Instituto Universitário de Lisboa

IUL Business School

Department of Finance

Essays on Option Pricing, with Applications on Interest Rates, Equities and Credit Derivatives

Pedro Miguel Silva Prazeres

Thesis specially presented for the fulfillment of the Degree of Doctor in Finance

Supervisor:

Doutor João Pedro Vidal Nunes, Professor Catedrático, ISCTE-IUL

May, 2017

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Resume

This thesis is devoted to option pricing, with applications on interest rates, equities and credit derivatives, and is comprised of three separate and self-contained essays:

A Pricing Swaptions under Multifactor Gaussian HJM Models

Several approximations have been proposed in the literature for the pricing of European-style swaptions under multifactor term structure models. However, none of them provides an estimate for the inherent approximation error. Until now, only the Edgeworth expansion technique of Collin-Dufresne and Goldstein (2002) is able to characterize the order of the approximation error. Under a multifactor Heath, Jarrow, and Morton (1992) Gaussian framework, this paper proposes a new approximation for European-style swaptions, which is able to set bounds on the magnitude of the approximation error and is based on the *conditioning approach* initiated by Curran (1994) and Rogers and Shi (1995). All the proposed pricing bounds will arise as a simple by-product of the Nielsen and Sandmann (2002) setup, and will be shown to provide a better accuracy-efficiency trade-off than all the approximations already proposed in the literature.

B Pricing of European-style Barrier Options under Stochastic Interest Rates

This paper offers an extremely fast and accurate novel methodology for the pricing of (long-term) European-style single barrier options on underlying spot prices driven by a geometric Brownian motion and under the stochastic interest rates framework of Vasiček (1977). The proposed valuation methodology extends the *stopping time approach* of Kuan and Webber (2003) to a more general setting, and expresses the price of a European-style barrier option in terms of the first passage time density of the underlying asset price to the barrier level. Using several model parameter constellations and option maturities, our numerical results show that the proposed pricing approach is much more accurate and faster than the two-dimensional *extended Fortet method* of Bernard et al. (2008).

C Pricing Credit and Equity Default Swaps under the Jump to Default Extended CEV Model

This paper offers a novel methodology for the pricing of credit and equity default swaps under the *jump to default extended constant elasticity of variance* (JDCEV) model of Carr and Linetsky (2006). The proposed method extends the *stopping time approach* of Kuan and Webber (2003), and expresses the value of the building blocks of both contracts in terms of the first passage time density of the underlying asset price to the contract triggering level. The numerical results show that the proposed pricing methodology is extremely accurate and much faster than the Laplace transform approach of Mendoza-Arriaga and Linetsky (2011).

JEL Classification: G13

Keywords: European-style swaptions; European-style barrier options; credit default swaps; equity default swaps; Gaussian HJM multifactor models; CEV model; JDCEV model.

Resumo

Esta tese dedica-se ao tema da avaliação de opções, com aplicações a taxas de juro, ações e derivados de crédito, e é composta por três artigos distintos:

A Pricing Swaptions under Multifactor Gaussian HJM Models

Várias aproximações foram já propostas na literatura para a avaliação de *swaptions* de estilo Europeu, no âmbito de modelos de taxa de juro multi-fator. Contudo, nenhuma delas fornece uma estimativa para o erro de aproximação subjacente. Até agora, apenas a *Edgeworth expansion technique* de Collin-Dufresne e Goldstein (2002) é capaz de caracterizar a ordem do erro de aproximação. No âmbito de um modelo Heath, Jarrow e Morton (1992) Gaussiano multi-fator, este artigo propõe uma nova aproximação para *swaptions* de estilo Europeu, que é capaz de estabelecer limites para a magnitude do erro de aproximação e é baseada na *conditioning approach* iniciada por Curran (1994) e Rogers e Shi (1995). Todos os limites de preço propostos surgirão como um simples sub-produto da estrutura de Nielsen e Sandmann (2002), e será demonstrado que estes proporcionam um melhor equilíbrio entre precisão e eficiência do que todas as aproximações já propostas na literatura.

B Pricing of European-style Barrier Options under Stochastic Interest Rates

Este artigo oferece uma nova metodologia, extremamente rápida e precisa, para a avaliação de opções de estilo Europeu com barreira sobre ativos subjacentes caracterizados por um *geometric Brownian motion* e no âmbito do modelo de taxas de juro estocásticas de Vasiček (1977). A metodologia de avaliação proposta estende a *stopping time approach* de Kuan e Webber (2003) a uma configuração mais geral, e expressa o preço de uma opção de estilo Europeu com barreira em termos da densidade de probabilidade do primeiro tempo de passagem do preço do ativo subjacente pelo nível da barreira. Utilizando várias configurações de parâmetros e maturidades de opções, os nossos resultados numéricos mostram que a metodologia de avaliação proposta é muito mais precisa e rápida do que o *extended Fortet method* bi-dimensional de Bernard et al. (2008).

C Pricing Credit and Equity Default Swaps under the Jump to Default Extended CEV Model

Este artigo oferece uma nova metodologia para a avaliação de *credit* e *equity default swaps* no âmbito do modelo *jump to default extended constant elasticity of variance* (JDCEV) de Carr e Linetsky (2006). A abordagem proposta estende a *stopping time approach* de Kuan e Webber (2003), e expressa o valor das componentes de ambos os contratos em termos da densidade de probabilidade do primeiro tempo de passagem do preço do ativo subjacente pelo nível de acionamento do contrato. Os resultados numéricos mostram que a abordagem de avaliação proposta é precisa e muito mais rápida do que a *Laplace transform approach* de Mendoza-Arriaga e Linetsky (2011).

Classificaçao JEL: G13

Palavras-chave: *swaptions* de estilo Europeu; opções de estilo Europeu com barreira; *credit default swaps*; *equity default swaps*; modelos HJM Gaussianos multi-fator; modelo CEV; modelo JDCEV.

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1 Introduction

This thesis is devoted to option pricing, with applications on interest rates, equities and credit derivatives, and is comprised of three separate and self-contained essays.

The main purpose of the first paper is to offer a fast and extremely accurate analytical approximation for European-style swaptions under a multifactor Gaussian Heath, Jarrow, and Morton (1992)—HJM, hereafter—framework.

European-style swaptions are essentially options on coupon-bearing bonds, that is on a portfolio of pure discount bonds. Under several single-factor term structure models, a European-style swaptions can be valued analytically through its decomposition into a portfolio of options on zero-coupon bonds—see, for instance, Jamshidian (1989) under the Vasiček (1977) model, or Longstaff (1993) for the Cox et al. (1985) setup. However, under a (more realistic) multifactor term structure framework, no exact closed-form solution has ever been found for European-style swaptions, because the optimal exercise boundary involves a nonlinear function of several random variables, whose joint probability density is unknown.

The novel approximation for European-style swaptions proposed in this paper is based on the *conditioning approach* initiated by Curran (1994) and Rogers and Shi (1995) in the context of Asian option pricing, and extended by Nielsen and Sandmann (2002) to a stochastic interest rate setting. This new pricing approach is restricted to a multifactor HJM Gaussian setup, but should be faster to implement than the Edgeworth expansion technique, and will provide explicit (and tight) bounds for the approximation error.

