}{\Delta \vdash \tilde{\sim}\Gamma}\ gal_{\mathsf{N}} \qquad res_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Sigma}{\Delta \vdash \tilde{\sim}\Gamma \mathbin{\check{\cup}} \Sigma} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Sigma}{\tilde{\sim}\Delta \mathbin{\hat{\cap}} \Gamma \vdash \Sigma}\ res_{\mathsf{N}}
$$

- Pure S-type structural rules:

$$
cont_{\mathsf{S}}\ \frac{X \vdash Y}{\tilde{\neg} Y \vdash \tilde{\neg} X} \qquad \hat{\top}\ \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} \hat{\top} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} \check{\bot}}\ \check{\bot}
$$

$$
W_{\mathsf{S}}\ \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} Z}\ W_{\mathsf{S}} \qquad C_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Y}{X \vdash Y}\ C_{\mathsf{S}}
$$

$$
E_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \mathbin{\hat{\wedge}} X \vdash Y} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{X \vdash Z \mathbin{\check{\vee}} Y}\ E_{\mathsf{S}} \qquad A_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} (Y \mathbin{\hat{\wedge}} Z) \vdash W}{(X \mathbin{\hat{\wedge}} Y) \mathbin{\hat{\wedge}} Z \vdash W} \qquad \frac{W \vdash X \mathbin{\hat{\wedge}} (Y \mathbin{\hat{\wedge}} Z)}{W \vdash (X \mathbin{\hat{\wedge}} Y) \mathbin{\hat{\wedge}} Z}\ A_{\mathsf{S}}
$$

- Pure N-type structural rules:

$$
cont_{\mathsf{N}}\ \frac{\Gamma \vdash \Delta}{\tilde{\sim}\Delta \vdash \tilde{\sim}\Gamma} \qquad \hat{1}\ \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \hat{1} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \check{0}}\ \check{0}
$$

$$
W_{\mathsf{N}}\ \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \Pi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Pi}\ W_{\mathsf{N}} \qquad C_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Delta}{\Gamma \vdash \Delta}\ C_{\mathsf{N}}
$$

$$
E_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Pi}{\Delta \mathbin{\hat{\cap}} \Gamma \vdash \Pi} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Pi}{\Gamma \vdash \Pi \mathbin{\check{\cup}} \Delta}\ E_{\mathsf{N}} \qquad A_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} (\Delta \mathbin{\hat{\cap}} \Pi) \vdash \Sigma}{(\Gamma \mathbin{\hat{\cap}} \Delta) \mathbin{\hat{\cap}} \Pi \vdash \Sigma} \qquad \frac{\Sigma \vdash \Gamma \mathbin{\hat{\cap}} (\Delta \mathbin{\hat{\cap}} \Pi)}{\Sigma \vdash (\Gamma \mathbin{\hat{\cap}} \Delta) \mathbin{\hat{\cap}} \Pi}\ A_{\mathsf{N}}
$$

- Pure S-type logical rules:

$$
\neg\ \frac{\tilde{\neg} A \vdash X}{\neg A \vdash X} \qquad \frac{X \vdash \tilde{\neg} A}{X \vdash \neg A}\ \neg \qquad \wedge\ \frac{A \mathbin{\hat{\wedge}} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \quad Y \vdash B}{X \mathbin{\hat{\wedge}} Y \vdash A \wedge B}\ \wedge
$$

Monotonic modal logic. D.MT∇ also includes the rules listed below.

- Multi-type display rules:

$$
{\scriptstyle \langle\hat{\nu}\rangle[\check{\lambda}]}\,\frac{\langle\hat{\nu}\rangle\Gamma\vdash X}{\Gamma\vdash[\check{\lambda}]X}
\qquad
{\scriptstyle \langle\hat{\lambda}^c\rangle[\check{\nu}^c]}\,\frac{\langle\hat{\lambda}^c\rangle X\vdash\Gamma}{X\vdash[\check{\nu}^c]\Gamma}
\qquad
{\scriptstyle \langle\hat{\in}\rangle[\check{\ni}]}\,\frac{\langle\hat{\in}\rangle\Gamma\vdash X}{\Gamma\vdash[\check{\ni}]X}
\qquad
{\scriptstyle \langle\hat{\not\ni}\rangle[\check{\notin}]}\,\frac{\langle\hat{\not\ni}\rangle X\vdash\Gamma}{X\vdash[\check{\notin}]\Gamma}
$$

- Logical rules for multi-type connectives:

$$
{\scriptstyle \langle\nu\rangle}\,\frac{\langle\hat{\nu}\rangle\alpha\vdash X}{\langle\nu\rangle\alpha\vdash X}
\qquad
\frac{\Gamma\vdash\alpha}{\langle\hat{\nu}\rangle\Gamma\vdash\langle\nu\rangle\alpha}\,{\scriptstyle \langle\nu\rangle}
\qquad
{\scriptstyle [\nu^c]}\,\frac{\alpha\vdash\Gamma}{[\nu^c]\alpha\vdash[\check{\nu}^c]\Gamma}
\qquad
\frac{X\vdash[\check{\nu}^c]\alpha}{X\vdash[\nu^c]\alpha}\,{\scriptstyle [\nu^c]}
$$

$$
{\scriptstyle \langle\not\ni\rangle}\,\frac{\langle\hat{\not\ni}\rangle A\vdash\Gamma}{\langle\not\ni\rangle A\vdash\Gamma}
\qquad
\frac{X\vdash A}{\langle\hat{\not\ni}\rangle X\vdash\langle\not\ni\rangle A}\,{\scriptstyle \langle\not\ni\rangle}
\qquad
{\scriptstyle [\ni]}\,\frac{A\vdash X}{[\ni]A\vdash[\check{\ni}]X}
\qquad
\frac{\Gamma\vdash[\check{\ni}]A}{\Gamma\vdash[\ni]A}\,{\scriptstyle [\ni]}
$$

Conditional logic. D.MT> includes left and right logical rules for $[\ni]$, the display postulates $\langle\hat{\in}\rangle[\check{\ni}]$ and the rules listed below.

- Multi-type display rules:

$$
{\scriptstyle \hat{\blacktriangle}\,\check{\triangleright}}\,\frac{X\vdash\Gamma\mathbin{\check{\triangleright}}Y}{\Gamma\mathbin{\hat{\blacktriangle}}X\vdash Y}
\qquad
\frac{\Gamma\vdash X\mathbin{\check{\blacktriangleright}}Y}{X\vdash\Gamma\mathbin{\check{\triangleright}}Y}\,{\scriptstyle \check{\blacktriangleright}\,\check{\triangleright}}
\qquad
\frac{X\vdash[\check{\notin}\rangle\Gamma}{\Gamma\vdash[\check{\not\ni}\rangle X}\,{\scriptstyle [\check{\notin}\rangle[\check{\not\ni}\rangle}
$$

- Logical rules for multi-type connectives and pure G-type logical rules:

$$
{\scriptstyle \triangleright}\,\frac{\Gamma\vdash\alpha\qquad A\vdash X}{\alpha\triangleright A\vdash\Gamma\mathbin{\check{\triangleright}}X}
\qquad
\frac{X\vdash\alpha\mathbin{\check{\triangleright}}A}{X\vdash\alpha\triangleright A}\,{\scriptstyle \triangleright}
\qquad
{\scriptstyle [\not\ni\rangle}\,\frac{X\vdash A}{[\not\ni\rangle A\vdash[\check{\not\ni}\rangle X}
\qquad
\frac{\Gamma\vdash[\check{\not\ni}\rangle A}{\Gamma\vdash[\not\ni\rangle A}\,{\scriptstyle [\not\ni\rangle}
$$

$$
{\scriptstyle \cap}\,\frac{\alpha\mathbin{\hat{\cap}}\beta\vdash\Gamma}{\alpha\cap\beta\vdash\Gamma}
\qquad
\frac{\Gamma\vdash\alpha\qquad\Delta\vdash\beta}{\Gamma\mathbin{\hat{\cap}}\Delta\vdash\alpha\cap\beta}\,{\scriptstyle \cap}
$$

Axiomatic extensions of monotonic modal logic. Each rule is labelled with the name of its corresponding axiom.

$$
\mathrm{N}\,\frac{\langle\hat{\not\ni}\rangle\hat{\top}\vdash\Gamma}{\hat{\top}\vdash[\check{\nu}^c]\Gamma}
\qquad
\mathrm{C}\,\frac{\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle\Gamma\mathbin{\hat{\wedge}}\langle\hat{\in}\rangle\Delta)\vdash\Theta}{\langle\hat{\nu}\rangle\Gamma\mathbin{\hat{\wedge}}\langle\hat{\nu}\rangle\Delta\vdash[\check{\nu}^c]\Theta}
\qquad
\mathrm{D}\,\frac{\Gamma\vdash[\check{\ni}]\,\tilde{\neg}\langle\hat{\in}\rangle\Delta}{\langle\hat{\nu}\rangle\Delta\vdash\tilde{\neg}\langle\hat{\nu}\rangle\Gamma}
$$

$$
\mathrm{M}\,\frac{\langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle\Gamma\vdash\Delta}{\langle\hat{\lambda}^c\rangle\langle\hat{\nu}\rangle\Gamma\vdash\Delta}
\qquad
\mathrm{P}\,\frac{\Gamma\vdash[\check{\ni}]\check{\bot}}{\hat{\top}\vdash\tilde{\neg}\langle\hat{\nu}\rangle\Gamma}
\qquad
\mathrm{T}\,\frac{\Gamma\vdash[\check{\ni}]X}{\langle\hat{\nu}\rangle\Gamma\vdash X}
$$

Axiomatic extensions of conditional logic. Each rule is labelled with the name of its corresponding axiom.

$$
\mathrm{ID}\,\frac{\Delta\vdash[\check{\not\ni}\rangle\langle\hat{\in}\rangle\Gamma\qquad\langle\hat{\in}\rangle\Gamma\vdash X}{\hat{\top}\vdash(\Gamma\mathbin{\hat{\cap}}\Delta)\mathbin{\check{\triangleright}}X}
\qquad
\mathrm{CS}\,\frac{\Gamma\vdash[\check{\ni}][\check{\notin}\rangle\Delta\qquad X\vdash[\check{\notin}\rangle\Delta\qquad Y\vdash Z}{X\mathbin{\hat{\wedge}}Y\vdash(\Gamma\mathbin{\hat{\cap}}\Delta)\mathbin{\check{\triangleright}}Z}
$$

$$
\mathrm{CEM}\,\frac{\Pi\vdash[\check{\not\ni}\rangle\langle\hat{\in}\rangle\Gamma\quad\Pi\vdash[\check{\not\ni}\rangle\langle\hat{\in}\rangle\Theta\quad\Delta\vdash[\check{\not\ni}\rangle\langle\hat{\in}\rangle\Gamma\quad\Delta\vdash[\check{\not\ni}\rangle\langle\hat{\in}\rangle\Theta\quad Y\vdash X}{\hat{\top}\vdash((\Gamma\mathbin{\hat{\cap}}\Delta)\mathbin{\check{\triangleright}}X)\mathbin{\check{\vee}}((\Theta\mathbin{\hat{\cap}}\Pi)\mathbin{\check{\triangleright}}\tilde{\neg}Y)}
$$

$$
\mathrm{CN}\,\frac{\Gamma\vdash[\check{\ni}][\check{\notin}\rangle\Delta\quad\Gamma\vdash[\check{\ni}]Y\quad\Theta\vdash[\check{\ni}][\check{\notin}\rangle\Pi\quad\Theta\vdash[\check{\ni}]X}{\hat{\top}\vdash((\Gamma\mathbin{\hat{\cap}}\Delta)\mathbin{\check{\triangleright}}X)\mathbin{\check{\vee}}((\Theta\mathbin{\hat{\cap}}\Pi)\mathbin{\check{\triangleright}}Y)}
$$

3.8 Properties

In this section we discuss the properties of the display calculi presented in the section above. Proofs of the following results for the display calculi associated with the basic logics of arbitrary D.LE languages are discussed in [35, Section 4.2]. They straightforwardly apply the basic proper display calculi associated with basic logics of the multi-type languages discussed in the present chapter since their proof only relies on the order theoretic properties of the interpretation of the logical connectives. Below we only expand on the properties of the calculi for the relevant axiomatic extensions.

The display calculi introduced in the section above are proper (cf. [67, 35]), and hence the general theory of proper multi-type display calculi guarantees that they enjoy *cut elimination* and *subformula property* [20].[6]

In [35, Section 7], it is shown that any analytic inductive inequality can be equivalently transformed via ALBA into a set of quasi-inequalities each of which corresponds to an analytic rule of the corresponding display calculus. Instantiating this result to the calculi of Section 3.7, let H_m (resp. H_c) be the class of all perfect heterogeneous m-algebras (resp. perfect heterogeneous c-algebras). Given a set of analytic inductive sequents R, the extension of D.MT∇ (resp. D.MT>) with inference rules obtained by running ALBA on R is denoted by D.MT∇R (resp. D.MT> R). The subclass of H_m (resp. H_c) defined by R is denoted by $H_m(R)$ (resp. $H_c(R)$).

3.8.1 Soundness

To show the soundness of the rules of D.MT∇R (resp. D.MT> R) w.r.t. $H_m(R)$ (resp. $H_c(R)$), it suffices to show that the interpretation of each rule[7] in D.MT∇R (resp. D.MT> R) is valid in $H_m(R)$ (resp. $H_c(R)$). The soundness of the rules in D.MT∇ and D.MT> follows from the definitions of H_m and H_c, respectively. And the soundness of the rules from R follows from the soundness of ALBA rules on members of H_m (resp. H_c), and the ALBA runs reported in the appendix. Specifically, in what follows, for any perfect m-algebra (resp. c-algebra) $\mathbb{H} := (\mathbb{A}, \mathbb{B}, ...)$, let x range over $\mathbb{A}$ and γ, δ, θ range over $\mathbb{B}$. Then the rules on the left-hand side of the squiggly arrows below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{\langle \hat{\in} \rangle \Gamma \vdash X}{\Gamma \vdash [\check{\ni}] X} \quad \rightsquigarrow \quad \forall \gamma \forall x [\langle \in \rangle \gamma \leq x \Leftrightarrow \gamma \leq [\ni] x]$$

$$\frac{\langle \hat{\not\ni} \rangle (\langle \hat{\in} \rangle \Gamma \mathbin{\hat{\wedge}} \langle \hat{\in} \rangle \Delta) \vdash \Theta}{\langle \hat{\nu} \rangle \Gamma \mathbin{\hat{\wedge}} \langle \hat{\nu} \rangle \Delta \vdash [\check{\nu^c}] \Theta} \quad \rightsquigarrow \quad \forall \gamma \forall \delta \forall \theta [\langle \not\ni \rangle (\langle \in \rangle \gamma \wedge \langle \in \rangle \delta) \leq \theta \Rightarrow \langle \nu \rangle \gamma \wedge \langle \nu \rangle \delta \leq [\nu^c] \theta]$$

$$\frac{\Gamma \vdash [\check{\ni}] \check{\bot}}{\hat{\top} \vdash \tilde{\neg} \langle \hat{\nu} \rangle \Gamma} \quad \rightsquigarrow \quad \forall \gamma [\gamma \leq [\ni] \bot \Rightarrow \top \leq \neg \langle \nu \rangle \gamma]$$

[6] As also observed in [11], the adjective 'proper' singles out a subclass of Belnap's display calculi [4] identified by Wansing in [67, Section 4.1]. A display calculus is *proper* if every structural rule is closed under uniform substitution. This requirement strengthens Belnap's conditions C_6 and C_7. In [20], this requirement is extended to *multi-type* display calculi. A logic is *(properly) displayable* if it can be captured by some (proper) display calculus (see [35, Section 2.2]).

[7] A rule is interpreted as an implication of the inequalities that correspond to the assumptions and conclusions of the rule. For a precise definition we refer the reader to [35, Section 4.2.1].

