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Slanted Canonicity of Analytic Inductive Inequalities

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We prove an algebraic canonicity theorem for normal LE-logics of arbitrary signature, in a generalized setting in which the non-lattice connectives are interpreted as operations mapping tuples of elements of the given lattice to closed or open elements of its canonical extension. Interestingly, the syntactic shape of LE-inequalities which guarantees their canonicity in this generalized setting turns out to coincide with the syntactic shape of *analytic inductive inequalities*, which guarantees LE-inequalities to be equivalently captured by analytic structural rules of a proper display calculus. We show that this canonicity result connects and strengthens a number of recent canonicity results in two different areas: subordination algebras, and transfer results via Gödel-McKinsey-Tarski translations.

CCS Concepts: • **Theory of computation → Modal and temporal logics;**

Additional Key Words and Phrases: Sahlqvist canonicity, algorithmic correspondence and canonicity, non-distributive lattices, analytic inductive inequalities, subordination algebras, transfer results via Gödel-McKinsey-Tarski translations

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18

1 INTRODUCTION

This article addresses the connection between canonicity problems in two seemingly unrelated areas, namely subordination algebras and transfer results for nonclassical modal logics via Gödel-McKinsey-Tarski translations or variations thereof (GMT-type translations). Subordination algebras were introduced in [1] as a generalization of de Vries' compingent algebras [18] and are equivalent presentations of pre-contact algebras [19], proximity algebras [21], and quasi-modals algebras [3, 4]. Canonicity for subordination algebras has been studied in [17] using topological techniques, in the context of a Sahlqvist-type result obtained in the setting of classical modal logic for a proper subclass of Sahlqvist formulas, referred to as *s-Sahlqvist* formulas. The syntactic shape of s-Sahlqvist formulas guarantees key algebraic/topological properties to their algebraic interpretation, which compensate for the fact that the semantic modal operations on subordination algebras are not defined on its original algebra, but might map elements of it to closed or open elements of its canonical extension.

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As to the problem of obtaining Sahlqvist-type results for certain non-classical logics by reduction to classical Sahlqvist theory by means of GMT-type translations, in [24], the correspondence-via-translation problem has been completely solved for Sahlqvist inequalities in the signature of Distributive Modal Logic, but the corresponding canonicity-via-translation problem, reported to be much harder, was not addressed there, and the canonicity result was obtained following the methodology introduced by Jónsson [31]. In [16], results on both correspondence-via-translation and canonicity-via-translation for inductive inequalities in arbitrary signatures of normal distributive lattice expansions (aka normal DLE-logics) are presented, but the canonicity via translation is restricted to arbitrary normal expansions of bi-Heyting algebras. The source of the additional difficulties was identified in the fact that the algebraic interpretations of the S4-modal operators used to define the GMT-type translations are not defined on each original algebra but might map elements of it to closed or open elements of its canonical extension.

The two independent problems described above have hence a common root in their involving operations on canonical extensions of distributive lattice expansions that do not in general restrict to clopen elements but map clopens to open or to closed elements. These maps, which we refer to as *slanted* maps (cf. Definition 3.1), have been considered in [23, Section 2.3] in the context of a characterization of canonical extensions of maps as continuous extensions w.r.t. certain topologies, but the canonicity theory of term inequalities involving these maps was not developed there; interestingly, examples of maps endowed with these weaker topological properties are the adjoints/residuals of the σ- or π-extensions of normal modal expansions, and their key role in achieving canonicity results, and specifically in extending Jónsson's methodology for canonicity from Sahlqvist to inductive inequalities, was emphasised in [34].

In the present article, we develop the *generalized Sahlqvist-type canonicity* theory for normal LE-logics of arbitrary signature, in a setting in which the algebraic interpretations of the connectives of the expanded signature map elements of the given algebra to closed or open elements of its canonical extension. Interestingly, the class of formulas/inequalities for which this result holds is the class of *analytic inductive* LE-inequalities, introduced in [29] in the context of the theory of analytic calculi in structural proof theory, to characterize the logics which can be presented by means of proper display calculi [37].

Perhaps surprisingly, far from being hard, this generalized canonicity result is obtained as a very smooth refinement of extant generalized Sahlqvist-type canonicity results for LE-logics (cf. [12, 13]), established within unified correspondence theory [9]. One of the main contributions of unified correspondence theory is the introduction of an algebraic and algorithmic approach to the proof of canonicity (and correspondence) results that unifies and uniformly generalizes the methodologies developed by Jónsson [31], Ghilardi-Meloni [27], Sambin-Vaccaro [36], and Conradie-Goranko-Vakarelov [10]. The fact that the algebraic and algorithmic approach extends so smoothly to the present setting (a step-by-step comparison with the algorithmic canonicity results in standard algebras of [12] and [13] is discussed in Section A) is further evidence of its robustness.

The generalized canonicity result obtained in this article is then applied to the two canonicity problems mentioned above. Namely, a strengthening of the canonicity result for subordination algebras of [17] is obtained as a direct application, simply by recognizing that the s-Sahlqvist formulas exactly coincide with the analytic 1-Sahlqvist formulas in the classical normal modal/tense logic signature. Moreover, the canonicity-via-translation result of [16] is extended to normal DLE-logics in arbitrary signatures for a subclass of analytic inductive inequalities referred to as *transferable* (cf. Definition 5.1); the syntactic shape of the formulas in this subclass guarantees that the suitable parametric translation of each formula in this class is analytic inductive, so that the generalized canonicity result obtained in this article applies to them.

Structure of This Article. In Section 2, we collect preliminary notions, facts, and notation on LE-logics, their standard[1] algebraic semantics, canonical extensions of normal LEs, (analytic) Sahlqvist and inductive LE-inequalities, and the algorithm ALBA on analytic inductive LE-inequalities. In Section 3, we introduce slanted LE-algebras and their canonical extensions, define how these structures can serve as a semantic environment for normal LE-logics, and introduce the notion of *slanted canonicity* (or *s-canonicity*, cf. Definition 3.8). In Section 4, we prove the main result of this article, namely that analytic inductive LE-inequalities are s-canonical. In Section 5, we apply the main result of the previous section to extend the transfer result of canonicity to the class of transferable analytic inductive DLE-inequalities. In Section 6, we apply the main result to the setting of subordination algebras to strengthen the canonicity result of [17]. In Section 7, we discuss further directions stemming from the present results. In Section A, we collect the technical lemmas intervening in the proof of our main result.

2 PRELIMINARIES

In this section, we recall the definition of normal LE-logics (for *Lattice Expansion*, see Definition 2.4) and various notions and facts about their algebraic semantics and algorithmic correspondence and canonicity theory. The material presented here re-elaborates [12, Sections 1, 3, 4], [29, Section 3], and [6, Section 5].

2.1 Basic Normal LE-logics

Throughout this article, we will make heavy use of the following auxiliary definition: an *order-type*[2] over $n \in \mathbb{N}$ is an n-tuple $\varepsilon \in \{1, \partial\}^n$. For every order-type ε, we denote its *opposite* order-type by ε^∂, that is, $\varepsilon_i^\partial = 1$ iff $\varepsilon_i = \partial$ for every $1 \leq i \leq n$. For any lattice $\mathbb{A}$, we let $\mathbb{A}^1 := \mathbb{A}$ and $\mathbb{A}^\partial$ be the dual lattice, that is, the lattice associated with the converse partial order of $\mathbb{A}$. For any order-type ε over n, we let $\mathbb{A}^\varepsilon := \Pi_{i=1}^n \mathbb{A}^{\varepsilon_i}$. The language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, from now on abbreviated as $\mathcal{L}_{\mathrm{LE}}$, takes as parameters: (1) a denumerable set PROP of proposition letters, elements of which are denoted p, q, r, possibly with indexes; and (2) disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$. Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (respectively, $n_g \in \mathbb{N}$) and is associated with some order-type ε_f over n_f (respectively, ε_g over n_g).[3] The terms (formulas) of $\mathcal{L}_{\mathrm{LE}}$ are defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi}),$$

where $p \in \mathrm{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$. Note that, to simplify notations, for $\circ \in \mathcal{F} \cup \mathcal{G}$, we will sometimes write $\circ(\overline{\varphi}, \overline{\psi})$ where $\overline{\varphi}$ is used in the coordinates whose order-type is 1 of $\circ$ and $\overline{\psi}$ in the ones whose order-type is ∂. Terms in $\mathcal{L}_{\mathrm{LE}}$ are denoted either by s, t, or by lowercase Greek letters such as φ, ψ, γ, etc. We let $\mathcal{L}_{\mathrm{LE}}^{\leq}$ denote the set of $\mathcal{L}_{\mathrm{LE}}$-*inequalities*, i.e., expressions of the form $\varphi \leq \psi$ where φ, ψ are $\mathcal{L}_{\mathrm{LE}}$-terms, and $\mathcal{L}_{\mathrm{LE}}^{quasi}$ denote the set of $\mathcal{L}_{\mathrm{LE}}$-*quasi-inequalities*, i.e., expressions of the form $(\varphi_1 \leq \psi_1 \,\&\, \cdots \,\&\, \varphi_n \leq \psi_n) \Rightarrow \varphi \leq \psi$ where $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n, \varphi, \psi \in \mathcal{L}_{\mathrm{LE}}$.

Remark 2.1. We assume that the families F and G are disjoint for the sake of generality. As done e.g. in [25], elements in $F \cap G$ can be duplicated, and the two copies treated separately, on the basis of different order-theoretic properties (cf. Remark 2.2).

[1]We warn the reader of a possible clash in terminology with the fuzzy logic literature, where the expression "standard algebras" has a technical meaning. Throughout this article, "standard algebras" refers to the well-known universal-algebraic definition of algebras, and the adjective "standard" is used to emphasise the distinction between algebras as they are usually defined and the slanted algebras introduced in the present article.

[2]Throughout this article, order-types will be typically associated with arrays of variables $\overline{p} := (p_1, \ldots, p_n)$. When the order of the variables in $\overline{p}$ is not specified, we will sometimes abuse notation and write $\varepsilon(p) = 1$ or $\varepsilon(p) = \partial$.

[3]Unary f (respectively, g) connectives will be typically denoted $\Diamond$ (respectively, $\Box$) if their order-type is 1, and $\lhd$ (respectively, $\rhd$) if their order-type is ∂.

Remark 2.2. The purpose of grouping LE-connectives in the families $\mathcal{F}$ and $\mathcal{G}$ is to identify–and refer to–the two types of order-theoretic behaviour which will be relevant for the development of this theory and are specified in Definition 2.4. The order-theoretic properties defining membership in these families are dual to each other, and are such that, roughly speaking, connectives in $\mathcal{F}$ (respectively, $\mathcal{G}$) can be thought of generalized *operators* (respectively, *dual operators*), of which diamond (respectively, box) in modal logic and fusion in substructural logic (respectively, intuitionistic implication) are prime examples. We refer to [12] for an extensive illustration of how this classification can be instantiated in several well known LE-signatures. The order-duality of this classification is, of course, very convenient for presentation purposes, since it allows to recover one half of the relevant proofs from the other half, simply by invoking order-duality. However, there is more to it: as discussed more in detail before Definition 2.8, the order-theoretic properties underlying the definition of the family $\mathcal{F}$ (respectively, $\mathcal{G}$) are strongly linked with the order-theoretic properties of the σ-extensions (respectively, π-extensions) of the algebraic interpretations of connectives in $\mathcal{F}$ (respectively, $\mathcal{G}$), and especially with their properties of *adjunction/residuation*, which in turn are key and have been exploited in other articles (cf., e.g., [29]) also from a *proof-theoretic* perspective for guaranteeing certain key results about proper display calculi (e.g., conservativity, cut elimination) to hold uniformly. Specific to the results we are presently after, these order-theoretic properties also guarantee that these σ- and π-extensions will have sufficient *topological* properties for our main canonicity result to go through.

Definition 2.3. For any language $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, an $\mathcal{L}_{\mathrm{LE}}$-*logic* is a set of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in \mathcal{L}_{\mathrm{LE}}$, which contains the following axioms:

- Sequents for lattice operations:

$$p \vdash p, \qquad \bot \vdash p, \qquad p \vdash \top,$$
$$p \vdash p \vee q, \qquad q \vdash p \vee q, \qquad p \wedge q \vdash p, \qquad p \wedge q \vdash q,$$

- Sequents for each connective $f \in \mathcal{F}$ and $g \in \mathcal{G}$ with $n_f, n_g \geq 1$:

$$f(p_1, \ldots, \bot, \ldots, p_{n_f}) \vdash \bot, \text{ for } \varepsilon_f(i) = 1,$$
$$f(p_1, \ldots, \top, \ldots, p_{n_f}) \vdash \bot, \text{ for } \varepsilon_f(i) = \partial,$$
$$\top \vdash g(p_1, \ldots, \top, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1,$$
$$\top \vdash g(p_1, \ldots, \bot, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial,$$
$$f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \text{ for } \varepsilon_f(i) = 1,$$
$$f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \text{ for } \varepsilon_f(i) = \partial,$$
$$g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1,$$
$$g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \vee q, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial,$$

and is closed under the following inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

where $\varphi(\chi/p)$ denotes uniform substitution of χ for p in φ, and for each connective $f \in \mathcal{F}$ and $g \in \mathcal{G}$,

$$\frac{\varphi \vdash \psi}{f(\varphi_1, \ldots, \varphi, \ldots, \varphi_n) \vdash f(\varphi_1, \ldots, \psi, \ldots, \varphi_n)} \ (\varepsilon_f(i) = 1)$$
$$\frac{\varphi \vdash \psi}{f(\varphi_1, \ldots, \psi, \ldots, \varphi_n) \vdash f(\varphi_1, \ldots, \varphi, \ldots, \varphi_n)} \ (\varepsilon_f(i) = \partial)$$

$$\frac{\varphi \vdash \psi}{g(\varphi_1, \ldots, \varphi, \ldots, \varphi_n) \vdash g(\varphi_1, \ldots, \psi, \ldots, \varphi_n)} \ (\varepsilon_g(i) = 1)$$

$$\frac{\varphi \vdash \psi}{g(\varphi_1, \ldots, \psi, \ldots, \varphi_n) \vdash g(\varphi_1, \ldots, \varphi, \ldots, \varphi_n)} \ (\varepsilon_g(i) = \partial).$$

The minimal $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$-logic is denoted by $\mathbf{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, or simply by $\mathbf{L}_{\mathrm{LE}}$ when $\mathcal{F}$ and $\mathcal{G}$ are clear from the context.

The standard algebraic semantics of LE-logics is given as follows:

Definition 2.4. For any LE-signature $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, an $\mathcal{L}_{\mathrm{LE}}$-*algebra* is a tuple $\mathbb{A} = (A, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ such that A is a bounded lattice, $\mathcal{F}^{\mathbb{A}} = \{f^{\mathbb{A}} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^{\mathbb{A}} = \{g^{\mathbb{A}} \mid g \in \mathcal{G}\}$, such that every $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (respectively, $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is an n_f-ary (respectively, n_g-ary) operation on $\mathbb{A}$. A lattice expansion[4] is *normal* if every $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (respectively, $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) preserves finite (hence also empty) joins (respectively, meets) in each coordinate with $\varepsilon_f(i) = 1$ (respectively, $\varepsilon_g(i) = 1$) and reverses finite (hence also empty) meets (respectively, joins) in each coordinate with $\varepsilon_f(i) = \partial$ (respectively, $\varepsilon_g(i) = \partial$).

In what follows, we will generically refer to algebras in the definition above as LEs when it is not important to emphasize the specific signature, and as $\mathcal{L}_{\mathrm{LE}}$-algebras when it is. Standard LEs as defined above are not the main focus of this article, which is rather the *non-standard* algebraic semantics of normal LE-logics, which we discuss in Section 3.

2.2 Perfect LEs and Standard Canonical Extensions

Definition 2.5. Let A be a (bounded) sublattice of a complete lattice A'.

(1) A is *dense* in A' if every element of A' can be expressed both as a join of meets and as a meet of joins of elements from A.
(2) A is *compact* in A' if, for all $S, T \subseteq A$, if $\bigwedge S \leq \bigvee T$, then $\bigwedge S' \leq \bigvee T'$ for some finite $S' \subseteq S$ and $T' \subseteq T$.
(3) The *canonical extension* of a lattice A is a complete lattice A^δ containing A as a dense and compact sublattice.

For any lattice $\mathbb{A}$, its canonical extension, besides being unique up to an isomorphism fixing $\mathbb{A}$, always exists (cf. [22, Propositions 2.6 and 2.7][5]).

Definition 2.6. A complete lattice A is *perfect* if A is both completely join-generated by the set $J^\infty(A)$ of the completely join-irreducible elements of A, and completely meet-generated by the set $M^\infty(A)$ of the completely meet-irreducible elements of A.

---

[4] Normal LEs are sometimes referred to as *lattices with operators* (LOs). This terminology derives from the setting of Boolean algebras with operators, in which operators are understood as operations which preserve finite (hence also empty) joins in each coordinate. Thanks to the Boolean negation, operators are typically taken as primitive connectives, and all the other modal operations are reduced to these. However, this terminology is somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as $\mathbb{A}^\varepsilon \to \mathbb{A}^\eta$ for some order-type ε on n and some order-type $\eta \in \{1, \partial\}$. Rather than speaking of lattices with (ε, η)-operators, we then speak of normal LEs. This terminology is also used in other articles, e.g., [16]. For the sake of internal consistency, we stick with the name "Boolean Algebra Expansion" in Section 6.

[5] In [22], the proof of the existence of the canonical extension is constructive, and is based on the complete lattice of Galois-stable sets of the polarity (A, X, I), where A and X, respectively, are the sets of filters and ideals of the given lattice, and I the relation of having non-empty intersection, as in the lattice representation of Hartonas and Dunn [30].

Denseness implies that $J^\infty(A^\delta)$ is contained in the meet closure $K(A^\delta)$ of A in A^δ and that $M^\infty(A^\delta)$ is contained in the join closure $O(A^\delta)$ of A in A^δ [20]. The elements of $K(A^\delta)$ are referred to as *closed* elements, and elements of $O(A^\delta)$ as *open* elements. The canonical extension of an LE $\mathbb{A}$ will be defined as a suitable expansion of the canonical extension of the underlying lattice of $\mathbb{A}$. Before turning to this definition, recall that taking the canonical extension of a lattice commutes with taking order-duals and products, namely: $(A^\partial)^\delta = (A^\delta)^\partial$ and $(A_1 \times A_2)^\delta = A_1^\delta \times A_2^\delta$ (cf. [20, Theorem 2.8]). Hence, $(A^n)^\delta$ can be identified with $(A^\delta)^n$ and $(A^\varepsilon)^\delta$ with $(A^\delta)^\varepsilon$ for any order-type ε.

Thanks to these identifications, in order to extend operations of any arity which are monotone or antitone in each coordinate from a lattice $\mathbb{A}$ to its canonical extension, treating the case of *monotone* and *unary* operations suffices:

Definition 2.7. For every unary, order-preserving operation $f : \mathbb{A} \to \mathbb{A}$, the *σ-extension* of f is defined first by declaring, for every $k \in K(A^\delta)$,

$$f^\sigma(k) := \bigwedge\{f(a) \mid a \in \mathbb{A} \text{ and } k \leq a\},$$

and then, for every $u \in A^\delta$,

$$f^\sigma(u) := \bigvee\{f^\sigma(k) \mid k \in K(A^\delta) \text{ and } k \leq u\}.$$

The *π-extension* of f is defined first by declaring, for every $o \in O(A^\delta)$,

$$f^\pi(o) := \bigvee\{f(a) \mid a \in \mathbb{A} \text{ and } a \leq o\},$$

and then, for every $u \in A^\delta$,

$$f^\pi(u) := \bigwedge\{f^\pi(o) \mid o \in O(A^\delta) \text{ and } u \leq o\}.$$

Key to the use of canonical extensions in logic (e.g., for proving semantic completeness via canonicity, as well as for the proof-theoretic results mentioned in Remark 2.2) are certain basic desiderata that canonical extensions must satisfy. These desiderata start from the condition that the canonical extension of a given $\mathcal{L}_{\mathrm{LE}}$-algebra must also be an $\mathcal{L}_{\mathrm{LE}}$-algebra, and that, moreover, the canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra must be a perfect (respectively, complete, in the constructive setting) $\mathcal{L}_{\mathrm{LE}}$-algebra (cf. Definition 2.9 below). The first desideratum is met immediately whenever all connectives in the given signature $\mathcal{L}_{\mathrm{LE}}$ are *smooth*, i.e., their σ- and π-extensions coincide. While *unary* normal connectives are smooth, (see, e.g., [22, Lemma 4.4]), connectives with arity greater than 1 are typically non-smooth (see, e.g., [26, Example 4.6]). Hence, whenever $\mathcal{L}_{\mathrm{LE}}$ includes non-smooth connectives, to satisfy the first desideratum one needs to decide whether to define the canonical extensions of $\mathcal{L}_{\mathrm{LE}}$-algebras by taking, for each connective in $\mathcal{L}_{\mathrm{LE}}$, either its σ- or its π-extension. Our choice is guided by the second desideratum: it is easy to see that the σ- and π-extensions of ε-monotone maps are ε-monotone; moreover, the σ-extension of a map which coordinate-wise preserves *finite* joins or reverses *finite* meets will coordinate-wise preserve *arbitrary* joins or reverse *arbitrary* meets, and dually, the π-extension of a map which coordinate-wise preserves *finite* meets or reverses *finite* joins will coordinate-wise preserve *arbitrary* meets or reverse *arbitrary* joins (see [22, Lemma 4.6]). Therefore, defining the canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra by choosing the σ-extensions of operations in $\mathcal{F}^{\mathbb{A}}$ and the π-extensions of operations in $\mathcal{G}^{\mathbb{A}}$ will guarantee both the first and the second desideratum.[6] These considerations motivate the following:

[6]For some discussion on this and the importance of choosing the appropriate extension, we refer the reader to [24, Section 7].

Definition 2.8. The canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (A, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}^\delta := (A^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ such that $f^{\mathbb{A}^\delta}$ and $g^{\mathbb{A}^\delta}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

As mentioned in the previous discussion, defined as indicated above, the canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ can be shown to be a *perfect* $\mathcal{L}_{\mathrm{LE}}$-algebra:

Definition 2.9. An LE $\mathbb{A} = (A, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is perfect if A is a perfect lattice (cf. Definition 2.6), and moreover the following infinitary distribution laws are satisfied for each $f \in \mathcal{F}$, $g \in \mathcal{G}$, $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$: for every $S \subseteq A$,

$$\begin{array}{llr}
f(x_1, \dots, \bigvee S, \dots, x_{n_f}) = \bigvee\{f(x_1, \dots, x, \dots, x_{n_f}) \mid x \in S\} & \text{if } \varepsilon_f(i) = 1;\\
f(x_1, \dots, \bigwedge S, \dots, x_{n_f}) = \bigvee\{f(x_1, \dots, x, \dots, x_{n_f}) \mid x \in S\} & \text{if } \varepsilon_f(i) = \partial;\\
g(x_1, \dots, \bigwedge S, \dots, x_{n_g}) = \bigwedge\{g(x_1, \dots, x, \dots, x_{n_g}) \mid x \in S\} & \text{if } \varepsilon_g(i) = 1;\\
g(x_1, \dots, \bigvee S, \dots, x_{n_g}) = \bigwedge\{g(x_1, \dots, x, \dots, x_{n_g}) \mid x \in S\} & \text{if } \varepsilon_g(i) = \partial.
\end{array}$$

Before finishing the present subsection, let us spell out and further simplify the definitions of the extended operations. First of all, we recall that taking the order-dual interchanges closed and open elements: $K((A^\delta)^\partial) \cong O(A^\delta)$ and $O((A^\delta)^\partial) \cong K(A^\delta)$; similarly, $K((A^n)^\delta) \cong K(A^\delta)^n$, and $O((A^n)^\delta) \cong O(A^\delta)^n$. Hence, $K((A^\delta)^\varepsilon) \cong \prod_i K(A^\delta)^{\varepsilon(i)}$ and $O((A^\delta)^\varepsilon) \cong \prod_i O(A^\delta)^{\varepsilon(i)}$ for every LE $\mathbb{A}$ and every order-type ε on any $n \in \mathbb{N}$, where

$$K(A^\delta)^{\varepsilon(i)} := \begin{cases} K(A^\delta) & \text{if } \varepsilon(i) = 1 \\ O(A^\delta) & \text{if } \varepsilon(i) = \partial \end{cases} \qquad O(A^\delta)^{\varepsilon(i)} := \begin{cases} O(A^\delta) & \text{if } \varepsilon(i) = 1 \\ K(A^\delta) & \text{if } \varepsilon(i) = \partial. \end{cases}$$

Denoting by $\leq^\varepsilon$ the product order on $(A^\delta)^\varepsilon$, we have for every $f \in \mathcal{F}$, $g \in \mathcal{G}$, $\overline{k} \in K((A^\delta)^{\varepsilon_f})$, $\overline{o} \in O((A^\delta)^{\varepsilon_f})$ $\overline{u} \in (A^\delta)^{n_f}$ and $\overline{v} \in (A^\delta)^{n_g}$,

$$\begin{array}{ll}
f^\sigma(\overline{k}) := \bigwedge\{f(\overline{a}) \mid \overline{a} \in A^{\varepsilon_f} \text{ and } \overline{k} \leq^{\varepsilon_f} \overline{a}\} & f^\sigma(\overline{u}) := \bigvee\{f^\sigma(\overline{k}) \mid \overline{k} \in K((A^\delta)^{\varepsilon_f}) \text{ and } \overline{k} \leq^{\varepsilon_f} \overline{u}\}\\
g^\pi(\overline{o}) := \bigvee\{g(\overline{a}) \mid \overline{a} \in A^{\varepsilon_g} \text{ and } \overline{a} \leq^{\varepsilon_g} \overline{o}\} & g^\pi(\overline{v}) := \bigwedge\{g^\pi(\overline{o}) \mid \overline{o} \in O((A^\delta)^{\varepsilon_g}) \text{ and } \overline{v} \leq^{\varepsilon_g} \overline{o}\}.
\end{array}$$

The algebraic completeness of $\mathbf{L}_{\mathrm{LE}}$ and the canonical embedding of LEs into their canonical extensions immediately yield completeness of $\mathbf{L}_{\mathrm{LE}}$ w.r.t. the appropriate class of perfect LEs.