The analytical tractability provided by the multifactor Gaussian—but not necessarily Markovian or time-homogeneous—HJM term structure model proposed is obtained at the expense of an important theoretical drawback: Interest rates are assumed to be normally distributed, and can therefore attain negative values with positive probability. But even though the no-arbitrage Gaussian setup adopted in this paper is more restrictive than, for instance, the more general affine framework used by Collin-Dufresne and Goldstein (2003), the extremely accurate pricing solutions to be proposed in this paper can always be used as control variates for more general diffusion pricing models. Moreover and following, for instance, Nunes et al. (1999, Theorem 1) or Kristensen and Mele (2011, Definition 1), the Gaussian pricing formulae offered in this paper can also be used as the (most accurate) zero-order term of the perturbed or Taylor series expansion pricing solution associated to a more general affine term structure model.

The Gaussian framework adopted offers also a common ground for the comparison of all alternative pricing methods. Since the conditioning approach proposed will provide extremely tight bounds for the approximation error, it will be possible to compare rigorously the accuracy and efficiency of all the approximations already proposed in the literature for European-style swaptions. The alternative approximations have been compared in the literature against benchmark prices obtained through Monte Carlo studies that involve different levels of accuracy. For instance, Collin-Dufresne and Goldstein (2002, Page 16) run 2×10^6 simulations, whereas Schrager and Pelsser (2006, Page 689) simulate only $500,000$ paths, all using standard variance reduction techniques. Based on a much more demanding setting—involving 10^9 simulations, coupled with antithetic variates—to produce the Monte Carlo proxy of the exact swaption price, this paper will show that the conditioning approach significantly improves upon the existing literature in both speed and accuracy.

The main objective of the second paper is to offer a fast and accurate novel methodology for the pricing of European-style single barrier options on asset prices driven by a geometric Brownian motion and under the stochastic interest rates framework of Vasiček (1977). This paper generalizes the *stopping time approach* (ST approach, hereafter) first proposed by Kuan and Webber (2003) for options on pure discount bonds, under single-factor term structure models, and later extended by Dias et al. (2014) to the pricing of European-style single and double barrier options under the *jump to default extended constant elasticity of variance* (JDCEV) framework of Carr and Linetsky (2006).

European-style barrier options are path-dependent contingent claims, which are characterized by a strike price and an upper or lower barrier level. These contracts become standard European-style options if the barrier level is—for *knock-in options*—or is not—for *knock-out* options—breached by the underlying spot price, during the option lifetime. If not, the option expires worthless, in which case a cash rebate may be received by the option holder. The existence of a barrier makes these contracts cheaper than their standard counterparts, and allow investors to better express their views about the future evolution of the underlying spot price.

To the authors' knowledge, the pricing of European-style barrier options in the context of a stochastic interest rate framework has only been pursued through the *extended Fortet method* of Bernard et al. (2008). These authors extend the Fortet (1943) method and offer a two-dimensional Markovian pricing approximation. In different contexts, the Fortet (1943) method has also been adopted, for instance, by Longstaff and Schwartz (1995), for the pricing of risky debt, and by Collin-Dufresne and Goldstein (2001), for credit risk modelling.

Our pricing methodology extends the ST approach of Kuan and Webber (2003) to a two-factor option pricing model with stochastic interest rates, and expresses the European-style barrier option price in terms of the density function of the first passage time of the underlying asset price to the barrier level. Using the standard partition method of Park and Schuurmann (1976), we are able to recover this hitting density as the implicit solution of a non-linear integral equation. However, and since we are working under a two-factor model, our valuation approach involves a double integral, in both time and interest rate dimensions. We will show that our proposed pricing solution can be simplified to require only one integration with respect to time, because the probability density function of the short-term interest rate, conditional on the knock-in or knock-out event, will be obtained in closed-form. Therefore, the ST approach will be shown to be much more accurate and efficient than the extended Fortet method of Bernard et al. (2008).

The main purpose of the third paper is to offer a novel valuation methodology for credit default swaps (CDSs) and equity default swaps (EDSs) under the *jump to default* *extended constant elasticity of variance* model proposed by Carr and Linetsky (2006). This paper generalizes the *stopping time approach* first proposed by Kuan and Webber (2003) for options on pure discount bonds, under single-factor term structure models, and later extended by Dias et al. (2014) to the pricing of European-style single and double barrier options under the JDCEV framework.

With the global financial crisis of 2007-09, CDSs became the most widely traded credit derivative in financial markets. These securities can be thought of as an insurance contract, which provides its buyer compensation in the case of a credit event of a reference entity. A credit event can encompass, but is not limited to, bankruptcy of the reference entity, failure to pay, or a debt restructuring. In return, the credit default swap (CDS) seller receives a series of periodic payments, up to the credit event or the contract maturity, whichever occurs first. EDSs are hybrid credit-equity securities, which combine characteristics of CDS contracts and equity barrier derivatives. These instruments, originally launched around fifteen years ago, allow investors to simultaneously hedge the equity and credit risk associated with a reference entity. Similarly to CDSs, the equity default swap (EDS) pays its buyer a pre-determined amount in the case of a triggering event, which in this case is defined as a sharp decrease (typically of 50% to 70%) in the underlying stock of the reference entity. Conversely, the EDS seller also receives regular payments through the life of the contract, up to the triggering event, if it occurs. Hence, a CDS can be understood as an EDS with a triggering level equal to zero, since it is expected that, in the event of a default, the stock price trades near zero.

To the authors' knowledge, the valuation of EDSs under the JDCEV framework has only been pursued by Mendoza-Arriaga and Linetsky (2011). In their paper, the authors offer pricing formulae for the building blocks of an EDS contract (protection leg, premium leg and accrued interest) via the inversion of Laplace transforms of several expectations containing the first passage time of the underlying price process through the contract triggering level. These authors are also able to price CDS contracts, by considering the limit when the triggering level tends to zero.

Our faster pricing methodology extends the ST approach of Kuan and Webber (2003), and expresses the value of the building blocks of CDS and EDS contracts in terms of the density function of the first passage time of the underlying asset price to the contract triggering level. Through the standard partition method of Park and Schuurmann (1976), this hitting density is recovered as the implicit solution of a non-linear integral equation. We note that the ST approach is able to accommodate the valuation of CDS and EDS contracts under the *constant elasticity of variance* (CEV) model of Cox (1975), as a special case. Moreover, we show that when the contract triggering level is set to zero, our ST approach nests the CDS pricing solutions already offered, under the JDCEV model, by Carr and Linetsky (2006), which do not depend on the first passage time density.

The remainder of this thesis is organized as follows. Chapter 2 presents the first paper. Chapter 3 presents the second paper. Chapter 4 presents the third paper. Chapter 5 summarizes the main conclusions.