The validity of $\forall\gamma\forall x[\langle\in\rangle\gamma \leq x \Leftrightarrow \gamma \leq [\ni]x]$ follows from the fact that $\langle\in\rangle$ and $[\ni]$ form a residuation pair in $\mathbb{H}$. The validity of the quasi-inequalities corresponding to axioms C and P in $H_m(\{C\})$ and $H_m(\{P\})$ respectively follows from the validity-preserving ALBA runs reported in the appendix. We report below on the validity-preserving ALBA run for C.

$$\begin{array}{lll} \text{C.} & \mathbb{H} \models \nabla p \wedge \nabla q \to \nabla(p \wedge q) \ \rightsquigarrow\ \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q) & \\ \hline & \mathbb{H} \models \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q) & \\ \text{iff} & \mathbb{H} \models \forall\gamma\forall\delta\forall\theta\forall pq[\gamma \subseteq [\ni]p \ \&\ \delta \subseteq [\ni]q \ \&\ \langle\not\ni\rangle(p \wedge q) \subseteq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \subseteq [\nu^c]\theta] & \text{first approx.} \\ \text{iff} & \mathbb{H} \models \forall\gamma\forall\delta\forall\theta\forall pq[\langle\in\rangle\gamma \subseteq p \ \&\ \langle\in\rangle\delta \subseteq q \ \&\ \langle\not\ni\rangle(p \wedge q) \subseteq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \subseteq [\nu^c]\theta] & \text{Residuation} \\ \text{iff} & \mathbb{H} \models \forall\gamma\forall\delta\forall\theta[\langle\not\ni\rangle(\langle\in\rangle\gamma \wedge \langle\in\rangle\delta) \subseteq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \subseteq [\nu^c]\theta] & (\star)\ \text{Ackermann} \end{array}$$

3.8.2 Completeness

As discussed above, the algorithmic correspondence perspective on the theory of analytic calculi (here in their incarnation as "proper display calculi") allows for a uniform justification of the soundness of analytic rules in terms of the soundness of the algorithm ALBA used to generate them. These benefits extend also to the uniform justification of the *completeness* of proper display calculi w.r.t. the logics they are intended to capture.

First let us show the completeness w.r.t. the basic monotonic modal logic and conditional logic. The (translations of the) rules M, RCEA and RCK_n are derivable as follows.

$$\text{M.}\ \frac{A \vdash B}{\nabla A \vdash \nabla B} \ \rightsquigarrow\ \frac{A \vdash B}{\langle\nu\rangle[\ni]A \vdash [\nu^c]\langle\not\ni\rangle B}$$

$$\begin{array}{rc} & A \vdash B \\ \hline & [\ni]A \vdash [\check{\ni}]B \\ \hline & \langle\hat{\in}\rangle[\ni]A \vdash B \\ \hline & \langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A \vdash \langle\not\ni\rangle B \\ \hline \text{M} & \langle\hat{\lambda}^c\rangle\langle\hat{\nu}\rangle[\ni]A \vdash \langle\not\ni\rangle B \\ \hline & \langle\hat{\nu}\rangle[\ni]A \vdash [\check{\nu}^c]\langle\not\ni\rangle B \\ \hline & \langle\hat{\nu}\rangle[\ni]A \vdash [\nu^c]\langle\not\ni\rangle B \\ \hline & \langle\nu\rangle[\ni]A \vdash [\nu^c]\langle\not\ni\rangle B \end{array}$$

$$\text{RCEA.}\ \frac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)} \ \rightsquigarrow\ \frac{A \vdash B \qquad B \vdash A}{([\ni]A \cap [\not\ni\rangle A) \rhd C \vdash ([\ni]B \cap [\not\ni\rangle B) \rhd C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} B \vdash A \\ \hline [\ni]B \vdash [\check{\ni}]A \\ \hline [\ni]B \vdash [\ni]A \end{array} \qquad \begin{array}{c} A \vdash B \\ \hline [\not\ni\rangle B \vdash [\check{\not\ni}\rangle A \\ \hline [\not\ni\rangle B \vdash [\not\ni\rangle A \end{array}}{[\ni]B \,\hat{\cap}\, [\not\ni\rangle B \vdash [\ni]\varphi \cap [\not\ni\rangle A}}{[\ni]B \cap [\not\ni\rangle B \vdash [\ni]\varphi \cap [\not\ni\rangle A} \qquad C \vdash C}{([\ni]A \cap [\not\ni\rangle A) \rhd C \vdash ([\ni]B \cap [\not\ni\rangle B) \,\check{\rhd}\, C}}{([\ni]A \cap [\not\ni\rangle A) \rhd C \vdash ([\ni]B \cap [\not\ni\rangle B) \rhd C}$$

$$RCK_n.\ \frac{A_1 \wedge \ldots \wedge A_n \to B}{(C > A_1) \wedge \ldots \wedge (C > A_n) \to (C > B)}$$

$$\rightsquigarrow\ \frac{A_1 \wedge \ldots \wedge A_n \vdash B}{([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge \ldots \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_n \vdash ([\ni]C \cap [\not\ni\rangle C) \rhd B}$$

To show that the translation of RCK_n is derivable, let us preliminarily show that $([\ni]C \cap [\not\ni\rangle C) \,\hat{\blacktriangle}\, ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash A_1 \wedge A_2$ is derivable.

$$
\begin{prooftree}
\AxiomC{$C \vdash C$}
\UnaryInfC{$[\ni]C \vdash [\check{\ni}]C$}
\UnaryInfC{$[\ni]C \vdash [\ni]C$}
\AxiomC{$C \vdash C$}
\UnaryInfC{$[\not\ni\rangle C \vdash [\check{\not\ni}\rangle C$}
\UnaryInfC{$[\not\ni\rangle C \vdash [\not\ni\rangle C$}
\BinaryInfC{$[\ni]C \mathbin{\hat{\cap}} [\not\ni\rangle C \vdash [\ni]C \cap [\not\ni\rangle C$}
\UnaryInfC{$[\ni]C \cap [\not\ni\rangle C \vdash [\ni]C \cap [\not\ni\rangle C$}
\AxiomC{$A_1 \vdash A_1$}
\BinaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} A_1$}
\LeftLabel{W_S}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \mathbin{\hat{\wedge}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} A_1$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} A_1$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash A_1$}
\AxiomC{$C \vdash C$}
\UnaryInfC{$[\ni]C \vdash [\check{\ni}]C$}
\UnaryInfC{$[\ni]C \vdash [\ni]C$}
\AxiomC{$C \vdash C$}
\UnaryInfC{$[\not\ni\rangle C \vdash [\check{\not\ni}\rangle C$}
\UnaryInfC{$[\not\ni\rangle C \vdash [\not\ni\rangle C$}
\BinaryInfC{$[\ni]C \mathbin{\hat{\cap}} [\not\ni\rangle C \vdash [\ni]C \cap [\not\ni\rangle C$}
\UnaryInfC{$[\ni]C \cap [\not\ni\rangle C \vdash [\ni]C \cap [\not\ni\rangle C$}
\AxiomC{$A_2 \vdash A_2$}
\BinaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} A_2$}
\LeftLabel{W_S}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \mathbin{\hat{\wedge}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} A_2$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} A_2$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash A_2$}
\BinaryInfC{$((([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2) \mathbin{\hat{\wedge}} ((([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2) \vdash A_1 \wedge A_2$}
\LeftLabel{C_S}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash A_1 \wedge A_2$}
\end{prooftree}
$$

Iterating the previous derivation $n-1$ times (where the specific instantiation of W_S is suitably chosen so as to derive the specific instantiation of the end sequent), we obtain the left premise of the following derivation, which provides the required derivation of the conclusion of RCK_n from its premise.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} (([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge \ldots \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_n) \vdash A_1 \wedge \ldots \wedge A_n$}
\AxiomC{$A_1 \wedge \ldots \wedge A_n \vdash B$}
\RightLabel{Cut_{S}}
\BinaryInfC{$([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} (([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge \ldots \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_n) \vdash B$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge \ldots \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_n \vdash ([\ni]C \cap [\not\ni\rangle C) \mathbin{\check{\rhd}} B$}
\UnaryInfC{$([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge \ldots \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_n \vdash ([\ni]C \cap [\not\ni\rangle C) \rhd B$}
\end{prooftree}
$$

As for the completeness of the axiomatic extensions, in [11] an effective procedure is introduced for generating cut free derivations of the translations of each rule and analytic inductive axiom (of any normal lattice expansion signature) in the corresponding proper display calculus. Below, we illustrate this effective procedure by applying it to the analytic axioms of the present setting.

N. $\nabla\top \rightsquigarrow [\nu^c]\langle\not\ni\rangle\top$ P. $\neg\nabla\bot \rightsquigarrow \neg\langle\nu\rangle[\ni]\bot$ T. $\nabla A \rightarrow A \rightsquigarrow \langle\nu\rangle[\ni]A \vdash A$

$$
\begin{prooftree}
\AxiomC{$\hat{\top} \vdash \top$}
\UnaryInfC{$\langle\hat{\ni}\rangle\hat{\top} \vdash \langle\ni\rangle\top$}
\LeftLabel{N}
\UnaryInfC{$\hat{\top} \vdash [\check{\nu}^c]\langle\ni\rangle\top$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\bot \vdash \check{\bot}$}
\UnaryInfC{$[\ni]\bot \vdash [\check{\ni}]\check{\bot}$}
\LeftLabel{P}
\UnaryInfC{$\hat{\top} \vdash \tilde{\neg}[\ni]\bot$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\LeftLabel{T}
\UnaryInfC{$\langle\hat{\nu}\rangle[\ni]A \vdash A$}
\end{prooftree}
$$

ID. $A > A \rightsquigarrow ([\ni]A \cap [\not\ni\rangle A) \rhd A$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\LeftLabel{ID}
\BinaryInfC{$\hat{\top} \vdash ([\check{\ni}]A \mathbin{\hat{\cap}} [\check{\not\ni}\rangle A) \mathbin{\check{\rhd}} A$}
\end{prooftree}
$$

CS. $(A \wedge B) \rightarrow (A > B) \rightsquigarrow A \wedge B \vdash ([\ni]A \cap [\not\ni\rangle A) \rhd B$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\AxiomC{$B \vdash B$}
\LeftLabel{CS}
\TrinaryInfC{$A \mathbin{\hat{\wedge}} B \vdash ([\check{\ni}]A \mathbin{\hat{\cap}} [\check{\not\ni}\rangle A) \mathbin{\check{\rhd}} B$}
\end{prooftree}
$$

CEM. $(A > B) \vee (A > \neg B) \;\rightsquigarrow\; ([\ni]A \cap [\not\ni\rangle A) \vartriangleright B \vee ([\ni]A \cap [\not\ni\rangle A) \vartriangleright \neg B$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$B \vdash B$}
\LeftLabel{CEM}
\QuinaryInfC{$\hat{\top} \vdash ([\ni]A \mathbin{\hat{\cap}} [\not\ni\rangle A) \mathbin{\check{\vartriangleright}} B \mathbin{\check{\vee}} ([\ni]A \mathbin{\hat{\cap}} [\not\ni\rangle A) \mathbin{\check{\vartriangleright}} \tilde{\neg} B$}
\end{prooftree}
$$

C. $\nabla A \wedge \nabla B \to \nabla(A \wedge B) \rightsquigarrow \langle\nu\rangle[\ni]A \wedge \langle\nu\rangle[\ni]B \vdash [\nu^c]\langle\not\ni\rangle(A \wedge B)$

D. $\nabla A \to \neg\nabla\neg A \rightsquigarrow \langle\nu\rangle[\ni]A \vdash \neg\langle\nu\rangle[\ni]\neg A$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\ni]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\AxiomC{$B \vdash B$}
\UnaryInfC{$[\ni]B \vdash [\ni]B$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]B \vdash B$}
\BinaryInfC{$\langle\hat{\in}\rangle[\ni]A \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle[\ni]B \vdash A \wedge B$}
\UnaryInfC{$\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle[\ni]A \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle[\ni]B) \vdash \langle\not\ni\rangle(A \wedge B)$}
\LeftLabel{C}
\UnaryInfC{$\langle\hat{\nu}\rangle[\ni]A \mathbin{\hat{\wedge}} \langle\hat{\nu}\rangle[\ni]B \vdash [\check{\nu}^c]\langle\not\ni\rangle(A \wedge B)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\UnaryInfC{$\neg A \vdash \tilde{\neg}\langle\hat{\in}\rangle[\ni]A$}
\UnaryInfC{$[\ni]\neg A \vdash [\check{\ni}]\,\tilde{\neg}\langle\hat{\in}\rangle[\ni]A$}
\LeftLabel{D}
\UnaryInfC{$\langle\hat{\nu}\rangle[\ni]A \vdash \tilde{\neg}\langle\hat{\nu}\rangle[\ni]\neg A$}
\end{prooftree}
$$

CN. $(A > B) \vee (B > A) \;\rightsquigarrow\; ([\ni]A \cap [\not\ni\rangle A) \vartriangleright B \vee ([\ni]B \cap [\not\ni\rangle B) \vartriangleright A$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\AxiomC{$B \vdash B$}
\UnaryInfC{$[\not\ni\rangle B \vdash [\check{\not\ni}\rangle B$}
\UnaryInfC{$B \vdash [\check{\notin}\rangle[\not\ni\rangle B$}
\UnaryInfC{$[\ni]B \vdash [\check{\ni}][\check{\notin}\rangle[\not\ni\rangle B$}
\AxiomC{$B \vdash B$}
\UnaryInfC{$[\ni]B \vdash [\check{\ni}]B$}
\LeftLabel{CN}
\QuaternaryInfC{$\hat{\top} \vdash (([\ni]A \mathbin{\hat{\cap}} [\not\ni\rangle A) \mathbin{\check{\vartriangleright}} B) \mathbin{\check{\vee}} (([\ni]B \mathbin{\hat{\cap}} [\not\ni\rangle B) \mathbin{\check{\vartriangleright}} A)$}
\end{prooftree}
$$

3.8.3 Conservativity

To argue that the calculi introduced in Section 3.7 conservatively extend their corresponding Hilbert systems, we follow the standard proof strategy discussed in [35, 36]. Let $\vdash_{\mathbf{L}}$ denote the syntactic consequence relation arising from Hilbert systems presented in Section 3.2.1, and $\models_{\mathrm{H}}$ denote the semantic consequence relation arising from heterogeneous Kripke frames and their complex (heterogeneous) algebras. We need to show that, for all formulas A and B of the original language of the Hilbert system, if $\tau(A \vdash B)$ is derivable in a display calculus, then $A \vdash_{\mathbf{L}} B$. This claim can be proved using the following facts: (a) the rules of display calculi are sound w.r.t. heterogeneous Kripke frames and their complex (heterogeneous) algebras (cf. Section 3.8.1); (b) Hilbert systems are complete w.r.t. their respective class of algebras; and (c) homogenous algebras are equivalently presented as heterogeneous algebras (cf. Section 3.3.2), so that the semantic consequence relations arising from each type of structures preserve and reflect the translation (cf. Proposition 3.4.1). Then, let $A \vdash B$ be an entailment between formulas of the language of the original Hilbert systems. If $\tau(A \vdash B)$ is derivable in a display calculus, then, by (a), $\models_{\mathrm{H}} \tau(A \vdash B)$. By (c), this implies that $A \models_{V} B$, where $\models_{\mathrm{V}}$ denotes the semantic consequence relation arising from m-algebras or c-algebras. By (b), this implies that $A \vdash_{\mathbf{L}} B$, as required.

3.9 Conclusions and further directions

Present contributions. In the present chapter, we have proposed a semantic analysis of two well-known non-normal logics (monotonic modal logic and conditional logic), and used it to introduce both a uniform correspondence-theoretic framework encompassing and significantly extending various well-known Sahlqvist-type results for these logics, and a proof-theoretic framework modularly capturing not only the basic logics but also an infinite class of axiomatic extensions of the basic monotonic modal logic and conditional logic. The correspondence-theoretic and the proof-theoretic frameworks are closely connected with each other, both because they stem from the same semantic analysis, and because, more fundamentally, they instantiate results, tools and insights developed at the interface of correspondence theory and structural proof theory [35]. This line of research can be naturally extended in various ways, and in what follows we list some natural further directions.