2.3 Inductive and Sahlqvist (Analytic) LE-Inequalities

In this section, we recall the definitions of inductive and Sahlqvist LE-inequalities introduced in [12] and their corresponding "analytic" restrictions introduced in [29] in the distributive setting and then generalized to the setting of LEs of arbitrary signatures in [28]. Each inequality in any of these classes is canonical and elementary (cf. [12, Theorems 8.8 and 8.9]).

Definition 2.10 (Signed Generation Tree). The *positive* (respectively, *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{LE}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (respectively, $-$), and then propagating the labelling on each remaining node as follows:

- For any node labelled with $\vee$ or $\wedge$, assign the same sign to its children nodes.
- For any node labelled with $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$, and for any $1 \leq i \leq n_h$, assign the same (respectively, the opposite) sign to its ith child node if $\varepsilon_h(i) = 1$ (respectively, if $\varepsilon_h(i) = \partial$).

Nodes in signed generation trees are *positive* (respectively, *negative*) if are signed $+$ (respectively, $-$).

Signed generation trees will be mostly used in the context of term inequalities $s \leq t$. In this context, we will typically consider the positive generation tree $+s$ for the left-hand side and the

Table 1. Skeleton and PIA Nodes for LE

Skeleton	PIA
Δ-adjoints	Syntactically Right Adjoint (SRA)
$+$ $\vee$	$+$ $\wedge$ g with $n_g = 1$
$-$ $\wedge$	$-$ $\vee$ f with $n_f = 1$
Syntactically Left Residual (SLR)	Syntactically Right Residual (SRR)
$+$ f with $n_f \geq 1$	$+$ g with $n_g \geq 2$
$-$ g with $n_g \geq 1$	$-$ f with $n_f \geq 2$

negative one $-t$ for the right-hand side. We will also say that a term-inequality $s \leq t$ is *uniform* in a given variable p if all occurrences of p in both $+s$ and $-t$ have the same sign, and that $s \leq t$ is ε*-uniform* in a (sub)array $\overline{p}$ of its variables if $s \leq t$ is uniform in every p in $\overline{p}$, occurring with the sign indicated by ε. With a routine proof by induction, one can show that if a term-inequality $s \leq t$ is 1-uniform (respectively, ∂-uniform) in a given variable p, then the term-function associated with s (see Definitions 2.4 and 3.1) is order-preserving (respectively, order-reversing) in p, and the term-function associated with t is order-reversing (respectively, order-preserving) in p. Therefore, the validity of $s(p) \leq t(p)$ is equivalent to the validity of $s(\top) \leq t(\top)$ (respectively, $s(\bot) \leq t(\bot)$).[7] This observation is easily generalised to an arbitrary subarray of variables of $s \leq t$.

For any term $s(p_1, \ldots p_n)$, any order-type ε over n, and any $1 \leq i \leq n$, an ε*-critical node* in a signed generation tree of s is a leaf node $+p_i$ if $\varepsilon(i) = 1$ and $-p_i$ if $\varepsilon(i) = \partial$. An ε*-critical branch* in the tree is a branch from an ε-critical node. Variable occurrences corresponding to ε-critical nodes are to be solved for (cf. Section 2.6).

For every term $s(p_1, \ldots p_n)$ and every order-type ε, we say that $+s$ (respectively, $-s$) *agrees with* ε, and write $\varepsilon(+s)$ (respectively, $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (respectively, $-s$) is ε-critical. We will also write $+s' \prec *s$ (respectively, $-s' \prec *s$) to indicate that the subterm s' inherits the positive (respectively, negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(\gamma) \prec *s$ (respectively, $\varepsilon^\partial(\gamma) \prec *s$) to indicate that the signed subtree γ, with the sign inherited from $*s$, agrees with ε (respectively, with ε^∂).

We will write $\varphi(!x)$ (respectively, $\varphi(!\overline{x})$) to indicate that the variable x (respectively, each variable x in $\overline{x}$) occurs exactly once in φ. Accordingly, we will write $\varphi[\gamma/!x]$ (respectively, $\varphi[\overline{\gamma}/!\overline{x}]$) to indicate the formula obtained by substituting γ (respectively, each term γ in $\overline{\gamma}$) for the unique occurrence of (its corresponding variable) x in φ.

Definition 2.11. Nodes in signed generation trees will be called Δ*-adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes. A branch is *excellent* if it is good and in P_1 there are only SRA-nodes. A good branch is *Skeleton* if the length of P_1 is 0 (hence, Skeleton branches are excellent), and is *SLR*, or *definite*, if P_2 only contains SLR nodes.

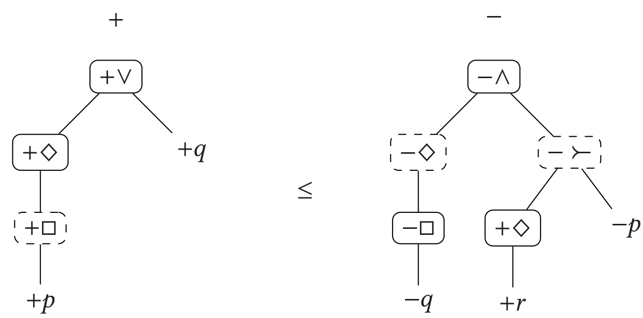
We refer to [12, Remark 3.3] and [24, Section 3] for a discussion about the notational conventions and terminology.

[7] As noticed in [11, Remark 6.3], these equivalences are in fact instances of the Ackermann Lemmas where the p-variant assignment v' is such that $v'(p) = \top$ (if $s \leq t$ is 1-uniform) or $v'(p) = \bot$ (if $s \leq t$ is ∂-uniform).

Example 2.12. The language $\mathcal{L}_{\text{LE}}$ of bi-intuitionistic modal logic is obtained by instantiating $\mathcal{F} = \{\Diamond, \succ\}$ and $\mathcal{G} = \{\Box, \to\}$ with $n_\Diamond = n_\Box = 1$, $n_\succ = n_\to = 2$ and $\varepsilon_\Diamond = \varepsilon_\Box = 1$, $\varepsilon_\succ = \varepsilon_\to = (\partial, 1)$. In this language, the signed generation trees associated with the inequality

$$\Diamond\Box p \vee q \leq \Diamond\Box q \wedge (\Diamond r \succ p)$$

are represented in the following diagram, where PIA nodes occur inside dashed rectangles and Skeleton nodes inside continuous ones.



The inequality above is not uniform in p and q (each of the variables has a positive occurrence and a negative one), but is uniform in r (in this particular case because r occurs only once). If we consider the order-type ε on (p, q, r) given by $\varepsilon = (1, 1, \partial)$, the critical nodes in the generations trees are $+p$ and $+q$. There is no ε-critical occurrence of r. The term $+s := +(\Diamond\Box p \vee q)$ agrees with ε, while the term $-t = -(\Diamond\Box q \wedge (\Diamond r \succ p))$ agrees with ε^∂. Now, for the subterms $t' := \Diamond r$ and $t'' := \Diamond r \succ p$ of t, we have $+t' \prec -t$ and $-t'' \prec -t$. Moreover, $\varepsilon^\partial(t') \prec -t$ and $\varepsilon^\partial(t'') \prec -t$.

The branches which end in $+p$, $+q$, and $-p$ are good since, traversing the corresponding branches starting from the root, we first encounter Skeleton nodes and then only PIA nodes. The branches ending in $+p$ and $+q$ are also excellent because they do not contain SRR nodes (the only SRR node occurring in this example is $-\succ$). The branch which ends in $+q$ is in particular Skeleton since it contains no occurrences of PIA nodes (the length of P_1 is 0). Finally, the branches which end in $-q$ and $+r$ are not good, since (again starting from the root) a Skeleton node ($-\Box$, $+\Diamond$) occurs in the scope of a PIA node ($-\Diamond$, $-\succ$).

Definition 2.13 (Inductive Inequalities). For any order-type ε and any irreflexive and transitive relation (i.e., strict partial order) Ω on $p_1, \ldots, p_n$, the signed generation tree $*s$ ($* \in \{-, +\}$) of a term $s(p_1, \ldots, p_n)$ is (Ω, ε)-*inductive* if

(1) for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good (cf. Definition 2.11);
(2) every m-ary SRR-node occurring in the critical branch is of the form

$$\circledast(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1} \ldots, \gamma_m),$$

where for any $h \in \{1, \ldots, m\} \setminus j$:
   (a) $\varepsilon^\partial(\gamma_h) \prec *s$ (cf. discussion before Definition 2.11), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in γ_h and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω, ε)-*inductive* if the signed generation trees $+s$ and $-t$ are (Ω, ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε)-inductive for some Ω and ε.

In what follows, we refer to formulas φ such that only PIA nodes occur in $+\varphi$ (respectively, $-\varphi$) as *positive* (respectively, *negative*) *PIA-formulas*, and to formulas ξ such that only Skeleton nodes occur in $+\xi$ (respectively, $-\xi$) as *positive* (respectively, *negative*) *Skeleton-formulas*. PIA formulas $*\varphi$ in which no nodes $+\wedge$ and $-\vee$ occur are referred to as *definite*. Skeleton formulas $*\xi$ in which no nodes $-\wedge$ and $+\vee$ occur are referred to as *definite*.

Definition 2.14. For an order-type ε, the signed generation tree $*s$, $* \in \{-,+\}$, of a term $s(p_1,\ldots,p_n)$ is *ε-Sahlqvist* if every ε-critical branch is excellent. An inequality $s \leq t$ is *ε-Sahlqvist* if the trees $+s$ and $-t$ are both ε-Sahlqvist. An inequality $s \leq t$ is *Sahlqvist* if it is ε-Sahlqvist for some ε.

Definition 2.15 (Analytic inductive and analytic Sahlqvist inequalities). For every order-type ε and every irreflexive and transitive relation Ω on the variables $p_1,\ldots,p_n$, the signed generation tree $*s$ ($* \in \{+,-\}$) of a term $s(p_1,\ldots,p_n)$ is *analytic* (Ω,ε)*-inductive* (respectively, *analytic ε-Sahlqvist*) if

(1) $*s$ is (Ω,ε)-inductive (respectively, ε-Sahlqvist);
(2) every branch of $*s$ is good (cf. Definition 2.11).

An inequality $s \leq t$ is *analytic* (Ω,ε)*-inductive* (respectively, *analytic ε-Sahlqvist*) if $+s$ and $-t$ are both analytic (Ω,ε)-inductive (respectively, analytic ε-Sahlqvist). An inequality $s \leq t$ is *analytic inductive* (respectively, *analytic Sahlqvist*) if is analytic (Ω,ε)-inductive (respectively, analytic ε-Sahlqvist) for some Ω and ε (respectively, for some ε).

Example 2.16. In light of the previous definitions and the discussion in Example 2.12, the modal bi-intuitionistic inequality $\Diamond\Box p \vee q \leq \Diamond\Box q \wedge (\Diamond r \succ p)$ is ε-Sahlqvist (and hence inductive) for $\varepsilon(p,q,r) = (1,1,\partial)$. However, it is not analytic since the negative generation tree of its right-hand side contains branches which are not good.

Notation 2.17. Following [6], we will sometimes represent (Ω,ε)-analytic inductive inequalities as follows:

$$(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}],$$

where $(\varphi \leq \psi)[!\overline{x}, !\overline{y}, !\overline{z}, !\overline{w}]$ is the Skeleton of the given inequality, $\overline{\alpha}$ (respectively, $\overline{\beta}$) denotes the positive (respectively, negative) maximal PIA-subformulas, i.e., each α in $\overline{\alpha}$ and β in $\overline{\beta}$ contains at least one ε-critical occurrence of some propositional variable, and moreover:

(1) for each $\alpha \in \overline{\alpha}$, either $+\alpha \prec +\varphi$ or $+\alpha \prec -\psi$;
(2) for each $\beta \in \overline{\beta}$, either $-\beta \prec +\varphi$ or $-\beta \prec -\psi$,

and $\overline{\gamma}$ (respectively, $\overline{\delta}$) denotes the positive (respectively, negative) maximal ε^∂-subformulas, i.e.:

(1) for each $\gamma \in \overline{\gamma}$, either $+\gamma \prec +\varphi$ or $+\gamma \prec -\psi$;
(2) for each $\delta \in \overline{\delta}$, either $-\delta \prec +\varphi$ or $-\delta \prec -\psi$.

For the sake of a more compact notation, in what follows we sometimes write $(\varphi \leq \psi)[\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}]$ in place of

$$(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}].$$

Remark 2.18 (The Distributive Setting). When interpreting LE-languages on perfect distributive lattice expansions (DLEs), the logical disjunction is interpreted by means of the coordinate-wise completely $\wedge$-preserving join operation of the lattice, and the logical conjunction with the coordinate-wise completely $\vee$-preserving meet operation of the lattice. Hence, we are justified in listing $+\wedge$ and $-\vee$ among the SLRs, and $+\vee$ and $-\wedge$ among the SRRs, as is done in Table 2.

Consequently, we obtain enlarged classes of Sahlqvist and inductive inequalities by simply applying Definitions 2.11, 2.14, and 2.13, with respect to Table 2.

Table 2. Skeleton and PIA Nodes for $\mathcal{L}_{\mathrm{DLE}}$

Skeleton	PIA
Δ-adjoints	SRA
$+\ \ \vee$	$+\ \ \wedge\ \ g$ with $n_g = 1$
$-\ \ \wedge$	$-\ \ \vee\ \ f$ with $n_f = 1$
SLR	SRR
$+\ \ \wedge\ \ f$ with $n_f \geq 1$	$+\ \ \vee\ \ g$ with $n_g \geq 2$
$-\ \ \vee\ \ g$ with $n_g \geq 1$	$-\ \ \wedge\ \ f$ with $n_f \geq 2$

2.4 Basic LE-language Expanded with Residuals

We now introduce an expansion of the language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ with connectives which are to be interpreted as the residuals in each coordinate (cf. Definition 2.20) of the connectives in $\mathcal{F}$ and $\mathcal{G}$. This is the first of two expansion steps (the second of which being described in Section 2.5) which lead to the language $\mathcal{L}^+_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ (see Section 2.5) in which the ALBA-reductions take place.

Formally, any given language $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ can be associated with the language $\mathcal{L}^*_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}^*, \mathcal{G}^*)$, where $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{G}^* \supseteq \mathcal{G}$ are obtained by expanding $\mathcal{L}_{\mathrm{LE}}$ with the following connectives:

(1) the n_f-ary connective $f^\sharp_i$ for $1 \leq i \leq n_f$, the intended interpretation of which is the right residual of $f \in \mathcal{F}$ in its ith coordinate if $\varepsilon_f(i) = 1$ (respectively, its Galois-adjoint if $\varepsilon_f(i) = \partial$);
(2) the n_g-ary connective $g^\flat_i$ for $1 \leq i \leq n_g$, the intended interpretation of which is the left residual of $g \in \mathcal{G}$ in its ith coordinate if $\varepsilon_g(i) = 1$ (respectively, its Galois-adjoint if $\varepsilon_g(i) = \partial$).

We stipulate that $f^\sharp_i \in \mathcal{G}^*$ if $\varepsilon_f(i) = 1$, and $f^\sharp_i \in \mathcal{F}^*$ if $\varepsilon_f(i) = \partial$. Dually, $g^\flat_i \in \mathcal{F}^*$ if $\varepsilon_g(i) = 1$, and $g^\flat_i \in \mathcal{G}^*$ if $\varepsilon_g(i) = \partial$. The order-type assigned to the additional connectives is predicated on the order-type of their intended interpretations. That is, for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$,

(1) if $\varepsilon_f(i) = 1$, then $\varepsilon_{f^\sharp_i}(i) = 1$ and $\varepsilon_{f^\sharp_i}(j) = \varepsilon^\partial_f(j)$ for any $j \neq i$;
(2) if $\varepsilon_f(i) = \partial$, then $\varepsilon_{f^\sharp_i}(i) = \partial$ and $\varepsilon_{f^\sharp_i}(j) = \varepsilon_f(j)$ for any $j \neq i$;
(3) if $\varepsilon_g(i) = 1$, then $\varepsilon_{g^\flat_i}(i) = 1$ and $\varepsilon_{g^\flat_i}(j) = \varepsilon^\partial_g(j)$ for any $j \neq i$;
(4) if $\varepsilon_g(i) = \partial$, then $\varepsilon_{g^\flat_i}(i) = \partial$ and $\varepsilon_{g^\flat_i}(j) = \varepsilon_g(j)$ for any $j \neq i$.

For instance, if f and g are binary connectives such that $\varepsilon_f = (1, \partial)$ and $\varepsilon_g = (\partial, 1)$, then $\varepsilon_{f^\sharp_1} = (1, 1)$, $\varepsilon_{f^\sharp_2} = (1, \partial)$, $\varepsilon_{g^\flat_1} = (\partial, 1)$ and $\varepsilon_{g^\flat_2} = (1, 1)$.

Remark 2.19. We warn the reader that the notation introduced above depends on which connective is taken as primitive, and needs to be carefully adapted to well-known cases. For instance, consider the "usion" connective $\circ$ (which, when denoted as f, is such that $\varepsilon_f = (1, 1)$). Its residuals $f^\sharp_1$ and $f^\sharp_2$ are commonly denoted / and \, respectively. However, if \ is taken as the primitive connective g, then $g^\flat_2$ is $\circ = f$, and $g^\flat_1(x_1, x_2) := x_2/x_1 = f^\sharp_1(x_2, x_1)$. This example shows that, when identifying $g^\flat_1$ and $f^\sharp_1$, the conventional order of the coordinates is not preserved, and depends on which connective is taken as primitive.

Definition 2.20. For any language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, the *basic $\mathcal{L}_{\mathrm{LE}}$-logic with residuals* is defined by specializing Definition 2.3 to the language $\mathcal{L}^*_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}^*, \mathcal{G}^*)$ and closing under the following residuation rules for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ with $n_f, n_g \geq 1$:

$$(\varepsilon_f(i)=1)\ \frac{(\varphi_1,\ldots,\varphi,\ldots,\varphi_{n_f})\vdash\psi}{\varphi\vdash f_i^\sharp(\varphi_1,\ldots,\psi,\ldots,\varphi_{n_f})}\qquad\frac{\varphi\vdash g(\varphi_1,\ldots,\psi,\ldots,\varphi_{n_g})}{\varphi\vdash g(\varphi_1,\ldots,\psi,\ldots,\varphi_{n_g})}\ (\varepsilon_g(i)=1)$$

$$(\varepsilon_f(i)=\partial)\ \frac{f(\varphi_1,\ldots,\varphi,\ldots,\varphi_{n_f})\vdash\psi}{f_i^\sharp(\varphi_1,\ldots,\psi,\ldots,\varphi_{n_f})\vdash\varphi}\qquad\frac{\varphi\vdash g(\varphi_1,\ldots,\psi,\ldots,\varphi_{n_g})}{\psi\vdash g_i^\flat(\varphi_1,\ldots,\varphi,\ldots,\varphi_{n_g})}\ (\varepsilon_g(i)=\partial).$$

The double line in each rule above indicates that the rule should be read both top-to-bottom and bottom-to-top. Let $\mathbf{L}_{\mathrm{LE}}^*$ be the minimal basic $\mathcal{L}_{\mathrm{LE}}$-logic with residuals. For any language $\mathcal{L}_{\mathrm{LE}}$, by an $\mathcal{L}_{\mathrm{LE}}$*-logic with residuals* we understand any axiomatic extension of the basic $\mathcal{L}_{\mathrm{LE}}$-logic with residuals in $\mathcal{L}_{\mathrm{LE}}^*$.

The algebraic semantics of $\mathbf{L}_{\mathrm{LE}}^*$ is given by the class of $\mathcal{L}_{\mathrm{LE}}$-algebras with residuals, defined as tuples $\mathbb{A}=(A,\mathcal{F}^*,\mathcal{G}^*)$ such that A is a lattice, and moreover,

(1) for every $f\in\mathcal{F}$ s.t. $n_f\geq 1$, all $a_1,\ldots,a_{n_f}\in A$ and $b\in A$, and each $1\leq i\leq n_f$,
   - if $\varepsilon_f(i)=1$, then $f(a_1,\ldots,a_i,\ldots a_{n_f})\leq b$ iff $a_i\leq f_i^\sharp(a_1,\ldots,b,\ldots,a_{n_f})$;
   - if $\varepsilon_f(i)=\partial$, then $f(a_1,\ldots,a_i,\ldots a_{n_f})\leq b$ iff $a_i\leq^\partial f_i^\sharp(a_1,\ldots,b,\ldots,a_{n_f})$.

   We say that $f_i^\sharp$ is the *right residual* of f in its ith coordinate.

(2) for every $g\in\mathcal{G}$ s.t. $n_g\geq 1$, any $a_1,\ldots,a_{n_g}\in A$ and $b\in A$, and each $1\leq i\leq n_g$,
   - if $\varepsilon_g(i)=1$, then $b\leq g(a_1,\ldots,a_i,\ldots a_{n_g})$ iff $g_i^\flat(a_1,\ldots,b,\ldots,a_{n_g})\leq a_i$.
   - if $\varepsilon_g(i)=\partial$, then $b\leq g(a_1,\ldots,a_i,\ldots a_{n_g})$ iff $g_i^\flat(a_1,\ldots,b,\ldots,a_{n_g})\leq^\partial a_i$.

   We say that $g_i^\flat$ is the *left residual* of g in its ith coordinate.

It is also routine to prove using the Lindenbaum-Tarski construction that $\mathbf{L}_{\mathrm{LE}}^*$ (as well as any of its axiomatic extensions) is sound and complete w.r.t. the class of $\mathcal{L}_{\mathrm{LE}}$-algebras with residuals (w.r.t. the suitably defined equational subclass, respectively).

Definition 2.21. For every definite positive PIA $\mathcal{L}_{\mathrm{LE}}$-formula $\varphi=\varphi(!x,\overline{z})$, and any definite negative PIA $\mathcal{L}_{\mathrm{LE}}$-formula $\psi=\psi(!x,\overline{z})$ such that x occurs in them exactly once, the $\mathcal{L}_{\mathrm{LE}}^*$-formulas $\mathsf{LA}(\varphi)(u,\overline{z})$ and $\mathsf{RA}(\psi)(u,\overline{z})$ (for $u\in Var-(x\cup\overline{z})$) are defined by simultaneous recursion as follows:

$$\begin{array}{rcl}
\mathsf{LA}(x) & = & u;\\
\mathsf{LA}(g(\overline{\varphi_{-j}(\overline{z})},\varphi_j(x,\overline{z}),\overline{\psi(\overline{z})})) & = & \mathsf{LA}(\varphi_j)(g_j^\flat(\overline{\varphi_{-j}(\overline{z})},u,\overline{\psi(\overline{z})}),\overline{z});\\
\mathsf{LA}(g(\overline{\varphi(\overline{z})},\overline{\psi_{-j}(\overline{z})},\psi_j(x,\overline{z}))) & = & \mathsf{RA}(\psi_j)(g_j^\flat(\overline{\varphi(\overline{z})},\overline{\psi_{-j}(\overline{z})},u),\overline{z});\\
\\
\mathsf{RA}(x) & = & u;\\
\mathsf{RA}(f(\overline{\psi_{-j}(\overline{z})},\psi_j(x,\overline{z}),\overline{\varphi(\overline{z})})) & = & \mathsf{RA}(\psi_j)(f_j^\sharp(\overline{\psi_{-j}(\overline{z})},u,\overline{\varphi(\overline{z})}),\overline{z});\\
\mathsf{RA}(f(\overline{\psi(\overline{z})},\overline{\varphi_{-j}(\overline{z})},\varphi_j(x,\overline{z}))) & = & \mathsf{LA}(\varphi_j)(f_j^\sharp(\overline{\psi(\overline{z})},\overline{\varphi_{-j}(\overline{z})},u),\overline{z}).
\end{array}$$

Above, $\overline{\varphi_{-j}}$ denotes the vector obtained by removing the jth coordinate of $\overline{\varphi}$.