2 Pricing Swaptions under Multifactor Gaussian HJM Models

Abstract

Several approximations have been proposed in the literature for the pricing of European-style swaptions under multifactor term structure models. However, none of them provides an estimate for the inherent approximation error. Until now, only the Edgeworth expansion technique of Collin-Dufresne and Goldstein (2002) is able to characterize the order of the approximation error. Under a multifactor Heath, Jarrow, and Morton (1992) Gaussian framework, this paper proposes a new approximation for European-style swaptions, which is able to set bounds on the magnitude of the approximation error and is based on the *conditioning approach* initiated by Curran (1994) and Rogers and Shi (1995). All the proposed pricing bounds will arise as a simple by-product of the Nielsen and Sandmann (2002) setup, and will be shown to provide a better accuracy-efficiency trade-off than all the approximations already proposed in the literature.

This paper is based on Nunes and Prazeres (2013), published in *Mathematical Finance*.

JEL Classification: G13

Keywords: Gaussian HJM multifactor models; European-style swaptions; conditioning approach; rank 1 approximation; lognormal approximation; stochastic duration; Edgeworth expansion; hyperplane approximation; low-variance martingale approximation.

2.1 Introduction

The main purpose of the present paper is to offer a fast and extremely accurate analytical approximation for European-style swaptions under a multifactor Gaussian Heath, Jarrow, and Morton (1992)—HJM, hereafter—framework.

European-style swaptions are essentially options on coupon-bearing bonds, that is on a portfolio of pure discount bonds. Under several single-factor term structure models,[2.1] a European-style swaptions can be valued analytically through its decomposition into a portfolio of options on zero-coupon bonds—see, for instance, Jamshidian (1989) under the Vasiček (1977) model, or Longstaff (1993) for the Cox et al. (1985) setup. However, under a (more realistic) multifactor term structure framework, no exact closed-form solution has ever been found for European-style swaptions, because the optimal exercise boundary involves a nonlinear function of several random variables, whose joint probability density is unknown.

Since European-style swaptions are amongst the most widely traded fixed-income derivatives, it is not surprising that several pricing approximations have been proposed in the literature. El Karoui and Rochet (1989) obtained an analytical approximation for European-style options on coupon-bearing bonds, under a multifactor Gaussian HJM model, by using a *proportionality assumption*. Such assumption enables the exercise boundary to be expressed as a monotonic function of a univariate normal random variable and is equivalent to the *rank 1* approximation suggested by Brace and Musiela (1994). Still under the same framework, Pang (1996) approximates the probability distribution of the underlying coupon-bearing bond by a lognormal distribution, which has the same first two moments. Although it is well known that the sum of lognormal random variables is not lognormally distributed, the price of a coupon bond weighs mostly its last pure discount bond price component (i.e. the one associated with the redemption of the bond's face value and with the payment of the last coupon). Therefore, the intuition

[2.1] As long as discount factors are monotonic functions of the single state variable, and if closed-form solutions exist for options on zero-coupon bonds.

behind the lognormal approximation proposed by Pang (1996) is that the probability distribution of the coupon-bearing bond price should essentially depend upon the probabilistic behaviour of its last component, which is lognormally distributed for the Gaussian framework considered.

A completely different approach was undertaken by Wei (1997), for single-factor models, and developed by Munk (1999), for any multifactor term structure model. These authors approximate the price of a European-style option on a coupon-bearing bond by a multiple of the price of a European-style option on a zero-coupon bond with maturity equal to the *stochastic duration*[2.2] of the coupon-bearing bond (and with an adjusted strike price). A similar approach is also pursued by Schrager and Pelsser (2006), since these authors approximate the affine dynamics of the swap rate (under the relevant swap measure) by replacing some (low variance) martingales by their expectations.

Under the Duffie and Kan (1996) general affine class of interest rate models, Singleton and Umantsev (2002) approximate directly the optimal exercise boundary through a linear function of the model's factors, which enables all the relevant exercise probabilities to be computed through the Fourier transform method of Duffie et al. (2000, Proposition 2).[2.3] The basic idea is to use an hyperplane to approximate only the segment of the concave exercise boundary where the density of the state variables is mostly concentrated. For an $\mathbb{A}_2(2)$ specification, these authors outperform the stochastic duration approach, both in terms of accuracy and speed.[2.4]

All the previous approximation schemes are uncontrolled in the sense that the order or magnitude of the approximation error is unknown. The only exception corresponds to the Edgeworth expansion technique, applied by Collin-Dufresne and Goldstein (2002) to the

[2.2] The stochastic duration of a coupon-bearing bond can be defined as the maturity of a pure discount bond with the same instantaneous variance of relative price changes.

[2.3] Moreover, the numerical (and time-consuming) solution of the complex-valued ordinary differential equations stated in Duffie et al. (2000, Equations (2.5) and (2.6)) can be also avoided through the Gaussian approximation and the Gauss-Hermite quadrature approach proposed by Joslin (2010, Appendix B). However, under the Gaussian framework adopted in this paper, such approximation is unnecessary because the transition density function of the model's factors is known in closed-form.

[2.4] Note that any n-factor affine term structure model can be cast (through an appropriate invariant transformation) into the $\mathbb{A}_m(n)$ canonical formulation of Dai and Singleton (2000, Definition 1), where m $(\leq n)$ is the number of state variables driving the factor's variances.

family of affine term structure models, and extended by Collin-Dufresne and Goldstein (2003) to the HJM and to the random field *affine* frameworks. As long as the moments of the underlying coupon-bearing bond can be obtained analytically (under all the necessary forward measures), the corresponding probability density functions can be approximated through a (truncated) cumulant expansion, whose highest order term characterizes the order of the approximation error. Through Monte Carlo experiments, these authors have reported an accuracy level which is much higher than the one associated with earlier applications of Edgeworth series expansions to the pricing of Asian options—see, for example, Turnbull and Wakeman (1991). Since the Edgeworth expansion is a series expansion about the normal distribution, the authors argue that it seems natural that its accuracy increases for underlying assets characterized by lower volatility regimes, as it is the case for interest rates (when compared against the equity market).

The novel approximation for European-style swaptions proposed in this paper is based on the *conditioning approach* initiated by Curran (1994) and Rogers and Shi (1995) in the context of Asian option pricing, and extended by Nielsen and Sandmann (2002) to a stochastic interest rate setting. This new pricing approach is restricted to a multifactor HJM Gaussian setup, but should be faster to implement than the Edgeworth expansion technique, and will provide explicit (and tight) bounds for the approximation error.