A modular framework for classical modal logic. In the present chapter, we have considered monotonic modal logic and conditional logic because this choice made it possible to address a significant diversity of order-theoretic behaviour of the non-normal connectives with a minimal set of examples: namely a unary monotone operator and a binary operator which is normal (finitely meet-preserving) in its second coordinate and arbitrary in the first coordinate. A natural further direction concerns the systematic application of these techniques to wider classes of non-normal logics. Even restricting attention to the signature of $\mathcal{L}_\nabla$, a natural direction concerns developing a modular account of classical modal logic [4] and its (monotone, regular) extensions up to normal modal logic. Of course the translations employed in the present chapter for monotonic modal logic do not account for classical modal logic, because monotonicity is in-built in these translations. The question is then whether one can express monotonicity as an (analytic) inductive condition under a translation similar to the one used in the non-normal coordinate of the conditional logic operator $>$.

From Boolean to distributive lattice-based non-normal logics. The semantic analysis of the present chapter hinges on the embedding of well-known state-based semantics (monotone neighbourhood frames, selection functions) into two-sorted classical Kripke frames and their discrete dualities with perfect (heterogeneous) Boolean algebras. Pivoting on more general discrete dualities, such as Birkhoff's discrete duality between perfect distributive lattices and posets, one can develop the systematic theory of e.g. the *non-normal* counterparts of positive modal logic [17, 7] or intuitionistic modal logics [18, 19, 59]. In particular, it would be interesting to investigate the applicability of the present approach for capturing the lattice of non-normal intuitionistic modal logics introduced in [14].

Neighbourhood and selection functions as formal tools for context-relativization and category-formation. We plan to investigate alternative (intuitive) interpretations of neighbourhood and conditional frames in order to expand the realm of possible applications.
A natural option would be to consider a neighbourhood as a context relativising the interpretation of a term. An obvious application would be in lexical semantics (see e.g. [2])

where the meaning of a word is often context-dependent.

A second option would be to consider neighbourhoods as categories. Again, an obvious application would be in computational linguistics (see e.g. [47]) where each word is assigned to a syntactical category depending on the role it plays in the formation of grammatically correct sentences or phrases.

Notice that a word can occur in different contexts or it can be assigned to different categories. Therefore, one may consider generalizations of the framework with multiple (weighed) neighbourhood functions or relations as a way to represent (probabilistic) distributions in a data set.

In many machine learning approaches, a system needs both positive and negative evidence. For example, a classification system needs examples for each class that it is capable of predicting; if the classification is binary (e.g. the system tries to decide whether an email is spam or not), it needs to have positive and negative examples. This generalises to multiple classes (e.g. given a music song, predict the genre of that song). Therefore, one may consider (generalisations of) bi-neighbourhood frames (see e.g. [15]), in which sets of pairs of neighbourhoods provide independent positive and negative evidence.

Finally, each neighbourhood can be endowed with additional structure in order to capture specific behaviour. This refinement would build a bridge between the literature in non-normal modal logics and the literature on so-called modal logics for structural control in linguistics and logic (see e.g. [46, 52, 30, 36]).

Appendix: Algorithmic proof of Theorem 3.2.3

In what follows, we show that the correspondence results collected in Theorem 3.2.3 can be retrieved as instances of a suitable multi-type version of algorithmic correspondence for normal logics (cf. [12, 13]), hinging on the usual order-theoretic properties of the algebraic interpretations of the logical connectives, while admitting nominal variables of two sorts. For the sake of enabling a swift translation into the language of m-frames and c-frames, we write nominals directly as singletons, and, abusing notation, we quantify over the elements defining these singletons. These computations also serve to prove that each analytic structural rule is sound on the heterogeneous perfect algebras validating its correspondent axiom. In the computations relative to each analytic axiom, the line marked with ($\star$) marks the quasi-inequality that interprets the corresponding analytic rule. This computation does *not* prove the equivalence between the axiom and the rule, since the variables occurring in each starred quasi-inequality are restricted rather than arbitrary. However, the proof of soundness is completed by observing that all ALBA rules in the steps above the marked inequalities are (inverse) Ackermann and adjunction rules, and hence are sound also when arbitrary variables replace (co-)nominal variables.

N. $\mathbb{H} \models \nabla\top \rightsquigarrow \top \subseteq [\nu^c]\langle\not\ni\rangle\top$

$\top \subseteq [\nu^c]\langle\not\ni\rangle\top$

iff $\forall X\forall w[\langle\not\ni\rangle\top \subseteq \{X\}^c \Rightarrow \{w\} \subseteq [\nu^c]\{X\}^c]$ (⋆) first. app.

iff $\forall X\forall w[X = W \Rightarrow \{w\} \subseteq [\nu^c]\{X\}^c)$ ($\langle\ni\rangle\top = \{W\}^c$)

iff $\forall w[\{w\} \subseteq [\nu^c]\{W\}^c]$

iff $\forall w[\{w\} \subseteq (R_{\nu^c}^{-1}[W])^c]$

iff $\forall w[\{w\} \subseteq R_{\nu}^{-1}[W]]$

iff $\forall w[W \in \nu(w)]$

P. $\mathbb{H} \models \neg\nabla\bot \rightsquigarrow \top \subseteq \neg\langle\nu\rangle[\ni]\bot$

$\top \subseteq \neg\langle\nu\rangle[\ni]\bot$

iff $\forall X[X \subseteq [\ni]\bot \Rightarrow T \subseteq \neg\langle\nu\rangle X]$ (⋆) first. app.

iff $W \subseteq \neg\langle\nu\rangle[\ni]\emptyset$

iff $W \subseteq \neg\langle\nu\rangle\{\emptyset\}$ $[\ni]\emptyset = \{Z \subseteq W \mid Z \subseteq \emptyset\}$

iff $W \subseteq \{w \in W \mid wR_\nu\emptyset\}^c$

iff $\forall w[\emptyset \notin \nu(w)]$.

C. $\mathbb{H} \models \nabla p \wedge \nabla q \to \nabla(p \wedge q) \rightsquigarrow \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$

$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$

iff $\forall Z_1Z_2Z_3\forall pq[\{Z_1\} \subseteq [\ni]p \;\&\; \{Z_2\} \subseteq [\ni]q \;\&\; \langle\not\ni\rangle(p \wedge q) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ first approx.

iff $\forall Z_1Z_2Z_3\forall pq[\langle\in\rangle\{Z_1\} \subseteq p \;\&\; \langle\in\rangle\{Z_2\} \subseteq q \;\&\; \langle\not\ni\rangle(p \wedge q) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ Residuation

iff $\forall Z_1\forall Z_2\forall Z_3[\langle\not\ni\rangle(\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\}) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ (⋆) Ackermann

iff $\forall Z_1\forall Z_2\forall Z_3[(\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\}) \subseteq [\notin]\{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ Residuation

iff $\forall Z_1\forall Z_2\forall Z_3[\forall x(xR_\in Z_1 \;\&\; xR_\in Z_2 \Rightarrow \neg xR_\notin Z_3) \Rightarrow \forall x(xR_\nu Z_1 \;\&\; xR_\nu Z_2 \Rightarrow \neg xR_{\nu^c} Z_3)]$ Standard translation

iff $\forall Z_1\forall Z_2\forall Z_3[\forall x(x \in Z_1 \;\&\; x \in Z_2 \Rightarrow x \in Z_3) \Rightarrow \forall x(Z_1 \in \nu(x) \;\&\; Z_2 \in \nu(x) \Rightarrow Z_3 \in \nu(x))]$ Relations interpretation

iff $\forall Z_1\forall Z_2\forall Z_3[Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow \forall x(Z_1 \in \nu(x) \;\&\; Z_2 \in \nu(x) \Rightarrow Z_3 \in \nu(x))]$

iff $\forall Z_1\forall Z_2\forall x[Z_1 \in \nu(x) \;\&\; Z_2 \in \nu(x) \Rightarrow Z_1 \cap Z_2 \in \nu(x))]$. Monotonicity

4'. $\mathbb{H} \models \nabla p \to \nabla\nabla p \rightsquigarrow \langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$

$\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$

iff $\forall Z_1\forall x'\forall p[\{Z_1\} \subseteq [\ni]p \;\&\; [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p \subseteq \{x'\}^c) \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ first approx.

iff $\forall Z_1\forall x'\forall p[\langle\in\rangle\{Z_1\} \subseteq p \;\&\; [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p \subseteq \{x'\}^c) \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ Residuation

iff $\forall Z_1\forall x'[[\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle\langle\in\rangle\{Z_1\} \subseteq \{x'\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ Ackermann

iff $\forall Z_1[\langle\nu\rangle\{Z_1\} \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle\langle\in\rangle\{Z_1\}]$

iff $\forall Z_1\forall x[xR_\nu Z_1 \Rightarrow \forall Z_2(xR_{\nu^c}Z_2 \Rightarrow \exists y(Z_2R_{\not\ni}y \;\&\; \forall Z_3(yR_{\nu^c}Z_3 \Rightarrow \exists w(Z_3R_{\not\ni}w \;\&\; wR_\in Z_1)))))]$ Standard translation

iff $\forall Z_1\forall x[x \in \nu(Z) \Rightarrow \forall Z_2(Z_2 \notin \nu(x) \Rightarrow \exists y(y \notin Z_2 \;\&\; \forall Z_3(Z_2 \notin \nu(y) \Rightarrow \exists w(w \notin Z_3 \;\&\; w \in Z_1)))))]$ Relations translation

iff $\forall Z_1\forall x[x \in \nu(Z) \Rightarrow \forall Z_2(Z_2 \notin \nu(x) \Rightarrow \exists y(y \notin Z_2 \;\&\; \forall Z_3(Z_2 \notin \nu(y) \Rightarrow Z_1 \not\subseteq Z_3)))]$ Relations translation

iff $\forall Z_1\forall x[x \in \nu(Z) \Rightarrow (\forall Z_2(\forall y(\forall Z_3(Z_1 \subseteq Z_3 \Rightarrow Z_3 \in \nu(y)) \Rightarrow y \in Z_2) \Rightarrow Z_2 \in \nu(x)))]$ Contraposition

iff $\forall Z_1\forall x[x \in \nu(Z) \Rightarrow (\forall Z_2(\forall y(Z_1 \in \nu(y)) \Rightarrow y \in Z_2) \Rightarrow Z_2 \in \nu(x)))]$ Monotonicity

iff $\forall Z_1\forall x[x \in \nu(Z) \Rightarrow \{y \mid Z_1 \in \nu(y)\} \in \nu(x)]$. Monotonicity

4. $\mathbb{H} \models \nabla\nabla p \to \nabla p \rightsquigarrow \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle p$

	$\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle p$	
iff	$\forall x\forall Z_1\forall p[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \;\&\; \langle\not\ni\rangle p \subseteq \{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$	first approx.
iff	$\forall x\forall Z_1\forall p[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \;\&\; p \subseteq [\notin]\{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$	Adjunction
iff	$\forall x\forall Z_1[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni][\notin]\{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$	Ackermann
iff	$\forall x\forall Z_1[(\exists Z_2(xR_\nu Z_2 \;\&\; \forall y(Z_2R_\ni y \Rightarrow \exists Z_3(yR_\nu Z_3 \;\&\; \forall w(Z_3R_\ni w \Rightarrow \neg wR_\notin Z_1))))) \Rightarrow \neg xR_{\nu^c}Z_1]$	Standard translation
iff	$\forall x\forall Z_1[((\exists Z_2 \in \nu(x))(\forall y \in Z_2)(\exists Z_3 \in \nu(y))(\forall w \in Z_3)(w \in Z_1)) \Rightarrow Z_1 \in \nu(x)]$	Relation translation
iff	$\forall x\forall Z_1[((\exists Z_2 \in \nu(x))(\forall y \in Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_1 \in \nu(x)]$	
iff	$\forall x\forall Z_1\forall Z_2[(Z_2 \in \nu(x) \;\&\; (\forall y \in Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_1 \in \nu(x)]$	
iff	$\forall x\forall Z_1\forall Z_2[(Z_2 \in \nu(x) \;\&\; (\forall y \in Z_2)(Z_1 \in \nu(y))) \Rightarrow Z_1 \in \nu(x)]$	Monotonicity

5. $\mathbb{H} \models \neg\nabla\neg p \to \nabla\neg\nabla\neg p \rightsquigarrow \neg[\nu^c]\langle\not\ni\rangle\neg p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$

	$\neg[\nu^c]\langle\not\ni\rangle\neg p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	
iff	$\forall x\forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p \subseteq \{x\}^c \;\&\; \langle\not\ni\rangle\neg p \subseteq \{Z_1\}^c \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$	first approx.
iff	$\forall x\forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p \subseteq \{x\}^c \;\&\; \neg[\notin]\{Z_1\}^c \subseteq p \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$	Residuation
iff	$\forall x\forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\neg[\notin]\{Z_1\}^c \subseteq \{x\}^c \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$	Ackermann
iff	$\forall Z_1[\neg[\nu^c]\{Z_1\}^c \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\neg[\notin]\{Z_1\}^c]$	
iff	$\forall Z_1\forall x[xR_{\nu^c}Z_1 \Rightarrow \forall Z_2(xR_{\nu^c}Z_2 \Rightarrow \exists y(Z_2R_{\not\ni}y \;\&\; \forall Z_3(yR_\nu Z_3 \Rightarrow \exists w(Z_3R_\ni w \;\&\; wR_\notin Z_1))))]$	Standard translation
iff	$\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow (\forall Z_2 \notin \nu(x))(\exists y \notin Z_2)(\forall Z_3 \in \nu(y))(\exists w \in Z_3)(w \notin Z_1)]$	Relation translation
iff	$\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow (\forall Z_2 \notin \nu(x))(\exists y \notin Z_2)(\forall Z_3 \in \nu(y))(Z_3 \nsubseteq Z_1)]$	
iff	$\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow \forall Z_2(((\forall y \notin Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_2 \in \nu(x))]$	Contraposition
iff	$\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow \forall Z_2((\forall y \notin Z_2)(Z_1 \in \nu(y)) \Rightarrow Z_2 \in \nu(x))]$	Monotonicity
iff	$\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow \{y \mid Z_1 \in \nu(y)\}^c \in \nu(x))]$	Monotonicity

B. $\mathbb{H} \models p \to \nabla\neg\nabla\neg p \rightsquigarrow p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$

	$p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	
iff	$\forall x\forall p[\{x\} \subseteq p \Rightarrow \{x\} \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p]$	first approx.
iff	$\forall x[\{x\} \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\{x\}]$	Ackermann
iff	$\forall x[\{x\} \subseteq [\nu^c]\langle\not\ni\rangle[\nu]\langle\ni\rangle\{x\}]$	
iff	$\forall x[\forall Z_1(xR_{\nu^c}Y \Rightarrow \exists y(YR_{\not\ni}x \;\&\; \forall Z_2(yR_\nu Z_2 \Rightarrow Z_2R_\ni x)))]$	Standard translation
iff	$\forall x[\forall Z_1(Z_1 \notin \nu(x) \Rightarrow \exists y(x \notin Z_1 \;\&\; \forall Z_2(Z_2 \in \nu(y) \Rightarrow x \in Z_2)))]$	Relations translation
iff	$\forall x[\forall Z_1(\forall y(\forall Z_2(x \notin Z_2 \Rightarrow Z_2 \notin \nu(y)) \Rightarrow y \in Z_1) \Rightarrow Z_1 \in \nu(x))]$	Contrapositive
iff	$\forall x[\forall Z_1(\forall y(\{x\}^c \notin \nu(y_1)) \Rightarrow y \in Z_1) \Rightarrow Z_1 \in \nu(x))]$	Monotonicity
iff	$\forall x[\{y \mid \{x\}^c \notin \nu(y)\} \in \nu(x))]$	Monotonicity
iff	$\forall x\forall X[x \in X \Rightarrow \{y \mid X^c \notin \nu(y)\} \in \nu(x)]$	Monotonicity