Lemma 2.22. *For every definite positive PIA $\mathcal{L}_{\mathrm{LE}}$-formula $\varphi=\varphi(!x,\overline{z})$, and any definite negative PIA $\mathcal{L}_{\mathrm{LE}}$-formula $\psi=\psi(!x,\overline{z})$ such that x occurs in them exactly once,*

(1) *if $+x\prec+\varphi$, then $\mathsf{LA}(\varphi)(u,\overline{z})$ is monotone in u and for each z in $\overline{z}$, $\mathsf{LA}(\varphi)(u,\overline{z})$ has the opposite polarity to the polarity of φ in z;*
(2) *if $-x\prec+\varphi$, then $\mathsf{LA}(\varphi)(u,\overline{z})$ is antitone in u and for each z in $\overline{z}$, $\mathsf{LA}(\varphi)(u,\overline{z})$ has the same polarity as φ in z;*

(3) *if* $+x \prec +\psi$, *then* $\mathsf{RA}(\psi)(u, \overline{z})$ *is monotone in* u *and for each* z *in* $\overline{z}$, $\mathsf{RA}(\psi)(u, \overline{z})$ *has the opposite polarity to the polarity of* ψ *in* z*;*

(4) *if* $-x \prec +\psi$, *then* $\mathsf{RA}(\psi)(u, \overline{z})$ *is antitone in* u *and for each* z *in* $\overline{z}$, $\mathsf{RA}(\psi)(u, \overline{z})$ *has the same polarity as* ψ *in* z*.*

Proof. By simultaneous induction on φ and ψ. If $\varphi = \psi = x$, then the assumptions of items (1) and (3) are satisfied; then $\mathsf{RA}(\psi) = \mathsf{LA}(\varphi) = u$ is clearly monotone in u and the second part of the statement is vacuously satisfied. As to the inductive step, if $\varphi(!x, \overline{z}) = g(\overline{\varphi'_{-j}(\overline{z})}, \varphi'_j(x, \overline{z}), \overline{\psi'(\overline{z})})$, with each φ' in $\overline{\varphi'}$ being positive PIA and each ψ' in $\overline{\psi'}$ being negative PIA, then $g^\flat_j \in \mathcal{F}^*$ is monotone in its jth coordinate and has the opposite polarity of ε_g in all the other coordinates. Hence, $g^\flat_j(\overline{\varphi'_{-j}(\overline{z})}, u, \overline{\psi'(\overline{z})})$ has the opposite polarity of $\varphi(!x, \overline{z})$ in each z in $\overline{z}$. Two cases can occur: (a) If $+x \prec +\varphi_j$, then by induction hypothesis, $\mathsf{LA}(\varphi_j)(u', \overline{z})$ is monotone in u', and has the opposite polarity of φ_j in every z in $\overline{z}$. Hence,

$$\mathsf{LA}(\varphi) = \mathsf{LA}(\varphi_j)(g^\flat_j(\overline{\varphi'_{-j}(\overline{z})}, u, \overline{\psi'(\overline{z})})/u', \overline{z})$$

is monotone in u and has the opposite polarity to the polarity of φ in each z in $\overline{z}$. (b) If $-x \prec +\varphi_j$, then by induction hypothesis, $\mathsf{LA}(\varphi_j)(u', \overline{z})$ is antitone in u', and has the same polarity as φ_j in every z in $\overline{z}$. Hence,

$$\mathsf{LA}(\varphi) = \mathsf{LA}(\varphi_j)(g^\flat_j(\overline{\varphi'_{-j}(\overline{z})}, u, \overline{\psi'(\overline{z})})/u', \overline{z})$$

is antitone in u and has the same polarity as φ in each z in $\overline{z}$. The remaining cases are $\varphi := g(\overline{\varphi'(\overline{z})}, \overline{\psi'_{-h}(\overline{z})}, \psi_h(x, \overline{z}))$, $\psi := f(\overline{\varphi'_{-j}(\overline{z})}, \varphi'_j(x, \overline{z}), \overline{\psi'(\overline{z})})$, and $\psi := f(\overline{\varphi'(\overline{z})}, \overline{\psi'_{-h}(\overline{z})}, \psi'_h(x, \overline{z}))$ and are shown in a similar way. ◻

2.5 The Language of Non-Distributive ALBA

The expanded language of perfect LEs will include the connectives corresponding to all the residual of the original connectives, as well as a denumerably infinite set of sorted variables NOM called *nominals*, ranging over the completely join-irreducible elements of perfect LEs (or, constructively, on the closed elements of the constructive canonical extensions, as in [13]), and a denumerably infinite set of sorted variables CO-NOM, called *co-nominals*, ranging over the completely meet-irreducible elements of perfect LEs (or, constructively on the open elements of the constructive canonical extensions). The elements of NOM will be denoted with $\mathbf{i}, \mathbf{j}$, possibly indexed, and those of CO-NOM with $\mathbf{m}, \mathbf{n}$, possibly indexed.

Let us introduce the expanded language formally: the *formulas* φ of $\mathcal{L}^+_{\mathrm{LE}}$ are given by the following recursive definition:

$$\varphi ::= \quad \mathbf{j} \mid \mathbf{m} \mid \psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

with $\psi \in \mathcal{L}_{\mathrm{LE}}$, $\mathbf{j} \in \mathsf{NOM}$ and $\mathbf{m} \in \mathsf{CO\text{-}NOM}$, $f \in \mathcal{F}^*$ and $g \in \mathcal{G}^*$. As in the case of $\mathcal{L}_{\mathrm{LE}}$, we can form inequalities and quasi-inequalities based on $\mathcal{L}^+_{\mathrm{LE}}$. If $\mathbb{A}$ is a perfect LE, then an *assignment* for $\mathcal{L}^+_{\mathrm{LE}}$ on $\mathbb{A}$ is a map $V : \mathsf{PROP} \cup \mathsf{NOM} \cup \mathsf{CO\text{-}NOM} \to \mathbb{A}$ sending propositional variables to elements of $\mathbb{A}$, sending nominals to $J^\infty(\mathbb{A})$ and co-nominals to $M^\infty(\mathbb{A})$. For any LE $\mathbb{A}$, an *admissible assignment* for $\mathcal{L}^+_{\mathrm{LE}}$ on $\mathbb{A}$ is an assignment V for $\mathcal{L}^+_{\mathrm{LE}}$ on $\mathbb{A}^\delta$, such that $V(p) \in \mathbb{A}$ for each $p \in \mathsf{PROP}$. In other words, the assignment V sends propositional variables to elements of the subalgebra $\mathbb{A}$, while nominals and co-nominals get sent to the completely join-irreducible (respectively, closed) and the completely meet-irreducible (respectively, open) elements of $\mathbb{A}^\delta$, respectively.

2.6 Non-Distributive ALBA on Analytic Inductive LE-inequalities

In this subsection, we describe a successful ALBA-run on an analytic (Ω, ε)-inductive $\mathcal{L}_{\mathrm{LE}}$-inequality $\varphi \leq \psi$. The procedure described below serves *both* to compute the first order correspondent of the given inequality in various semantic settings, as discussed, e.g., in [8], [11], [12], and [14], *and* to compute the shape of the analytic structural rules corresponding to the given inequality, as discussed in [6] and [29] .

The run proceeds in three stages. The first stage preprocesses $\varphi \leq \psi$ by eliminating all uniformly occurring propositional variables, and applying distribution and splitting rules exhaustively. This produces a finite set of inequalities, $\varphi_i' \leq \psi_i'$, $1 \leq i \leq n$, from which ALBA forms the *initial quasi-inequalities*.

The second stage (called the *reduction stage*) transforms the quasi-inequalities through the application of transformation rules, which are listed below. The aim is to eliminate all propositional variables in favour of terms built from constants, nominals, and co-nominals (for an expanded discussion on the general reduction strategy, the reader is referred to [9] and [15]). A system for which this has been done will be called *pure* or *purified*. The actual eliminations are effected through the Ackermann-rules, while the other rules are used to bring the quasi-inequalities into the appropriate shape which make these applications possible.

The third stage either reports failure if some system could not be purified, or else returns the conjunction of the pure quasi-inequalities which we denote by ALBA($\varphi \leq \psi$). We now outline each of the three stages in more detail.

2.7 Stage 1: Preprocessing and Initialization

ALBA Receives an Analytic (Ω, ε)-inductive $\mathcal{L}_{\mathrm{LE}}$-inequality $\varphi \leq \psi$ as input. It applies the following *rules for elimination of monotone variables* to $\varphi \leq \psi$ exhaustively, in order to eliminate any propositional variables which occur uniformly:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\top) \leq \beta(\top)} \qquad \frac{\gamma(p) \leq \delta(p)}{\gamma(\bot) \leq \delta(\bot)}$$

for $\alpha(p) \leq \beta(p)$ 1-uniform in p and $\gamma(p) \leq \delta(p)$ ∂-uniform in p, respectively (see the discussion after Definition 2.10).

Next, ALBA exhaustively distributes $f \in \mathcal{F}$ over $+\vee$ in its positive coordinates and over $-\wedge$ in its negative coordinates, and $g \in \mathcal{G}$ over $-\wedge$ in its positive coordinates and over $+\vee$ in its negative coordinates, so as to bring occurrences of $+\vee$ and $-\wedge$ to the surface wherever this is possible, and then eliminate them via exhaustive applications of *splitting* rules.

Splitting-Rules.

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

This gives rise to a set of *definite* analytic inductive inequalities $\{\varphi_i' \leq \psi_i' \mid 1 \leq i \leq n\}$, each of which will be treated separately.

Next, in each PIA-subformula of each such definite analytic inductive inequality, ALBA exhaustively distributes $-f \in \mathcal{F}$ over $-\vee$ in its positive coordinates and over $+\wedge$ in its negative coordinates, and $+g \in \mathcal{G}$ over $+\wedge$ in its positive coordinates and over $-\vee$ in its negative coordinates, so as to bring occurrences of $-\vee$ and $+\wedge$ as close as possible to the root of each PIA subformula. Let $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$ denote one of the inequalities resulting from this step (we suppress the indices). Now ALBA transforms $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$ into the following *initial quasi-inequality* (the soundness of these steps on perfect LEs, or constructive canonical extensions, has been discussed in [12, Section 6] and [13, Section 5]):

$$\forall \bar{\mathbf{j}} \forall \overline{\mathbf{m}} \forall \bar{\mathbf{i}} \forall \overline{\mathbf{n}}((\bar{\mathbf{j}} \leq \overline{\alpha} \mathbin{\&} \overline{\beta} \leq \overline{\mathbf{m}} \mathbin{\&} \bar{\mathbf{i}} \leq \overline{\gamma} \mathbin{\&} \overline{\delta} \leq \overline{\mathbf{n}}) \Rightarrow (\varphi \leq \psi)[!\bar{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\bar{\mathbf{i}}/!\overline{z}, !\overline{\mathbf{n}}/!\overline{w}]). \tag{1}$$

In the quasi-inequality above, symbols such as $\bar{\mathbf{j}} \leq \overline{\alpha}$ denote the conjunction of inequalities of the form $\mathbf{j}_k \leq \alpha_k$ for each $\mathbf{j}_k$ in $\bar{\mathbf{j}}$ and α_k in $\overline{\alpha}$. Before passing each initial quasi-inequality separately to stage 2 (described below), by exhaustively applying splitting rules to the top-most nodes of the formulas in $\overline{\alpha}$ and $\overline{\beta}$, we transform each quasi-inequality into one of similar shape as (1) and in which each α in $\overline{\alpha}$ and each β in $\overline{\beta}$ contains at most one critical occurrence. Hence, w.l.o.g. we can assume that each α in $\overline{\alpha}$ and β in $\overline{\beta}$ contains exactly one ε-critical occurrence (since in case any of them does not, the corresponding inequality will be ε^∂-uniform, and hence it can be assimilated to the inequalities $\bar{\mathbf{i}} \leq \overline{\gamma}$ or $\overline{\delta} \leq \overline{\mathbf{n}}$). Hence, we can represent the resulting quasi-inequality as follows:

$$\begin{array}{c}\forall \bar{\mathbf{j}} \forall \overline{\mathbf{m}} \forall \bar{\mathbf{i}} \forall \overline{\mathbf{n}}((\bar{\mathbf{j}} \leq \overline{\alpha_p} \mathbin{\&} \bar{\mathbf{j}} \leq \overline{\alpha_q} \mathbin{\&} \overline{\beta_p} \leq \overline{\mathbf{m}} \mathbin{\&} \overline{\beta_q} \leq \overline{\mathbf{m}} \mathbin{\&} \bar{\mathbf{i}} \leq \overline{\gamma} \mathbin{\&} \overline{\delta} \leq \overline{\mathbf{n}}) \\ \Rightarrow (\varphi \leq \psi)[!\bar{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\bar{\mathbf{i}}/!\overline{z}, !\overline{\mathbf{n}}/!\overline{w}]),\end{array} \tag{2}$$

where $\overline{p}$ (respectively, $\overline{q}$) is the vector of the atomic propositions in $\varphi \leq \psi$ such that $\varepsilon(p) = 1$ (respectively, $\varepsilon(q) = \partial$), and the subscript in each PIA-formula in $\overline{\alpha}$ and $\overline{\beta}$ indicates the unique ε-critical propositional variable occurrence contained in that formula.

2.8 Stage 2: Reduction and Elimination

The aim of this stage is to eliminate all occurring propositional variables from a given initial quasi-inequality (1). This is done by means of the splitting rules, introduced above, as well as the following *residuation rules* and *Ackermann-rules.* The rules applied in this subsection are collectively called *reduction rules.* The terms and inequalities in this subsection are from $\mathcal{L}^+_{\mathrm{LE}}$.

Residuation Rules. These rules operate on the inequalities in S, by rewriting a chosen inequality in S into another inequality. For every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, and any $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$,

$$\frac{f(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_f}) \leq \psi}{\varphi_i \leq f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f})}\ \varepsilon_f(i) = 1 \qquad \frac{f(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_f}) \leq \psi}{f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f}) \leq \varphi_i}\ \varepsilon_f(i) = \partial$$

$$\frac{\psi \leq g(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_g})}{g_i^\flat(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g}) \leq \varphi_i}\ \varepsilon_g(i) = 1 \qquad \frac{\psi \leq g(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_g})}{\varphi_i \leq g_i^\flat(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}\ \varepsilon_g(i) = \partial$$

Right Ackermann-Rule.

$$\frac{(\{\alpha_i \leq p \mid 1 \leq i \leq n\} \cup \{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\},\ \ \mathsf{Ineq})}{(\{\beta_j(\bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\bigvee_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\},\ \ \mathsf{Ineq})}\ (RAR)$$

where:

- p does not occur in $\alpha_1, \ldots, \alpha_n$ or in Ineq,
- $\beta_1(p), \ldots, \beta_m(p)$ are positive in p, and
- $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p.

Left Ackermann-Rule.

$$\frac{(\{p \leq \alpha_i \mid 1 \leq i \leq n\} \cup \{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\},\ \ \mathsf{Ineq})}{\{\beta_j(\bigwedge_{i=1}^n \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\},\ \ \mathsf{Ineq})}\ (LAR)$$

where:

- p does not occur in $\alpha_1, \ldots, \alpha_n$ or in Ineq,
- $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and
- $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p.

By applying adjunction and residuation rules on all PIA-formulas α and β, the antecedent of (2) can be equivalently written as follows (cf. Definition 2.21):

$$\begin{aligned}&\overline{\mathsf{LA}(\alpha_p)[\mathbf{j}/u,\overline{p},\overline{q}]} \leq \overline{p}\ \&\ \overline{\mathsf{RA}(\beta_p)[\mathbf{m}/u,\overline{p},\overline{q}]} \leq \overline{p}\ \&\ \overline{q} \leq \overline{\mathsf{LA}(\alpha_q)[\mathbf{j}/u,\overline{p},\overline{q}]}\\ &\quad\&\ \overline{q} \leq \overline{\mathsf{RA}(\beta_q)[\mathbf{m}/u,\overline{p},\overline{q}]}\ \&\ \overline{\mathbf{i}} \leq \overline{\gamma}\ \&\ \overline{\delta} \leq \overline{\mathbf{n}}.\end{aligned} \tag{3}$$

Notice that the "parametric"(i.e., non-critical) variables in $\overline{p}$ and $\overline{q}$ actually occurring in each formula $\mathsf{LA}(\alpha_p)[\mathbf{j}/u,\overline{p},\overline{q}]$, $\mathsf{RA}(\beta_p)[\mathbf{m}/u,\overline{p},\overline{q}]$, $\mathsf{LA}(\alpha_q)[\mathbf{j}/u,\overline{p},\overline{q}]$, and $\mathsf{RA}(\beta_q)[\mathbf{m}/u,\overline{p},\overline{q}]$ are those that are strictly $<_\Omega$-smaller than the (critical) variable indicated in the subscript of the given PIA-formula. After applying adjunction and residuation as indicated above, the resulting quasi-inequality is in Ackermann shape relative to the $<_\Omega$-minimal variables.

For every $p \in \overline{p}$ and $q \in \overline{q}$, let us define the sets $\mathsf{Mv}(p)$ and $\mathsf{Mv}(q)$ by recursion on $<_\Omega$ as follows:

- $\mathsf{Mv}(p) := \{\mathsf{LA}(\alpha_p)[\mathbf{j}_k/u,\overline{\mathsf{mv}(p)}/\overline{p},\overline{\mathsf{mv}(q)}/\overline{q}],\mathsf{RA}(\beta_p)[\mathbf{m}_h/u,\overline{\mathsf{mv}(p)}/\overline{p},\overline{\mathsf{mv}(q)}/\overline{q}] \mid 1 \leq k \leq n_{i_1}, 1 \leq h \leq n_{i_2}, \overline{\mathsf{mv}(p)} \in \prod_p \mathsf{Mv}(p), \overline{\mathsf{mv}(q)} \in \prod_q \mathsf{Mv}(q)\}$
- $\mathsf{Mv}(q) := \{\mathsf{LA}(\alpha_q)[\mathbf{j}_h/u,\overline{\mathsf{mv}(p)}/\overline{p},\overline{\mathsf{mv}(q)}/\overline{q}],\mathsf{RA}(\beta_q)[\mathbf{m}_k/u,\overline{\mathsf{mv}(p)}/\overline{p},\overline{\mathsf{mv}(q)}/\overline{q}] \mid 1 \leq h \leq m_{j_1}, 1 \leq k \leq m_{j_2}, \overline{\mathsf{mv}(p)} \in \prod_p \mathsf{Mv}(p), \overline{\mathsf{mv}(q)} \in \prod_q \mathsf{Mv}(q)\}$

where, n_{i_1} (respectively, n_{i_2}) is the number of occurrences of p in αs (respectively, in βs) for every $p \in \overline{p}$, and m_{j_1} (respectively, m_{j_2}) is the number of occurrences of q in αs (respectively, in βs) for every $q \in \overline{q}$. By induction on $<_\Omega$, we can apply the Ackermann rule exhaustively so as to eliminate all variables $\overline{p}$ and $\overline{q}$. Then, the antecedent of the resulting *purified* quasi-inequality has the following form:

$$\overline{\mathbf{i}} \leq \overline{\gamma}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] \qquad \overline{\delta}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] \leq \overline{\mathbf{n}}. \tag{4}$$

Up to now, we have only made use of the assumption that the initial inequality is inductive, and not also analytic. The next step is not needed for the elimination of propositional variables, since we have already reached a successful elimination. However, it will turn out to be useful when discussing canonicity.

By assumption, $\varepsilon(p) = 1$ for every p in $\overline{p}$ and $\varepsilon(q) = \partial$ for every q in $\overline{q}$; recalling that every $+\gamma$ (respectively, $-\delta$) agrees with ε^∂ and that γ (respectively, δ) is positive (respectively, negative) PIA for every $\gamma \in \overline{\gamma}$ (respectively, $\delta \in \overline{\delta}$) (this is precisely what the analiticity assumption yields), the following semantic equivalences hold for each γ in $\overline{\gamma}$ and δ in $\overline{\delta}$:

$$\gamma\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] = \bigwedge\left\{\gamma\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right] \mid \overline{\mathsf{mv}(p)} \in \prod_p \mathsf{Mv}(p), \overline{\mathsf{mv}(q)} \in \prod_q \mathsf{Mv}(q)\right\}.$$

$$\delta\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right] = \bigvee\left\{\delta\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right] \mid \overline{\mathsf{mv}(p)} \in \prod_p \mathsf{Mv}(p), \overline{\mathsf{mv}(q)} \in \prod_q \mathsf{Mv}(q)\right\}.$$

Hence, by applying splitting, for every γ in $\overline{\gamma}$ and δ in $\overline{\delta}$, the corresponding inequalities in (4) can be equivalently replaced by (at most) $\Sigma_{n,m}(n_i m_j)$ inequalities of the form

$$\mathbf{i} \leq \gamma\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right] \qquad \delta\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right] \leq \mathbf{n}, \tag{5}$$

where $\gamma\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right]$ is strictly syntactically open and $\delta\left[\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right]$ is strictly syntactically closed (cf. Definition 4.3 and Lemma 4.8).

3 SLANTED LE-ALGEBRAS AND THEIR CANONICAL EXTENSIONS

3.1 Basic Definitions and Properties

Definition 3.1. Let A, B be lattices. For any $n_f \in \mathbb{N}$ and any order-type ε_f on n_f, a coordinate-wise finitely join-preserving n_f-ary map $f : B^{\varepsilon_f} \to A^\delta$ is *c-slanted* if its range is included in $K(A^\delta)$. When $B = A$, the map f is a *c-slanted operation on A*. For any $n_g \in \mathbb{N}$ and any order-type ε_g on n_g, a coordinate-wise finitely meet-preserving n_g-ary map $g : B^{\varepsilon_g} \to A^\delta$ is *o-slanted* if its range is included in $O(A^\delta)$. When $B = A$, the map g is an *o-slanted operation on A*.

By definition, slanted maps are normal, in the sense of Definition 2.4, as maps $B^\varepsilon \to A^\delta$. Examples of (properly) c-slanted (respectively, o-slanted) operations arise as the restrictions to the original algebra of the left (respectively, right) adjoints and residuals of the π-extensions (respectively, σ-extensions) of standard normal g-type (respectively, f-type) operations (cf. [12, Lemma 10.6]) when the signature $(\mathcal{F}, \mathcal{G})$ is not closed under adjoints and residuals.

Definition 3.2. For any LE-signature $(\mathcal{F}, \mathcal{G})$, a *slanted (distributive) lattice expansion* (abbreviated as slanted (D)LE or s-(D)LE) is a tuple $\mathbb{A} = (A, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A})$ such that A is a bounded (distributive) lattice, $\mathcal{F}^\mathbb{A} = \{f^\mathbb{A} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\mathbb{A} = \{g^\mathbb{A} \mid g \in \mathcal{G}\}$, such that every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (respectively, $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) is an n_f-ary (respectively, n_g-ary) c-slanted (respectively, o-slanted) operation on A. A *slanted Boolean algebra expansion* (abbreviated as *slanted BAE* or *s-BAE*) is a structure $\mathbb{A} = (A, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A})$ such that $\mathcal{F}^\mathbb{A}$ and $\mathcal{G}^\mathbb{A}$ are as above, and A is a Boolean algebra.

Slanted LEs generalise the standard notion of normal LE (cf. Definition 2.4), as follows: Via the canonical embedding $e : A \to A^\delta$, and using compactness, it is not difficult to see that $e[A] = K(A^\delta) \cap O(A^\delta)$ for any lattice A. Hence, any standard normal operation h on A gives rise to a slanted operation $e \cdot h$ on A which will be c-slanted if h is coordinate-wise finitely join-preserving or meet-reversing, and o-slanted if if h is coordinate-wise finitely meet-preserving or join-reversing. Conversely, any slanted operation on A the range of which is included in $K(A^\delta) \cap O(A^\delta) = e[A]$ gives rise to a normal operation on A in the standard sense. Hence, any standard LE $\mathbb{A}$ can be "lifted"to a slanted LE $\mathbb{A}^\star$ in the same signature by pre-composing all operations of $\mathbb{A}$ with e, and any slanted LE $\mathbb{S}$ based on A such that all its operations target $K(A^\delta) \cap O(A^\delta) = e[A]$ gives rise to a standard LE $\mathbb{S}_\star$ in the same signature, and moreover, $(\mathbb{A}^\star)_\star = \mathbb{A}$ and $(\mathbb{S}_\star)^\star = \mathbb{S}$.

In the remainder of the paper, we will abuse notation and write e.g. f for $f^\mathbb{A}$ when this causes no confusion. Slanted LEs constitute the main semantic environment of this article.