The analytical tractability provided by the multifactor Gaussian—but not necessarily Markovian or time-homogeneous—HJM term structure model proposed is obtained at the expense of an important theoretical drawback: Interest rates are assumed to be normally distributed, and can therefore attain negative values with positive probability. But even though the no-arbitrage Gaussian setup adopted in this paper is more restrictive than, for instance, the more general affine framework used by Collin-Dufresne and Goldstein (2003), the extremely accurate pricing solutions to be proposed in this paper can always be used as control variates for more general diffusion pricing models. Moreover and following, for instance, Nunes et al. (1999, Theorem 1) or Kristensen and Mele (2011, Definition 1), the Gaussian pricing formulae offered in this paper can also be used as the (most accurate) zero-order term of the perturbed or Taylor series expansion pricing solution associated to a more general affine term structure model. However, the extension to stochastic volatility term structure models is outside the scope of the present paper, whose contribution is, nevertheless, the derivation of an extremely accurate and controlled approximation for European-style swaptions under a multifactor Gaussian HJM framework.[2.5]

The Gaussian framework adopted offers also a common ground for the comparison of all alternative pricing methods. Since the conditioning approach proposed will provide extremely tight bounds for the approximation error, it will be possible to compare rigorously the accuracy and efficiency of all the approximations already proposed in the literature for European-style swaptions. The alternative approximations have been compared in the literature against benchmark prices obtained through Monte Carlo studies that involve different levels of accuracy. For instance, Collin-Dufresne and Goldstein (2002, Page 16) run 2×10^6 simulations, whereas Schrager and Pelsser (2006, Page 689) simulate only $500,000$ paths, all using standard variance reduction techniques. Based on a much more demanding setting—involving 10^9 simulations, coupled with antithetic variates—to produce the Monte Carlo proxy of the exact swaption price, this paper will show that the conditioning approach significantly improves upon the existing literature in both speed and accuracy.

Next sections are organized as follows. Section 2.2 summarizes the multifactor Gaussian HJM model adopted. Section 2.3 uses the conditioning approach to derive explicit lower and upper bounds for the price of European-style swaptions. Section 2.4 runs several Monte Carlo experiments to compare the accuracy and efficiency of these explicit pricing solutions against all the approximations already proposed in the literature. The main conclusion, stated in Section 2.5, is that the conditioning approach offers the best accuracy-efficiency trade-off for the pricing of European-style swaptions.

[2.5] Following Nunes et al. (1999, Equation (14)) or Kristensen and Mele (2011, Equation (17)), the higher order (although less significant) terms of the perturbed or Taylor series stochastic volatility expansion must be obtained through the nontrivial differentiation of the proposed Gaussian pricing solution with respect to the vector of state variables. Such extension is left for future research.

2.2 Multifactor Gaussian HJM model

We consider a stochastic intertemporal economy defined on a finite trading interval $\mathcal{T} = [t_0, \tau]$, for some fixed time $\tau > t_0$. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where all the information accruing to all the agents in the economy is described by the augmented, right continuous, and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ generated through the standard Brownian motion $W^{\mathbb{Q}}(t) \in \mathbb{R}^n$, initialized at zero and defined under $\mathbb{Q}$. The probability measure $\mathbb{Q}$ represents the martingale measure obtained when the "money market account" is taken as the numéraire of the economy underlying the model under analysis.[2.6]

The Gaussian HJM model under use can be formulated in terms of pure discount bond prices, which are assumed to evolve through time (under measure $\mathbb{Q}$) according to the following stochastic differential equation:

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sigma(t,T)' \cdot dW^{\mathbb{Q}}(t), \tag{2.1}$$

where $P(t,T)$ represents the time-t price of a (unit face value) zero-coupon bond expiring at time T, for all $T \in \mathcal{T}$ and $t \in [t_0, T]$, $r(t)$ is the time-t instantaneous spot rate, and "$\cdot$" denotes the inner product in $\mathbb{R}^n$. The n-dimensional adapted volatility function $\sigma(\cdot, T) : [t_0, T] \to \mathbb{R}^n$ is assumed to be deterministic and to satisfy the usual mild measurability and integrability requirements—as stated, for instance, in Lamberton and Lapeyre (1996, Theorem 3.5.5)—as well as the "pull-to-par" boundary condition $\sigma(u,u) = 0 \in \mathbb{R}^n, \forall u \in [t_0, T]$.

Using, for instance, Nunes (2004, Proposition 2.2), it is well known that equation (2.1)

[2.6] Meaning that the relative prices of all assets with respect to the numéraire given by a "money market account" are $\mathbb{Q}$-martingales.

yields the following solution, for any arbitrary *forward measure* $\mathbb{Q}^c$:

$$\begin{aligned} P(T_a, T_b) &= \frac{P(t_0, T_b)}{P(t_0, T_a)} \exp\left\{-\frac{1}{2} g(t_0, T_a, T_b) + l(t_0, T_a, T_b, T_c)\right. \\ &\quad \left. + \int_{t_0}^{T_a} [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot dW^{\mathbb{Q}_c}(s)\right\}, \end{aligned} \tag{2.2}$$

for $t_0 \le T_a \le T_b$ and $T_c \ge t_0$, with[2.7]

$$g(t_0, T_a, T_b) := \int_{t_0}^{T_a} \|\sigma(s, T_b) - \sigma(s, T_a)\|^2 \, ds, \tag{2.3}$$

$$l(t_0, T_a, T_b, T_c) := \int_{t_0}^{T_a} [\sigma(s, T_b) - \sigma(s, T_a)]' \cdot [\sigma(s, T_c) - \sigma(s, T_a)] \, ds, \tag{2.4}$$

and where

$$dW^{\mathbb{Q}_c}(t) = dW^{\mathbb{Q}}(t) - \sigma(t, T_c) \, dt$$

is also a vector of standard Brownian motion increments in $\mathbb{R}^n$—with the same standard filtration as $dW^{\mathbb{Q}}(t)$—but defined under the $\mathbb{Q}^c$ forward measure, which arises if the numéraire is changed to a zero-coupon bond with maturity at time T_c, and is defined through the Radon-Nikodým derivative

$$\left.\frac{d\mathbb{Q}_c}{d\mathbb{Q}}\right| \mathcal{F}_t = \exp\left[\int_{t_0}^{t} \sigma(s, T_c)' \cdot dW^{\mathbb{Q}}(s) - \frac{1}{2} \int_{t_0}^{t} \|\sigma(s, T_c)\|^2 \, ds\right].$$

2.3 Conditioning approach

This section adapts the *conditioning approach* of Nielsen and Sandmann (2002) to the valuation of European-style receiver swaptions, that is European-style put options on a swap rate. More precisely,[2.8]

Definition 2.1. *The terminal payoff of a European-style receiver swaption with strike rate*

[2.7] $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

[2.8] European-style payer swaptions can be priced through the *payer-receiver swaption parity* stated, for instance, in Longstaff and Schwartz (2001, Page 2073): a long position in a European-style receiver swaption and a short position in a European-style payer swaption (with the same strike rate and tenor structure) is equivalent to a receiver forward swap with start date at the maturity date of the swaptions and fixed interest rate equal to the strike rate.

C, maturity at date T_0 ($\geq t_0$), and on an interest rate swap (IRS) with a unit nominal value and reset dates $T_0 < \ldots < T_{N_0 - 1}$ is equal to

$$[C - y_{0,N_0}(T_0)]^+ \sum_{i=1}^{N_0} \tau_i P(T_0, T_i), \tag{2.5}$$

where τ_i is the year fraction between T_{i-1} and T_i (under some market daycount convention), and

$$y_{0,N_0}(T_0) = \frac{1 - P(T_0, T_{N_0})}{\sum_{i=1}^{N_0} \tau_i P(T_0, T_i)} \tag{2.6}$$

is the time-T_0 spot swap rate.