D. $\mathbb{H} \models \nabla p \to \neg\nabla\neg p \ \rightsquigarrow\ \langle\nu\rangle[\ni]p \subseteq \neg\langle\nu\rangle[\ni]\neg p$

	$\langle\nu\rangle[\ni]p \subseteq \neg\langle\nu\rangle[\ni]\neg p$	
iff	$\forall Z\forall Z'[\{Z\} \subseteq [\ni]p \ \&\ Z' \subseteq [\ni]\neg p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle Z']$	first approx.
iff	$\forall Z\forall Z'[\langle\in\rangle\{Z\} \subseteq p \ \&\ \{Z'\} \subseteq [\ni]\neg p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle\{Z'\}]$	Residuation
iff	$\forall Z\forall Z'[\{Z'\} \subseteq [\ni]\neg\langle\in\rangle\{Z\} \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle\{Z'\}]$	$(\star)$ Ackermann
iff	$\forall Z[\langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle[\ni]\neg\langle\in\rangle\{Z\}]$	
iff	$\forall Z[\langle\nu\rangle\{Z\} \subseteq [\nu]\langle\ni\rangle\langle\in\rangle\{Z\}]$	
iff	$\forall Z\forall x[xR_\nu Z \Rightarrow \forall Y(xR_\nu Y \Rightarrow \exists w(YR_\ni w \ \&\ wR_\in Z))]$	Standard Translation
iff	$\forall Z\forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \in \nu(x) \Rightarrow \exists w(w \in Y \ \&\ w \in Z))]$	Relation translation
iff	$\forall Z\forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \in \nu(x) \Rightarrow Y \not\subseteq Z^c)]$	
iff	$\forall Z\forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \subseteq Z^c \Rightarrow Y \notin \nu(x))]$	Contrapositive
iff	$\forall Z\forall x\forall Y[Z \in \nu(x) \Rightarrow Z^c \notin \nu(x)]$	Monotonicity

CS. $\mathbb{H} \models (p \wedge q) \to (p > q) \ \rightsquigarrow\ (p \wedge q) \subseteq ([\ni]p \cap [\not\ni\rangle p) \rhd q$

	$(p \wedge q) \subseteq ([\ni]p \cap [\not\ni\rangle p) \rhd q$	
iff	$\forall x\forall Z\forall x'\forall pq[\{x\} \subseteq p \wedge q \ \&\ \{Z\} \subseteq [\ni]p \cap [\not\ni\rangle p \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\} \rhd \{x'\}^c]$	first. approx.
iff	$\forall x\forall Z\forall x\forall p\forall q[\{x\} \subseteq p \ \&\ \{x\} \subseteq q \ \&\ \{Z\} \subseteq [\ni]p \ \&\ \{Z\} \subseteq [\not\ni\rangle p \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\} \rhd \{x'\}^c]$	Splitting rule
iff	$\forall x\forall Z\forall x'\forall p\forall q[\{x\} \subseteq p \ \&\ \{x\} \subseteq q \ \&\ \{Z\} \subseteq [\ni]p \ \&\ p \subseteq [\notin\rangle\{Z\} \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\} \rhd \{x'\}^c]$	Residuation
iff	$\forall x\forall Z\forall x'\forall q[\{x\} \subseteq [\notin\rangle\{Z\} \ \&\ \{x\} \subseteq q \ \&\ \{Z\} \subseteq [\ni][\notin\rangle\{Z\} \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\} \rhd \{x'\}^c]$	Ackermann
iff	$\forall x\forall Z\forall x'[\{x\} \subseteq [\notin\rangle\{Z\} \ \&\ \{Z\} \subseteq [\ni][\notin\rangle\{Z\} \ \&\ \{x\} \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\} \rhd \{x'\}^c]$	$(\star)$ Ackermann
iff	$\forall x\forall Z[\{x\} \subseteq [\notin\rangle\{Z\} \ \&\ \{Z\} \subseteq [\ni][\notin\rangle\{Z\} \Rightarrow \{x\} \subseteq \{Z\} \rhd \{x\}]$	
iff	$\forall x\forall Z[\neg xR_\notin Z \ \&\ \forall y(ZR_\ni y \Rightarrow \neg yR_\notin Z) \Rightarrow \forall y(T_f(x, Z, y) \Rightarrow y = x)]$	Standard translation
iff	$\forall x\forall Z[x \in Z \ \&\ \forall y(y \in Z \Rightarrow Z \in y) \Rightarrow \forall y(y \in f(x, Z) \Rightarrow y = x)]$	Relation interpretation
iff	$\forall x\forall Z[x \in Z \ \Rightarrow\ \forall y(y \in f(x, Z) \Rightarrow y = x)]$	
iff	$\forall x\forall Z[x \in Z \Rightarrow f(x, Z) \subseteq \{x\}]$	

ID. $\mathbb{H} \models p > p \ \rightsquigarrow\ ([\ni]p \cap [\not\ni\rangle p) \rhd p$

	$\top \subseteq ([\ni]p \cap [\not\ni\rangle p) \rhd p$	
iff	$\forall ZZ'\forall x'p[(\{Z\} \subseteq [\ni]p \ \&\ \{Z'\} \subseteq [\not\ni\rangle p \ \&\ p \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c]$	first approx.
iff	$\forall ZZ'\forall x'p[(\langle\in\rangle\{Z\} \subseteq p \ \&\ \{Z'\} \subseteq [\not\ni\rangle p \ \&\ p \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c]$	Adjunction
iff	$\forall Z\forall Z'\forall x'[(\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \ \&\ \langle\in\rangle\{Z\} \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c$	Ackermann
iff	$\forall Z\forall Z'[\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \ \Rightarrow \forall x'[\langle\in\rangle\{Z\} \subseteq \{x'\}^c \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c]]$	Currying
iff	$\forall Z\forall Z'[\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \ \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \langle\in\rangle\{Z\}]$	$(\star)$ Ackermann
iff	$\forall x\forall Z\forall Z'[\forall w(Z'R_{\not\ni} w \Rightarrow \neg wR_\in Z) \Rightarrow \forall y(T_f(x, Z, y) \ \&\ Z = Z' \Rightarrow y \in Z)]$	Standard Translation
iff	$\forall x\forall Z\forall Z'\forall y[\forall w(Z'R_{\not\ni} w \Rightarrow \neg wR_\in Z) \ \&\ (T_f(x, Z, y) \ \&\ Z = Z' \Rightarrow y \in Z)]$	
iff	$\forall x\forall Z\forall Z'\forall y[\forall w(w \notin Z' \Rightarrow w \notin Z) \ \&\ (y \in f(x, Z) \ \&\ Z = Z' \Rightarrow y \in Z)]$	Relation interpretation
iff	$\forall x\forall Z\forall Z'\forall y[Z \subseteq Z' \ \&\ (y \in f(x, Z) \ \&\ Z = Z' \Rightarrow y \in Z)]$	
iff	$\forall x\forall Z\forall y[(y \in f(x, Z) \Rightarrow y \in Z)]$	
iff	$\forall x\forall Z[f(x, Z) \subseteq Z]$	

T. $\mathbb{H} \models \nabla p \to p \;\rightsquigarrow\; \langle \nu \rangle [\ni] p \subseteq p$

	$\langle \nu \rangle [\ni] p \subseteq p$	
iff	$\forall x \forall Z \forall p [p \subseteq \{x\}^c \;\&\; \{Z\} \subseteq [\ni] p \Rightarrow \langle \nu \rangle \{Z\} \subseteq \{x\}^c]$	first approx.
iff	$\forall x \forall Z \forall p [p \subseteq \{x\}^c \;\&\; \langle \in \rangle \{Z\} \subseteq p \Rightarrow \langle \nu \rangle \{Z\} \subseteq \{x\}^c]$	Adjunction
iff	$\forall x \forall Z [\langle \in \rangle \{Z\} \subseteq \{x\}^c \Rightarrow \langle \nu \rangle \{Z\} \subseteq \{x\}^c]$	($\star$) Ackermann
iff	$\forall Z [\langle \nu \rangle \{Z\} \subseteq \langle \ni \rangle \{Z\}]$	inverse approx.
iff	$\forall x \forall Z [x R_\nu Z \Rightarrow x R_\ni Z]$	Standard translation
iff	$\forall x \forall Z [Z \in \nu(x) \Rightarrow x \in Z]$.	Relation translation

CEM. $\mathbb{H} \models (p > q) \vee (p > \neg q) \;\rightsquigarrow\; (([\ni] p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni] p \cap [\not\ni\rangle p) \rhd \neg q)$

	$\top \subseteq (([\ni] p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni] p \cap [\not\ni\rangle p) \rhd \neg q)$	
iff	$\forall p \forall q \forall X \forall Y \forall x \forall y (\{X\} \subseteq [\ni] p \cap [\not\ni\rangle p \;\&\; \{Y\} \subseteq [\ni] p \cap [\not\ni\rangle p \;\&\; q \subseteq \{x\}^c \;\&\; \{y\} \subseteq q$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg \{y\})$	first approx.
iff	$\forall p \forall q \forall X \forall Y \forall x \forall y (\{X\} \subseteq [\ni] p \;\&\; \{X\} \subseteq [\not\ni\rangle p \;\&\; \{Y\} \subseteq [\ni] p \;\&\; \{Y\} \subseteq [\not\ni\rangle p \;\&\; q \subseteq \{x\}^c \;\&\; \{y\} \subseteq q$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg \{y\})$ ($\star$)	Splitting
iff	$\forall p \forall q \forall X \forall Y \forall x \forall y (\{X\} \subseteq [\ni] p \;\&\; p \subseteq [\notin\rangle \{X\} \;\&\; \{Y\} \subseteq [\ni] p \;\&\; p \subseteq [\notin\rangle \{Y\} \;\&\; q \subseteq \{x\}^c \;\&\; \{y\} \subseteq q$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg \{y\})$	Residuation
iff	$\forall X \forall Y \forall x \forall y (\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle \{X\} \wedge [\notin\rangle \{Y\}) \;\&\; \{y\} \subseteq \{x\}^c$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg \{y\})$	Ackermann
iff	$\forall X \forall Y \forall x (\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle \{X\} \wedge [\notin\rangle \{Y\}) \Rightarrow \forall y (\{y\} \subseteq \{x\}^c \Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg \{y\}))$	Currying
iff	$\forall X \forall Y \forall x (\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle \{X\} \wedge [\notin\rangle \{Y\}) \Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg \{x\}^c))$	
iff	$\forall X \forall Y \forall x [(\forall y (X R_\ni y \text{ or } Y R_\ni y) \Rightarrow \neg y R_{\notin} X \;\&\; \neg y R_{\notin} Y)$ $\Rightarrow \forall y (\neg T_f(y, X, x) \text{ or } (\forall z (T_f(y, Y, z) \Rightarrow z = x)))]$	Standard translation
iff	$\forall X \forall Y \forall x [(\forall y (y \in X \text{ or } y \in Y) \Rightarrow y \in X \;\&\; y \in Y)$ $\Rightarrow \forall y (x \notin f(y, X) \text{ or } (\forall z (z \in f(y, Y) \Rightarrow z = x)))]$	Relation interpretation
iff	$\forall X \forall Y \forall x [(X \cup Y \subseteq X \cap Y) \Rightarrow \forall y (x \notin f(y, X) \text{ or } (\forall z (z \in f(y, Y) \Rightarrow z = x)))]$	
iff	$\forall X \forall Y \forall x [X = Y \Rightarrow \forall y (x \notin f(y, X) \text{ or } (\forall z (z \in f(y, Y) \Rightarrow z = x)))]$	
iff	$\forall X \forall x \forall y [(x \notin f(y, X) \text{ or } (\forall z (z \in f(y, X) \Rightarrow z = x)))]$	
iff	$\forall X \forall x \forall y [(x \in f(y, X) \Rightarrow f(y, X) = \{x\})]$	
iff	$\forall X \forall y [\lvert f(y, X) \rvert \leq 1]$.	

CN. $\mathbb{H} \models (p > q) \vee (q > p) \;\rightsquigarrow\; ([\ni] p \wedge [\not\ni\rangle p) \rhd q) \vee (([\ni] q \wedge [\not\ni\rangle q) \rhd p$

	$\top \subseteq (([\ni] p \wedge [\not\ni\rangle p) \rhd q) \vee (([\ni] q \wedge [\not\ni\rangle q) \rhd p)$	
iff	$\forall p \forall q \forall X \forall Y (\{X\} \subseteq [\ni] p \;\&\; \{Y\} \subseteq [\ni] q \Rightarrow \top \subseteq ((\{X\} \cap [\not\ni\rangle p) \rhd q) \vee ((\{Y\} \cap [\not\ni\rangle q) \rhd p)$	Approx.
iff	$\forall p \forall q \forall X \forall Y (\langle \in \rangle \{X\} \subseteq p \;\&\; \langle \in \rangle \{Y\} \subseteq q \Rightarrow \top \subseteq ((\{X\} \cap [\not\ni\rangle p) \rhd q) \vee ((\{Y\} \cap [\not\ni\rangle q) \rhd p)$	Residuation
iff	$\forall X \forall Y (\top \subseteq ((\{X\} \cap [\not\ni\rangle \langle \in \rangle \{X\}) \rhd \langle \in \rangle \{Y\}) \vee ((\{Y\} \cap [\not\ni\rangle \langle \in \rangle \{Y\}) \rhd \langle \in \rangle \{X\}))$	Ackermann
iff	$\forall X \forall Y \forall z [z \in (T_f^{(0)}[\{X\} \cap (R_{\not\ni}^{-1}[R_\ni[\{X\}]])^c, (R_\ni[\{Y\}])^c])^c \cup (T_f^{(0)}[\{Y\} \cap (R_{\not\ni}^{-1}[R_\ni[\{Y\}]])^c, (R_\ni[\{X\}])^c])^c]$	Standard translation
iff	$\forall X \forall Y \forall z [z \in (T_f^{(0)}[\{X\} \cap (R_{\not\ni}^{-1}[X])^c, Y^c])^c \cup (T_f^{(0)}[\{Y\} \cap (R_{\not\ni}^{-1}[Y])^c, X^c])^c]$	
iff	$\forall X \forall Y \forall z [z \in (T_f^{(0)}[\{X\} \cap \{Z' \mid X \subseteq Z'\}, Y^c])^c \cup (T_f^{(0)}[\{Y\} \cap \{Z' \mid Y \subseteq Z'\}, X^c])^c]$	
iff	$\forall X \forall Y \forall z [z \in (T_f^{(0)}[\{X\}, Y^c])^c \cup (T_f^{(0)}[\{Y\}, X^c])^c]$	
iff	$\forall X \forall Y \forall z \forall u [(u \in f(z, X) \Rightarrow u \in Y) \text{ or } (u \in f(z, Y) \Rightarrow u \in X)]$	
iff	$\forall X \forall Y \forall z [(f(z, X) \subseteq Y) \text{ or } (f(z, Y) \subseteq X)]$.	

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Chapter 4

Neighbourhood semantics for graded modal logic

4.1 Introduction

Graded modal logic **GrK** is an extension of propositional logic with graded modalities $\Diamond_n (n \in \mathbb{N})$ that count the number of successors of a given state. The interpretation of formula $\Diamond_n \varphi$ in a Kripke model is that the number of successors that satisfy φ is at least n. Originally introduced in Goble [9], the notion of a graded modality is developed so that 'propositions can be distinguished by degrees or grades of necessity or possibility' [9, Page 1]. This language was studied in Kaplan [11] as an extension of **S5**. Fine [8], De Caro [6] and Cerrato [2] investigated the completeness of **GrK** and its extensions. Van der Hoek [15] investigated the expressibility, decidability and definability of graded modal logic and also correspondence theory. Cerrato [3] proved the decidability by filtration for graded modal logic.