Example 3.3. Examples of slanted BAEs and LEs arise in connection with subordination algebras [1], quasi-modal algebras [3] and generalized implication lattices [2]. The slanted algebras arising from subordination and quasi-modal algebras will be described in detail in Section 6. Let us consider here the case of generalized implications.

A *generalized implication lattice* [2] is a pair $\mathbb{L} = (L, \Rightarrow)$ such that L is a bounded distributive lattice, and $\Rightarrow: L \times L \to \mathcal{I}(L)$ (where $\mathcal{I}(L)$ denotes the set of the ideals of L) satisfies the following conditions: for every a, b and $c \in L$,

(1) $(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \wedge c)$;
(2) $(a \Rightarrow b) \cap (b \Rightarrow c) = (a \vee b) \Rightarrow c$;
(3) $(a \Rightarrow b) \cap (b \Rightarrow c) \subseteq a \Rightarrow c$;
(4) $a \Rightarrow a = L$.

For every generalized implication lattice $\mathbb{L}$, let $\mathbb{L}^* := (L, g_\Rightarrow)$ be its associated slanted algebra, where $g_\Rightarrow : L \times L \to L^\delta$ is defined by the assignment $(a, b) \mapsto \bigvee\{c \in L \mid c \in a \Rightarrow b\}$. It can be readily verified that $g_\Rightarrow$ is a binary o-slanted operator of order-type $(\partial, 1)$ satisfying the inequalities

$1 \leq g_{\Rightarrow}(a,a)$ and $g_{\Rightarrow}(a,b) \wedge g_{\Rightarrow}(b,c) \leq g_{\Rightarrow}(a,c)$ for every $a,b,c \in L$ which are analytic Sahlqvist and analytic inductive respectively. Conversely, if $\mathbb{A} = (L,g)$ is an s-DLE s.t. $\mathcal{F} = \varnothing$ and $\mathcal{G} := \{g\}$ with $n_g = 2$ and $\varepsilon_g = (\partial, 1)$ satisfying the properties verified by $g_{\Rightarrow}$, then $\mathbb{A}_* := (L, \Rightarrow_g)$, where $\Rightarrow_g \colon L \times L \longrightarrow \mathcal{I}(L)$ is defined by the assignment $(a,b) \mapsto \{c \in L \mid c \leq g(a,b)\}$, is a generalized implication lattice. It is routine to show that $(\mathbb{L}^*)_* = \mathbb{L}$ for every generalized implication lattice $\mathbb{L}$, and $(\mathbb{A}_*)^* = \mathbb{A}$ for every s-DLE as above.

As is done in [23, Section 2.3] and [34, Section 5], the σ- and π-extensions of slanted n-ary operations of a given bounded lattice $\mathbb{A}$ are defined not as maps $(\mathbb{A}^n)^\delta \longrightarrow (\mathbb{A}^\delta)^\delta$ as in the standard definition (cf. [22, Definition 4.1]), but as maps $(\mathbb{A}^n)^\delta \longrightarrow \mathbb{A}^\delta$. Towards the formal definition, recall (cf. Section 2.2) that in order to extend operations of any arity which are monotone or antitone in each coordinate from a lattice $\mathbb{A}$ to its canonical extension, treating the case of *monotone* and *unary* operations suffices:

Definition 3.4. Let A, B be bounded lattices. For every unary, c-slanted map $f : B \longrightarrow A^\delta$, the σ*-extension* of f is the map $f^\sigma : B^\delta \longrightarrow A^\delta$ defined first by declaring, for every $k \in K(B^\delta)$,

$$f^\sigma(k) := \bigwedge\{f(a) \mid a \in B \text{ and } k \leq a\},$$

and then, for every $u \in B^\delta$,

$$f^\sigma(u) := \bigvee\{f^\sigma(k) \mid k \in K(B^\delta) \text{ and } k \leq u\}.$$

For every unary, o-slanted map $g : B \longrightarrow A^\delta$, the π*-extension* of g is the map $g^\pi : B^\delta \longrightarrow A^\delta$ defined first by declaring, for every $o \in O(B^\delta)$,

$$g^\pi(o) := \bigvee\{g(a) \mid a \in B \text{ and } a \leq o\},$$

and then, for every $u \in B^\delta$,

$$g^\pi(u) := \bigwedge\{g^\pi(o) \mid o \in O(B\delta) \text{ and } u \leq o\}.$$

It immediately follows by denseness and the definition above that, if $e : A \longrightarrow A^\delta$ is the canonical embedding, then $e^\sigma = e^\pi = id_{A^\delta}$. Likewise, it can be readily verified that, for every (standard) map $h : B \longrightarrow A$ which is coordinatewise finitely join-preserving or meet-reversing (respectively, coordinate-wise finitely meet-preserving or join-reversing), $(e \cdot h)^\sigma = h^\sigma$ (respectively, $(e \cdot h)^\pi = h^\pi$). Conversely, as discussed above, any c-slanted (respectively, o-slanted) map $h : B \longrightarrow A^\delta$ the range of which is included in $K(A^\delta) \cap O(A^\delta) = e[A]$ gives rise to a normal map $h_\star : B \longrightarrow A$ to which the standard definitions of σ- and π-extensions apply, and it can be readily verified that $h^\sigma = (h_\star)^\sigma$ (respectively, $h^\pi = (h_\star)^\pi$).

Let us spell out and further simplify the definition above when $B := A^\varepsilon$ for any order-type ε on $n \geq 1$. First, recall that taking the order-dual interchanges closed and open elements: $K((A^\delta)^\partial) = O(A^\delta)$ and $O((A^\delta)^\partial) = K(A^\delta)$; similarly, $K((A^n)^\delta) = K(A^\delta)^n$, and $O((A^n)^\delta) = O(A^\delta)^n$. Hence, $K((A^\delta)^\varepsilon) = \prod_i K(A^\delta)^{\varepsilon(i)}$ and $O((A^\delta)^\varepsilon) = \prod_i O(A^\delta)^{\varepsilon(i)}$ for every LE $\mathbb{A}$ and every ε, where

$$K(A^\delta)^{\varepsilon(i)} := \begin{cases} K(A^\delta) & \text{if } \varepsilon(i) = 1 \\ O(A^\delta) & \text{if } \varepsilon(i) = \partial \end{cases} \quad O(A^\delta)^{\varepsilon(i)} := \begin{cases} O(A^\delta) & \text{if } \varepsilon(i) = 1 \\ K(A^\delta) & \text{if } \varepsilon(i) = \partial. \end{cases}$$

Letting $\leq^\varepsilon$ denote the product order on $(A^\delta)^\varepsilon$, we have for every $f \in \mathcal{F}$, $g \in \mathcal{G}$, $\overline{k} \in K((A^\delta)^{\varepsilon_f})$, $\overline{o} \in O((A^\delta)^{\varepsilon_f})$, $\overline{u} \in (A^\delta)^{n_f}$, and $\overline{v} \in (A^\delta)^{n_g}$,

$$f^\sigma(\overline{k}) := \bigwedge\{f(\overline{a}) \mid \overline{a} \in A^{\varepsilon_f} \text{ and } \overline{k} \leq^{\varepsilon_f} \overline{a}\} \quad f^\sigma(\overline{u}) := \bigvee\{f^\sigma(\overline{k}) \mid \overline{k} \in K((A^\delta)^{\varepsilon_f}) \text{ and } \overline{k} \leq^{\varepsilon_f} \overline{u}\}$$

$$g^\pi(\overline{o}) := \bigvee\{g(\overline{a}) \mid \overline{a} \in A^{\varepsilon_g} \text{ and } \overline{a} \leq^{\varepsilon_g} \overline{o}\} \quad g^\pi(\overline{v}) := \bigwedge\{g^\pi(\overline{o}) \mid \overline{o} \in O((A^\delta)^{\varepsilon_g}) \text{ and } \overline{v} \leq^{\varepsilon_g} \overline{o}\}.$$

Lemma 3.5. *For every lattice $\mathbb{A}$, any c-slanted operation f on $\mathbb{A}$ of arity n_f and order-type ε_f, and any o-slanted operation g on $\mathbb{A}$ of arity n_g and order-type ε_g,*

(1) *f^σ is ε_f-monotone and g^π is ε_g-monotone;*
(2) *f^σ is completely join-preserving in all coordinates i such that $\varepsilon_f(i) = 1$ and completely meet-reversing in all coordinates i such that $\varepsilon_f(i) = \partial$;*
(3) *g^π is completely meet-preserving in all coordinates i such that $\varepsilon_g(i) = 1$ and completely join-reversing in all coordinates i such that $\varepsilon_g(i) = \partial$.*

Proof. As to item 1, let $u, v \in (\mathbb{A}^\delta)^{\varepsilon_f}$. If $u \leq v$, then by denseness, for every $k \in K((\mathbb{A}^\delta)^{\varepsilon_f})$), if $k \leq u$ then $k \leq v$. Hence $f^\sigma(u) := \bigvee\{f^\sigma(k) \mid k \in K((\mathbb{A}^\delta)^{\varepsilon_f}) \text{ and } k \leq u\} \leq \bigvee\{f^\sigma(k) \mid k \in K((\mathbb{A}^\delta)^{\varepsilon_f}) \text{ and } k \leq v\} := f^\sigma(v)$. The proof of the ε_g-monotonicity of g^π is dual.

The arguments for proving the remaining items in the standard setting (cf. [22, Lemma 4.6]) can be straightforwardly generalized to the present setting. However, we are going to adopt a simpler method, which is constructive and for which we do not need to appeal to the restricted distributive law. Namely, since $\mathbb{A}^\delta$ is a complete lattice, it is enough to show that the right residuals (resp. Galois residuals) of f^σ exist in each coordinate. For the sake of keeping the notation simple, let us show that if f is binary and of order-type $\varepsilon_f = \varepsilon = (1, \partial)$, the right residual of f in the first coordinate (which needs to be of order-type $(1, 1)$) exists. Let $g_1 : \mathbb{A}^\delta \times \mathbb{A}^\delta \to \mathbb{A}^\delta$ be defined as follows: $g_1(o, o') := \bigvee\{a \in A \mid f^\sigma(a, o') \leq o\}$ for all $o, o' \in O(\mathbb{A}^\delta)$ and $g_1(v_1, v_2) := \bigwedge\{g(o_1, o_2) \mid o_i \in O(A^\delta) \text{ and } v_i \leq o_i\}$ for all $v_1, v_2 \in \mathbb{A}^\delta$.[8] Let us show that, for every $k \in K(\mathbb{A}^\delta)$ and all $o, o' \in O(A^\delta)$,

$$f^\sigma(k, o') \leq o \quad \text{iff} \quad k \leq g_1(o, o'). \tag{6}$$

From left to right, if $\bigwedge\{f(a, b) \mid a, b \in A \text{ and } k \leq a \text{ and } b \leq o'\} =: f^\sigma(k, o') \leq o$, then by compactness (recall that $f(a, b) \in K(A^\delta)$) this implies that $f(a_1, b_1) \wedge \cdots \wedge f(a_n, b_n) \leq o$ for some $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in A$ such that $k \leq a_i$ and $b_i \leq o'$ for every $1 \leq i \leq n$. Since f is ε-monotone, letting $b := b_1 \vee \cdots \vee b_n$ and $a := a_1 \wedge \cdots \wedge a_n$, this implies that $k \leq a$, $b \leq o'$ and $f(a, b) \leq f(a_1, b_1) \wedge \cdots \wedge f(a_n, b_n) \leq o$. Hence, $f^\sigma(a, o') := \bigwedge\{f(a, b) \mid b \in A \text{ and } b \leq o'\} \leq f(a, b) \leq o$, and hence $k \leq a \leq \bigvee\{a \in A \mid f^\sigma(a, o') \leq o\} =: g_1(o, o')$, as required.

For the converse direction, if $k \leq g_1(o, o') := \bigvee\{a \in A \mid f^\sigma(a, o') \leq o\}$, then, by compactness, $k_1 \leq a_1 \vee \cdots \vee a_n$ for some $a_1, \ldots, a_n \in A$ such that $\bigwedge\{f(a_i, b) \mid b \in A \text{ and } b \leq o'\} =: f^\sigma(a_i, o') \leq o$ for every $1 \leq i \leq n$. Hence, by compactness, for every $1 \leq i \leq n$, there exist some $b_i^1, \ldots, b_i^{n_i} \in A$ such that $b_i^j \leq o'$ for every $1 \leq j \leq n_i$ and

$$f(a_i, b_i^1) \wedge \cdots \wedge f(a_i, b_i^{n_i}) \leq o.$$

For each $1 \leq i \leq n$, let $b_i := b_i^1 \vee \cdots \vee b_i^{n_i}$. Hence, $b_i \leq o'$ and, by the antitonicity of f in its second coordinate, $f(a_i, b_i) \leq f(a_i, b_i^1) \wedge \cdots \wedge f(a_i, b_i^{n_i}) \leq o$ for every $1 \leq i \leq n$. Hence, letting $b := b_1 \vee \cdots \vee b_n$, and $a := a_1 \vee \cdots \vee a_n$, we have $b \leq o'$ and $k \leq a$, and moreover,

[8] If $f : \mathbb{A}^{\varepsilon_f} \to \mathbb{A}^\delta$, then, for every $1 \leq i \leq n_f$ such that $\varepsilon_f(i) = 1$, we let $g_i : (\mathbb{A}^\delta)^{\varepsilon_{g_i}} \to \mathbb{A}^\delta$ be defined as follows: $g_i(\overline{o}) := \bigvee\{a \in A \mid f^\sigma(\overline{o}[a/o_i]) \leq o_i\}$ for every $\overline{o} \in O(\mathbb{A}^\delta)^{\varepsilon_{g_i}}$ and $g_i(\overline{v}) := \bigwedge\{g_i(\overline{o}) \mid o \in O(A^\delta)^{\varepsilon_{g_i}} \text{ and } \overline{v} \leq^{\varepsilon_{g_i}} \overline{o}\}$ for any $v \in \mathbb{A}^\delta$, where $\varepsilon_{g_i}(i) = 1$ and $\varepsilon_{g_i}(j) = \varepsilon_f^\partial(j)$ if $j \neq i$. For every $1 \leq i \leq n_f$ such that $\varepsilon_f(i) = \partial$, we let $g_i : (\mathbb{A}^\delta)^{\varepsilon_{g_i}} \to \mathbb{A}^\delta$ be defined as follows: $g_i(\overline{k}) := \bigwedge\{a \in A \mid f^\sigma(\overline{k}[a/o_i]) \leq o_i\}$ for every $\overline{k} \in K(\mathbb{A}^\delta)^{\varepsilon_{g_i}}$ and $g_i(\overline{v}) := \bigvee\{g_i(\overline{k}) \mid k \in K(A^\delta)^{\varepsilon_{g_i}} \text{ and } \overline{k} \leq^{\varepsilon_{g_i}} \overline{v}\}$ for any $v \in \mathbb{A}^\delta$, where $\varepsilon_{g_i}(j) = \varepsilon_f(j)$ for every $1 \leq j \leq n_f$.

$$
\begin{array}{lll}
 & f^{\sigma}(k,o') & \\
:= & \bigwedge\{f(a,b) \mid a,b \in A \text{ and } k \leq a \text{ and } b \leq o'\} & \\
\leq & f(a,b) & \\
= & f(a_1 \vee \cdots \vee a_n, b) & a := a_1 \vee \cdots \vee a_n \\
= & f(a_1,b) \vee \cdots \vee f(a_n,b) & f \text{ finitely join preserving in its first coord.} \\
\leq & f(a_1,b_1) \vee \cdots \vee f(a_n,b_n) & b := b_1 \vee \cdots \vee b_n \text{ and } \varepsilon_f(2) = \partial \\
\leq & o & f(a_i,b_i) \leq o \text{ for every } 1 \leq i \leq n
\end{array}
$$

as required. Let us show that, for all $u, u', v \in \mathbb{A}^{\delta}$,

$$f^{\sigma}(u,u') \leq v \quad \text{iff} \quad u \leq g_1(v,u').$$

Let $u, u', v \in \mathbb{A}^{\delta}$. From left to right, if $f^{\sigma}(u,u') \leq v$, to show that $u_1 \leq g(v,u') := \bigwedge\{g(o,o') \mid o,o' \in O(A^{\delta}) \text{ and } v \leq o \text{ and } u' \leq o'\}$, it is enough to show that $k \leq g(o,o')$ for all $k \in K(\mathbb{A}^{\delta})$ such that $k \leq u$ and for all $o, o' \in O(\mathbb{A}^{\delta})$ such that $v \leq o$ and $u' \leq o'$. Since f^{σ} is ε-monotone, for any such k, o and o', we have $f^{\sigma}(k,o') \leq f^{\sigma}(u,u') \leq v \leq o$. By (6), this implies that $k \leq g_1(o,o')$ as required. From right to left, if $u \leq g_1(v,u') := \bigwedge\{g(o,u') \mid o,o' \in O(A^{\delta}) \text{ and } v \leq o \text{ and } u' \leq o'\}$, to show that $f^{\sigma}(u,u') \leq v$, we need to show that $f^{\sigma}(k,o') \leq o$ for every $k \in K(A^{\delta})$ s.t. $k \leq u$ and every $o, o' \in O(A^{\delta})$ s.t. $v \leq o$ and $u' \leq o'$. By assumption, $k \leq u \leq g(v,u') \leq g(o,o')$ which, by (6), implies $f^{\sigma}(k,o') \leq o$, as required. □

As in the standard case (cf. discussion before Definition 2.8), we define the canonical extension of a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra so as to meet the desideratum that it be a perfect (resp. complete, in the constructive setting) *standard* $\mathcal{L}_{\mathrm{LE}}$-algebra. In light of the lemma above, we are again justified in choosing the σ-extensions of connectives in $\mathcal{F}$ and the π-extensions of connectives in $\mathcal{G}$, which motivates the following:

Definition 3.6. The canonical extension of a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (A, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}^{\delta} := (A^{\delta}, \mathcal{F}^{\mathbb{A}^{\delta}}, \mathcal{G}^{\mathbb{A}^{\delta}})$ such that, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$, the operations $f^{\mathbb{A}^{\delta}}$ and $g^{\mathbb{A}^{\delta}}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$, respectively, as in Definition 3.4.

It immediately follows from the definition above and Lemma 3.5 that the canonical extension of a slanted LE $\mathbb{A}$ is a perfect LE (cf. Definition 2.9) (respectively, complete LE, in the constructive setting) in the standard sense.

Also, from the discussions after Definitions 3.2 and 3.4, it readily follows that $(\mathbb{A}^{\star})^{\delta} = \mathbb{A}^{\delta}$ for every standard LE $\mathbb{A}$, and that $(\mathbb{S}_{\star})^{\delta} = \mathbb{S}^{\delta}$ for every slanted LE $\mathbb{S}$ based on a bounded lattice A and such that all its operations target $K(A^{\delta}) \cap O(A^{\delta}) = e[A]$.

3.2 Slanted LE-Algebras as Models of LE-Inequalities

Fix an arbitrary LE-signature $(\mathcal{F}, \mathcal{G})$. From the discussion of the previous section, it is clear that, for any slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$, any *assignment* into $\mathbb{A}$, i.e., any map $v : \mathsf{PROP} \to \mathbb{A}$, uniquely extends to an $\mathcal{L}_{\mathrm{LE}}$-homomorphism $v : \mathbf{Fm} \to \mathbb{A}^{\delta}$ (abusing notation, the same symbol for the given assignment also denotes its homomorphic extension). Hence,

Definition 3.7. An $\mathcal{L}_{\mathrm{LE}}$-inequality $\varphi \leq \psi$ is *satisfied* in a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ under the assignment v (notation: $(\mathbb{A}, v) \models \varphi \leq \psi$) if $(\mathbb{A}^{\delta}, e \cdot v) \models \varphi \leq \psi$ in the usual sense, where $e \cdot v$ is the assignment on $\mathbb{A}^{\delta}$ obtained by composing the canonical embedding $e : \mathbb{A} \to \mathbb{A}^{\delta}$ to the assignment $v : \mathsf{PROP} \to \mathbb{A}$.

Moreover, $\varphi \leq \psi$ is *valid* in $\mathbb{A}$ (notation: $\mathbb{A} \models \varphi \leq \psi$) if $(\mathbb{A}^{\delta}, e \cdot v) \models \varphi \leq \psi$ for every assignment v into $\mathbb{A}$ (notation: $\mathbb{A}^{\delta} \models_{\mathbb{A}} \varphi \leq \psi$). We will often refer to assignments into $\mathbb{A}$ as *admissible* assignments.

From the definition above, and the discussion after Definition 3.6, it immediately follows that any LE-inequality $\varphi \leq \psi$ is valid in $\mathbb{A}$ iff $\varphi \leq \psi$ is valid in $\mathbb{A}^\star$ for every standard LE $\mathbb{A}$, and that $\varphi \leq \psi$ is valid in $\mathbb{S}$ iff $\varphi \leq \psi$ is valid in $\mathbb{S}_\star$ for every slanted LE $\mathbb{S}$ based on a bounded lattice A and such that all its operations target $K(A^\delta) \cap O(A^\delta) = e[A]$. This shows that the notion of validity on slanted algebras generalizes standard validity in the appropriate way.

Notice that, whether constructive or non-constructive, the canonical extension of any slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ is an $\mathcal{L}^+_{\mathrm{LE}}$-algebra in the standard sense. Hence, given the definition above, any slanted $\mathcal{L}_{\mathrm{LE}}$-algebra is also a *slanted* $\mathcal{L}^+_{\mathrm{LE}}$-algebra, in the sense that the definition above makes the machinery of $\mathbb{A}^\delta$ available for the interpretation of the language $\mathcal{L}^+_{\mathrm{LE}}$ on $\mathbb{A}$ in the sense specified in the definition above.

Recall that by definition, $f^{\mathbb{A}^\delta} = (f^{\mathbb{A}})^\sigma$ for each $f \in \mathcal{F}$, and $g^{\mathbb{A}^\delta} = (g^{\mathbb{A}})^\pi$ for each $g \in \mathcal{G}$. We are now in a position to define the notion of *slanted canonicity* (abbreviated as *s-canonicity*) for $\mathcal{L}_{\mathrm{LE}}$-sequents/inequalities:

Definition 3.8 (Slanted Canonicity of $\mathcal{L}_{\mathrm{LE}}$-inequalities). An $\mathcal{L}_{\mathrm{LE}}$-inequality $\varphi \leq \psi$ is *s-canonical* if for every slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$,

$$\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi \quad \text{implies} \quad \mathbb{A}^\delta \models \varphi \leq \psi.$$

From the definition above, and the discussion after Definition 3.7, it immediately follows that standard canonicity translates into a *relativized* notion of slanted canonicity: namely, slanted canonicity relative to the slanted algebras in the range of the constructor $(-)^\star$. Conversely, slanted canonicity *relativized* to the domain of definition of the constructor $(-)_\star$ corresponds to the notion of standard canonicity.

4 SLANTED CANONICITY OF ANALYTIC INDUCTIVE LE-INEQUALITIES

This section is aimed at showing that every analytic inductive formula is s-canonical (in the sense of Definition 3.8). We first give the statement of the canonicity theorem and its proof, and subsequently prove the proposition needed in the previously stated proof and its requisite preliminaries.

Theorem 4.1. *For any language $\mathcal{L}_{\mathrm{LE}}$, all analytic inductive $\mathcal{L}_{\mathrm{LE}}$-inequalities are s-canonical.*

Proof. Let $\varphi \leq \psi$ be an analytic inductive $\mathcal{L}_{\mathrm{LE}}$-inequality, fix a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$, and let $\mathbb{A}^\delta$ be its canonical extension. As discussed in Section 2.6, ALBA succeeds in reducing $\varphi \leq \psi$ to a set $\mathsf{ALBA}(\varphi \leq \psi)$ of pure quasi-inequalities in the expanded language $\mathcal{L}^+_{\mathrm{LE}}$. The required canonicity proof is summarised in the following U-shaped diagram:

$$\begin{array}{lcl}
\mathbb{A} \ \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\quad\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \quad\Updownarrow \\
\quad\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \mathsf{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \models \mathsf{ALBA}(\varphi \leq \psi)
\end{array}$$

The upper bi-implication on the left is due to the definition of validity on slanted LEs (cf. Definition 3.7). The lower bi-implication on the left is given by Proposition 4.9 below. The horizontal bi-implication follows from the facts that, by assumption, $\mathsf{ALBA}(\varphi \leq \psi)$ is pure, and that, when restricted to pure formulas, the ranges of admissible and arbitrary assignments coincide. The bi-implication on the right is due to [12, Theorem 6.1] (which can be applied since the canonical extension of a slanted LE is a standard LE). ∎

Towards the proof of Proposition 4.9, the following definitions and lemmas will be useful:

Definition 4.2. The sets SC and SO of *syntactically closed* and *syntactically open* $\mathcal{L}_{\mathrm{LE}}^{+}$-terms are defined simultaneously as follows: for every $f^* \in \mathcal{F}^*$, $f \in \mathcal{F}$, $g^* \in \mathcal{G}^*$, and $g \in \mathcal{G}$,

$$\mathrm{SC} \ni \varphi ::= p \mid \mathbf{j} \mid \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid f^*(\overline{\varphi}, \overline{\psi}) \mid g(\overline{\varphi}, \overline{\psi})$$

$$\mathrm{SO} \ni \psi ::= p \mid \mathbf{m} \mid \top \mid \bot \mid \psi \vee \psi \mid \psi \wedge \psi \mid g^*(\overline{\psi}, \overline{\varphi}) \mid f(\overline{\psi}, \overline{\varphi}).$$

Recall that, when writing $h(\overline{\chi}, \overline{\xi})$, we let $\overline{\chi}$ represent all the coordinates of h such that $\varepsilon_h(i) = 1$ and $\overline{\xi}$ represent all the coordinates of h such that $\varepsilon_h(i) = \partial$.