Definition 2.1 assumes, as is common market practice, that the swaption maturity coincides with the first reset date of the underlying IRS (i.e. time T_0). Therefore, combining expressions (2.5) and (2.6), and as shown, for instance, by Singleton and Umantsev (2002, Equation (5.2)), the terminal payoff of the $T_0 \times T_{N_0}$ receiver swaption can be rewritten as

$$\left[C \sum_{i=1}^{N_0} \tau_i P(T_0, T_i) + P(T_0, T_{N_0}) - 1\right]^+, \tag{2.7}$$

which corresponds to the terminal value of a European-style call with strike equal to the unit nominal value, with maturity at time T_0 , and on a coupon-bearing bond promising N_0 cash flows of value $C\tau_i + \mathbb{I}_{\{i=N_0\}}$ at times T_i (with $i = 1, \ldots, N_0$), where $\mathbb{I}_{\{A\}}$ is the indicator function of set A.

For simplicity, the pricing bounds derived in the next lines will be specified for European-style calls on a coupon-bearing bond. Given the equivalence between expressions (2.5) and (2.7), the valuation formulas obtained are also applied, in Section 2.4, to the pricing of European-style receiver swaptions.

2.3.1 Lower Bound

Denote by $c_{t_0}[B(t_0); X; T_0]$ the time-t_0 fair price of a European-style call with strike X, expiry date at time T_0 ($\geq t_0$), and on a coupon-bearing bond with present value $B(t_0)$.

Following Geman et al. (1995), and since $\mathbb{Q}_0$ is assumed to be a martingale measure with respect to the numéraire $P(t_0, T_0)$, then[2.9]

$$c_{t_0}[B(t_0); X; T_0] = P(t_0, T_0)\,\mathbb{E}_{\mathbb{Q}_0}\left\{ [B(T_0) - X]^+ \middle| \mathcal{F}_{t_0} \right\}. \tag{2.8}$$

As in Rogers and Shi (1995), let $Z \in \mathbb{R}$ be any $\mathcal{F}_{T_0}$-measurable random variable. From the law of iterative expectations and using Jensen's inequality,

$$\begin{aligned} c_{t_0}[B(t_0); X; T_0] &= P(t_0, T_0)\,\mathbb{E}_{\mathbb{Q}_0}\left\{ \mathbb{E}_{\mathbb{Q}_0}\left[(B(T_0) - X)^+ \middle| Z \right] \middle| \mathcal{F}_{t_0} \right\} \\ &\geq c^l_{t_0}[B(t_0); X; T_0], \end{aligned} \tag{2.9}$$

where

$$c^l_{t_0}[B(t_0); X; T_0] := P(t_0, T_0)\,\mathbb{E}_{\mathbb{Q}_0}\left\{ [\mathbb{E}_{\mathbb{Q}_0}(B(T_0)| Z) - X]^+ \middle| \mathcal{F}_{t_0} \right\} \tag{2.10}$$

defines a lower bound for the true call option price. The next proposition provides an explicit solution for the conditional expectation contained on the right-hand side of equation (2.10), by assuming a standard normal distribution for the conditioning variable. Later—in Proposition 2.4—the conditioning variable Z will be completely defined in order to minimize the inherent approximation error.

Proposition 2.1. *Under the Gaussian HJM model* (2.1)*, the time-t_0 price of a European-style call with strike X, with maturity at time T_0 ($\geq t_0$), and on a coupon-bearing bond with present value $B(t_0)$ and generating N_0 cash flows of value k_i ($i = 1, \ldots, N_0$) at times $T_1 < \ldots < T_{N_0}$ (with $T_1 \geq T_0$), is bounded from below by*

$$\begin{aligned} & c^l_{t_0}[B(t_0); X; T_0] \\ = \; & P(t_0, T_0) \\ & \times \mathbb{E}_{\mathbb{Q}_0}\left\{ \left[\sum_{i=1}^{N_0} k_i \frac{P(t_0, T_i)}{P(t_0, T_0)} \exp\left(-\frac{1}{2} m(t_0, T_0, T_i)^2 + m(t_0, T_0, T_i) Z \right) - X \right]^+ \middle| \mathcal{F}_{t_0} \right\}, \end{aligned} \tag{2.11}$$

[2.9] $\mathbb{E}_{\mathbb{Q}_c}(Y | \mathcal{F}_{t_0})$ denotes the expected value of the random variable Y, conditional on $\mathcal{F}_{t_0}$, and computed under the equivalent martingale measure $\mathbb{Q}_c$.

where $Z \sim N^1(0,1)$ *and*[2.10]

$$m(t_0, T_0, T_i) := \mathbb{E}_{\mathbb{Q}_0}\left\{ Z \int_{t_0}^{T_0} [\sigma(s, T_i) - \sigma(s, T_0)]' \cdot dW^{\mathbb{Q}_0}(s) \middle| \mathcal{F}_{t_0} \right\}. \tag{2.12}$$

Proof. Since N_0 cash flows k_i ($i = 1, \ldots, N_0$) will be generated by the underlying coupon bond between the option's expiry date (T_0) and the bond's maturity date (T_{N_0}), then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_0}[B(T_0)| Z] &= \mathbb{E}_{\mathbb{Q}_0}\left[\sum_{i=1}^{N_0} k_i P(T_0, T_i) \middle| Z \right] \\ &= \sum_{i=1}^{N_0} k_i \mathbb{E}_{\mathbb{Q}_0}[P(T_0, T_i)| Z]. \end{aligned} \tag{2.13}$$

The conditional expectation of each discount factor follows from equation (2.2), with $T_a = T_c = T_0$ and $T_b = T_i$:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_0}[P(T_0, T_i)| Z] = & \frac{P(t_0, T_i)}{P(t_0, T_0)} \exp\left[-\frac{1}{2} g(t_0, T_0, T_i) \right] \\ & \mathbb{E}_{\mathbb{Q}_0}\left\{ \exp\left[\int_{t_0}^{T_0} (\sigma(s, T_i) - \sigma(s, T_0))' \cdot dW^{\mathbb{Q}_0}(s) \right] \middle| Z \right\}. \end{aligned} \tag{2.14}$$

Assuming that Z possesses a standard univariate normal distribution, since

$$\int_{t_0}^{T_0} [\sigma(s, T_i) - \sigma(s, T_0)]' \cdot dW^{\mathbb{Q}_0}(s) \sim N^1(0, g(t_0, T_0, T_i)),$$

and following, for instance, Mood et al. (1974, Page 167), then

$$\int_{t_0}^{T_0} [\sigma(s, T_i) - \sigma(s, T_0)]' \cdot dW^{\mathbb{Q}_0}(s) \middle| Z \sim N^1\left(m(t_0, T_0, T_i) Z, v(T_i, T_i)^2 \right), \tag{2.15}$$

with

$$v(T_i, T_i)^2 := g(t_0, T_0, T_i) - m(t_0, T_0, T_i)^2, \tag{2.16}$$

and where the deterministic function $m(t_0, T_0, T_i)$ is defined by the covariance (2.12).