De Rijke [7] introduced graded tuple bisimulation for graded modal logic. Using this he proved the finite model property (which was first proved in Cerrato [3] via filtration) and that a first-order formula is invariant under graded bisimulation iff it is equivalent to a graded modal formula. Aceto, Ingolfsdottir and Sack [1] showed that resource bisimulation and graded bisimulation coincide over image-finite Kripke frames. Van der Hoek and Meyer [16] proposed a graded modal logic **GrS5**, which is seen as a graded epistemic logic and is able to express 'accepting φ if there are at most n exceptions to φ'. Ma and van Ditmarsch [13] developed dynamic extensions of graded epistemic logics.

Monotonic modal logics are weakenings of normal modal logics in which the *additivity* ($\Diamond\bot \leftrightarrow \bot$ and $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$) of the diamond modality has been weakened to *monotonicity* ($\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$), which can also be formulated as a derivation rule: from $\vdash \varphi \to \psi$ infer $\vdash \Diamond\varphi \to \Diamond\psi$. Monotonic modal logics are interpreted over monotonic neighbourhood frames, that is neighbourhood frames where the collection of neighbourhoods of a point is closed under supersets. There have been many results about monotonic modal logics and monotonic neighbourhood frames [4, 10, 14], including model constructions, definability, correspondence theory, canonical model constructions, algebraic duality, coalgebraic semantics, interpolation, simulations of monotonic modal logics by bimodal normal logics, etc.

In this chapter, we propose a neighbourhood semantics for graded modal logic. We define an operation $(.)^\bullet$ (Def. 4.4.2) to obtain a class of monotonic neighbourhood frames on which graded modal logic is interpreted. This class of neighbourhood frames is shown to be first-order definable in Section 4.5 and modally undefinable in Section 4.6. In Section 4.7 we obtain a new definition of graded bisimulation with respect to Kripke frames by modifying the definition of monotonic bisimulation and show that it is equivalent to the one proposed in [7]. Our results show that techniques for monotonic modal logics can be successfully applied to graded modal logic.

4.2 Preliminaries

4.2.1 Graded modal logic

Language. Let Prop be a set of proposition letters. Language $\mathcal{L}_g$ is defined by induction as follows:

$$\mathcal{L}_g \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond_n\varphi$$

where $p \in \mathsf{Prop}$ and $n \in \mathbb{N}$. We recall that $\mathbb{N}$ is the set of natural numbers. The *complexity* of a formula $\varphi \in \mathcal{L}_g$ is the number of connectives occurring in φ. Other propositional connectives $\bot$, $\top$, $\wedge$, $\to$, $\leftrightarrow$ are defined as usual. The *dual* of $\Diamond_n\varphi$ is defined as $\Box_n\varphi := \neg\Diamond_n\neg\varphi$. Further, define $\Diamond\varphi := \Diamond_1\varphi$ and $\Diamond_{!n}\varphi := \Diamond_n\varphi \wedge \neg\Diamond_{n+1}\varphi$. The interpretation of a formula $\Diamond_n\varphi$ in a Kripke model is that the number of successors that satisfy φ is at least n. The interpretation of formula $\Diamond_{!n}\varphi$ is that the number of successors that satisfy φ is exactly n.

Kripke semantics. A *Kripke frame* is a pair (W, R), denoted $\mathcal{F}$, where W is a set of states and R is a binary relation on W. Denote by F_K the class of all Kripke frames. A *Kripke model* is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F}$ is a Kripke frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a *valuation*. For model $\mathcal{M} = (W, R, V)$ and $w \in W$, we call $\mathcal{M}, w$ a *pointed model*.

Given a set X, denote by $|X|$ the *cardinality* of X. Suppose that w is a state in a Kripke model $\mathcal{M} = (W, R, V)$. The *truth* of a $\mathcal{L}_g$-formula φ at w in $\mathcal{M}$, notation $\mathcal{M}, w \Vdash \varphi$, is defined inductively as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & p \in V(p) \\
\mathcal{M}, w \Vdash \neg\psi & \text{iff} & \mathcal{M}, w \nVdash \psi \\
\mathcal{M}, w \Vdash \psi_1 \vee \psi_2 & \text{iff} & \mathcal{M}, w \Vdash \psi_1 \text{ or } \mathcal{M}, w \Vdash \psi_2 \\
\mathcal{M}, w \Vdash \Diamond_n\psi & \text{iff} & |R[w] \cap [\![\psi]\!]_{\mathcal{M}}| \geq n
\end{array}$$

where $R[w] = \{v \in W : Rwv\}$ is the set of w-successors and $[\![\psi]\!]_{\mathcal{M}} = \{v \in W : \mathcal{M}, v \Vdash \psi\}$ is the *truth set* of φ in $\mathcal{M}$. For a set Γ of $\mathcal{L}_g$-formulas, we write $\mathcal{M}, w \Vdash \Gamma$ if $\mathcal{M}, w \Vdash \varphi$ for all $\varphi \in \Gamma$. Pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are said to be *modally equivalent* (notation: $\mathcal{M}, w \equiv_k \mathcal{M}', w'$) if for all $\mathcal{L}_g$-formulas φ, we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', w' \Vdash \varphi$.

A formula φ is *valid at a state w in a frame $\mathcal{F}$*, notation $\mathcal{F}, w \Vdash \varphi$, if φ is true at w in every model $(\mathcal{F}, V)$ based on $\mathcal{F}$; φ is *valid in a frame $\mathcal{F}$*, notation $\mathcal{F} \Vdash \varphi$, if it is valid at every state in $\mathcal{F}$; φ is *valid in a class of frames S_K*, notation $\Vdash_{S_K} \varphi$, if $\mathcal{F} \Vdash \varphi$ for all $\mathcal{F} \in S_K$.

Let S_K be a class of Kripke frames and $\Gamma \cup \{\varphi\}$ a set of $\mathcal{L}_g$-formulas. We say that φ is a *(local) semantic consequence of Γ over S_K*, notation $\Gamma \Vdash_{S_K} \varphi$, if for all models $\mathcal{M}$ based on frames in S_K, and all states in $\mathcal{M}$, if $\mathcal{M}, w \Vdash \Gamma$ then $\mathcal{M}, w \Vdash \varphi$.

Graded semantics. In this subsection, we recall the graded semantics from Ma and van Ditmarsch [13]. The sum operation and the 'greater than or equal to' relation ($\geq$) are defined over natural numbers $\mathbb{N}$ plus ω, the least ordinal number greater than any natural number, i.e., $\forall n \in \mathbb{N}, n < \omega$. Variables n, m, i, j range over the natural numbers $\mathbb{N}$, not over $\mathbb{N} \cup \{\omega\}$.

A *graded frame* is a pair $\mathfrak{f} = (W, \sigma)$, where W is a set of states and $\sigma : W \times W \to \mathbb{N} \cup \{\omega\}$ is a function assigning a natural number or ω to each pair of states. Denote by F_G the class of all graded frames. A *graded model* is a pair $\mathfrak{M} = (\mathfrak{f}, V)$ where $\mathfrak{f}$ is a graded frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation.

For $X \subseteq W$ and $w \in W$, define $\sigma(w, X)$ as $\Sigma_{u \in X} \sigma(w, u)$, the sum of $\sigma(w, u)$ for all $u \in X$. In particular, we define $\sigma(w, \emptyset) = 0$. The notation $X \subseteq_{<\omega} W$ represents that X is a finite subset of W and $\mathcal{P}_{<\omega}(W)$ is the set of finite subsets of W.

Suppose that w is a state in a graded model $\mathfrak{M} = (W, \sigma, V)$. The *truth* of a $\mathcal{L}_g$-formula φ at w in $\mathfrak{M}$, notation $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows:

$$
\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \Vdash \neg\psi & \text{iff} & \mathfrak{M}, w \nVdash \psi \\
\mathfrak{M}, w \Vdash \psi_1 \vee \psi_2 & \text{iff} & \mathfrak{M}, w \Vdash \psi_1 \text{ or } \mathfrak{M}, w \Vdash \psi_2 \\
\mathfrak{M}, w \Vdash \Diamond_n \psi & \text{iff} & \exists X \subseteq_{<\omega} W\ (\sigma(w, X) \geq n \ \&\ X \subseteq [\![\psi]\!]_{\mathfrak{M}})
\end{array}
$$

To our knowledge, graded frames first appeared in [6] as an intermediate structure to prove completeness of **GrK** with respect to Kripke frames. They are called *multiframes* in [1]. Graded frames are alternative semantics for graded modal logic, indeed each graded frame can be associated with a Kripke frame validating the same formulas, and vice versa as follows (cf. [13, Proposition 2.12]): Given a Kripke frame $\mathcal{F} = (W, R)$, the associated graded frame $\mathcal{F}^\circ = (W, \sigma)$ is defined by setting $\sigma(w, u) = 1$ if wRu, and $\sigma(w, u) = 0$ otherwise; given a graded frame $\mathcal{F} = (W, \sigma)$, the associated Kripke frame $\mathcal{F}_\circ = (W_\circ, R)$ is defined by setting $W_\circ = \{(w, i) \mid w \in W \ \&\ i \in \mathbb{N} \cup \{\omega\}\}$ and $(w, i)R(u, j)$ iff $\sigma(w, u) \geq j > 0$.

Axiomatization. The *minimal graded modal logic* **GrK** consists of the following axiom schemas and inference rules:

$$
\begin{array}{ll}
(Ax1) & \text{all instances of propositional tautologies} \\
(Ax2) & \Diamond_0 \varphi \leftrightarrow \top \\
(Ax3) & \Diamond_n \bot \leftrightarrow \bot \qquad (n > 0) \\
(Ax4) & \Diamond_{n+1} \varphi \to \Diamond_n \varphi \\
(Ax5) & \Box(\varphi \to \psi) \to (\Diamond_n \varphi \to \Diamond_n \psi) \\
(Ax6) & \neg\Diamond(\varphi \wedge \psi) \wedge \Diamond_{!m} \varphi \wedge \Diamond_{!n} \psi \to \Diamond_{!(m+n)}(\varphi \vee \psi) \\
(MP) & \text{from } \varphi \text{ and } \varphi \to \psi \text{ infer } \psi \\
(Gen) & \text{from } \varphi \text{ infer } \Box\varphi
\end{array}
$$

The set of theorems derivable in the system **GrK** is also called **GrK**. A *graded modal logic* is a set Λ of $\mathcal{L}_g$-formulas with **Grk** $\subseteq \Lambda$. If $\varphi \in \Lambda$, we write $\vdash_\Lambda \varphi$.

Theorem 4.2.1 ([6])**.** **GrK** *is sound and complete with respect to the class of all Kripke frames.*

Theorem 4.2.2 (Theorem 3.2 of [13])**.** **GrK** *is sound and complete with respect to the class of all graded frames.*

4.2.2 Monotonic modal logic

We consider monotonic modal logic with modalities parametrized by natural numbers, i.e. $\Diamond_n$ and $\Box_n$ with $n \in \mathbb{N}$ instead of the usual single modality. As there is no interaction between different $\Diamond_n$ and $\Diamond_m$, the logic for such modalities is not essentially different from the logic for a single modality $\Diamond$ that was originally proposed.

First, a word on notation. In graded modal logic $\Diamond_n$ denotes the existence of at least n worlds. So in particular $\Diamond$ denotes the existence of at least one world. Whereas in monotonic logic the existence of a neighbourhood is denoted by $\Box$ [4] or ∇ [10]. We prefer to stick to the notation matching usage in graded modal logic. Therefore also in monotonic modal logic write $\Diamond$ (or $\Diamond_n$) to denote the existence of a neighbourhood instead of $\Box$ or ∇ ($\Box_n$ or ∇_n). Consequently, the duals of modalities are also swapped.

Neighbourhood Semantics. A *neighbourhood frame* is a tuple $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ where W is a set of states and each $\nu_n : W \to \mathcal{PP}(W)$, called *neighbourhood function*. Denote by F_N the class of all neighbourhood frames. A *neighbourhood model* is a pair $\mathbb{M} = (\mathbb{F}, V)$, where $\mathbb{F}$ is a neighbourhood frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation.

The *truth of a $\mathcal{L}_g$-formula φ at a state w of a neighbourhood model* $\mathbb{M} = (\mathbb{F}, V)$, notation, $\mathbb{M}, w \Vdash \varphi$, is defined inductively as follows, where $n \in \mathbb{N}$:

$$\begin{array}{lll} \mathbb{M}, w \Vdash p & \text{iff} & p \in V(p) \\ \mathbb{M}, w \Vdash \neg\psi & \text{iff} & \mathbb{M}, w \nVdash \psi \\ \mathbb{M}, w \Vdash \psi_1 \vee \psi_2 & \text{iff} & \mathbb{M}, w \Vdash \psi_1 \text{ or } \mathbb{M}, w \Vdash \psi_2 \\ \mathbb{M}, w \Vdash \Diamond_n\psi & \text{iff} & [\![\psi]\!]_{\mathbb{M}} \in \nu_n(w) \end{array}$$

As an example, Figure 4.1 depicts a Kripke model, graded model and a neighbourhood model which all make $\Diamond_3 p$ true.

A neighbourhood function $\nu : W \to \mathcal{PP}(W)$ is *supplemented* or *closed under supersets* if for all $w \in W$ and $X \subseteq W$, $X \in \nu(w)$ and $X \subseteq Y$ imply $Y \in \nu(w)$. A neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is *monotonic* if each ν_n is supplemented. A neighbourhood model $\mathbb{M} = (\mathbb{F}, V)$ is *monotonic* if $\mathbb{F}$ is monotonic. Denote by F_M the class of all monotonic neighbourhood frames. Monotonic pointed models $\mathbb{M}, w$ and $\mathbb{M}', w'$ are said to be *modally equivalent* if for all $\mathcal{L}_g$-formulas φ, we have $\mathbb{M}, w \Vdash \varphi$ iff $\mathbb{M}', w' \Vdash \varphi$. For monotonic model $\mathbb{M}$, we have

$$\mathbb{M}, w \Vdash \Diamond_n\varphi \quad \text{iff} \quad \exists X (X \in \nu_n(w) \;\&\; X \subseteq [\![\varphi]\!]_{\mathbb{M}}).$$

Axiomatization. The *minimal monotonic modal logic* $\mathbf{M}_{\mathbb{N}}$ consists of the following axioms and inference rules, where $n \in \mathbb{N}$:

$(Ax1)$ all instances of propositional tautologies
(MP) from φ and $\varphi \to \psi$ infer ψ
(RM_n) from $\varphi \to \psi$ infer $\Diamond_n\varphi \to \Diamond_n\psi$

The set of theorems derivable in the system $\mathbf{M}_{\mathbb{N}}$ is also called $\mathbf{M}_{\mathbb{N}}$. A *monotonic modal logic* is a set Λ of $\mathcal{L}_{\mathbb{N}}$-formulas with $\mathbf{M}_{\mathbb{N}} \subseteq \Lambda$. If $\varphi \in \Lambda$, we write $\vdash_\Lambda \varphi$.

Theorem 4.2.3 (Theorem 2.41 of [14]). $\mathbf{M}_{\mathbb{N}}$ *is sound and strongly complete with respect to* F_M.