The previous definition identifies the syntactic shape of the terms, the (formal) topological[9] properties of which guarantee the soundness of the Ackermann rules under admissible assignments in the setting of standard (i.e., non-slanted) LEs. The following definition identifies a more restricted syntactic shape of LE-terms which aims at guaranteeing the soundness of the Ackermann rules under admissible assignments in the setting of *slanted* LEs; this restriction consists in imposing the same constraints both to the connectives of the original language and to those of the expanded language.

Definition 4.3. The sets SSC and SSO of *strictly syntactically closed* (ssc) and *strictly syntactically open* (sso) $\mathcal{L}_{\mathrm{LE}}^{+}$-terms are defined simultaneously as follows: for every $f^* \in \mathcal{F}^*$, and $g^* \in \mathcal{G}^*$,

$$\mathrm{SSC} \ni \varphi ::= p \mid \mathbf{j} \mid \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid f^*(\overline{\varphi}, \overline{\psi}),$$

$$\mathrm{SSO} \ni \psi ::= p \mid \mathbf{m} \mid \top \mid \bot \mid \psi \vee \psi \mid \psi \wedge \psi \mid g^*(\overline{\psi}, \overline{\varphi}).$$

From the definition above, it immediately follows that

Lemma 4.4. *For all ssc formulas $\varphi(\overline{!x}, \overline{!y})$ and all sso formulas $\psi(\overline{!x}, \overline{!y})$ which are positive in any x in $\overline{!x}$ and negative in any y in $\overline{!y}$, and all tuples $\overline{\varphi'}$ and $\overline{\psi'}$ of ssc formulas and sso formulas, respectively,*

(1) $\varphi[\overline{\varphi'}/\overline{!x}, \overline{\psi'}/\overline{!y}]$ *is ssc;*
(2) $\psi[\overline{\psi'}/\overline{!x}, \overline{\varphi'}/\overline{!y}]$ *is sso.*

Lemma 4.5. *If $\alpha(!x)$ is a definite positive PIA $\mathcal{L}_{\mathrm{LE}}$-formula and $\beta(!x)$ is a definite negative PIA $\mathcal{L}_{\mathrm{LE}}$-formula, then*

(1) *α is sso and β is ssc.*
(2) *If $+x \prec +\alpha$ and $+x \prec +\beta$, then $\mathsf{LA}(\alpha)[\mathbf{j}/!u]$ is ssc and $\mathsf{RA}(\beta)[\mathbf{m}/!u]$ is sso.*
(3) *If $-x \prec +\alpha$ and $-x \prec +\beta$, then $\mathsf{LA}(\alpha)[\mathbf{j}/!u]$ is sso and $\mathsf{RA}(\beta)[\mathbf{m}/!u]$ is ssc.*

Proof.
(1) Straightforward by simultaneous induction on α and β.
(2) and (3) We proceed by simultaneous induction on α and β.
If $\alpha = \beta = x$, then the assumptions of item (2) are satisfied; then $\mathsf{LA}(\alpha)[\mathbf{j}/!u] = \mathbf{j}/u$ is clearly ssc and $\mathsf{RA}(\beta)[\mathbf{m}/!u] = \mathbf{m}/u$ is clearly sso.

As to the inductive step, if $\alpha = g(\overline{\varphi}, \overline{\psi})$, with each φ in $\overline{\varphi}$ positive PIA (hence, by item (1), sso) and each ψ in $\overline{\psi}$ negative PIA (hence, by item (1), ssc), and the only occurrence of x is in φ_h, then φ_h is positive PIA, and moreover, $g_h^\flat \in \mathcal{F}^*$ is positive in its hth coordinate and has the opposite polarity of ε_g in all the other coordinates. Hence, $g_h^\flat(\overline{\varphi_{-h}}, \mathbf{j}/!u, \overline{\psi})$ is ssc. Two cases can occur: (a) If $+x \prec +\alpha$, then $+x \prec +\varphi_h$; hence, by induction hypothesis, $\mathsf{LA}(\varphi_h)[\mathbf{i}/!u']$ is ssc, and moreover, $+u' \prec \mathsf{LA}(\varphi_h)(u')$ (cf. Lemma 2.22). Hence,

$$\mathsf{LA}(\alpha)[\mathbf{j}/!u] = \mathsf{LA}(\varphi_h)[g_h^\flat(\overline{\varphi_{-h}}, \mathbf{j}/!u, \overline{\psi})/!u']$$

[9] This denomination stems from the original definition of canonical extensions as algebraic structures encoding information about Stone duals of Boolean algebras [32].

is ssc (cf. Lemma 4.4). (b) If $-x \prec +\alpha$, then $-x \prec +\varphi_h$, hence by induction hypothesis, $\mathsf{LA}(\varphi_h)[\mathbf{i}/!u']$ is sso, and moreover, $-u' \prec \mathsf{LA}(\varphi_h)(u')$ (cf. Lemma 2.22). Hence,

$$\mathsf{LA}(\alpha)[\mathbf{j}/!u] = \mathsf{LA}(\varphi_h)[g_h^{\flat}(\overline{\varphi_{-h}}, \mathbf{j}/!u, \overline{\psi})/!u']$$

is sso (cf. Lemma 4.4). The remaining cases are $\alpha = g(\overline{\varphi}, \overline{\psi})$ such that the only occurrence of x is in ψ_h, $\beta = f(\overline{\varphi}, \overline{\psi})$ with x occurring in φ_h or ψ_h, and are shown in a similar way. □

In the following two lemmas, α, $\beta_1, \ldots, \beta_n$ and $\gamma_1, \ldots, \gamma_n$ are $\mathcal{L}_{\mathrm{LE}}^+$-terms. We work under the assumption that the values of all parameters occurring in them (propositional variables, nominals and conominals) are given by some fixed admissible assignment. Recall that every slanted $\mathcal{L}_{\mathrm{LE}}$-algebra is also an slanted $\mathcal{L}_{\mathrm{LE}}^+$-algebra (cf. discussion after Definition 3.7).

Lemma 4.6 (RightHanded Ackermann Lemma for Admissible Assignments). *Let α be ssc, $p \notin \mathsf{PROP}(\alpha)$, let $\beta_1(p), \ldots, \beta_n(p)$ be ssc and positive in p, and let $\gamma_1(p), \ldots, \gamma_n(p)$ be sso and negative in p. Then, for every slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ and every admissible assignment v into $\mathbb{A}$,*

$$(\mathbb{A}, v) \models \beta_i(\alpha/p) \leq \gamma_i(\alpha/p) \text{ for all } 1 \leq i \leq n$$

iff there exists some p-variant v' of v into $\mathbb{A}$ such that

$$(\mathbb{A}, v') \models \alpha \leq p \text{ and } (\mathbb{A}, v') \models \beta_i(p) \leq \gamma_i(p) \text{ for all } 1 \leq i \leq n.$$

Lemma 4.7 (LeftHanded Ackermann Lemma for Admissible Assignments). *Let α be sso, $p \notin \mathsf{PROP}(\alpha)$, let $\beta_1(p), \ldots, \beta_n(p)$ be ssc and negative in p, and let $\gamma_1(p), \ldots, \gamma_n(p)$ be sso and positive in p. Then, for every slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ and every admissible assignment v into $\mathbb{A}$,*

$$(\mathbb{A}, v) \models \beta_i(\alpha/p) \leq \gamma_i(\alpha/p) \text{ for all } 1 \leq i \leq n$$

iff there exists some admissible p-variant v' of v into $\mathbb{A}$ such that

$$(\mathbb{A}, v') \models p \leq \alpha \text{ and } (\mathbb{A}, v') \models \beta_i(p) \leq \gamma_i(p) \text{ for all } 1 \leq i \leq n.$$

The two lemmas above are proved in Section A.2.

Lemma 4.8. *Executing ALBA on an analytic inductive $\mathcal{L}_{\mathrm{LE}}$-inequality $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$, as indicated in Section 2.6, we obtain quasi-inequalities each of which is such that each inequality in its antecedent, which as discussed at the end of Section 2.8, is of either of the following forms:*

$$\mathbf{i} \leq \gamma\left(\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right) \qquad \delta\left(\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right) \leq \mathbf{n}, \tag{7}$$

is such that its left-hand side is ssc and its right-hand side is sso.

Proof. Clearly, $\mathbf{i}$ is ssc and $\mathbf{n}$ is sso. Let us show that the formula $\gamma\left(\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right)$ is sso while $\delta\left(\overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}\right)$ is ssc. Recall from Notation 2.17 that $\gamma(\overline{p}, \overline{q})$ (respectively, $\delta(\overline{p}, \overline{q})$) is a positive (respectively, negative) PIA term, and both γ and δ are ε^{∂}-uniform as subterms of the original analytic inductive inequality. Recall that $\varepsilon(p) = 1$ for every variable p in $\overline{p}$ and $\varepsilon(q) = \partial$ for each q in $\overline{q}$. Hence, $-p \prec +\gamma$ and $+q \prec +\gamma$, and $-p \prec -\delta$ and $+q \prec -\delta$ for each p in $\overline{p}$ and each q in $\overline{q}$. Lemma 4.5 implies that $\gamma(\overline{p}, \overline{q})$ is sso and $\delta(\overline{p}, \overline{q}))$ is ssc. Hence, the proof is complete if we show that $\mathsf{mv}(p)$ is ssc for every variable p such that $\varepsilon(p) = 1$ and $\mathsf{mv}(q)$ is sso for every variable q such that $\varepsilon(q) = \partial$. Recall (cf. Subsection 2.6) that for every p in $\overline{p}$, the formula $\mathsf{mv}(p)$ is either of the form $\mathsf{LA}(\alpha_p)[\mathbf{j}_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}]$ for some definite positive PIA formula α_p (and hence $+p \prec +\alpha_p$), or is of the form $\mathsf{RA}(\beta_p)[\mathbf{m}_h/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}]$ for some definite negative PIA formula β_p (and hence $+p \prec -\beta_p$). Likewise, for every q in $\overline{q}$, the formula $\mathsf{mv}(q)$ is either of the form $\mathsf{LA}(\alpha_q)[\mathbf{j}_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}]$ for some definite positive PIA formula α_q (and hence $-q \prec +\alpha_q$), or is of the form $\mathsf{RA}(\beta_q)[\mathbf{m}_h/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}]$ for some definite negative PIA formula β_q (and hence $-q \prec -\beta_q$). The proof proceeds by induction on $<_\Omega$. If p is $<_\Omega$-minimal, then the form of $\mathsf{mv}(p)$ simplifies to either $\mathsf{LA}(\alpha_p)[\mathbf{j}_k/u]$ for some positive PIA formula α_p such that $+p \prec +\alpha_p$, or to $\mathsf{RA}(\beta_p)[\mathbf{m}_h/u]$ for some negative PIA formula β_p such that $+p \prec -\beta_p$. In either case, items (2) and (3) of Lemma 4.5 guarantee that $\mathsf{mv}(p)$ is ssc. Similarly, items (2) and (3) of Lemma 4.5 guarantee that $\mathsf{mv}(q)$ is sso when q is $<_\Omega$-minimal. The inductive step follows from items (2) and (3) of Lemma 4.5, the inductive hypothesis, and the polarities of the coordinates of the formulas $\mathsf{LA}(\alpha_p)$, $\mathsf{LA}(\alpha_q)$, $\mathsf{RA}(\beta_p)$, and $\mathsf{RA}(\beta_q)$ (cf. Lemma 2.22); as an example, consider the case in which $\mathsf{mv}(q)$ is of the form $\mathsf{LA}(\alpha_q)[\mathbf{j}_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}]$ for some positive PIA formula α_q (and hence $-q \prec +\alpha_q$). Then by Lemma 4.5.3, the formula $\mathsf{LA}(\alpha_q)[\mathbf{j}_k/!u, \overline{p}, \overline{q}]$, which, by Lemma 2.22 is antitone in u and $\overline{p}$ and monotone in $\overline{q}$, is sso; hence, by induction hypothesis and Lemma 4.4, $\mathsf{mv}(q) := \mathsf{LA}(\alpha_q)[\mathbf{j}_k/!u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}]$ is sso. □

Proposition 4.9 (Correctness of Executions of ALBA on Analytic Inductive Inequalities under Admissible Assignments into Slanted Algebras). *For any analytic inductive $\mathcal{L}_{\mathrm{LE}}$-inequality $\varphi \leq \psi$, if* $\mathsf{ALBA}(\varphi \leq \psi)$ *denotes the set of pure $\mathcal{L}^+_{\mathrm{LE}}$-quasi-inequalities generated by the ALBA-runs discussed in Section 2.6, then for every slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$,*

$$\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi \quad \textit{iff} \quad \mathbb{A}^\delta \models_{\mathbb{A}} \mathsf{ALBA}(\varphi \leq \psi).$$

Proof. The proof is similar to the correctness proof of ALBA runs under arbitrary assignments in the standard setting (see, e.g., [11, Correctness Theorem] and [12, Correctness Theorem]). The only significant difference is that the Ackermann rules are generally not invertible under admissible assignments, not even on standard algebras (cf. [11, Example 9.1]), which, as discussed after Definition 3.2, correspond to a proper subclass of slanted algebras. However, by Lemmas 4.7 and 4.6, when the left-hand and right-hand sides of all non-pure inequalities involved in the application of an Ackermann rule are, respectively, ssc and sso, the rule is sound and invertible under admissible assignments. By Lemma 4.8, this requirement on the syntactic shape is always satisfied when the rule is applied in the ALBA-runs discussed in Section 2.6. □

5 TRANSFER OF CANONICITY FOR DLE-INEQUALITIES

In [16], Gödel-McKinsey-Tarski type translations (GMT-type translations) are used to obtain Sahlqvist correspondence and canonicity as transfer results in a number of settings. Specifically, GMT-type translations τ_ε are defined parametrically in each order-type on a set PROP of propositional variables so as to preserve the syntactic shape of (Ω, ε)-inductive inequalities in passing from arbitrary DLE-languages to corresponding target Boolean algebra expansion languages (BAE-languages) enriched with additional S4-modalities $\Diamond_{\geq}$ and $\Box_{\leq}$. While *correspondence* via translation is obtained in full generality for inductive inequalities in arbitrary DLE-languages (cf. [16, Theorem 6.1]), the *canonicity* via translation of inductive inequalities is obtained in [16] only in the restricted setting of normal modal expansions of *bi-Heyting* algebras (bHAEs) (cf. [16, Theorem 7.1]). The argument can be summarized by means of the following diagram: for every bHAE $\mathbb{A}$ and every (Ω, ε)-inductive inequality $\varphi \leq \psi$ of compatible signature, a BAE $\mathbb{B}$ exists such that the vertical bi-implications hold. Hence, the canonicity of $\varphi \leq \psi$ follows from the fact that the BAE-inequality $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive inequality, and that every such inequality has been shown to be canonical within generalized Sahlqvist theory in the framework of classical (i.e., Boolean) modal logic (cf. [10]).

$$
\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^{\delta} \models \varphi \leq \psi \\
\Updownarrow & & \Updownarrow \\
\mathbb{B} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi) & \Leftrightarrow & \mathbb{B}^{\delta} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)
\end{array}
$$

As explained in [16, Section 7.2], this argument could not be carried beyond the setting of bHAEs only because, although $\tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$ has the appropriate (inductive) syntactic shape, if $\mathbb{A}$ is not a bHAE, the algebraic interpretation of the S4-modalities $\Diamond_{\geq}$ and $\Box_{\leq}$ in $\mathbb{B}$ turns out to be *slanted* (according to the terminology introduced in this article), and the then state-of-the-art theory of canonicity would not account for inequalities between terms built out of slanted connectives. However, we are now in a position to apply Theorem 4.1 to justify the horizontal bi-implication of the diagram above, and hence to obtain the canonicity of a restricted class of analytic inductive inequalities in arbitrary DLE-signatures as a transfer result of the *slanted canonicity* of analytic inductive BAE-inequalities. In what follows, we recall the definition of τ_{ε}, and then define the class of analytic inductive DLE-inequalities $\varphi \leq \psi$ such that $\tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$ is analytic inductive.

Parametrized Translation. Recall from [16, Section 5.2.1] that, for any normal DLE-signature $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the signature of the target language of the parametric GMT-type translations τ_{ε} is the normal BAE-signature $\mathcal{L}_{\mathrm{BAE}}^{\circ} = \mathcal{L}_{\mathrm{BAE}}(\mathcal{F}^{\circ}, \mathcal{G}^{\circ})$ where $\mathcal{F}^{\circ} := \{\Diamond_{\geq}\} \cup \{f^{\circ} \mid f \in \mathcal{F}\}$, and $\mathcal{G}^{\circ} := \{\Box_{\leq}\} \cup \{g^{\circ} \mid g \in \mathcal{G}\}$, and for every $f \in \mathcal{F}$ (respectively, $g \in \mathcal{G}$), the connective f° (respectively, g°) is such that $n_{f^{\circ}} = n_f$ (respectively, $n_{g^{\circ}} = n_g$) and $\varepsilon_{f^{\circ}}(i) = 1$ for each $1 \leq i \leq n_f$ (respectively, $\varepsilon_{g^{\circ}}(i) = 1$ for each $1 \leq i \leq n_g$).

The target language for the parametrized GMT translations over Prop is given by

$$\mathcal{L}_{\mathrm{BAE}}^{\circ} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid f^{\circ}(\overline{\alpha}) \mid g^{\circ}(\overline{\alpha}) \mid \Diamond_{\geq}\alpha \mid \Box_{\leq}\alpha.$$

For any order-type ε on PROP, the translation $\tau_{\varepsilon} : \mathcal{L}_{\mathrm{DLE}} \rightarrow \mathcal{L}_{\mathrm{BAE}}^{\circ}$ is defined by the following recursion:

$$
\tau_{\varepsilon}(p) = \begin{cases} \Box_{\leq} p & \text{if } \varepsilon(p) = 1 \\ \Diamond_{\geq} p & \text{if } \varepsilon(p) = \partial, \end{cases}
\qquad
\begin{array}{rcl}
\tau_{\varepsilon}(\bot) & = & \bot \\
\tau_{\varepsilon}(\top) & = & \top \\
\tau_{\varepsilon}(\varphi \wedge \psi) & = & \tau_{\varepsilon}(\varphi) \wedge \tau_{\varepsilon}(\psi) \\
\tau_{\varepsilon}(\varphi \vee \psi) & = & \tau_{\varepsilon}(\varphi) \vee \tau_{\varepsilon}(\psi) \\
\tau_{\varepsilon}(f(\overline{\varphi})) & = & \Diamond_{\geq} f^{\circ}\left(\overline{\tau_{\varepsilon}(\varphi)}^{\varepsilon_f}\right) \\
\tau_{\varepsilon}(g(\overline{\varphi})) & = & \Box_{\leq} g^{\circ}\left(\overline{\tau_{\varepsilon}(\varphi)}^{\varepsilon_g}\right)
\end{array}
$$

where for each order-type η on n and any n-tuple $\overline{\psi}$ of $\mathcal{L}_{\mathrm{BAE}}^{\circ}$-formulas, $\overline{\psi}^{\eta}$ denotes the n-tuple $(\psi_i')_{i=1}^{n}$, where $\psi_i' = \psi_i$ if $\eta(i) = 1$ and $\psi_i' = \neg\psi_i$ if $\eta(i) = \partial$.

It is clear from its definition that τ_{ε} is intended to preserve the (good or excellent) shape of the ε-critical branches of (Ω, ε)-inductive inequalities; however, τ_{ε} will systematically destroy the good shape of non-critical branches (i.e., ε^{∂}-critical branches) by inserting Skeleton nodes $+\Diamond_{\geq}$ and $-\Box_{\leq}$ in the scope of PIA nodes, whenever the given ε^{∂}-critical variable originally occurs in the scope of a PIA-connective. This motivates the following

Definition 5.1. For every order-type ε on PROP, an (Ω, ε)-analytic inductive inequality $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$ (cf. Notation 2.17) is τ_{ε}-*transferable* if for every maximal positive (respectively, negative) ε^{∂}-uniform PIA-subformula γ in $\overline{\gamma}$ (respectively, δ in $\overline{\delta}$), either $\gamma = q$ (respectively, $\delta = p$) for some $q \in$ PROP (respectively, $p \in$ PROP) such that $\varepsilon(q) = \partial$ (respectively, $\varepsilon(p) = 1$), or γ (respectively, δ) does not contain atomic propositions at all.

Example 5.2. The inequality $\Diamond(\Box p_1 \wedge \Box\Box p_2) \leq \Box(\Diamond\top \vee p_2) \wedge \Box(p_1 \vee \Diamond\Diamond\top)$ is τ_{ε}-transferable analytic ε-Sahlqvist for $\varepsilon(p_1, p_2) = (1, 1)$. Its τ_{ε}-translation is the following analytic ε-Sahlqvist inequality:

$$\Diamond_{\geq}\Diamond^{\circ}(\Box_{\leq}\Box^{\circ}\Box_{\leq}p_1 \wedge \Box_{\leq}\Box^{\circ}\Box_{\leq}\Box^{\circ}\Box_{\leq}p_2) \leq \Box_{\leq}\Box^{\circ}(\Diamond_{\geq}\Diamond^{\circ}\top \vee \Box_{\leq}p_2) \wedge \Box_{\leq}\Box^{\circ}(\Box_{\leq}p_1 \vee \Diamond_{\geq}\Diamond^{\circ}\Diamond_{\geq}\Diamond^{\circ}\top).$$

From the definition above, it immediately follows that

Proposition 5.3. *For every τ_ε-transferable (Ω, ε)-analytic inductive $\mathcal{L}_{\mathrm{DLE}}$-inequality $\varphi \leq \psi$, the $\mathcal{L}_{\mathrm{BAE}}$-inequality $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is analytic inductive, and hence s-canonical (cf. Theorem 4.1).*

Hence, we can extend [16, Theorem 7.1] as follows:

Theorem 5.4 (Canonicity via Translation). *For any order-type ε and any strict order Ω on* PROP*, the slanted canonicity theorem of analytic (Ω, ε)-inductive $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-inequalities transfers to the standard canonicity of τ_ε-transferable analytic (Ω, ε)-inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities.*

6 CANONICITY IN THE SETTING OF SUBORDINATION ALGEBRAS

In [17], the canonicity of a subclass of Sahlqvist formulas (the so-called s-Sahlqvist formulas, cf. Definition 6.14) in the signature of tense modal logic is shown w.r.t. the semantics of subordination algebras and their canonical extensions. In this section, we obtain a strengthening of this result as a consequence of Theorem 4.1, via the following steps: (a) Equivalently presenting subordination algebras as a class of *slanted BAEs* (cf. Definitions 6.6 and 6.3, and Proposition 6.7); (b) Verifying that satisfaction and validity of tense formulas/inequalities are preserved and reflected across this equivalent presentation (cf. Proposition 6.11); (c) Verifying that the algebraic canonicity of tense formulas in the setting of subordination algebras can be reduced to their *slanted* canonicity (cf. Proposition 6.12); and (d) Recognizing s-Sahlqvist formulas as a proper subclass of analytic inductive formulas of classical tense logic (cf. Proposition 6.15). Having understood the canonicity of s-Sahlqvist formulas in the setting of subordination algebras as an instance of slanted canonicity makes it possible to consider various extensions of this result which we discuss in the conclusions.

Definition 6.1 (Subordination Algebra). A *subordination algebra* is a pair $\mathbb{S} = (A, \prec)$ where A is a Boolean algebra and $\prec$ is a binary relation on A verifying the following conditions for all $a, b, c, d \in A$:

S1. $0 \prec 0$ and $1 \prec 1$;
S2. $a, b \prec c$ implies $a \vee b \prec c$;
S3. $a \prec b, c$ implies $a \prec b \wedge c$;
S4. $a \prec b \leq c \prec d$ implies $a \prec d$.