[2.10] Hereafter, the notation $Y \sim N^1(\mu, \sigma^2)$ is intended to mean that the one-dimensional random variable Y is normally distributed, with mean μ and variance σ^2.

Applying result (2.15) and attending to the definition of the moment generating function of a normal random variable, equation (2.14) becomes

$$\mathbb{E}_{\mathbb{Q}_0}\left[P\left(T_0, T_i\right) \middle| Z \right] = \frac{P\left(t_0, T_i\right)}{P\left(t_0, T_0\right)} \exp\left[-\frac{1}{2} m\left(t_0, T_0, T_i\right)^2 + m\left(t_0, T_0, T_i\right) Z \right]. \qquad (2.17)$$

Combining equations (2.10), (2.13) and (2.17), the lower bound (2.11) follows immediately for the call price. □

Note that the quasi-analytical pricing solution (2.11) still involves a single integration over the domain of Z. In order to obtain an explicit solution for equation (2.11), and following Nielsen and Sandmann (2002), consider the following family $\{\mathcal{P}, \mathcal{N}, \mathcal{M}\}$ of disjoint sets, such that $\mathcal{P} \cup \mathcal{N} \cup \mathcal{M} = \{1, \ldots, N_0\} \equiv \mathcal{D}$:

$$\mathcal{P} := \left\{ i \in \mathcal{D} : m\left(t_0, T_0, T_i\right) > 0 \right\}, \qquad (2.18)$$

$$\mathcal{N} := \left\{ i \in \mathcal{D} : m\left(t_0, T_0, T_i\right) < 0 \right\}, \qquad (2.19)$$

and

$$\mathcal{M} := \left\{ i \in \mathcal{D} : m\left(t_0, T_0, T_i\right) = 0 \right\}. \qquad (2.20)$$

Hence, equation (2.11) can be rewritten as

$$c_{t_0}^{l}\left[B\left(t_0\right); X; T_0\right] = P\left(t_0, T_0\right) \mathbb{E}_{\mathbb{Q}_0}\left\{ \left[\sum_{i \in \mathcal{P} \cup \mathcal{N}} k_i \frac{P\left(t_0, T_i\right)}{P\left(t_0, T_0\right)} f_i\left(Z\right) - \hat{X} \right]^+ \middle| \mathcal{F}_{t_0} \right\}, \qquad (2.21)$$

where

$$\hat{X} := X - \sum_{i \in \mathcal{M}} k_i \frac{P\left(t_0, T_i\right)}{P\left(t_0, T_0\right)}, \qquad (2.22)$$

and

$$f_i\left(Z\right) := \exp\left(-\frac{1}{2} m\left(t_0, T_0, T_i\right)^2 + m\left(t_0, T_0, T_i\right) Z \right). \qquad (2.23)$$

Since k_i and $\frac{P(t_0,T_i)}{P(t_0,T_0)}$ are positive and because all functions $f_i\left(Z\right)$ are convex, for all values of i, then equation

$$\sum_{i\in\mathcal{P}\cup\mathcal{N}} k_i \frac{P(t_0,T_i)}{P(t_0,T_0)} f_i(Z) = \hat{X} \tag{2.24}$$

possesses either zero, one or two solutions in Z. Similarly to Nielsen and Sandmann (2002, Definition 1),

Definition 2.2. *Let the two possible solutions of equation* (2.24) *be represented by* z_* *and* z^*, *where* $z_* \leq z^*$.

- *If* $\mathcal{P} \neq \emptyset$ *but* $\mathcal{N} = \emptyset$, *then define the unique solution by* z^* *and set* $z_* = -\infty$.
- *If* $\mathcal{P} = \emptyset$ *but* $\mathcal{N} \neq \emptyset$, *then define the unique solution by* z_* *and set* $z^* = \infty$.
- *If* $\mathcal{P} \neq \emptyset$ *and* $\mathcal{N} \neq \emptyset$, *then either two solutions* z_* *and* z^* *exist or no solution exists and* $z_* = z^* = \infty$.

Proposition 2.2. *Under the assumptions of Proposition 2.1,*

$$\begin{aligned} c^l_{t_0}[B(t_0);X;T_0] &= \sum_{i\in\mathcal{P}\cup\mathcal{N}} k_i P(t_0,T_i)\,\Phi[z_* - m(t_0,T_0,T_i)] - \hat{X}P(t_0,T_0)\,\Phi(z_*) \\ &\quad + \sum_{i\in\mathcal{P}\cup\mathcal{N}} k_i P(t_0,T_i)\,\Phi[m(t_0,T_0,T_i) - z^*] - \hat{X}P(t_0,T_0)\,\Phi(-z^*), \end{aligned} \tag{2.25}$$

where $\Phi(\cdot)$ *represents the cumulative density function of the univariate standard normal distribution,* $\hat{X}$ *is defined by equation* (2.22), *while* z_* *and* z^* *are given by Definition 2.2.*

Proof. Applying Definition 2.2, equation (2.21) can be restated as

$$\begin{aligned} & c^l_{t_0}[B(t_0);X;T_0] \\ =\; & P(t_0,T_0)\,\mathbb{E}_{\mathbb{Q}_0}\left\{\left[\sum_{i\in\mathcal{P}\cup\mathcal{N}} k_i \frac{P(t_0,T_i)}{P(t_0,T_0)} f_i(Z) - \hat{X}\right]\mathbb{I}_{\{Z\leq z_*\}}\middle|\,\mathcal{F}_{t_0}\right\} \\ & + P(t_0,T_0)\,\mathbb{E}_{\mathbb{Q}_0}\left\{\left[\sum_{i\in\mathcal{P}\cup\mathcal{N}} k_i \frac{P(t_0,T_i)}{P(t_0,T_0)} f_i(Z) - \hat{X}\right]\mathbb{I}_{\{Z\geq z^*\}}\middle|\,\mathcal{F}_{t_0}\right\}. \end{aligned} \tag{2.26}$$

Since $Z \sim N^1(0,1)$ and using equation (2.23), equation (2.26) yields

$$\begin{aligned} & c_{t_0}^l \left[B(t_0); X; T_0\right] \\ = & \sum_{i \in \mathcal{P} \cup \mathcal{N}} k_i P(t_0, T_i) \int_{-\infty}^{z_*} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[z - m(t_0, T_0, T_i)\right]^2\right\} dz - \hat{X} P(t_0, T_0) \Phi(z_*) \\ & + \sum_{i \in \mathcal{P} \cup \mathcal{N}} k_i P(t_0, T_i) \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[z - m(t_0, T_0, T_i)\right]^2\right\} dz - \hat{X} P(t_0, T_0) \Phi(-z^*), \end{aligned}$$

and equation (2.25) follows immediately. □

Instead of the method proposed by Nielsen and Sandmann (2002) that relies on the numerical solution of equation (2.24), one could also have solved equation (2.11) through Lord (2006, Theorem 1). However, and if the covariance function $m(t_0, T_0, T_i)$ is not monotone in T_i, Lord (2006, Theorem 1) would require the numerical and time-consuming minimization of the function $\mathbb{E}_{\mathbb{Q}_0}(B(T_0)|Z)$.