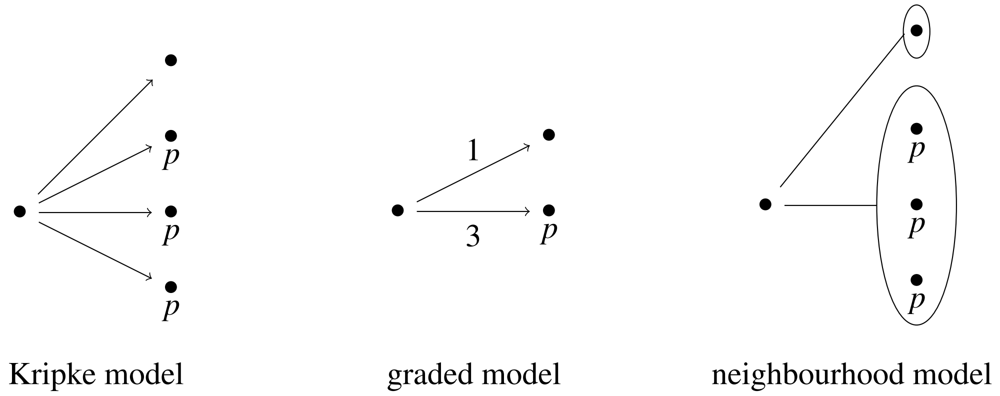

Figure 4.1: Three different ways to make $\Diamond_3 p$ true

4.3 Graded modal logics are monotonic modal logics

In this section we show that graded modal logics are monotonic modal logics. Let **G** be a graded modal logic.

Proposition 4.3.1. *Graded modal logics are monotonic modal logics.*

Proof. Let **G** be a graded modal logic. To show that **G** is a monotonic modal logic, it suffices to show that (i) **G** is closed under (MP) and (ii) for all $n \in \mathbb{N}$, **G** is closed under (RM_n). Item (i) is immediate. We now show item (ii). We distinguish the case $n = 0$ from the case $n > 0$.

Let $n = 0$. Assume that $\mathbf{G} \vdash \varphi \to \psi$. By $(Ax2)$, we have $\Diamond_0\varphi \leftrightarrow \top$ and $\Diamond_0\psi \leftrightarrow \top$ and hence $\Diamond_0\varphi \to \top$ and $\top \to \Diamond_0\psi$. It follows that $\mathbf{G} \vdash \Diamond_0\varphi \to \Diamond_0\psi$.

Let now $n > 0$. Assume that $\mathbf{G} \vdash \varphi \to \psi$. By (Gen), $\mathbf{G} \vdash \Box(\varphi \to \psi)$. Then by $(Ax5)$, $\mathbf{G} \vdash \Box(\varphi \to \psi) \to (\Diamond_n\varphi \to \Diamond_n\psi)$. Finally, by (MP) we get $\mathbf{G} \vdash \Diamond_n\varphi \to \Diamond_n\psi$. □

Corollary 4.3.2. **GrK** *is a monotonic modal logic.*

We now define axiomatization $\mathbf{GrK}_{Mon}$ as the extension of $\mathbf{M}_{\mathbb{N}}$ with $(Ax2)$ – $(Ax6)$ of **GrK** and the novel axiom $(Ax7)$ $\Diamond(\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Diamond\psi$. We show that **GrK** and $\mathbf{GrK}_{Mon}$ derive the same theorems.

Proposition 4.3.3. *For any formula φ,* $\mathbf{GrK} \vdash \varphi$ *iff* $\mathbf{GrK}_{Mon} \vdash \varphi$.

Proof. $(\Leftarrow)$ (Gen) is derivable in $\mathbf{GrK}_{Mon}$ as follows:

1	φ	assumption
2	$\varphi \to (\neg\varphi \to \bot)$	Duns Scotus law
3	$\neg\varphi \to \bot$	1,2 (MP)
4	$\Diamond\neg\varphi \to \Diamond\bot$	3 by (RM_1)
5	$\Diamond\neg\varphi \to \bot$	4 by $(Ax3)$
6	$\top \to \neg\Diamond\neg\varphi$	5 by contraposition
7	$\Box\varphi$	6 by def. of $\Box$ and $(Ax1)$

$(\Rightarrow)$ It suffices to show that $(Ax7)$ is derivable and (RM_n) is admissible rule in **GrK**. The latter follows from Proposition 4.3.1. $(Ax7)$ is equivalent to (i) $\Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$ and (ii) $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$. (i) and (ii) are derivable as follows:

1	$\Box(\varphi \to \varphi \vee \psi)$	by $(Ax1)$ and (Gen)
2	$\Diamond\varphi \to \Diamond(\varphi \vee \psi)$	1 and (Ax5) by (MP)
3	$\Box(\psi \to \varphi \vee \psi)$	by $(Ax1)$ and (Gen)
4	$\Diamond\psi \to \Diamond(\varphi \vee \psi)$	3 and (Ax5) by (MP)
5	$\Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$	2 and 4 by (Ax1)

1	$\neg\Diamond(\varphi \wedge \psi) \wedge \Diamond_0\varphi \wedge \neg\Diamond\varphi \wedge \Diamond_0\psi \wedge \neg\Diamond\psi$ $\to \Diamond_0(\varphi \vee \psi) \wedge \neg\Diamond(\varphi \vee \psi)$	(Ax6) with $m = n = 0$
2	$\neg\Diamond(\varphi \wedge \psi) \wedge \neg\Diamond\varphi \wedge \neg\Diamond\psi \to \neg\Diamond(\varphi \vee \psi)$	1 by $(Ax2)$ and $\top \wedge \varphi \leftrightarrow \varphi$
3	$\Diamond(\varphi \vee \psi) \to \Diamond(\varphi \wedge \psi) \vee \Diamond\varphi \vee \Diamond\psi$	2 by contraposition, De Morgan and double negation
4	$\varphi \wedge \psi \to \varphi$	classical tautology
5	$\Diamond(\varphi \wedge \psi) \to \Diamond\varphi$	4, RM_1
6	$\Diamond(\varphi \wedge \psi) \to \Diamond\varphi \vee \Diamond\psi$	5, property of $\vee$
7	$\Diamond\varphi \to \Diamond\varphi \vee \Diamond\psi$	classical tautology
8	$\Diamond\psi \to \Diamond\varphi \vee \Diamond\psi$	classical tautology
9	$\Diamond(\varphi \wedge \psi) \vee \Diamond\varphi \vee \Diamond\psi \to \Diamond\varphi \vee \Diamond\psi$	6, 7, 8, property of $\vee$
10	$\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$	3, 9, hypothetical syllogism

$\Box$

Another interesting question is whether there exists a class of neighbourhood frames with respect to which **GrK** is sound and complete. In monotonic neighbourhood frames the class of so-called *KW-formulas* ([10, Definition 5.13]) is elementary ([10, Theorem 5.14] and canonical ([10, Theorem 10.34]). Therefore, a presentation where each axiom is a KW-formula would make it straightforward to prove soundness and strong completeness. Unfortunately, (Ax5) and (Ax6) are not KW-formulas, since they have $\neg$ inside the scope of $\Diamond$, which is forbidden in KW-formulas. Therefore we can not prove completeness of **GrK** indirectly via a reference to KW-formulas.

If we adopt a more direct method to prove the completeness, we need to show that the properties defined by (Ax2)-(Ax7) holds in the canonical frame of monotonic modal logic containing them. Axioms (Ax5) and (Ax6) resp. correspond to the properties:

$$\forall w \forall X \forall Y (X \cap (W \setminus Y) \notin \nu_1(w) \ \& \ X \in \nu_n(w) \Rightarrow Y \in \nu_n(w))$$

$$\forall w \forall X \forall Y (X \cap Y \notin \nu_1(w) \ \& \ X \in \nu_m(w) \ \& \ X \notin \nu_{m+1}(w) \ \& \ Y \in \nu_n(w) \ \& \ Y \notin \nu_{n+1}(w) \Rightarrow X \cup Y \in \nu_{m+n}(w) \ \& \ X \cup Y \notin \nu_{(m+n+1)}(w))$$

The difficulty lies at showing that (Ax5) and (Ax6) are valid in the canonical frame of monotonic modal logic containing (Ax5) and (Ax6). For canonical frames of monotonic modal logics, we refer to [4, Def. 9.3], [10, Def. 6.2] and [14, Def. 2.37].

In the next section, we identify a class of complete neighbourhood frames via an operation $(.)^\bullet$, which is shown to be first-order definable in Section 4.5 and modally undefinable in Section 4.6.

4.4 Graded neighbourhood frames

Given a set X, denote by $\mathcal{P}_{\geq n}(X)$ the set of subsets of X such that the cardinality of each subset is at least n, in other words, $\mathcal{P}_{\geq n}(X) = \{X' \subseteq X \mid |X'| \geq n\}$. For $\Gamma \subseteq \mathcal{P}(W)$, define $\Uparrow\Gamma$

to be the up-set generated by Γ, that is, $\Uparrow\Gamma := \{Y \in \mathcal{P}(W) \mid \exists X(X \in \Gamma \,\&\, X \subseteq Y)\}$.

Definition 4.4.1. A neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is a *graded neighbourhood frame* if for all $w \in W$, there exists an $A \subseteq W$ such that for all $n \in \mathbb{N}$, $\nu_n(w) = \Uparrow\mathcal{P}_{\geq n}(A)$.

Definition 4.4.2. For a Kripke frame $\mathcal{F} = (W, R)$, the associated graded neighbourhood frame of $\mathcal{F}$ is $\mathcal{F}^\bullet = (W, \{\nu_n\}_{n\in\mathbb{N}})$, where for $w \in W$ and $n \in \mathbb{N}$, $\nu_n(w) = \Uparrow\mathcal{P}_{\geq n}(R[w])$.

That each ν_n in $\mathcal{F}^\bullet = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is monotonic follows directly from the definition. Then we have the following result:

Proposition 4.4.3. *Let $\mathcal{F} = (W, R)$ be a Kripke frame and V a valuation on $\mathcal{F}$. Then for all $w \in W$ and all formulas φ*

$$(\mathcal{F}, V), w \Vdash \varphi \quad \textit{iff} \quad (\mathcal{F}^\bullet, V), w \Vdash \varphi.$$

Proof. The proof is by induction on φ. The propositional cases follows from the definition and induction hypothesis.

As for the modal case, let φ be $\Diamond_n\psi$, $n \in \mathbb{N}$, we have

$$\begin{array}{rlll} (\mathcal{F}, V), w \Vdash \Diamond_n\psi & \text{iff} & |R[w] \cap [\![\psi]\!]_{(\mathcal{F},V)}| \geq n & \\ & \text{iff} & |R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}| \geq n & \text{(IH)} \\ & \text{iff} & \exists X \subseteq W\ (X \in \nu_n(w) \,\&\, X \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}) & (*) \\ & \text{iff} & (\mathcal{F}^\bullet, V), w \Vdash \Diamond_n\psi & \end{array}$$

Here is the proof for the equivalence marked by $(*)$. First assume that $|R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}| \geq n$. Then $R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \in \mathcal{P}_{\geq n}(R[w])$. By definition, $\nu_n(w) = \Uparrow\mathcal{P}_{\geq n}(R[w])$. Hence, $R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \in \nu_n(w)$. We also have $R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$, which completes the proof of this direction. Now assume that $X \in \nu_n(w)$ and $X \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$. Since $\nu_n(w) = \Uparrow\mathcal{P}_{\geq n}(R[w])$, $X \in \Uparrow\mathcal{P}_{\geq n}(R[w])$. Then there exists $Y \in \mathcal{P}_{\geq n}(R[w])$ and $Y \subseteq X$. It follows that $Y \subseteq R[w]$ and $|Y| \geq n$. Since $X \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$, $Y \subseteq [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$. Hence, $Y = Y \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)} \subseteq R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}$ and therefore $|R[w] \cap [\![\psi]\!]_{(\mathcal{F}^\bullet,V)}| \geq |Y| \geq n$. □

Given a graded neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ with $\nu_n(w) = \Uparrow\mathcal{P}_{\geq n}(A_w)$, we can associate it with a Kripke frame $\mathbb{F}_\bullet = (W, R)$ with $R[w] = A_w$. It follows from definitions that $(\mathbb{F}_\bullet)^\bullet = \mathbb{F}$ and $(\mathcal{F}^\bullet)_\bullet = \mathcal{F}$.

For a class of Kripke frames S_K, let $S_K^\bullet = \{\mathcal{F}^\bullet \mid \mathcal{F} \in S_K\}$. Recall that F_K is the class of all Kripke frames. Since $(\mathbb{F}_\bullet)^\bullet = \mathbb{F}$ for any graded neighbourhood frame $\mathbb{F}$, $\mathsf{F}_K^\bullet$ is equivalent to the class of all graded neighbourhood frames.

Theorem 4.4.4. **GrK** *is sound and strongly complete with respect to the class of graded neighbourhood frames.*

Proof. By Theorem 4.2.1, **GrK** is sound and strongly complete with respect to F_K. By Proposition 4.4.3, **GrK** is sound and strongly complete with respect to $\mathsf{F}_K^\bullet$. Then the claim follows from the fact that $\mathsf{F}_K^\bullet$ is equivalent to the class of all graded neighbourhood frames. □

4.5 Graded neighbourhood frames are first-order definable

A class S_N of neighbourhood frames is *first-order definable* if there exists a set of first-order formulas Γ such that $\mathbb{F} \models \Gamma$ iff $\mathbb{F} \in S_N$. In this section, we show that the class of graded neighbourhood frames is(two-sorted) first-order definable in the (two-sorted) first-order language $\mathcal{L}_g^1$ of $\mathcal{L}_g$ defined below.

Each monotonic neighbourhood frame $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ can be seen as a two-sorted relational structure $(W, \mathcal{P}(W), \{R_{\nu_n}\}_{n\in\mathbb{N}}, R_\ni)$ where $R_{\nu_n} \subseteq W \times \mathcal{P}(W)$ and $R_\ni \subseteq \mathcal{P}(W) \times W$ such that $wR_{\nu_n}X$ iff $X \in \nu_n(w)$ and $XR_\ni w$ iff $w \in X$. Accordingly, the (two-sorted) first-order language $\mathcal{L}_g^1$ of $\mathcal{L}_g$ has equality $=$, first-order variables $w, u, v, \ldots$ over W, first-order variables $X, Y, Z, \ldots$ over $\mathcal{P}(W)$, binary symbols R_{ν_n} for $n \in \mathbb{N}$ and $R_\ni$, and unary relation symbols $P, Q, \ldots$ corresponding to $p, q, \ldots \in \mathsf{Prop}$.

In other words, given sets of variables Ψ and Φ, formulas in $\mathcal{L}_g^1$ are defined inductively as follows:

$$\mathcal{L}_g^1 \ni \chi ::= w = u \mid X = Y \mid Pw \mid R_{\nu_n}wX \mid R_\ni Xw \mid \neg\chi \mid \chi \vee \chi \mid \forall x\chi \mid \forall X\chi$$

where $w, u \in \Psi$, $X, Y \in \Phi$, P corresponds to $p \in \mathsf{Prop}$ and $n \in \mathbb{N}$.

A set A is called *atomic in* $\nu_1(w)$ if for all $a \in A$, $\{a\} \in \nu_1(w)$. Denote by $(\star)$ the following conditions: for all $w \in W$

($\star$1) $\nu_0(w) = \mathcal{P}(W)$.

($\star$2) $\nu_n(w)$ is closed under supersets for $n \in \mathbb{N}$.

($\star$3) $\emptyset \notin \nu_n(w)$ for $n \in \mathbb{N}$.

($\star$4) If $X \in \nu_n(w)$, then there exists a minimal $Y \in \nu_n(w)$ such that $Y \subseteq X$.

($\star$5) If Y is a minimal element in $\nu_n(w)$, then $|Y| = n$ and Y is atomic in $\nu_1(w)$.

($\star$6) If $\{y_1\}, \ldots, \{y_n\} \in \nu_1(w)$ and $y_1, \ldots, y_n$ are pairwise distinct, then $\bigcup_{1\le i\le n}\{y_i\}$ is a minimal element in $\nu_n(w)$.

Note that conditions $(\star)$ can be expressed in language $\mathcal{L}_g^1$. For example, $|\,Y\,| \geq n$ iff $y_1 \in Y \wedge \ldots \wedge y_n \in Y \wedge \bigwedge_{i\neq j} y_i \neq y_j$, and Y is atomic in $\nu_1(w)$ iff $\forall Z(\forall Z'(Z' \subseteq Z \Rightarrow Z' = \emptyset$ or $Z' = Z)$ & $Z \subseteq Y \Rightarrow Z \in \nu_1(w))$.