Properties S1–S4 imply that ${\prec}(a, -) := \{b \in A \mid a \prec b\}$ is a filter of A and ${\prec}(-, a) := \{b \in A \mid b \prec a\}$ is an ideal of A for every $a \in A$. In what follows, we will sometimes use the notations ${\prec}(S, -) := \bigcup\{{\prec}(a, -) \mid a \in S\}$ for any $S \subseteq A$, and ${\prec}(x, -) := \bigcup\{{\prec}(a, -) \mid x \leq a\}$ for any $x \in J^\infty(A^\delta)$.

Definition 6.2. A subordination algebra $\mathbb{S} = (A, \prec)$ is *complete* (respectively, *perfect*) if A is complete (respectively, complete and atomic), and $\prec$ satisfies the following infinitary versions of conditions S2 and S3: for all $a \in A$ and $S \subseteq A$,

S2$^\infty$. If $s \prec a$ for all $s \in S$, then $\bigvee S \prec a$;
S3$^\infty$. If $a \prec s$ for all $s \in S$, then $a \prec \bigwedge S$.

The *(constructive) canonical extension* of a subordination algebra $\mathbb{S} = (A, \prec)$ (cf. [17, Definition 1.10]) is the structure $\mathbb{S}^\delta := (A^\delta, \prec^\delta)$ such that A^δ is the canonical extension of A and $\prec^\delta$ is the binary relation defined as follows:

(1) If $k \in K(A^\delta)$ and $o \in O(A^\delta)$, then $k \prec^\delta o$ if $k \leq a \prec b \leq o$ for some $a, b \in A$;
(2) If $u, v \in A^\delta$, then $u \prec^\delta v$ if for all $k \in K(A^\delta)$ and $o \in O(A^\delta)$, $v \leq o$ and $k \leq u$ imply $k \prec^\delta o$.

The (constructive) canonical extension of a subordination algebra is a perfect (respectively, complete) subordination algebra (cf. [17, Definitions 1.7 and 1.10]).

Recall that a *tense BAE* is a BAE $\mathbb{A} = (A, \Diamond, \blacksquare)$ such that $\Diamond a \leq b$ iff $a \leq \blacksquare b$ for every $a, b \in A$. For any such tense BAE, we let $\Box$ and $\blacklozenge$ denote the modal operators dual to $\Diamond$ and $\blacksquare$, respectively. That is, $\Box a := \neg\Diamond\neg a$ and $\blacklozenge a := \neg\blacksquare\neg a$ for any $a \in A$. Perfect (respectively, complete) subordination algebras can be associated with perfect (respectively, complete) tense BAEs as follows:

Definition 6.3. For every perfect (respectively, complete) subordination algebra $\mathbb{S} = (A, \prec)$, its associated perfect (resp. complete) tense BAE is $\mathbb{S}^+ := (A, \Diamond^+, \blacksquare^+)$ where $\Diamond^+ : A \to A$ is defined by the assignment $u \mapsto \bigwedge\{v \in A \mid u \prec v\}$ and $\blacksquare^+ : A \to A$ is defined by the assignment $u \mapsto \bigvee\{v \in A \mid v \prec u\}$; for every perfect (respectively, complete) tense BAE $\mathbb{A} = (A, \Diamond, \blacksquare)$, we let $\mathbb{A}_+ := (A, \prec_+)$, where $u \prec_+ v$ iff $\Diamond u \leq v$, or equivalently, iff $u \leq \blacksquare v$ for all u, v in A.

Definition 6.4. A *tense slanted BAE* is a slanted BAE $\mathbb{S} = (A, \Diamond, \blacksquare)$ such that A is a Boolean algebra, $\Diamond : A \to A^\delta$ is a c-slanted finitely join-preserving map, $\blacksquare : A \to A^\delta$ is an o-slanted finitely join-preserving map and moreover, for every $a, b \in A$,

$$\Diamond a \leq b \quad \text{iff} \quad a \leq \blacksquare b.$$

For such an s-algebra, we let $\Box : A \to A^\delta$ denote the o-slanted operator defined by the assignment $a \mapsto \neg^{A^\delta}\Diamond\neg^A a$ and $\blacklozenge : A \to A^\delta$ denote the c-slanted operator defined by the assignment $a \mapsto \neg^{A^\delta}\blacksquare\neg^A a$. It is straightforward to show that $\blacklozenge a \leq b$ iff $a \leq \Box b$ for every $a, b \in A$.

Lemma 6.5. *If $\mathbb{S} = (A, \Diamond, \blacksquare)$ is a tense slanted BAE, then its canonical extension $\mathbb{S}^\delta = (A^\delta, \Diamond^\delta, \blacksquare^\delta)$ is a perfect tense BAE.*

Proof. Let $k \in K(A^\delta)$ and $o \in O(A^\delta)$ such that $\Diamond^\sigma k \leq o$, that is $\bigwedge\{\Diamond a \mid k \leq a \in A\} \leq \bigvee\{b \in A \mid b \leq o\}$. By compactness and the monotonicity of $\Diamond$, this implies that $\Diamond a_0 \leq b_0$ for some $a_0 \geq k$ and $b_0 \leq o$. So, by adjunction, $a_0 \leq \blacksquare b_0$. Hence

$$k = \bigwedge\{a \in A \mid k \leq a\} \leq a_0 \leq \blacksquare b_0 \leq \bigvee\{\blacksquare b \mid A \ni b \leq o\} = \blacksquare^\pi o.$$

Let $u, v \in A^\delta$ such that $\Diamond^\delta u \leq v$. Then $\Diamond^\sigma k \leq o$, and hence (cf. argument above) $k \leq \blacksquare^\pi o$, for all $K(A^\delta) \ni k \leq u$ and all $O(A^\delta) \ni o \geq v$. Therefore,

$$u = \bigvee\{k \in K(A^\delta) \mid k \leq u\} \leq \bigwedge\{\blacksquare^\pi o \mid v \leq o \in O(A^\delta)\} = \blacksquare^\delta o,$$

as required. Dually, one shows that $u \leq \blacksquare^\delta v$ implies $\Diamond^\delta u \leq v$ for all $u, v \in A^\delta$, which completes the proof that $\mathbb{S}^\delta$ is a tense algebra. □

Subordination algebras can be equivalently presented as tense slanted BAEs as follows:

Definition 6.6. For every subordination algebra $\mathbb{S} = (A, \prec)$, its associated tense slanted BAE is $\mathbb{S}^* := (A, \Diamond_\prec, \blacksquare_\prec)$ where $\Diamond_\prec : A \to A^\delta$ is defined by the assignment $a \mapsto \bigwedge\{b \in A \mid a \prec b\} \in K(A^\delta)$ and $\blacksquare_\prec : A \to A^\delta$ by the assignment $a \mapsto \bigvee\{b \in A \mid b \prec a\} \in O(A^\delta)$; for every tense slanted BAE $\mathbb{A} = (A, \Diamond, \blacksquare)$, its associated subordination algebra is $\mathbb{A}_* := (A, \prec_\Diamond)$, where $a \prec_\Diamond b$ iff $\Diamond a \leq b$ iff $a \leq \blacksquare b$.

Notice that the defining assignments of $\Diamond^+$ and $\Diamond_\prec$ (resp. of $\blacksquare^+$ and $\blacksquare_\prec$) are verbatim "the same"(however, the meets and joins are taken in different algebras) but the functional types of $\Diamond^+$ and $\Diamond_\prec$ (respectively, of $\blacksquare^+$ and $\blacksquare_\prec$) are different.

Proposition 6.7. *For every subordination algebra $\mathbb{S} = (A, \prec)$ and every tense slanted BAE $\mathbb{A} = (A, \Diamond, \blacksquare)$,*

(1) *$\mathbb{S}^*$ is a tense slanted BAE, and if $\mathbb{S}$ is perfect, then $\mathbb{S}^+$ is a perfect tense BAE in the standard sense;*
(2) *$\mathbb{A}_*$ is a subordination algebra, and if $\mathbb{A}$ is a perfect tense BAE in the standard sense, then $\mathbb{A}_+$ is a perfect subordination algebra;*
(3) *$(\mathbb{S}^*)_* = \mathbb{S}$ and if $\mathbb{S}$ is perfect, then $(\mathbb{S}^+)_+ = \mathbb{S}$;*
(4) *$(\mathbb{A}_*)^* = \mathbb{A}$ and if $\mathbb{A}$ is perfect, then $(\mathbb{A}_+)^+ = \mathbb{A}$;*
(5) *$(\Diamond_\prec)^\delta = \Diamond_{\prec^\delta}$ and $(\blacksquare_\prec)^\delta = \blacksquare_{\prec^\delta}$;*
(6) *$\prec_{\Diamond^\delta} = (\prec_\Diamond)^\delta$;*
(7) *$(\mathbb{S}^\delta)^+ = (\mathbb{S}^*)^\delta$;*
(8) *$(\mathbb{A}^\delta)_+ = (\mathbb{A}_*)^\delta$;*
(9) *if $\mathbb{S}$ is perfect, then $(\mathbb{S}^\delta)^+ = (\mathbb{S}^+)^\delta$;*
(10) *if $\mathbb{A}$ is perfect, then $(\mathbb{A}^\delta)_+ = (\mathbb{A}_+)^\delta$.*

Proof.

(1) By construction, $\Diamond_\prec$ and $\blacksquare_\prec$ are c-slanted and o-slanted respectively. Hence, it is enough to show that they are normal and satisfy the tense condition. The identities $\Diamond_\prec 0 = 0$ and $\blacksquare_\prec 1 = 1$ follow directly from S1. Moreover, for any $a, b \in A$, axiom S4 implies that ${\prec}(a, -) \cup {\prec}(b, -) \supseteq {\prec}(a \vee b, -)$, which implies that $\Diamond_\prec a \vee \Diamond_\prec b \leq \Diamond_\prec(a \vee b)$. Conversely, $\Diamond_\prec a \vee \Diamond_\prec b = \bigwedge\{c \vee d \mid a \prec c \text{ and } b \prec d\}$. From S2 and S4, if $a \prec c$ and $b \prec d$ then $a \vee b \prec c \vee d$. Hence, $\Diamond_\prec(a \vee b) \leq \Diamond_\prec a \vee \Diamond_\prec b$, as required. Similarly, one shows that $\blacksquare_\prec(a \wedge b) = \blacksquare_\prec a \wedge \blacksquare_\prec b$. Finally, for every $a, b \in A$,
$$\Diamond_\prec a \leq b \quad \text{iff} \quad a \prec b \quad \text{iff} \quad a \leq \blacksquare_\prec b. \tag{8}$$
Indeed, by construction, $a \prec b$ implies $\Diamond_\prec a \leq b$ and $a \leq \blacksquare_\prec b$. Moreover, if $\Diamond_\prec a \leq b$, then compactness and the definition of $\Diamond_\prec$ imply that $a \prec c \leq b$ for some $c \in A$, which implies $a \prec b$ by S4. Similarly, one shows that $a \leq \blacksquare_\prec b$ implies $a \prec b$, which completes the proof that $\Diamond_\prec$ and $\blacksquare_\prec$ satisfy the tense condition.
The proof of the second part of the statement (when $\mathbb{S}$ is perfect) is similar with a slight difference: the equivalence (8) arises from the completeness of $\prec$ rather than from the compactness of A^δ. Indeed, $\Diamond^+ a \leq b$ implies that $a \prec \bigwedge\{c \in A \mid a \prec c\} = \Diamond^+ a \leq b$, which implies $a \prec b$ by S4.
(2) It is routine to show that $\prec_\Diamond$ (respectively, $\prec_+$) satisfies conditions S1 to S4 (respectively, their infinitary versions).
(3) By definition, the underlying Boolean algebras of $\mathbb{S}$ and $(\mathbb{S}^*)_*$ (respectively, $(\mathbb{S}^+)_+$ if $\mathbb{S}$ is perfect) are identical. Moreover, equivalences (8), already proven in item 1, imply that the subordination relations of $\mathbb{S}$ and $(\mathbb{S}^*)_*$ (respectively, $(\mathbb{S}^+)_+$) coincide.
(4) The tense BAEs $(\mathbb{A}_*)^*$ and $\mathbb{A}$ share the same underlying Boolean algebra. Hence, $(\mathbb{A}_*)^* = \mathbb{A}$ if and only if $\Diamond_{\prec_\Diamond} a := \bigwedge\{b \in A \mid \Diamond a \leq b\} = \Diamond a$, and $\blacksquare_{\prec_\Diamond} a := \bigvee\{b \in A \mid \Diamond b \leq a\} = \bigvee\{b \in A \mid b \leq \blacksquare a\} = \blacksquare a$. These identities immediately follow from $\Diamond a \in K(A^\delta)$ and $\blacksquare a \in O(A^\delta)$.
For the perfect case, the equalities $\Diamond a = \Diamond_{\prec_\Diamond} a$ and $\blacksquare a = \blacksquare_{\prec_\Diamond} a$ are trivially verified, since $\Diamond a$ and $\blacksquare a$ are elements of A and the infimum and supremum are taken in A itself.
(5) Let us preliminarily show that $(\Diamond_\prec)^\sigma k = \Diamond_{\prec^\delta} k$ for any $k \in K(A^\delta)$. In order to show that $\bigwedge\{u \in A^\delta \mid k \prec^\delta u\} =: \Diamond_{\prec^\delta} k \leq (\Diamond_\prec)^\sigma k := \bigwedge\{\Diamond_\prec a \mid a \in A \text{ and } k \leq a\}$, it is enough to show that $k \prec^\delta \Diamond_\prec a$ for all $a \in A$ such that $k \leq a$. Since k is closed, by definition (cf. item 2 of Definition 6.2), this is equivalent to showing that $k \prec^\delta o$ for every $o \in O(A^\delta)$ such that

$\Diamond_{\prec} a \leq o$. By compactness, $\Diamond_{\prec} a := \bigwedge\{b \in A \mid a \prec b\} \leq o$ implies that $b_1 \wedge \ldots \wedge b_n \leq o$ for some $b_1, \ldots, b_n \in \prec(a, -)$. Hence, by axiom S3, $k \leq a \prec b_1 \wedge \ldots \wedge b_n \leq o$, which shows that $k \prec^\delta o$, as required.

Conversely, note first that, by denseness,

$$\Diamond_{\prec^\delta} k := \bigwedge\{u \in A^\delta \mid k \prec^\delta u\} = \bigwedge\{o \in O(A^\delta) \mid k \prec^\delta o\}.$$

Hence, to prove $(\Diamond_{\prec})^\sigma k \leq \Diamond_{\prec^\delta} k$, it is enough to show that $(\Diamond_{\prec})^\sigma k \leq o$ for every $o \in O(A^\delta)$ such that $k \prec^\delta o$. For such an o, by definition, $k \leq a \prec b \leq o$ for some $a, b \in A$. Hence, by definition, $(\Diamond_{\prec})^\sigma k \leq \Diamond_{\prec} a \leq b \leq o$, as required. The identity $(\Diamond_{\prec})^\sigma u = \Diamond_{\prec^\delta} u$ for all $u \in A^\delta$ follows straightforwardly from $(\Diamond_{\prec})^\sigma k = \Diamond_{\prec^\delta} k$ for all $k \in K(A^\delta)$ using the denseness of A^δ and the complete join-preservation of $\Diamond_{\prec^\delta}$ and $(\Diamond_{\prec})^\delta$.

At the same time, one shows that $(\blacksquare_{\prec})^\pi o = \blacksquare_{\prec^\delta} o$ for all $o \in O(A^\delta)$ and therefore, $(\blacksquare_{\prec})^\pi u = \blacksquare_{\prec^\delta} u$ for all $u \in A^\delta$.

(6) Let us preliminarily show that $k \prec_{\Diamond^\delta} o$ iff $k\ (\prec_\Diamond)^\delta\ o$ for every $k \in K(A^\delta)$ and $o \in O(A^\delta)$. If $k \prec_{\Diamond^\delta} o$, that is

$$\bigwedge\{\Diamond a \mid a \in A \text{ and } k \leq a\} =: \Diamond^\delta k \leq o = \bigvee\{b \in A \mid b \leq o\},$$

then, by compactness and since $\Diamond$ is monotone, $\Diamond a \leq b$ (i.e., $a \prec_\Diamond b$) for some $a \in A$ and $b \in A$ such that $k \leq a$ and $b \leq o$. Hence, $k\ (\prec_\Diamond)^\delta\ o$. Conversely, if $k\ (\prec_\Diamond)^\delta\ o$, i.e., if $\Diamond a \leq b$ for some $a, b \in A$ such that $k \leq a$ and $b \leq o$, then $\Diamond^\delta k \leq \Diamond a \leq b \leq o$, which yields $k \prec_{\Diamond^\delta} o$, as required. Let us show that $u \prec_{\Diamond^\delta} v$ iff $u\ (\prec_\Diamond)^\delta\ v$ for all $u, v \in A^\delta$. We have

$$u\ (\prec_\Diamond)^\delta\ v \tag{9}$$
$$\iff k\ (\prec_\Diamond)^\delta\ o \text{ for any } k \in K(A^\delta) \text{ and } o \in O(A^\delta) \text{ such that } k \leq u \text{ and } v \leq o \tag{10}$$
$$\iff k \prec_{\Diamond^\delta} o \text{ for any } k \in K(A^\delta) \text{ and } o \in O(A^\delta) \text{ such that } k \leq u \text{ and } v \leq o \tag{11}$$
$$\iff \bigvee\{k \in K(A^\delta) \mid k \leq u\} \prec_{\Diamond^\delta} \bigwedge\{o \in O(A^\delta) \mid v \leq o\} \tag{12}$$
$$\iff u \prec_{\Diamond^\delta} v \tag{13}$$

where (9) $\iff$ (10) is the definition of $(\prec_\Diamond)^\delta$, (10) $\iff$ (11) is obtained via the preliminary claim, (11) $\iff$ (12) follows from axioms S2$^\infty$ and S3$^\infty$ for $\prec_{\Diamond^\delta}$ (cf. item (2)) and finally (12) $\iff$ (13) is denseness.

(7) and (8) A^δ is the Boolean algebra underlying $(\mathbb{S}^\delta)^+$, $(\mathbb{S}^*)^\delta$, $(\mathbb{A}^\delta)_+$ and $(\mathbb{A}_*)^\delta$. Moreover, the modal operators of $(\mathbb{S}^\delta)^+$ and $(\mathbb{S}^*)^\delta$ are, respectively, $\Diamond_{\prec^\delta}$ and $\blacksquare_{\prec^\delta}$ and $(\Diamond_{\prec})^\delta$ and $(\blacksquare_{\prec})^\delta$ which coincide pairwise (cf. item 5). Finally, the subordination relations of $(\mathbb{A}^\delta)_+$ and $(\mathbb{A}_*)^\delta$ are, respectively, $\prec_{\Diamond^\delta}$ and $(\prec_\Diamond)^\delta$ which coincide, (cf. item (6)).

(9) and (10) The proofs are relatively similar to the ones of the non-perfect case with slightly different justifications: as in item (1), the completeness of $\prec$, $\Diamond$, and $\blacksquare$ is used instead of the compactness of A^δ. As an example, we prove item (9) and leave item (10) to the reader. As remarked above, $(\mathbb{S}^\delta)^+$ and $(\mathbb{S}^+)^\delta$ have A^δ as their underlying Boolean algebras. Hence, to finish the proof, let us show that the modal operators coincide. Since $(\Diamond^+)^\delta$ and $\Diamond_{\prec^\delta}$ are completely join-preserving, by denseness it enough to show that for every $k \in K(A^\delta)$,

$$\begin{aligned}\Diamond_{\prec^\delta} k &:= \bigwedge\{o \in O(A^\delta) \mid k \leq a \prec b \leq o \text{ for some } a, b \in A\}\\ &= \bigwedge\{\Diamond^+ a \mid k \leq a \in A\} := (\Diamond^+)^\delta k.\end{aligned}$$

If $k \leq a$, then $b := \Diamond^+ a \in A \subseteq O(A^\delta)$ and $k \leq a \prec \Diamond^+ a \leq \Diamond^+ a$, which implies that $\Diamond_{\prec^\delta} k \leq (\Diamond^+)^\delta k$. Conversely, if $o \in O(A^\delta)$ is such that $k \leq a \prec b \leq o$ for some $a, b \in A$, then

$\Diamond^+ a \leq b \leq o$ and hence $(\Diamond^+)^\delta k \leq \Diamond^+ a \leq b \leq o \leq \Diamond_{\prec^\delta} k$. At the same time, one shows that $\blacksquare_{\prec^\delta} o = (\blacksquare^+)^\delta o$ for every $o \in O(A^\delta)$, which is enough to prove that $\blacksquare_{\prec^\delta}$ and $(\blacksquare^+)^\delta$ coincide.

Remark 6.8. In Proposition 6.7, we showed that subordination algebras can be equivalently presented as tense slanted BAEs. In fact, subordination algebras can be also equivalently presented both as slanted BAEs of the form $\mathbb{A}_c = (A, \Diamond)$ (which we refer to as *closed slanted BAEs*), and as slanted BAEs of the form $\mathbb{A}_o = (A, \blacksquare)$ (which we refer to as *open slanted BAEs*). Hence, closed, open, and tense slanted BAEs are all equivalent presentations. These equivalences can of course be described without using subordination algebras as mediators. Namely, a slanted tense BAE $\mathbb{A} = (A, \Diamond, \blacksquare)$ is mapped to the closed slanted BAE $\mathbb{A}_c = (A, \Diamond)$ while a closed slanted BAE $\mathbb{A} = (A, \Diamond)$ is mapped to the tense algebra $\mathbb{A}_t = (A, \Diamond, \blacksquare_\Diamond)$ where $\blacksquare_\Diamond$ is the restriction to A of the adjoint of $\Diamond^\delta$. The equivalence between tense and open BAEs is defined similarly.

Remark 6.9. Open slanted BAEs (in the sense of Remark 6.8) are isomorphic to the quasi-modal algebras developed by Celani [3]. Recall that a *quasi-modal algebra* is a pair $\mathbb{Q} = (B, \Delta)$ where B is a Boolean algebra, $\Delta : B \longrightarrow \mathcal{I}(B)$, where $\mathcal{I}(B)$ denotes the set of the ideals of B, satisfying the following conditions: $\Delta(a \wedge b) = \Delta a \cap \Delta b$ and $\Delta 1 = A$. It is then clear that the order-isomorphism between the ideals of B and open elements of B^δ (cf. [20, Theorem 2.5]) can be used to establish an equivalence between quasi-modal algebras and open slanted BAEs (see Example 3.3). But this equivalence is not surprising, given that subordination algebras and quasi-modal algebras are known to be equivalent (cf., e.g., [4, Theorem 15]).

Let $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$ be the BAE language such that $\mathcal{F} = \{\Diamond, \blacklozenge\}$ and $\mathcal{G} = \{\Box, \blacksquare\}$, all modal connectives being unary and positive. Satisfaction and validity of $\mathcal{L}$-formulas/inequalities on subordination algebras can be defined in terms of Definition 6.3 as follows:

Definition 6.10. Definition 6.10. For every subordination algebra $\mathbb{S} = (A, \prec)$ every assignment $v : \mathsf{PROP} \longrightarrow A$, and every modal inequality $\varphi \leq \psi$,

$$(\mathbb{S}, v) \models \varphi \leq \psi \quad \text{iff} \quad ((\mathbb{S}^\delta)^+, e \cdot v) \models \varphi \leq \psi$$

where $e : A \longrightarrow A^\delta$ is the canonical embedding. As to validity,

$$\mathbb{S} \models \varphi \leq \psi \quad \text{iff} \quad (\mathbb{S}^\delta)^+ \models_{\mathbb{S}} \varphi \leq \psi.$$

Proposition 6.11. *For every (perfect) subordination algebra $\mathbb{S} = (A, \prec)$ every slanted (respectively, perfect) BAE $\mathbb{A} = (A, \Diamond)$, and every $\mathcal{L}$-inequality $\varphi \leq \psi$,*

(1) $\mathbb{S} \models \varphi \leq \psi$ *iff* $\mathbb{S}^* \models \varphi \leq \psi$;
(2) $\mathbb{A} \models \varphi \leq \psi$ *iff* $\mathbb{A}_* \models \varphi \leq \psi$;
(3) $\mathbb{S} \models \varphi \leq \psi$ *iff* $\mathbb{S}^+ \models \varphi \leq \psi$;
(4) $\mathbb{A} \models \varphi \leq \psi$ *iff* $\mathbb{A}_+ \models \varphi \leq \psi$.