The pricing solution (2.25) will only become a completely explicit solution after the specification of the conditioning random variable Z, which will define the deterministic function $m(t_0, T_0, T_i)$. For that purpose, and following Rogers and Shi (1995) and Nielsen and Sandmann (2002), the minimization of the approximation error will be pursued.

2.3.2 Upper Bound

Following Rogers and Shi (1995, Equation 3.5),

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_0}\left\{\mathbb{E}_{\mathbb{Q}_0}\left[\left(B(T_0) - X\right)^+ \middle| Z\right] - \left[\mathbb{E}_{\mathbb{Q}_0}\left(B(T_0)|Z\right) - X\right]^+ \middle| \mathcal{F}_{t_0}\right\} \\ \leq & \frac{1}{2}\mathbb{E}_{\mathbb{Q}_0}\left\{\sqrt{var\left[B(T_0)|Z\right]} \middle| \mathcal{F}_{t_0}\right\}, \end{aligned} \qquad (2.27)$$

where

$$var\left[B(T_0)|Z\right] := \mathbb{E}_{\mathbb{Q}_0}\left\{\left[B(T_0) - \mathbb{E}_{\mathbb{Q}_0}\left(B(T_0)|Z\right)\right]^2 \middle| Z\right\} \qquad (2.28)$$

represents the conditional variance of the time-T_0 underlying coupon bond price. Therefore,

Proposition 2.3. *Under the Gaussian HJM model* (2.1), *the time-*t_0 *price of a European-style call with strike* X*, with maturity at time* T_0 *(*$\geq t_0$*), and on a coupon-bearing bond with present value* $B(t_0)$ *and generating* N_0 *cash flows of value* k_i *(*$i = 1, \ldots, N_0$*) at times* $T_1 < \ldots < T_{N_0}$ *(with* $T_1 \geq T_0$*), is bounded from above by*

$$c_{t_0}^u [B(t_0); X; T_0] = c_{t_0}^l [B(t_0); X; T_0] + \varepsilon_{t_0} [B(t_0); X; T_0], \tag{2.29}$$

where the lower bound $c_{t_0}^l [B(t_0); X; T_0]$ *is given by Proposition 2.2 and the implicit approximation error is defined by*

$$\varepsilon_{t_0} [B(t_0); X; T_0] := \frac{1}{2} P(t_0, T_0) \sqrt{\mathbb{E}_{\mathbb{Q}_0} \{ var [B(T_0)| Z] | \mathcal{F}_{t_0} \}}, \tag{2.30}$$

with

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_0} \{ var [B(T_0)| Z] | \mathcal{F}_{t_0} \} \\ = \; & \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} k_i k_j \frac{P(t_0, T_i) P(t_0, T_j)}{P(t_0, T_0)^2} \\ & \times \{ \exp [l(t_0, T_0, T_i, T_j)] - \exp [m(t_0, T_0, T_i) m(t_0, T_0, T_j)] \} . \end{aligned} \tag{2.31}$$

Proof. Multiplying both sides of inequality (2.27) by the discount factor $P(t_0, T_0)$ and applying Cauchy-Schwarz inequality, as in Nielsen and Sandmann (2002), then

$$c_{t_0} [B(t_0); X; T_0] - c_{t_0}^l [B(t_0); X; T_0] \leq \frac{1}{2} P(t_0, T_0) \sqrt{\mathbb{E}_{\mathbb{Q}_0} \{ var [B(T_0)| Z] | \mathcal{F}_{t_0} \}},$$

and equations (2.29) and (2.30) follow.

Concerning the conditional variance, equations (2.2) and (2.17) yield

$$\begin{aligned} & B(T_0) - \mathbb{E}_{\mathbb{Q}_0} (B(T_0)| Z) \\ = \; & \sum_{i=1}^{N_0} k_i \frac{P(T_0, T_i)}{P(t_0, T_0)} \left\{ \exp \left[-\frac{1}{2} g(t_0, T_0, T_i) + \int_{t_0}^{T_0} (\sigma(s, T_i) - \sigma(s, T_0))' \cdot dW^{\mathbb{Q}_0}(s) \right] \right. \\ & \left. - \exp \left[-\frac{1}{2} m(t_0, T_0, T_i)^2 + m(t_0, T_0, T_i) Z \right] \right\} . \end{aligned}$$

Applying definition (2.28) and using the conditional probability distribution (2.15), then

$$
\begin{aligned}
& var\left[B\left(T_0\right)\right|Z] & (2.32)\\
= & \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} k_i k_j \frac{P\left(t_0,T_i\right)P\left(t_0,T_j\right)}{P\left(t_0,T_0\right)^2}\left\{\exp\left[v\left(T_i,T_j\right)^2\right]-1\right\} \\
& \times\exp\left\{\left[m\left(t_0,T_0,T_i\right)+m\left(t_0,T_0,T_j\right)\right]Z-\frac{1}{2}\left[m\left(t_0,T_0,T_i\right)^2+m\left(t_0,T_0,T_j\right)^2\right]\right\},
\end{aligned}
$$

where

$$
v\left(T_i,T_j\right)^2 := l\left(t_0,T_0,T_i,T_j\right)-m\left(t_0,T_0,T_i\right)m\left(t_0,T_0,T_j\right) \tag{2.33}
$$

corresponds to the conditional covariance between

$$
\int_{t_0}^{T_0}\left[\sigma\left(s,T_i\right)-\sigma\left(s,T_0\right)\right]'\cdot dW^{\mathbb{Q}_0}\left(s\right)
$$

and

$$
\int_{t_0}^{T_0}\left[\sigma\left(s,T_j\right)-\sigma\left(s,T_0\right)\right]'\cdot dW^{\mathbb{Q}_0}\left(s\right).
$$

Taking expectations of both sides of equation (2.32), and since $Z\sim N^1\left(0,1\right)$, the analytical solution (2.31) arises. □

Proposition 2.3 shows that, with the purpose of minimizing the option' approximation error, Z shall be chosen in order to reduce the quantity (2.31). Next proposition provides a first-order approximation for the (unknown) $\arg\min_{Z}\mathbb{E}_{\mathbb{Q}_0}\left\{var\left[B\left(T_0\right)\right|Z]\right|\mathcal{F}_{t_0}\}$.