Proposition 4.5.1. *Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame. Then $\mathbb{F}$ is graded iff $\mathbb{F}$ satisfies $(\star)$.*

Proof. For the left-to-right direction, assume that $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ is a graded neighbourhood frame, that is, for all $w \in W$, there exists some $A \subseteq W$ such that for all $n \in \mathbb{N}$, $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(A)$. Since $\uparrow\mathcal{P}_{\geq 0}(A) = \uparrow\mathcal{P}(A) = \mathcal{P}(W)$, item ($\star$1) holds. Item ($\star$2) and ($\star$3) also follow directly.

Now assume that $X \in \nu_n(w)$. Since $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(A)$, there exists $Y \in \mathcal{P}_{\geq n}(A)$ with $Y \subseteq X$. It follows that $|\,Y\,| \geq n$. Let Y' be a subset of Y containing exactly n-elements. Then Y' is a minimal element in $\nu_n(w)$ and $Y' \subseteq X$. Hence, item ($\star$4) follows.

Now assume that Y is a minimal element in $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(A)$. Then $Y \subseteq A$ and $|Y| = n$. Since $\nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$, for all $a \in A$, $\{a\} \in \nu_1(w)$. It follows that Y is atomic in $\nu_1(w)$. Hence, item ($\star$5) holds. For item ($\star$6), assume that $\{y_1\} \neq \ldots \neq \{y_n\} \in \nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$. Then $\{y_1, \ldots, y_n\} \in \uparrow\mathcal{P}_{\geq n}(A)$. It follows that $\{y_1, \ldots, y_n\}$ is a minimal element in $\nu_n(w)$. Hence, item ($\star$6) holds.

The right-to-left direction follows from Lemma 4.5.4 and 4.5.5 below. □

Lemma 4.5.2. *Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). If $X \in \nu_1(w)$, there exists $x \in X$ such that $\{x\} \in \nu_1(w)$.*

Proof. Assume that $X \in \nu_1(w)$. By ($\star$4), there exists a minimal $Y \in \nu_1(w)$ such that $Y \subseteq X$. By ($\star$3), $X \neq \emptyset$ and $Y \neq \emptyset$. By ($\star$5), Y is atomic in $\nu_1(w)$, i.e., for all $y \in Y$, $\{y\} \in \nu_1(w)$. It follows that there exists $x \in X$ such that $\{x\} \in \nu_1(w)$. □

Lemma 4.5.3. *Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). If $\nu_1(w) \neq \emptyset$, there exists a set $A \subseteq W$ such that A is the maximum atomic set in $\nu_1(w)$.*

Proof. Since $\nu_1(w) \neq \emptyset$, we assume $X \in \nu_1(w)$. By ($\star$3), $X \neq \emptyset$. By ($\star$4), there exists a minimal $X' \in \nu_1(w)$ such that $X' \subseteq A$. By ($\star$5), $|X'| = 1$ and X' is atomic in $\nu_1(w)$. Hence, we can assume $X' = \{a\}$. Let A be the union of all singletons in $\nu_1(w)$. Since $\{a\} \in \nu_1(w)$, $A \neq \emptyset$. Now we show that A is the maximum atomic set in $\nu_1(w)$. Since A is the union of all singletons in $\nu_1(w)$, A is atomic. Let B be an atomic set in $\nu_1(w)$. For any $b \in B$, by atomicity, $\{b\} \in \nu_1(w)$. It follows that $b \in A$. Therefore, $B \subseteq A$. Hence, A is the maximum atomic set in $\nu_1(w)$. □

Lemma 4.5.4. *Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). If $\nu_1(w) \neq \emptyset$, then $\nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$, where A is the maximum atomic set in $\nu_1(w)$.*

Proof. If $\nu_1(w) = \emptyset$, then $A = \emptyset$. Then $\nu_1(w) = \uparrow\mathcal{P}_{\geq 1}(A)$. If $\nu_1(w) \neq \emptyset$, assume that $X \in \nu_1(w)$. By Lemma 4.5.2, there exists an $x \in X$ such that $\{x\} \in \nu_1(w)$. Since A is the maximum atomic set in A, we have $x \in A$. It follows that $\{x\} \in \mathcal{P}_{\geq 1}(A)$. Since $x \in X$, $X \in \uparrow\mathcal{P}_{\geq 1}(A)$.

Assume that $X \in \uparrow\mathcal{P}_{\geq 1}(A)$. Then there exists $Y \in \mathcal{P}_1(A)$ such that $Y \subseteq X$. Since A is atomic in $\nu_1(w)$, for all $y \in Y$, $\{y\} \in \nu_1(w)$. By ($\star$2), $\nu_1(w)$ is monotonic. Therefore, $Y \in \nu_1(w)$. Since $Y \subseteq X$, $X \in \nu_1(w)$. □

Lemma 4.5.5. *Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ be a neighbourhood frame satisfying ($\star$). Then for $w \in W$,*

1. *If $\nu_1(w) = \emptyset$, then $\nu_n(w) = \emptyset$ for $n > 1$.*
2. *If $\nu_1(w) \neq \emptyset$, then $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(A)$ for $n > 1$, where A is the maximum atomic set in $\nu_1(w)$.*

Proof. For item 1, we prove by contradiction. Assume that $\nu_1(w) = \emptyset$ and for some $n > 1$, $X \in \nu_n(w)$. By ($\star$3), $X \neq \emptyset$. By ($\star$4) and ($\star$5), there exists $X' \subseteq X$ such that X' is atomic in $\nu_1(w)$. By ($\star$3), $X' \neq \emptyset$. By atomicity of X', $\nu_1(w) \neq \emptyset$, contradiction .

Now we prove item 2 and assume that $X \in \nu_n(w)$. By ($\star$4), there exists a minimal element of $\nu_n(w)$ such that $Y \subseteq X$. By ($\star$5), $|Y| \geq n$ and Y is atomic in $\nu_1(w)$. Since A

is the maximum atomic set of $\nu_1(w)$, $Y \subseteq A$. Since $| Y | \geq n$, $Y \in \mathcal{P}_{\geq n}(A)$. Since $Y \subseteq X$, $X \in \uparrow\mathcal{P}_{\geq n}(A)$.

Assume that $X \in \uparrow\mathcal{P}_{\geq n}(A)$. Then there exists $Y \in \mathcal{P}_{\geq n}(A)$ such that $Y \subseteq X$. It follows that $| Y | \geq n$. Since A is the maximum atomic set of $\nu_1(w)$, Y is atomic in $\nu_1(w)$. Hence, there exist distinct $y_1, \ldots, y_n \in Y$ such that $\{y_1\}, \ldots, \{y_n\} \in \nu_1(w)$ and $y_1 \neq \ldots \neq y_n$. By ($\star$6), $\bigcup_{1\leq i\leq n}\{y_i\}$ is a minimal element in $\nu_n(w)$. Since $\bigcup_{1\leq i\leq n}\{y_i\} \subseteq Y \subseteq X$ and $\nu_n(w)$ is monotonic by ($\star$2), $X \in \nu(w)$. □

4.6 Graded neighbourhood frames are not modally definable

A class S_N of neighbourhood frames is *modally definable* if there exists a set of modal formulas Δ such that $\mathbb{F} \Vdash \Delta$ iff $\mathbb{F} \in S_N$. In this section, we show that the class of graded neighbourhood frames is not modally definable. It is well known that if the class of neighbourhood frames is modally definable, then it is closed under bounded morphic images. Below we show that the class of graded neighbourhood frames is not closed under bounded morphic images (by exhibiting a counterexample), so we conclude that it is not modally definable.

Given a function $f : W \to W'$ and $X \subseteq W$, define $f[X] := \{f(x) : x \in X\}$.

Definition 4.6.1. Let $\mathbb{F} = (W, \{\nu_n\}_{n\in\mathbb{N}})$ and $\mathbb{F}' = (W, \{\nu'_n\}_{n\in\mathbb{N}})$ be neighbourhood frames. A *bounded morphism* from $\mathbb{F}$ to $\mathbb{F}'$ is a function $f : W \to W'$ satisfying for $n \in \mathbb{N}$

($BM1_n$) If $X \in \nu_n(w)$, then $f[X] \in \nu'_n(f(w))$.

($BM2_n$) If $X' \in \nu'_n(f(w))$, then there exists $X \subseteq W$ such that $f[X] \subseteq X'$ and $X \in \nu(w)$.

If there is a surjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$, we say that $\mathbb{F}'$ is a *bounded morphic image* of $\mathbb{F}$.

Proposition 4.6.2 (Prop. 5.3 of [10])**.** *Let $\mathbb{F}$ and $\mathbb{F}'$ be neighbourhood frames. If $\mathbb{F}'$ is a bounded morphic image of $\mathbb{F}$, then $\mathbb{F} \Vdash \varphi$ implies $\mathbb{F}' \Vdash \varphi$.*

Proposition 4.6.3. *If a class of neighbourhood frames is modally definable, then it is closed under bounded morphic images.*

Proof. Let S_N be a class of neighbourhood frames defined by a set of formulas Δ, $\mathbb{F} \in S_N$ and $\mathbb{F}'$ a bounded morphic image of $\mathbb{F}$. Since $\mathbb{F} \in S_N$, $\mathbb{F} \Vdash \Delta$. By Proposition 4.6.2, $\mathbb{F}' \Vdash \Delta$ and therefore $\mathbb{F}' \in S_N$. □

Example 4.6.4. Consider neighbourhood frames $\mathbb{F} = (\{a, b\}, \{\nu_n\}_{n\in\mathbb{N}})$ such that for $n \in \mathbb{N}$, $\nu_n(a) = \nu_n(b) = \uparrow\mathcal{P}_{\geq n}(\{a, b\})$ and $\mathbb{F}' = (\{c\}, \{\nu'_n\}_{n\in\mathbb{N}})$ such that $\nu'_0(c) = \{\emptyset, \{c\}\}$, $\nu'_1(c) = \nu'_2(c) = \{\{c\}\}$ and $\nu'_k(c) = \emptyset$ for $k > 2$. By Definition 4.4.1, $\mathbb{F}$ is a graded neighbourhood frame. As for $\mathbb{F}'$, we have $\nu_1(c) = \uparrow\mathcal{P}_{\geq 1}(\{c\})$ while $\nu_2(c) \neq \uparrow\mathcal{P}_{\geq 2}(\{c\})$. Therefore, $\mathbb{F}'$ is not a graded neighbourhood frame. It can be verified that function $f : \{a, b\} \to \{c\}$, with $f(a) = f(b) = c$, is a subjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$. Therefore, the class of graded neighbourhood frames is not closed under bounded morphic images.

Proposition 4.6.5. *The class of graded neighbourhood frames is not modally definable.*

Proof. It follows from Example 4.6.4 and the contraposition of Proposition 4.6.3. □

4.7 Bisimulation

The notion of graded tuple bisimulation was first proposed in de Rijke [7]. In this section, we obtain a new definition of graded bisimulation by substituting $\nu_n(w)$ with $\uparrow\mathcal{P}_{\geq n}(R[w])$ in the definition of monotonic bisimulation. And we prove that the new definition is equivalent to the old one (cf. Proposition 4.7.6 and 4.7.9).

4.7.1 From monotonic bisimulation to graded bisimulation

Definition 4.7.1 (Monotonic bisimulation, Def. 4.10 of [10]). Suppose that $\mathbb{M} = (W, \{\nu_n\}_{n\in\mathbb{N}}, V)$ and $\mathbb{M}' = (W', \{\nu'_n\}_{n\in\mathbb{N}}, V')$ are monotonic neighbourhood models. A non-empty relation $Z \subseteq W \times W'$ is a *monotonic bisimulation* (notation: $Z : \mathbb{M} \leftrightarroweq_m \mathbb{M}'$) provided that

- **(Prop)** If wZw', then w and w' satisfy the same proposition letters.
- **(Forth)** If wZw' and $X \in \nu_n(w)$, then there is $X' \subseteq W'$ such that $X' \in \nu'_n(w')$ and $\forall x' \in X' \exists x \in X : xZx'$.
- **(Back)** If wZw' and $X' \in \nu'_n(w')$, then there is $X \subseteq W$ such that $X \in \nu_n(w)$ and $\forall x \in X \exists x' \in X' : xZx'$.

If $w \in \mathbb{M}$ and $w' \in \mathbb{M}'$, then w and w' are *monotonic bisimilar states* (notation: $\mathbb{M}, w \leftrightarroweq_m \mathbb{M}', w'$) if there is a bisimulation $Z : \mathbb{M} \leftrightarroweq_m \mathbb{M}'$ with wZw'.

Proposition 4.7.2 (Prop. 4.11 of [10]). *Let $\mathbb{M} = (W, \{\nu_n\}_{n\in\mathbb{N}}, V)$ and $\mathbb{M}' = (W', \{\nu'_n\}_{n\in\mathbb{N}}, V')$ be monotonic neighbourhood models. If $\mathbb{M}, w \leftrightarroweq_m \mathbb{M}', w'$, then for $\mathcal{L}_g$-formula φ, $\mathbb{M}, w \Vdash \varphi$ iff $\mathbb{M}', w' \Vdash \varphi$.*

Substituting $\nu_n(w)$ in Definition 4.7.1 with $\uparrow\mathcal{P}_{\geq n}(R[w])$, we have:

Definition 4.7.3 (Graded bisimulation). Suppose that $\mathcal{F} = (W, R, V)$ and $\mathcal{M}' = (W', R', V)$ are Kripke models. A non-empty relation $Z \subseteq W \times W'$ is a *graded bisimulation* (notation: $Z : \mathcal{M} \leftrightarroweq_g \mathcal{M}'$) provided that

- **(Prop)** If wZw', then w and w' satisfy the same proposition letters.
- **(Forth)** If wZw' and $X \in \uparrow\mathcal{P}_{\geq n}(R[w])$, then there is an $X' \subseteq W'$ such that $X' \in \uparrow\mathcal{P}_{\geq n}(R'[w'])$ and $\forall x' \in X' \exists x \in X : xZx'$.
- **(Back)** If wZw' and $X' \in \uparrow\mathcal{P}_{\geq n}(R'[w'])$, then there is an $X \subseteq W$ such that $X \in \uparrow\mathcal{P}_{\geq n}(R[w])$ and $\forall x \in X \exists x' \in X' : xZx'$.

If $w \in \mathcal{M}$ and $w' \in \mathcal{M}'$, then w and w' are *graded bisimilar states* (notation: $\mathcal{M}, w \leftrightarroweq_g \mathcal{M}', w'$) if there is a bisimulation $Z : \mathcal{M} \leftrightarroweq_g \mathcal{M}'$ with wZw'.