Proof. For item (1), we recall that $\mathbb{S} \models \varphi \leq \psi$ if and only if $(\mathbb{S}^\delta)^+ \models_{\mathbb{S}} \varphi \leq \psi$. We also recall that, by Proposition 6.7, we have $(\mathbb{S}^\delta)^+ = (\mathbb{S}^*)^\delta$. Hence, we have $\mathbb{S} \models \varphi \leq \psi$ if and only if $(\mathbb{S}^*)^\delta \models_{\mathbb{S}} \varphi \leq \psi$. The conclusion now follows from the fact that $\mathbb{S}$ and $\mathbb{S}^*$ have the same underlying Boolean algebra. Items (2) to (4) are proved similarly. $\square$

Proposition 6.12. *For every s-canonical $\mathcal{L}$-inequality $\varphi \leq \psi$ and every subordination algebra $\mathbb{S}$,*

$$\mathbb{S} \models \varphi \leq \psi \quad \Leftrightarrow \quad \mathbb{S}^\delta \models \varphi \leq \psi.$$

Proof. The argument can be summarized by means of the following diagram:

$$\begin{array}{ccc}
\mathbb{S} \models \varphi \leq \psi & & \mathbb{S}^\delta \models \varphi \leq \psi \\
& & \Updownarrow \\
\Updownarrow & & ((\mathbb{S}^*)^\delta)_+ \models \varphi \leq \psi \\
& & \Updownarrow \\
\mathbb{S}^* \models \varphi \leq \psi & \Leftrightarrow & (\mathbb{S}^*)^\delta \models \varphi \leq \psi
\end{array}$$

The bi-implication on the left is due to Proposition 6.11.1; the horizontal bi-implication holds by assumption; the lower bi-implication on the right is due to Proposition 6.11.4; the upper bi-implication on the right is due to Proposition 6.7.7. □

Hence, as an immediate consequence Proposition 6.12 and Theorem 4.1, we get the following:

Corollary 6.13. *For every analytic inductive $\mathcal{L}$-inequality $\varphi \leq \psi$ and every subordination algebra $\mathbb{S}$,*

$$\mathbb{S} \models \varphi \leq \psi \quad \Leftrightarrow \quad \mathbb{S}^\delta \models \varphi \leq \psi.$$

Finally, we show that the corollary above strengthens [17, Corollary 3.8], by verifying that sub-Sahlqvist $\mathcal{L}$-formulas are a proper subclass of analytic inductive $\mathcal{L}$-formulas.

Definition 6.14.

(1) An $\mathcal{L}$-formula is *closed* (respectively, *open*) if it is built up from constants $\top$, $\bot$, propositional variables and their negations, by applying $\vee$, $\wedge$, $\Diamond$, and $\blacklozenge$ (respectively, $\vee$, $\wedge$, $\Box$, and $\blacksquare$);
(2) An $\mathcal{L}$-formula is *positive* (respectively, *negative*) if it is built up from constants $\top$, $\bot$, and propositional variables (respectively, negations of propositional variables) by applying $\wedge$, $\vee$, $\Diamond$, $\Box$, $\blacklozenge$, and $\blacksquare$;
(3) An $\mathcal{L}$-formula is *sub-positive* (respectively, *sub-negative*) if it is built up from closed positive formulas (respectively, open negative formulas) by applying $\vee$, $\wedge$, $\Box$, and $\blacksquare$ (respectively, $\vee$, $\wedge$, $\Diamond$, and $\blacklozenge$);
(4) A *boxed atom* is an $\mathcal{L}$-formula built up from propositional variables by applying $\Box$ and $\blacksquare$;
(5) An $\mathcal{L}$-formula is *strongly positive* if it is a conjunction of boxed atoms;
(6) An $\mathcal{L}$-formula is *untied* if it is built up from strongly positive and sub-negative formulas using only $\wedge$, $\Diamond$, and $\blacklozenge$; and
(7) A *sub-Sahlqvist formula* is an $\mathcal{L}$-formula of the form $\psi[(\varphi_1 \to \varphi_2)/!x]$ where $\psi(!x)$ is a boxed atom, φ_1 is untied, and φ_2 is sub-positive.

Proposition 6.15. *For every $\mathcal{L}$-formula φ, letting ε denote the order-type constantly equal to 1,*

(1) *If φ is closed (respectively, open), then $-\varphi$ is PIA (respectively, Skeleton);*
(2) *If φ is positive (respectively, negative), then $-\varphi$ (respectively, $+\varphi$) is ε^∂-uniform.*
(3) *If φ is closed positive (respectively, open negative), then $-\varphi$ (respectively, $+\varphi$) is ε^∂-uniform PIA.*
(4) *If φ is strongly positive, then $+\varphi$ is PIA, and each of its branches is excellent.*
(5) *If φ is sub-positive (respectively, sub-negative), then $-\varphi$ (respectively, $+\varphi$) is ε^∂-uniform and all of its branches are good.*
(6) *If φ is untied, then $+\varphi$ is analytic ε-Sahlqvist.*
(7) *If φ is sub-Sahlqvist, then $-\varphi$ is analytic ε-Sahlqvist, hence so is $\top \leq \varphi$.*

Proof. Items 1 and 2 immediately follow from the definitions involved. Item 3 is an immediate consequence of items 1 and 2. Item 4 immediately follows from item 3 and the definition of excellent branch. Item 5 follows from item 3 and the definition of a good branch. Item 6 follows from the fact that, by items 4 and 5, any untied formula is built up from positive PIA-formulas every branch of which is excellent and ε^∂-uniform formulas every branch of which is good, by applying Skeleton connectives. Clearly, this application will maintain both the good shape of ε^∂-critical branches, and the excellent shape of ε-critical branches. Finally, item (7) follows from the observation that if $\psi(!x)$ is a boxed atom, then $-\psi[(y \to z)/!x]$ is a Skeleton formula, and hence, replacing the placeholder variable z with the ε^∂-uniform formula φ_2 all of the branches of which are good, and the placeholder variable x with the analytic ε-Sahlqvist formula φ_1 will yield again an analytic ε-Sahlqvist formula. □

7 CONCLUSIONS

In this article, we have explored the *topological* properties of a class of LE-inequalities, the *analytic inductive inequalities*, which has been originally introduced in [29] as a concrete syntactic approximation of the *proof-theoretic* notion of analyticity in the context of proper display calculi [37]. The theoretical background in which this connection between topological and proof-theoretic properties could be established is *unified correspondence theory* [9], which applies algebraic and duality-theoretic techniques in the development of (generalized) Sahlqvist correspondence and canonicity results for nonclassical logics, and which has recently established systematic connections between generalized Sahlqvist theory and the core issue in structural proof theory of identifying large classes of analytic axioms and algorithmically computing their corresponding analytic structural rules, yielding precisely the notion of analytic inductive inequalities. The main result of the present paper is that the topological properties induced by the syntactic shape of analytic inductive LE-inequalities guarantee their algebraic canonicity in the setting of *slanted* LE-algebras of the appropriate signature (cf. Definition 3.2). This canonicity result connects and extends a number of recent canonicity results in very different areas: subordination algebras, quasi-modal algebras, and the transfer of canonicity via Gödel-McKinsey-Tarski translations.

Slanted LEs as a Comprehensive Mathematical Environment. In this article, we attributed a name to a notion (that of *slanted operations*, cf. Definition 3.1, from which the ensuing notion of *slanted algebra* derives) instances of which have cropped up in the literature in many contexts and with different angles, scopes, and motivations, spanning from the theory of (generalized) canonical extensions of maps [23] and their adjoints [34], to de Vries algebras [18] and their generalizations (in the equivalent forms of quasi-modal algebras, [3], pre-contact algebras [19] and subordination algebras [1]), and the Gödel-McKinsey-Tarski translation [16]. While the connection with duality-theoretic aspects is very much present in each of these contexts taken separately, the environment of slanted algebras as defined in this article makes it possible to provide a purely algebraic, modular, and uniform reformulation and generalization of extant results, and explore, as we have started to do, generalized settings, such as the (constructive) "non-distributive"one of the present paper, also paving the way towards their investigation with duality-theoretic and topological techniques on relational structures based, e.g., on polarities and reflexive graphs (cf. [14]). This line of investigation is ongoing.

Equivalence, Morphisms, and Duality. Related to the previous point, the environment of slanted LE lends itself naturally to be investigated with universal algebraic and category-theoretic tools, starting with the definition of slanted homomorphisms as lattice homomorphisms $h : \mathbb{A}_1 \to \mathbb{A}_2$ such as the following diagrams commute for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$
\begin{array}{ccc}
\mathbb{A}_1^\delta & \xrightarrow{h^\delta} & \mathbb{A}_2^\delta \\
f^{\mathbb{A}_1} \uparrow & & \uparrow f^{\mathbb{A}_2} \\
\mathbb{A}_1^{\varepsilon_f} & \xrightarrow{h^{\varepsilon_f}} & \mathbb{A}_2^{\varepsilon_f}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{A}_1^\delta & \xrightarrow{h^\delta} & \mathbb{A}_2^\delta \\
g^{\mathbb{A}_1} \uparrow & & \uparrow g^{\mathbb{A}_2} \\
\mathbb{A}_1^{\varepsilon_g} & \xrightarrow{h^{\varepsilon_g}} & \mathbb{A}_2^{\varepsilon_g}
\end{array}
$$

This line of investigation is ongoing.

From Normal to Non-Normal Settings. Although the best-known examples of applications of the theory of canonical extensions (e.g., [31]) concern logics in which the additional operations are all normal (i.e., coordinate-wise preserving or reversing all finite joins, for f-type operations, or meets, for g-type operations), the theory itself applies to arbitrary maps [23], and has already been applied to develop canonicity, correspondence and proof-theoretic results for non-normal logics in several settings, including the Boolean [5], the distributive [35] and the general lattice [7, 13]. In this article, we have addressed slanted canonicity in the setting of *normal* slanted LEs, in the sense indicated above (see also the discussion after Definition 3.1). A further direction that can be naturally pursued in this algebraic context concerns the development of (constructive) slanted canonicity results in the context of *non-normal* slanted algebras. This direction invests the study of the notion of *weakening relation* [33] as generalized subordination, and its possible applications in obtaining semantic cut elimination results generalizing those in, e.g., [28].

A APPENDIX

A.1 Topological Properties of Slanted Operations and Their Residuals

Fix a language $\mathcal{L}_{\mathrm{LE}}$, and a *slanted* $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (A, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ for the remainder of this section. This subsection collects the relevant order-theoretic and topological properties of the additional operations of $\mathbb{A}$ and their adjoints, which will be critical for the proof of the “topological versions” of the Ackermann lemmas in Section A.2. These results are the straightforward generalization to the setting of slanted algebras of properties that are well known to hold in the setting of normal LEs (e.g., [12, Section 10]). In what follows, we use the terminology ∂-monotone (respectively, ∂-antitone, ∂-positive, ∂-negative, ∂-open, ∂-closed) to mean its opposite, i.e., antitone (respectively, monotone, negative, positive, closed, open). By 1-monotone (respectively, antitone, positive, negative, open, closed) we simply mean monotone (antitone, positive, negative, open, closed). Also in symbols, for example, we will write $(O(\mathbb{A}^\delta))^1$ for $O(\mathbb{A}^\delta)$ and $(O(\mathbb{A}^\delta))^\partial$ for $K(\mathbb{A}^\delta)$, and similarly $(K(\mathbb{A}^\delta))^1$ for $K(\mathbb{A}^\delta)$ and $K(\mathbb{A}^\delta)^\partial$ for $O(\mathbb{A}^\delta)$. This convention generalizes to order-types and tuples in the obvious way. Thus, for example, $(O(\mathbb{A}^\delta))^\varepsilon$ is the Cartesian product of sets with $O(\mathbb{A}^\delta)$ as ith coordinate where $\varepsilon_i = 1$ and $K(\mathbb{A}^\delta)$ for jth coordinate where $\varepsilon_j = \partial$.

Lemma A.1. *For all $f \in \mathcal{F}^{\mathbb{A}}$, $g \in \mathcal{G}^{\mathbb{A}}$, $\overline{k} \in (K(\mathbb{A}^\delta))^{\varepsilon_f}$, and $\overline{o} \in (O(\mathbb{A}^\delta))^{\varepsilon_g}$,*

(1) $g(\overline{o}) \in O(\mathbb{A}^\delta)$,
(2) $f(\overline{k}) \in K(\mathbb{A}^\delta)$.

Proof. These facts straightforwardly follow from the fact that each $f \in \mathcal{F}^{\mathbb{A}^\delta}$ (respectively, $g \in \mathcal{G}^{\mathbb{A}^\delta}$) is the σ-extension (respectively, π-extension) of the corresponding operation in $\mathbb{A}$: for instance, $g(\overline{o}) = g^\pi(\overline{o}) = \bigvee\{g(\overline{a}) \mid \overline{a} \in \mathbb{A}^{\varepsilon_g} \text{ and } \overline{a} \leq^{\varepsilon_g} \overline{o}\}$, and $g(\overline{a}) \in O(\mathbb{A}^\delta)$ for each $\overline{a} \in \mathbb{A}^{\varepsilon_g}$. □

Remark A.2. In the standard setting in which any $f \in \mathcal{F}^{\mathbb{A}}$ and $g \in \mathcal{G}^{\mathbb{A}}$ maps tuples of clopen elements to clopen elements, it also holds (cf. [12, Lemma 10.2]) that, for all $\overline{k} \in (K(\mathbb{A}^\delta))^{\varepsilon_f}$, and $\overline{o} \in (O(\mathbb{A}^\delta))^{\varepsilon_g}$,

(1) If $\overline{o} \in (O(\mathbb{A}^\delta))^{\varepsilon_g}$, then $f(\overline{o}) \in O(\mathbb{A}^\delta)$,
(2) If $\overline{k} \in (K(\mathbb{A}^\delta))^{\varepsilon_f}$, then $g(\overline{k}) \in K(\mathbb{A}^\delta)$.

Clearly, these properties do not hold in the setting of slanted LEs, as, together with Lemma A.1 they would imply that any slanted operation maps tuples of clopen elements to clopen elements, which is not true. For a counterexample, let A be an infinite Boolean algebra and x_0 be an atom of A^δ which is not clopen. Then, we define the c-slanted operator $\Diamond$ on A defined by the assignment $\Diamond a := a \vee x_0$ for each $a \in A$. It is clear that $a \vee x_0$ is not open for every a with $x_0 \nleq a$, as it would imply that $x_0 = (a \vee x_0) \wedge \neg a$ is open. One can find a counterexample for a $g \in \mathcal{G}^{\mathbb{A}}$ in a similar fashion.

In the standard setting, these properties are used in the proofs of the counterparts of Lemmas A.6 and A.7 below (cf. Lemmas 10.6 and 10.7 of [12]). However, rather than being formulated in terms of syntactically open and closed formulas, Lemmas A.6 and A.7 are formulated in terms of the more restricted notions of ssc and sso, which is why their proofs go through nonetheless.

The proof of the following lemma is verbatim the same as the one of Lemma 10.3 in [12], since, in that proof, it is only needed that $g(\overline{a})$ is an open element and $f(\overline{a})$ is a closed element. For the sake of self-containdness, we report the proof.

Lemma A.3. *For all $f \in \mathcal{F}$, $g \in \mathcal{G}$, $1 \leq i \leq n_f$, and $1 \leq j \leq n_g$,*

(1) *If $\varepsilon_g(j) = 1$, then $g^\flat_j(\overline{k}) \in K(\mathbb{A}^\delta)$ for every $\overline{k} \in (K(\mathbb{A}^\delta))^{\varepsilon_{g^\flat_j}}$;*
(2) *If $\varepsilon_g(j) = \partial$, then $g^\flat_j(\overline{o}) \in O(\mathbb{A}^\delta)$ for every $\overline{o} \in (O(\mathbb{A}^\delta))^{\varepsilon_{g^\flat_j}}$;*
(3) *If $\varepsilon_f(i) = 1$, then $f^\sharp_i(\overline{o}) \in O(\mathbb{A}^\delta)$ for every $\overline{o} \in (O(\mathbb{A}^\delta))^{\varepsilon_{f^\sharp_i}}$;*
(4) *If $\varepsilon_f(i) = \partial$, then $f^\sharp_i(\overline{k}) \in K(\mathbb{A}^\delta)$ for every $\overline{k} \in (K(\mathbb{A}^\delta))^{\varepsilon_{f^\sharp_i}}$.*

Proof. (1) By denseness, $g^\flat_j(\overline{k}) = \bigwedge\{o \in O(\mathbb{A}^\delta) \mid g^\flat_j(\overline{k}) \leq o\}$. Let $Y := \{o \in O(\mathbb{A}^\delta) \mid g^\flat_j(\overline{k}) \leq o\}$ and $X := \{a \in \mathbb{A} \mid g^\flat_j(\overline{k}) \leq a\}$. To show that $g^\flat_j(\overline{k}) \in K(\mathbb{A}^\delta)$, it is enough to show that $\bigwedge X = \bigwedge Y$.

Since clopens are opens, $X \subseteq Y$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o \in Y$ there exists some $a \in X$ such that $a \leq o$. Let $o \in Y$, i.e., $g^\flat_j(\overline{k}) \leq o$. By residuation, $k_j \leq g(\overline{k}[o/k_j])$, where $\overline{k}[o/k_j]$ denotes the n_g-array obtained by replacing the jth coordinate of $\overline{k}$ by o. Notice that $\overline{k}[o/k_j] \in (O(\mathbb{A}^\delta))^{\varepsilon_g}$. This immediately follows from the fact that by assumption, $\varepsilon_{g^\flat_j}(l) = \varepsilon_g(l) = 1$ if $l = j$ and $\varepsilon_{g^\flat_j}(l) = \varepsilon^\partial_g(l)$ if $l \neq j$.

Since $k_j \in K(\mathbb{A}^\delta)$, and $g(\overline{k}[o/k_j]) = g^\pi(\overline{k}[o/k_j]) = \bigvee\{g(\overline{a}) \mid \overline{a} \in \mathbb{A}^{\varepsilon_g} \text{ and } \overline{a} \leq^{\varepsilon_g} \overline{k}[o/k_j]\}$ and $g(\overline{a}) \in O(\mathbb{A}^\delta)$, we may apply compactness and get that $k_j \leq g(\overline{a_1}) \vee \cdots \vee g(\overline{a_n})$ for some $\overline{a_1}, \ldots, \overline{a_n} \in \mathbb{A}^{\varepsilon_g}$ s.t. $\overline{a_1}, \ldots, \overline{a_n} \leq^{\varepsilon_g} \overline{k}[o/k_j]$. Let $\overline{a} = \overline{a_1} \vee^{\varepsilon_g} \cdots \vee^{\varepsilon_g} \overline{a_n}$. The ε_g-monotonicity of g implies that $k_j \leq g(\overline{a_1}) \vee \cdots \vee g(\overline{a_n}) \leq g(\overline{a})$, and hence $g^\flat_j(\overline{a}[k_j/a_j]) \leq a_j$. The proof is complete if we show that $g^\flat_j(\overline{k}) \leq g^\flat_j(\overline{a}[k_j/a_j])$. By the $\varepsilon_{g^\flat_j}$-monotonicity of $g^\flat_j$, it is enough to show that $\overline{k} \leq^{\varepsilon_{g^\flat_j}} \overline{a}[k_j/a_j]$. Since the two arrays coincide in their jth coordinate, we only need to check that this is true for every $l \neq j$. Recall that $\varepsilon_{g^\flat_j}(l) = \varepsilon^\partial_g(l)$ if $l \neq j$. Hence, the statement immediately follows from this and the fact that, by construction, $\overline{a} \leq^{\varepsilon_g} \overline{k}[o/k_j]$.

(2), (3), and (4) are order-variants of (1). □

The proofs of the following lemmas are verbatim the same as the ones of Lemmas 10.4 and 10.5 in [12].

Lemma A.4. *For all $f \in \mathcal{F}$ and $g \in \mathcal{G}$,*

(1) $g\left(\bigvee^{\varepsilon_g(1)} \mathcal{U}_1, \ldots, \bigvee^{\varepsilon_g(n_g)} \mathcal{U}_{n_g}\right) = \bigvee\{g(u_1, \ldots, u_{n_g}) \mid u_j \in \mathcal{U}_j$ *for every* $1 \leq j \leq n_g\}$ *for every n_g-tuple $(\mathcal{U}_1, \ldots, \mathcal{U}_{n_g})$ such that $\mathcal{U}_j \subseteq O(\mathbb{A}^\delta)^{\varepsilon_g(j)}$ and $\mathcal{U}_j$ is $\varepsilon_g(j)$-up-directed for each $1 \leq j \leq n_g$.*
(2) $f\left(\bigwedge^{\varepsilon_f(1)} \mathcal{D}_1, \ldots, \bigwedge^{\varepsilon_f(n_f)} \mathcal{D}_{n_f}\right) = \bigwedge\{f(d_1, \ldots, d_{n_f}) \mid d_j \in \mathcal{D}_j$ *for every* $1 \leq j \leq n_f\}$ *for every n_f-tuple $(\mathcal{D}_1, \ldots, \mathcal{D}_{n_f})$ such that $\mathcal{D}_j \subseteq K(\mathbb{A}^\delta)^{\varepsilon_f(j)}$ and $\mathcal{D}_j$ is $\varepsilon_f(j)$-down-directed for each $1 \leq j \leq n_f$.*

Proof. (1) The '$\geq$' direction easily follows from the ε_g-monotonicity of g. Conversely, by denseness, it is enough to show that if $c \in K(A^\delta)$ and $c \leq g(\bigvee^{\varepsilon_g(1)} \mathcal{U}_1, \ldots, \bigvee^{\varepsilon_g(n_g)} \mathcal{U}_{n_g})$, then $c \leq g(u_1, \ldots, u_{n_g})$ for some tuple $(u_1, \ldots, u_{n_g})$ such that $u_j \in \mathcal{U}_j$ for each $1 \leq j \leq n_g$. Hence, consider $c \leq g(\bigvee^{\varepsilon_g(1)} \mathcal{U}_1, \ldots, \bigvee^{\varepsilon_g(n_g)} \mathcal{U}_{n_g})$. Then, $g_1^\flat(c, \overline{\bigvee^{\varepsilon_g} \mathcal{U}}) \leq^{\varepsilon(1)} \bigvee^{\varepsilon_g(1)} \mathcal{U}_1$, where, to enhance readability, we suppress sub- and superscripts and write $\overline{\bigvee^{\varepsilon_g} \mathcal{U}}$ for $(\bigvee^{\varepsilon_g(2)} \mathcal{U}_2, \ldots, \bigvee^{\varepsilon_g(n_g)} \mathcal{U}_{n_g})$. If $\varepsilon_g(1) = 1$, then $\varepsilon_{g_1^\flat}(1) = 1$ and $\varepsilon_{g_1^\flat}(l) = \varepsilon_g^\partial(l)$ for every $2 \leq l \leq n_g$. Hence $\mathcal{U}_l \subseteq O(\mathbb{A}^\delta)^{\varepsilon_g(l)} = K(\mathbb{A}^\delta)^{\varepsilon_{g_1^\flat}(l)}$, hence $\bigvee^{\varepsilon_g(l)} \mathcal{U}_l = \bigwedge^{\varepsilon_{g_1^\flat}(l)} \mathcal{U}_l \in K(\mathbb{A}^\delta)^{\varepsilon_{g_1^\flat}(l)}$ for every $2 \leq l \leq n_g$. By Lemma A.3(1), this implies that $g_1^\flat(c, \overline{\bigvee^{\varepsilon_g} \mathcal{U}}) \in K(\mathbb{A}^\delta)$. Hence, by compactness, $g_1^\flat(c, \overline{\bigvee^{\varepsilon_g} \mathcal{U}}) \leq \bigvee_{i=1}^n o_i$ for some $o_1, \ldots, o_n \in \mathcal{U}_1$. Since $\mathcal{U}_1$ is up-directed, $\bigvee_{i=1}^n o_i \leq u_1$ for some $u_1 \in \mathcal{U}_1$. Hence $c \leq g(u_1, \overline{\bigvee^{\varepsilon_g} \mathcal{U}})$. The same conclusion can be reached via a similar argument if $\varepsilon_g(1) = \partial$. Therefore, $g_2^\flat(u_1, c, \overline{\bigvee^{\varepsilon_g} \mathcal{U}}) \leq^{\varepsilon_g(2)} \bigvee^{\varepsilon_g(2)} \mathcal{U}_2$, where $\overline{\bigvee^{\varepsilon_g} \mathcal{U}}$ now stands for $(\bigvee^{\varepsilon_g(3)} \mathcal{U}_3, \ldots, \bigvee^{\varepsilon_g(n_g)} \mathcal{U}_{n_g})$. By applying the same reasoning, we can conclude that $c \leq g(u_1, u_2, \overline{\bigvee^{\varepsilon_g} \mathcal{U}})$ for some $u_2 \in \mathcal{U}_2$, and so on. Hence, we can then construct a sequence $u_j \in \mathcal{U}_j$ for $1 \leq j \leq n_g$ such that $c \leq g(u_1, \ldots u_{n_g})$, as required.

(2) is order-dual to (1). □

Lemma A.5. *For all $f \in \mathcal{F}$, $g \in \mathcal{G}$, $1 \leq i \leq n_f$, and $1 \leq j \leq n_g$,*

(1) *If $\varepsilon_g(j) = 1$, then*

$$g_j^\flat\left(\bigwedge{}^{\varepsilon_{g_j^\flat}(1)} \mathcal{D}_1, \ldots, \bigwedge{}^{\varepsilon_{g_j^\flat}(n_g)} \mathcal{D}_{n_g}\right) = \bigwedge \{g_j^\flat(d_1, \ldots, d_{n_g}) \mid d_h \in \mathcal{D}_h \text{ for every } 1 \leq h \leq n_g\}$$

for every n_g-tuple $(\mathcal{D}_1, \ldots, \mathcal{D}_{n_g})$ such that $\mathcal{D}_h \subseteq K(\mathbb{A}^\delta)^{\varepsilon_{g_j^\flat}(h)}$ and $\mathcal{D}_h$ is $\varepsilon_{g_j^\flat}(h)$-down-directed for each $1 \leq h \leq n_g$.

(2) *If $\varepsilon_g(j) = \partial$, then*

$$g_j^\flat\left(\bigvee{}^{\varepsilon_{g_j^\flat}(1)} \mathcal{U}_1, \ldots, \bigvee{}^{\varepsilon_{g_j^\flat}(n_g)} \mathcal{U}_{n_g}\right) = \bigvee \{g_j^\flat(u_1, \ldots, u_{n_g}) \mid u_h \in \mathcal{U}_h \text{ for every } 1 \leq h \leq n_g\}$$

for every n_g-tuple $(\mathcal{U}_1, \ldots, \mathcal{U}_{n_g})$ such that $\mathcal{U}_h \subseteq O(\mathbb{A}^\delta)^{\varepsilon_{g_j^\flat}(h)}$ and $\mathcal{U}_h$ is $\varepsilon_{g_j^\flat}(h)$-up-directed for each $1 \leq h \leq n_g$.

(3) *If $\varepsilon_f(i) = 1$, then*

$$f_i^\sharp\left(\bigvee{}^{\varepsilon_{f_i^\sharp}(1)} \mathcal{U}_1, \ldots, \bigvee{}^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{U}_{n_f}\right) = \bigvee \{f_i^\sharp(u_1, \ldots, u_{n_f}) \mid u_h \in \mathcal{U}_h \text{ for every } 1 \leq h \leq n_f\}$$

for every n_f-tuple $(\mathcal{U}_1, \ldots, \mathcal{U}_{n_f})$ such that $\mathcal{U}_h \subseteq O(\mathbb{A}^\delta)^{\varepsilon_{f_i^\sharp}(h)}$ and $\mathcal{U}_h$ is $\varepsilon_{f_i^\sharp}(h)$-up-directed for each $1 \leq h \leq n_f$.

(4) *If $\varepsilon_f(i) = \partial$, then*

$$f_i^\sharp\left(\bigwedge{}^{\varepsilon_{f_i^\sharp}(1)} \mathcal{D}_1, \ldots, \bigwedge{}^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{D}_{n_f}\right) = \bigwedge \{f_i^\sharp(d_1, \ldots, d_{n_f}) \mid d_h \in \mathcal{D}_h \text{ for every } 1 \leq h \leq n_f\}$$

for every n_f-tuple $(\mathcal{D}_1, \ldots, \mathcal{D}_{n_f})$ such that $\mathcal{D}_h \subseteq K(\mathbb{A}^\delta)^{\varepsilon_{f_i^\sharp}(h)}$ and $\mathcal{D}_h$ is $\varepsilon_{f_i^\sharp}(h)$-down-directed for each $1 \leq h \leq n_f$.

Proof. 3. The '$\geq$' direction easily follows from the $\varepsilon_{f_i^\sharp}$-monotonicity of $f_i^\sharp$. For the converse inequality, by denseness it is enough to show that if we have $c \leq f_i^\sharp(\bigvee^{\varepsilon_{f_i^\sharp}(1)} \mathcal{U}_1, \ldots, \bigvee^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{U}_{n_f})$ for a closed element c, then $c \leq f_i^\sharp(u_1, \ldots, u_{n_f})$ for some tuple $(u_1, \ldots, u_{n_f})$ such that $u_h \in \mathcal{U}_h$ for every $1 \leq h \leq n_f$. By residuation, $c \leq f_i^\sharp(\bigvee^{\varepsilon_{f_i^\sharp}(1)} \mathcal{U}_1, \ldots, \bigvee^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{U}_{n_f})$ implies that we have the inequality $f(\bigvee^{\varepsilon_{f_i^\sharp}(1)} \mathcal{U}_1, \ldots, c, \ldots, \bigvee^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{U}_{n_f}) \leq \bigvee^{\varepsilon_{f_i^\sharp}(i)} \mathcal{U}_i$. The assumption $\varepsilon_f(i) = 1$ implies that $\varepsilon_{f_i^\sharp}(i) = 1$ and $\varepsilon_{f_i^\sharp}(l) = \varepsilon_f^\partial(l)$ for every $l \neq i$. Hence, $\mathcal{U}_l \subseteq O(\mathbb{A}^\delta)^{\varepsilon_{f_i^\sharp}(l)} = K(\mathbb{A}^\delta)^{\varepsilon_f(l)}$, and $\mathcal{U}_l$ is $\varepsilon_f(l)$-down-directed for every $l \neq i$. Recalling that $\bigvee^{\varepsilon_{f_i^\sharp}(l)}$ coincides with $\bigwedge^{\varepsilon_f(l)}$, we can apply Lemma A.4(2) and get:

$$f\left(\bigvee^{\varepsilon_{f_i^\sharp}(1)} \mathcal{U}_1, \ldots, c, \ldots, \bigvee^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{U}_{n_f}\right) = \bigwedge \{f(u_1, \ldots, c, \ldots, u_{n_f}) \mid u_l \in \mathcal{U}_l \text{ for every } l \neq i\}.$$

Hence, by compactness, $f(\bigvee^{\varepsilon_{f_i^\sharp}(1)} \mathcal{U}_1, \ldots, c, \ldots, \bigvee^{\varepsilon_{f_i^\sharp}(n_f)} \mathcal{U}_{n_f}) \leq \bigvee^{\varepsilon_{f_i^\sharp}(i)} \mathcal{U}_i$ implies that

$$\bigwedge_{1 \leq j \leq m} \{f(o_1^{(j)}, \ldots, c, \ldots, o_{n_f}^{(j)}) \mid o_l^{(j)} \in \mathcal{U}_l \text{ for all } l \neq i\} \leq o_i^{(1)} \vee \cdots \vee o_i^{(n)}$$

for some $o_i^{(1)}, \ldots, o_i^{(n)} \in \mathcal{U}_i$. The assumptions that $\varepsilon_f(i) = 1$ and that each $\mathcal{U}_h$ is $\varepsilon_{f_i^\sharp}(h)$-up-directed for every $1 \leq h \leq n_f$ imply that $\mathcal{U}_i$ is up-directed and $\mathcal{U}_l$ is $\varepsilon_f(l)$-down-directed for each $l \neq i$. Hence, some $u_1, \ldots, u_{n_f}$ exist such that $u_l \leq^{\varepsilon_f(l)} \bigwedge^{\varepsilon_f(l)}_{1 \leq j \leq m} o_l^{(j)}$ and $o_i^{(1)} \vee \cdots \vee o_i^{(n)} \leq u_i$. The ε_f-monotonicity of f implies the following chain of inequalities:

$$\begin{array}{rcl} f(u_1, \ldots, c, \ldots, u_{n_f}) & \leq & f\left(\bigwedge^{\varepsilon_f(1)}_{1 \leq j \leq m} o_1^{(j)}, \ldots, c, \ldots, \bigwedge^{\varepsilon_f(n_f)}_{1 \leq j \leq m} o_{n_f}^{(j)}\right) \\ & \leq & \bigwedge_{1 \leq j \leq m} \{f(o_1^{(j)}, \ldots, c, \ldots, o_{n_f}^{(j)}) \mid o_l^{(j)} \in \mathcal{U}_l \text{ for all } l \neq i\} \\ & \leq & o_i^{(1)} \vee \cdots \vee o_i^{(n)} \\ & \leq & u_i, \end{array}$$

which implies that $c \leq f_i^\sharp(u_1, \ldots, u_{n_f})$, as required.

(1), (2), and (4) are order-variants of (3). □

A.2 Proof of the Restricted Ackermann Lemmas (Lemmas 4.6 and 4.7)

For any $\mathcal{L}_{\mathrm{LE}}^+$-formula φ, any slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ and assignment V on $\mathbb{A}^\delta$, we write $\varphi(V)$ to denote the extension of φ in $\mathbb{A}^\delta$ under the assignment V. We remind the reader that, even when φ is in the basic signature and V is an admissible valuation, $\varphi(V)$ may fail to be an element of $\mathbb{A}$ (cf. Remark A.2 for a counterexample).

Let p be a propositional variable occurring in φ and V be any assignment. For any $x \in \mathbb{A}^\delta$, let $V[p := x]$ be the assignment which is identical to V except that it assigns x to p. Then $x \mapsto \varphi(V[p := x])$ defines an operation on $\mathbb{A}^\delta$, which we will denote $\varphi_p^V(x)$.

The proofs of the following two lemmas are more streamlined versions of those of Lemmas 10.6 and 10.7 of [12]. The modifications concern the differences between the notions of syntactically closed and open formulas (see Definition 4.2) and ssc and sso (see Definition 4.3).

Lemma A.6. *Let φ be ssc and ψ sso. Let V be an admissible assignment, $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$.*

(1)
  (a) *If $\varphi(p)$ is positive in p, then $\varphi_p^V(c) \in K(\mathbb{A}^\delta)$, and*
  (b) *if $\psi(p)$ is negative in p, then $\psi_p^V(c) \in O(\mathbb{A}^\delta)$.*

(2)
  (a) *If $\varphi(p)$ is negative in p, then $\varphi_p^V(o) \in K(\mathbb{A}^\delta)$, and*
  (b) *if $\psi(p)$ is positive in p, then $\psi_p^V(o) \in O(\mathbb{A}^\delta)$.*

Proof. We prove (1) by simultaneous induction on φ and ψ. Assume that $\varphi(p)$ is positive in p and $\psi(p)$ is negative in p. The base cases of the induction are those when φ is of the form $\top$, $\bot$, p, q (for propositional variables q different from p) or $\mathbf{i}$, and ψ is of the form $\top$, $\bot$, q (for propositional variables q different from p), or $\mathbf{m}$ (note that φ cannot be a co-nominal $\mathbf{m}$, since it is syntactically closed. Also, ψ cannot be p or a nominal $\mathbf{i}$, since ψ is negative in p and is syntactically open, respectively). These cases follow by noting (1) that $V[p := c](\bot) = 0 \in \mathbb{A}$, $V[p := c](\top) = 1 \in \mathbb{A}$, and $V[p := c](q) = V(q) \in \mathbb{A}$, (2) that $V[p := c](p) = c \in K(\mathbb{A}^\delta)$ and $V[p := c](\mathbf{i}) \in J^\infty(\mathbb{A}^\delta) \subset K(\mathbb{A}^\delta)$, and (3) that $V[p := c](\mathbf{m}) \in M^\infty(\mathbb{A}^\delta) \subset O(\mathbb{A}^\delta)$ (see discussion after Definition 2.6).

For the remainder of the proof we will not need to refer to the valuation V and will hence omit reference to it. We will accordingly write φ and ψ for φ_p^V and ψ_p^V, respectively.

In the cases $\varphi(p) = f^*(\overline{\varphi'(p)}, \overline{\psi'(p)})$ for $f^* \in \mathcal{F}^*$, $\varphi(p) = \varphi_1(p) \wedge \varphi_2(p)$, or $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$ both $\varphi_1(p)$ and $\varphi_2(p)$ are ssc and positive in p, and each $\varphi_i'(p)$ in $\overline{\varphi'(p)}$ is ssc and positive in p, and each $\psi_i'(p)$ in $\overline{\psi'(p)}$ is sso and negative in p. Hence, the claim follows by the inductive hypothesis, and Lemma A.1(2) if $f^* \in \mathcal{F}$, Lemma A.3 if $f^* \in \mathcal{F}^* \setminus \mathcal{F}$ and the fact that meets and finite joins of closed elements are closed, respectively.

Similarly, if $\psi(p) = g^*(\overline{\psi'(p)}, \overline{\varphi'(p)})$ for $g^* \in \mathcal{G}^*$, $\psi(p) = \psi_1(p) \vee \psi_2(p)$ or $\psi(p) = \psi_1(p) \wedge \psi_2(p)$, then both $\psi_1(p)$ and $\psi_2(p)$ are sso and negative in p, and each $\varphi_i'(p)$ in $\overline{\varphi'(p)}$ is ssc and positive in p, and each $\psi_i'(p)$ in $\overline{\psi'(p)}$ is sso and negative in p. Hence, the claim follows by the inductive hypothesis, and Lemma A.1(1) if $g^* \in \mathcal{G}$, Lemma A.3 if $g^* \in \mathcal{G}^* \setminus \mathcal{G}$ and the fact that joins and finite meets of open elements are open, respectively.

Item (2) can similarly be proved by simultaneous induction on negative φ and positive ψ. □

Lemma A.7. *Let $\varphi(p)$ be ssc, $\psi(p)$ sso, V an admissible assignment, $\mathcal{D} \subseteq K(\mathbb{A}^\delta)$ be down-directed, and $\mathcal{U} \subseteq O(\mathbb{A}^\delta)$ be up-directed.*

(1)
  (a) *If $\varphi(p)$ is positive in p, then $\varphi_p^V(\bigwedge \mathcal{D}) = \bigwedge\{\varphi_p^V(d) \mid d \in \mathcal{D}\}$, and*
  (b) *if $\psi(p)$ is negative in p, then $\psi_p^V(\bigwedge \mathcal{D}) = \bigvee\{\psi_p^V(d) \mid d \in \mathcal{D}\}$.*

(2)
  (a) *If $\varphi(p)$ is negative in p, then $\varphi_p^V(\bigvee \mathcal{U}) = \bigwedge\{\varphi_p^V(u) \mid u \in \mathcal{U}\}$, and*
  (b) *if $\psi(p)$ is positive in p, then $\psi_p^V(\bigvee \mathcal{U}) = \bigvee\{\psi_p^V(u) \mid u \in \mathcal{U}\}$.*

Proof. We prove (1) by simultaneous induction on φ and ψ. The base cases of the induction on φ are those when it is of the form $\top$, $\bot$, p, a propositional variable q other than p, or $\mathbf{i}$, and for ψ those when it is of the form $\top$, $\bot$, a propositional variable q other than p or $\mathbf{m}$. In each of these cases, the claim is trivial.

For the remainder of the proof, we will omit reference to the assignment V and simply write φ and ψ for φ_p^V and ψ_p^V, respectively.

In the cases in which $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$, $\varphi(p) = \varphi_1(p) \wedge \varphi_2(p)$, $\varphi(p) = f^*(\overline{\varphi'(p)}, \overline{\psi'(p)})$, $\psi(p) = \psi_1(p) \wedge \psi_2(p)$, $\psi(p) = \psi_1(p) \vee \psi_2(p)$, $\psi(p) = g^*(\overline{\psi'(p)}, \overline{\varphi'(p)})$, we have that φ_1 and φ_2 are ssc and positive in p and ψ_1 and ψ_2 are sso and negative in p, and moreover, each $\varphi_i'(p)$ in $\overline{\varphi'(p)}$ is ssc and positive in p, and each $\psi_j'(p)$ in $\overline{\psi'(p)}$ is sso and negative in p.

Hence, when $\varphi(p) = \varphi_1(p) \wedge \varphi_2(p)$ and $\psi(p) = \psi_1(p) \vee \psi_2(p)$, the claim follows by the inductive hypothesis and the associativity of, respectively, meet and join.

If $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$, then

$$\begin{array}{rcll}
\varphi(\bigwedge \mathcal{D}) & = & \varphi_1(\bigwedge \mathcal{D}) \vee \varphi_2(\bigwedge \mathcal{D}) & \\
& = & \bigwedge\{\varphi_1(c_i) \mid c_i \in \mathcal{D}\} \vee \bigwedge\{\varphi_2(c_i) \mid c_i \in \mathcal{D}\} & \text{(induction hypothesis)} \\
& = & \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_j) \mid c_i, c_j \in \mathcal{D}\} & (*) \\
& = & \bigwedge\{\varphi_1(c) \vee \varphi_2(c) \mid c \in \mathcal{D}\} & (\varphi \text{ monotone and } \mathcal{D} \text{ down-directed}) \\
& = & \bigwedge\{\varphi(c) \mid c \in \mathcal{D}\}, &
\end{array}$$

where the equality marked with $(*)$ follows from a restricted form of distributivity enjoyed by canonical extensions of general bounded lattices (cf. [22, Lemma 3.2]), applied to the family $\{A_1, A_2\}$ such that $A_i := \{\varphi_i(c_j) \mid c_j \in \mathcal{D}\}$ for $i \in \{1, 2\}$. Specifically, the monotonicity in p of $\varphi_i(p)$ and $\mathcal{D}$ being down-directed imply that A_1 and A_2 are down-directed subsets, which justifies the application of [22, Lemma 3.2].

If $\varphi(p) = f^*(\overline{\varphi'(p)}, \overline{\psi'(p)})$, with $f^* \in \mathcal{F}^*$, then

$$\varphi(\wedge\mathcal{D}) = f^*(\overline{\varphi'(\wedge\mathcal{D})}, \overline{\psi'(\wedge\mathcal{D})}) =$$
$$f^*(\textstyle\bigwedge_{d\in\mathcal{D}} \varphi'_1(d), \ldots, \bigwedge_{d\in\mathcal{D}} \varphi'_k(d), \bigvee_{d\in\mathcal{D}} \psi'_{k+1}(d), \ldots, \bigvee_{d\in\mathcal{D}} \psi'_{n_{f^*}}(d)).$$

The second equality above holds by the inductive hypothesis. To finish the proof, we need to show that

$$f^*(\textstyle\bigwedge_{d\in\mathcal{D}} \varphi'_1(d), \ldots, \bigvee_{d\in\mathcal{D}} \psi'_{n_{f^*}}(d)) = \bigwedge_{d\in\mathcal{D}} f^*(\overline{\varphi'(d)}, \overline{\psi'(d)}).$$

The '$\leq$' direction immediately follows from the monotonicity of f^*. For the converse inequality, by denseness, it is enough to show that if $o \in O(\mathbb{A}^\delta)$ and $f^*(\bigwedge_{d\in\mathcal{D}} \varphi'_1(d), \ldots, \bigvee_{d\in\mathcal{D}} \psi'_{f^*}(d)) \leq o$, then $\bigwedge_{d\in\mathcal{D}} f^*(\overline{\varphi'(d)}, \overline{\psi'(d)}) \leq o$. By Lemmas A.6(1) and A.4 or A.5, according to whether $f^* \in \mathcal{F}$ or $f^* \in \mathcal{F}^* \setminus \mathcal{F}$, we have:

$$f^*(\textstyle\bigwedge_{d\in\mathcal{D}} \varphi'_1(d), \ldots, \bigvee_{d\in\mathcal{D}} \psi'_{n_{f^*}}(d)) = \bigwedge\{f^*(\varphi'_1(d_1), \ldots, \psi'_{n_{f^*}}(d_{n_{f^*}})) \mid d_h \in \mathcal{D} \text{ for every } 1 \leq h \leq n_{f^*}\}.$$

By compactness (which can be applied by Lemmas A.1 or A.3, again according to the nature of f^*, and A.6(1)),

$$\textstyle\bigwedge\{f^*(\varphi'_1(d_1^{(i)}), \ldots, \psi'_{n_{f^*}}(d_{n_{f^*}}^{(i)})) \mid 1 \leq i \leq n\} \leq o.$$

Let $\mathcal{D}' := \{d_h^{(i)} \mid 1 \leq i \leq n \text{ and } 1 \leq h \leq n_{f^*}\}$. Since $\mathcal{D}$ is down-directed, $d^* \leq \bigwedge \mathcal{D}'$ for some $d^* \in \mathcal{D}$. Then, by monotonicity of f^* (Recall the notations at the beginning of 2.1) and since each $\varphi'_i(p)$ is positive in p and each $\psi'_j(p)$ is negative in p, the following chain of inequalities holds

$$\begin{array}{rcl}
\bigwedge_{d\in\mathcal{D}} f^*(\overline{\varphi'(d)}, \overline{\psi'(d)}) & \leq & f^*(\overline{\varphi'(d^*)}, \overline{\psi'(d^*)}) \\
& \leq & f^*(\varphi'_1(\bigwedge_{1\leq i\leq n} d_1^{(i)}), \ldots, \psi'_{n_{f^*}}(\bigwedge_{1\leq i\leq n} d_{n_{f^*}}^{(i)})) \\
& \leq & \bigwedge\{f^*(\varphi'_1(d_1^{(i)}), \ldots, \psi'_{n_{f^*}}(d_{n_{f^*}}^{(i)})) \mid 1 \leq i \leq n\} \\
& \leq & o.
\end{array}$$

The remaining cases are similar, and left to the reader.

Thus, the proof of item (1) is concluded. Item (2) can be proved similarly by simultaneous induction on φ negative in p and ψ positive in p. □

Proof of the Righthanded Ackermann Lemma for Admissible Assignments (Lemma 4.6). To keep the notation uncluttered, we will simply write β_i and γ_i for ${\beta_i}_p^V$ and ${\gamma_i}_p^V$, respectively. The implication from bottom-to-top follows by the monotonicity of the β_i and the antitonicity of the γ_i in p. Indeed, if $\alpha(V) \le u$, then, for each $1 \le i \le n$, $\beta_i(\alpha(V)) \le \beta_i(u) \le \gamma_i(u) \le \gamma_i(\alpha(V))$.

For the sake of the converse implication, assume that $\beta_i(\alpha(V)) \le \gamma_i(\alpha(V))$ for all $1 \le i \le n$. By Lemma A.6, $\alpha(V) \in K(A^\delta)$. Hence, $\alpha(V) = \bigwedge\{a \in \mathbb{A} \mid \alpha(V) \le a\}$, making it the meet of a down-directed subset of $K(A^\delta)$. Thus, for any $1 \le i \le n$, we have

$$\beta_i(\bigwedge\{a \in \mathbb{A} \mid \alpha(V) \le a\}) \le \gamma_i(\bigwedge\{a \in \mathbb{A} \mid \alpha(V) \le a\}).$$

Since γ_i is syntactically open and negative in p, and β_i is syntactically closed and positive in p, we may apply Lemma A.7 and equivalently obtain

$$\bigwedge\{\beta_i(a) \mid a \in \mathbb{A},\ \alpha(V) \le a\} \le \bigvee\{\gamma_i(a) \mid a \in \mathbb{A},\ \alpha(V) \le a\}.$$

By lemma A.6, $\beta_i(a) \in K(\mathbb{A}^\delta)$ and $\gamma_i(a) \in O(\mathbb{A}^\delta)$ for each $a \in \mathbb{A}$. Hence by compactness,

$$\beta_i(b_1) \wedge \cdots \beta_i(b_k) \le \gamma_i(a_1) \vee \cdots \vee \gamma_i(a_m).$$

for some $a_1, \ldots, a_m, b_1, \ldots b_k \in \mathbb{A}$ with $\alpha(V) \le a_j$, $1 \le j \le m$, and $\alpha(V) \le b_h$, $1 \le h \le k$. Let $a_i = b_1 \wedge \cdots \wedge b_k \wedge a_1 \wedge \cdots \wedge a_m$. Then, $\alpha(V) \le a_i \in \mathbb{A}$. By the monotonicity of β_i and the antitonicity of γ_i it follows that:

$$\beta_i(a_i) \le \gamma_i(a_i).$$

Now, letting $u = a_1 \wedge \cdots \wedge a_n$, we have $\alpha(V) \le u \in \mathbb{A}$, and by the monotonicity of the β_i and the antitonicity of the γ_i we get that

$$\beta_i(u) \le \gamma_i(u) \text{ for all } 1 \le i \le n.$$

Proof of the Lefthanded Ackermann Lemma for Admissible Assignments (Lemma 4.7). As in the previous lemma, we will write β_i and γ_i for ${\beta_i}_p^V$ and ${\gamma_i}_p^V$, respectively. The implication from bottom to top follows by the antitonicity of the β_i and the monotonicity of the γ_i.

For the sake of the converse implication, assume that ${\beta_i}_p^V(\alpha(V)) \le {\gamma_i}_p^V(\alpha(V))$ for all $1 \le i \le n$. But α is syntactically open and (trivially) negative in p, hence by Lemma A.6(2), $\alpha(V) \in O(\mathbb{A}^\delta)$, i.e., $\alpha(V) = \bigvee\{a \in \mathbb{A} \mid a \le \alpha(V)\}$. Thus, for any $1 \le i \le n$, it is the case that

$$\beta_i(\bigvee\{a \in \mathbb{A} \mid a \le \alpha(V)\}) \le \gamma_i(\bigvee\{a \in \mathbb{A} \mid a \le \alpha(V)\}).$$

Hence, by Lemma A.7(3) and 8.7(4):

$$\bigwedge\{\beta_i(a) \mid a \in \mathbb{A}, a \le \alpha(V)\} \le \bigvee\{\gamma_i(a) \mid a \in \mathbb{A}, a \le \alpha(V)\}.$$

The proof now proceeds like that of Lemma 4.6.

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