Proposition 4.7.4. *Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be Kripke models. If $\mathcal{M}, u \leftrightarroweq_g \mathcal{M}', u'$, then $\mathcal{M}, u \equiv_k \mathcal{M}', u'$.*

Proof. Since $\mathcal{M}, u \leftrightarrows_g \mathcal{M}', u'$, there exists a non-empty relation $Z \subseteq W \times W'$ such that $Z : \mathcal{M} \leftrightarrows_g \mathcal{M}'$ and uZu'. For neighbourhood frames $\mathcal{M}^\bullet = (W, \{\nu_n\}_{n\in\mathbb{N}}, V)$ and $\mathcal{M}'^\bullet = (W, \{\nu'_n\}_{n\in\mathbb{N}}, V')$, by definition, for $w \in W$ and $w' \in W'$, $\nu_n(w) = \uparrow\mathcal{P}_{\geq n}(R[w])$ and $\nu'_n(w') = \uparrow\mathcal{P}_{\geq n}(R'[w'])$. Substituting $\uparrow\mathcal{P}_{\geq n}(R[w])$ with $\nu_n(w)$ and $\uparrow\mathcal{P}_{\geq n}(R'[w'])$ with $\nu'_n(w')$ in the definition of $Z : \mathcal{M} \leftrightarrows_g \mathcal{M}'$, we have $Z : \mathcal{M}^\bullet, u \leftrightarrows_m \mathcal{M}'^\bullet, u'$ and uZu'. For all formulas φ, that $\mathcal{M}, u \Vdash \varphi$ iff $\mathcal{M}', u' \Vdash \varphi$ can be proved as follows:

$$\begin{array}{llll}
\mathcal{M}, u \Vdash \varphi & \text{iff} & \mathcal{M}^\bullet, u \Vdash \varphi & \text{Proposition 4.4.3} \\
& \text{iff} & \mathcal{M}'^\bullet, u' \Vdash \varphi & \text{Proposition 4.7.2} \\
& \text{iff} & \mathcal{M}', u' \Vdash \varphi & \text{Proposition 4.4.3}
\end{array}$$

□

4.7.2 Graded bisimulation is equivalent to graded tuple bisimulation

In the rest of this section, we recall the definition of graded tuple bisimulation in de Rijke [7] and show that it is equivalent to Definition 4.7.3. Given a set X, denote by $\mathcal{P}_{<\omega}(X)$ the set of finite subsets of X. We now get:

Definition 4.7.5 (Graded tuple bisimulation)**.** Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M} = (W', R', V')$ be two Kripke models. A tuple $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots)$ of relations is called *graded tuple bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $\mathcal{Z} : \mathcal{M} \leftrightarrows_{gt} \mathcal{M}'$) iff:

(1) $\mathcal{Z}_1$ is non-empty;

(2) for all i, $\mathcal{Z}_i \subseteq \mathcal{P}_{<\omega}(W_1) \times \mathcal{P}_{<\omega}(W_2)$;

(3) if $X\mathcal{Z}_iX'$, then $|X| = |X'| = i$;

(4) if $\{w\}\mathcal{Z}_1\{w'\}$, then w and w' satisfy the same proposition letters;

(5) if $\{w\}\mathcal{Z}_1\{w'\}$, $X \subseteq R[w]$ and $|X| = i \geq 1$, then there exists $X' \in \mathcal{P}_{<\omega}(W')$ with $X' \subseteq R'[w']$ and $X\mathcal{Z}_iX'$;

(6) if $\{w\}\mathcal{Z}_1\{w'\}$, $X' \subseteq R[w']$ and $|X'| = i \geq 1$, then there exists $X \in \mathcal{P}_{<\omega}(W)$ with $X \subseteq R[w]$ and $X\mathcal{Z}_iX'$;

(7) if $X\mathcal{Z}_iX'$, then (a) $\forall x \in X \exists x' \in X' : \{x\}\mathcal{Z}_1\{x'\}$, and (b) $\forall x' \in X' \exists x \in X : \{x\}\mathcal{Z}_1\{x'\}$.

Proposition 4.7.6. *Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be Kripke models and $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots)$ a tuple of relations such that $\mathcal{Z} : \mathcal{M} \leftrightarrows_{gt} \mathcal{M}'$. Define $Z \subseteq W \times W'$ to be a relation such that wZw' iff $\{w\}\mathcal{Z}_1\{w'\}$. Then $Z : \mathcal{M} \leftrightarrows_g \mathcal{M}'$.*

Proof. (**Prop**) follows from item (4) of Definition 4.7.5. As for (**Forth**), assume that wZw' and $X \in \uparrow\mathcal{P}_{\geq n}(R[w])$. Then there exists $Y \subseteq R[w]$ such that $Y \subseteq X$ and $|Y| = n$. Since $|Y| = n$ and $\{w\}\mathcal{Z}_1\{w'\}$, by items (5) and (3) there exists $Y' \subseteq R'[w']$, $|Y'| = n$ and $Y\mathcal{Z}_nY'$. It follows that $Y' \in \uparrow\mathcal{P}_{\geq n}(R'[w'])$. By item (7)(b), $\forall y' \in Y' \exists y \in Y : \{y\}\mathcal{Z}_1\{y'\}$. Since $Y \subseteq X$ and xZy iff $\{x\}\mathcal{Z}_1\{y\}$, we have $\forall y' \in Y' \exists x \in X : xZy'$, which completes the proof of that Z satisfies (**Forth**). That Z satisfies (**Back**) can be proved in a similar way. □

Now we show how to construct a graded tuple bisimulation out of a graded bisimulation, with the following lemmas:

Lemma 4.7.7. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke models and $Z : \mathcal{M}, w \leftrightarrows_g \mathcal{M}', w'$.*

(1) If $u \in R[w]$, then there exists $u' \in R'[w']$ with uZu'.

(2) If $u' \in R'[w']$, then there exists $u \in R[w]$ with uZu'.

Proof. (1) Since $u \in R[w]$, $\{u\} \in\uparrow \mathcal{P}_{\geq 1}(R[w])$. By (**Forth**), there exists $Y' \in \uparrow\mathcal{P}_{\geq 1}(R'[w'])$ such that $\forall y' \in Y' \exists x \in \{u\} : xZy'$. It follows that $\forall y' \in Y' : uZy'$. Since $Y' \in\uparrow\mathcal{P}_{\geq 1}(R'[w'])$, there exists $u' \in R'[w']$ such that $u' \in Y'$. It follows that uZu'.

Claim (2) can be proved in a similar way by using (**Back**). □

Let W and W' be sets, $X \subseteq W$, $X' \subseteq W'$ and $Z \subseteq W \times W'$. Sets X and X' are called a *Z-pair* if $\forall x \in X \exists x' \in X' : xZx'$ and $\forall x' \in X' \exists x \in X : xZx'$.

Lemma 4.7.8. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke models and $Z : \mathcal{M}, w \leftrightarrows_g \mathcal{M}', w'$.*

(1) If $X \subseteq R[w]$ and $|X| = i \geq 1$, then there exists $X' \subseteq R'[w']$ with $|X'| = i$ such that X and X' form a Z-pair.

(2) If $X' \subseteq R'[w']$ and $|X'| = i \geq 1$, then there exists $X \subseteq R[w]$ with $|X| = i$ such that X and X' form a Z-pair.

Proof. (1) The proof is by induction on i. If $i = 1$, we may assume that $X = \{u\}$. Since $X \subseteq R[w]$, we have $u \in R[w]$. By Lemma 4.7.7, there exists $u' \in R'[w']$ with uZu'. Let $X' = \{u'\}$. It follows that $|X'| = 1$ and that X and X' form a Z-pair.

Consider the case that $i > 1$. We may assume that $X = \{u\} \cup Y$, where $Y \subseteq R[w]$ and $u \notin Y$. It follows that $|Y| = i - 1 \geq 1$. By induction hypothesis, there exists an $Y' \subseteq R'[w']$ such that $|Y'| = i - 1$ and that Y and Y' forms a Z-pair. Since $u \in R[w]$, by Lemma 4.7.7, there exists $u' \in R'[w']$ with uZu'. If $u' \notin Y'$, let $X' = Y' \cup \{u'\}$. Then $|X'| = i$ and X and X' forms a Z-pair.

If $u' \in Y'$, there are two subcases: $\exists y \in Y \exists v' \in R'[w'] \backslash Y' : yZv'$ and for all $y \in Y$ and $v' \in R'[w'] \backslash Y'$, not yZv'.

Consider the case that $\exists y \in Y \exists v' \in R'[w'] \backslash Y' : yZv'$. Let $X' = Y' \cup \{v'\}$. Then $|X'| = i$. Since Y and Y' form a Z-pair, uZu' and yZv', X and X' form a Z-pair.

Consider the case that for all $y \in Y$ and $v' \in R'[w'] \backslash Y'$, not yZv'. Since $X \in\uparrow\mathcal{P}_{\geq i}(R[w])$, by (**Forth**), there exists $B' \in\uparrow \mathcal{P}_{\geq i}(R'[w'])$ such that $\forall b' \in B' \exists x \in X : xZb'$. Since $B' \in\uparrow \mathcal{P}_{\geq i}(R'[w'])$, there exists $B'' \subseteq B'$ such that $B'' \subseteq R'[w']$ and $|B''| \geq i$. Since $|Y'| = i - 1$, there exists $b'' \in B''$ such that $b'' \in R'[w'] \backslash Y'$. Since for all $y \in Y$ and $v' \in R'[w'] \backslash Y'$, not yZv', we have for all $y \in Y$, not yZb''. Since $\forall b' \in B' \exists x \in X : xZb'$ and $X = \{u\} \cup Y$, we have uZb''. Let $X' = Y' \cup \{b''\}$. Then $|X'| = i$. Since Y and Y' form a Z-pair and uZb'', X and X' form a Z-pair.

Claim (2) can be proved in a similar way by using (**Back**). □

Proposition 4.7.9. *Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be Kripke models and $Z \subseteq W \times W'$ a non-empty relation such that $Z : \mathcal{M} \leftrightarrows_g \mathcal{M}'$. Define a tuple of relations $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \dots)$ as: $\mathcal{Z}_1 = \{(\{w\}, \{w'\}) \mid wZw'\}$, and $\mathcal{Z}_n = \{(X, X') \mid\mid X| = |X'| = n$, X and X' form a Z-pair$\}$, for $n > 1$. Then $\mathcal{Z} : \mathcal{M} \leftrightarrows_{gt} \mathcal{M}'$.*

Proof. Since Z is non-empty, $\mathcal{Z}_1$ is non-empty. So item (1) in Definition 4.7.5 is satisfied. Items (2), (3) and (4) are satisfied by the definition of Z. Items (5) and (6) are satisfied by Lemma 4.7.8. Item (7) is satisfied by the definition of $\mathcal{Z}_i$ and the definition of Z-pairs. □

In summary, we showed how to construct a graded bisimulation out of a graded tuple bisimulation (Prop. 4.7.6), and vice versa (Prop. 4.7.9). Hence, graded bisimulation (Def. 4.7.3) and graded tuple bisimulation (Def. 4.7.5) are equivalent. Another notion of bisimulation called *resource bisimulation* was proposed in [1], which is very similar to the notion later proposed in [13]. A precise comparison of graded bisimulation to these notions is left for future research.

4.8 Conclusion

Inspired by graded models, we proposed a class of graded neighbourhood frames, and we showed that the axiomatiziation **GrK** is sound and strongly complete for this class. We further showed that graded neighbourhood frames are first-order definable but not modally definable. We also obtained a new definition of graded bisimulation building upon the notion of monotonic bisimulation, where some details concerning resource bisimulation are left for further research. Our results show that techniques for monotonic modal logics can be successfully applied to graded modal logics.

There are many options for further research:

(1) Using the approach developed in this chapter, updating neighbourhood models [12] can be compared to updated graded models [13].

(2) Building on multi-type display calculi for monotonic logics [5] we plan to introduce multi-type display calculi for graded modal logic.

(3) With yet another notion of bisimulation on graded frames, and algorithms to calculate two-sorted first-order correspondence on neighbourhood frames [10, 5], we plan to get two-sorted first-order correspondence on graded frames.

(4) Finally, given the logic **GrK** in Section 4.2 for n grades, and given its alternative incarnation as a monotonic modal logic in Section 4.3, we wish to find the axiomatization of the graded modal logic for one grade. In Proposition 4.3.1 we showed that (RM_n) is admissible in **GrK**. As **GrK** only has necessitation for $\Box$, this is indeed of some minor interest. We can also pose this question in the other direction: is **GrK** derivable in some extension of $\mathbf{M}_{\mathbb{N}}$, that makes the monotonic character of the logic clearer? Because of the axioms $(Ax4)$, $(Ax5)$ and $(Ax6)$, we should not expect this to be without interaction axioms for different modalities. However, an interesting case is graded modal logic for a single modality $\Diamond_n$: is there a monotonic modal logic axiomatizing this case, without interaction axioms? This logic should contain $\Diamond_n\bot \leftrightarrow \bot$, corresponding to the requirement that for all states w in the domain of a model, $\emptyset \notin \nu_n(w)$. Such a logic should also contain, for example, $(\Diamond_n\phi \wedge \Diamond_n\neg\phi) \rightarrow (\Diamond_n\psi \vee \Diamond_n\neg\psi)$. It is easy to see that this is valid in **GrK**. However, $(\Diamond_n\phi \wedge \Diamond_n\neg\phi) \rightarrow (\Diamond_n\psi \vee \Diamond_n\neg\psi)$ is not derivable in monotone modal logic, as there are models of monotone modal logic in which it is false. We leave the axiomatization of single-grade graded modal logic for future research.

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Chapter 5

Conclusion

In this thesis, we have explored various ways in which the algebraic and relational semantics of given logics can provide useful information for the design of analytic (multi-type) calculi for these logics, and their axiomatic extensions. In particular:

1. we have established a systematic connection between the syntactic shape of analytic inductive axioms and the generation of cut-free derivations of these axioms from their associated analytic structural rules, obtained via the semantically inspired algorithm ALBA;

2. we have introduced proper display calculi for monotone modal logic and conditional logic and a large family of their axiomatic extensions, thanks to a reformulation of their neighbourhood semantics in a suitable multi-type relational environment. Besides improving the proof theory of two well known logical frameworks and significantly expanding the range of their analytic (properly displayable) axiomatic extensions, this semantic analysis and ensuing calculus allows for a better connection of non-normal logics with the family of normal multi-type (D)LE-logics, and sets the stage for uniformly extending notions and techniques from the semantics and the proof theory of normal (D)LE-logics to these and other non-normal logics;

3. we have started extending the semantic analysis mentioned in the previous item to graded modal logic, which we have studied as a non-normal (monotone) modal logic.

The results above form a base for further investigations at the interface of syntax and semantics:

1. The semantic analysis of graded modal logic developed in the present dissertation offers a platform for the development of analytic multi-type proof calculi for graded modal logic and its axiomatic extensions;

2. other non-normal modal logics which naturally lend themselves to similar analyses include coalition logic [2], game logic [3] and probabilistic logics [1];

3. more in general, the question of which non-analytic logics allow for an equivalent multi-type analytic presentation is still open. Can we identify sufficient semantic

conditions guaranteeing the existence of such an equivalent multi-type presentation, which will in turn allow for the design of proper multi-type display calculi in an algorithmic way? The case studies mentioned in the previous item will be not only interesting in their own right, but will be also likely to provide more insights on this general issue.

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Explanation of contributions

The original contributions of this thesis are collected in Chapters 2, 3, and 4, which are respectively based on the articles listed below, which are written in collaboration with the members of my supervision team:

1. J. Chen, G. Greco, A. Palmigiano, and A. Tzimoulis. Syntactic completeness of proper display calculi. *ACM Transactions on Computational Logic* , 2022.

2. J. Chen, G. Greco, A. Palmigiano, and A. Tzimoulis. Non-normal modal logics and conditional logics: Semantic analysis and proof theory. *Information and Computation*, 2021.

3. J. Chen, H. van Ditmarsch, G. Greco, and A. Tzimoulis. Neighbourhood semantics for graded modal logic. *Bulletin of the Section of Logic*, 50(3):373–395, Jul. 2021.

The first article, written in collaboration with Giuseppe Greco, Apostolos Tzimoulis, and Alessandra Palmigiano, is motivated by an open problem in Apostolos Tzimoulis' dissertation. My coauthors proposed this problem to me at the beginning of my PhD trajectory, indicated the relevant literature, and gave me continuous feedback on my progress; furthermore, they helped me to structure the results, advised me on technical and presentation issues, and edited the final version of the paper.

The second article is written in collaboration with the same coauthors of the first one. Its topic was also proposed to me by them, in response to my interest in exploring the more semantic ramifications of the methodology developed in the first article. Also while working on this paper, I received continuous feedback by my coauthors, both on technical and presentation issues. Moreover, they edited the final version of the paper.

The topic of the third article, coauthored by Giuseppe Greco, Apostolos Tzimoulis, and Hans van Ditmarsch, was envisioned by me and proposed to them, as it was intended to bring together the lines of research in which the first two researchers are expert with the expertise of the third one. Also for this article, I benefited from very detailed and continuous feedback given by all my coauthors, who revised the final version of the paper.

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