Proposition 2.4. *Under the Gaussian HJM model* (2.1)*, if*

$$
Z := \frac{1}{\alpha}\sum_{i=1}^{N_0} k_i\frac{P\left(t_0,T_i\right)}{P\left(t_0,T_0\right)}\int_{t_0}^{T_0}\left[\sigma\left(s,T_i\right)-\sigma\left(s,T_0\right)\right]'\cdot dW^{\mathbb{Q}_0}\left(s\right), \tag{2.34}
$$

with

$$
\alpha^2 := \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} k_i k_j\frac{P\left(t_0,T_i\right)P\left(t_0,T_j\right)}{P\left(t_0,T_0\right)^2} l\left(t_0,T_0,T_i,T_j\right), \tag{2.35}
$$

then

$$Z \sim N^1(0,1), \tag{2.36}$$

$$m(t_0,T_0,T_i) = \frac{1}{\alpha}\sum_{j=1}^{N_0} k_j \frac{P(t_0,T_j)}{P(t_0,T_0)} l(t_0,T_0,T_j,T_i), \tag{2.37}$$

and

$$\mathbb{E}_{\mathbb{Q}_0}\left\{ var\left[B(T_0)\right| Z\right]\left|\mathcal{F}_{t_0}\right\} \approx 0. \tag{2.38}$$

Proof. From equation (2.35), since $\alpha^2 = \mathbb{E}_{\mathbb{Q}_0}\left[(\alpha Z)^2\right|\mathcal{F}_{t_0}\right]$, with Z defined through equation (2.34), then condition (2.36) is verified.

Using definition (2.12), then

$$\begin{aligned} m(t_0,T_0,T_i) &= \frac{1}{\alpha}\sum_{j=1}^{N_0} k_j \frac{P(t_0,T_j)}{P(t_0,T_0)} \mathbb{E}_{\mathbb{Q}_0}\left\{ \int_{t_0}^{T_0} \left[\sigma(s,T_j) - \sigma(s,T_0)\right]' \cdot dW^{\mathbb{Q}_0}(s) \right. \\ &\quad \left. \times \int_{t_0}^{T_0} \left[\sigma(s,T_i) - \sigma(s,T_0)\right]' \cdot dW^{\mathbb{Q}_0}(s) \right| \left. \mathcal{F}_{t_0} \right\}, \end{aligned}$$

which yields equation (2.37) after considering definition (2.4).

Finally, applying the first-order approximation $\exp(x) \approx 1 + x$ to equation (2.31), then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_0}\left\{ var\left[B(T_0)\right| Z\right]\left|\mathcal{F}_{t_0}\right\} &\approx \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} k_i k_j \frac{P(t_0,T_i)P(t_0,T_j)}{P(t_0,T_0)^2} \\ &\quad \times \left\{ l(t_0,T_0,T_i,T_j) - m(t_0,T_0,T_i)\, m(t_0,T_0,T_j)\right\}. \end{aligned} \tag{2.39}$$

Moreover and using the analytical solutions (2.35) and (2.37), it follows that

$$\begin{aligned} & m(t_0,T_0,T_i)\, m(t_0,T_0,T_j) \\ = & \frac{\left[\sum_{p=1}^{N_0} k_p \frac{P(t_0,T_p)}{P(t_0,T_0)} l(t_0,T_0,T_p,T_i)\right]\left[\sum_{q=1}^{N_0} k_q \frac{P(t_0,T_q)}{P(t_0,T_0)} l(t_0,T_0,T_q,T_j)\right]}{\sum_{p=1}^{N_0}\sum_{q=1}^{N_0} k_p k_q \frac{P(t_0,T_p)P(t_0,T_q)}{P(t_0,T_0)^2} l(t_0,T_0,T_p,T_q)}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
& \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} k_i k_j \frac{P(t_0,T_i)\,P(t_0,T_j)}{P(t_0,T_0)^2} m(t_0,T_0,T_i)\, m(t_0,T_0,T_j) \\
= & \frac{\sum_{p=1}^{N_0}\sum_{i=1}^{N_0} k_p k_i \frac{P(t_0,T_p)P(t_0,T_i)}{P(t_0,T_0)^2} l(t_0,T_0,T_p,T_i)}{\sum_{p=1}^{N_0}\sum_{q=1}^{N_0} k_p k_q \frac{P(t_0,T_p)P(t_0,T_q)}{P(t_0,T_0)^2} l(t_0,T_0,T_p,T_q)} \\
& \times \left[\sum_{q=1}^{N_0}\sum_{j=1}^{N_0} k_q k_j \frac{P(t_0,T_q)\,P(t_0,T_j)}{P(t_0,T_0)^2} l(t_0,T_0,T_q,T_j)\right] \\
= & \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} k_i k_j \frac{P(T_0,T_i)\,P(T_0,T_j)}{P(t_0,T_0)^2} l(t_0,T_0,T_i,T_j),
\end{aligned}
$$

where the second equality follows by replacing i by q in the numerator of the second line and q by i in the third line. Consequently, the right-hand side of equation (2.39) becomes equal to zero, i.e. result (2.38) is obtained. □

2.4 Numerical analysis

The purpose of the numerical experiments contained in this section is to compare the accuracy-efficiency performance of the conditioning approach against the rank 1 approximation of Brace and Musiela (1994), the lognormal approximation of Pang (1996), the stochastic duration approach of Munk (1999), the Edgeworth expansion technique of Collin-Dufresne and Goldstein (2002), the hyperplane approximation of Singleton and Umantsev (2002), and the low-variance martingale method of Schrager and Pelsser (2006).

Even though the pricing bounds proposed in Section 2.3 are applicable to the more general class of HJM Gaussian term structure models, and in order to compare all the approximations proposed in the literature for European-style swaptions, all numerical examples will be run under the nested n-factor affine $\mathbb{A}_0(n)$ specification that is required by both the hyperplane and the low-variance approximations. More precisely, a Gauss-Markov and time-inhomogeneous version of the Duffie and Kan (1996) model will be considered, which specifies the short-term interest rate $r(t)$ as an affine function of the model's factors:

$$r(t) = f + G' \cdot Y(t), \tag{2.40}$$

where $f \in \mathbb{R}$ and $G \in \mathbb{R}^n$ are model's parameters, while $Y(t) \in \mathbb{R}^n$ denotes the time-t vector of state variables. Additionally, the state variables are assumed to follow a multivariate and time-inhomogeneous elastic random walk:

$$dY(t) = (a \cdot Y(t) + b)\, dt + \Sigma \cdot dW^{\mathbb{Q}}(t), \tag{2.41}$$

where $a, \Sigma \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are model's parameters.

Given the affine specifications adopted for the drift and for the instantaneous variance of the stochastic differential equation (2.41), it is easy to show—see, for instance, Langetieg (1980, Equations (30), (32) and (33))—that pure discount bond prices are exponential-affine functions of the state variables:

$$P(t,T) = \exp\left[A(t,T) + B(t,T)' \cdot Y(t)\right], \tag{2.42}$$

where

$$B(t,T)' = G' \cdot a^{-1} \cdot \left[I_n - e^{a(T-t)}\right], \tag{2.43}$$

$$\begin{aligned} A(t,T) &= (T-t)\left(G' \cdot a^{-1} \cdot b - f\right) + B(t,T)' \cdot a^{-1} \cdot \left[b + \Sigma \cdot \Sigma' \cdot \left(a^{-1}\right)' \cdot G\right] \\ &\quad + \frac{1}{2} G' \cdot a^{-1} \cdot \left[(T-t)\, \Sigma \cdot \Sigma' + \Delta(t,T)\right] \cdot \left(a^{-1}\right)' \cdot G, \end{aligned} \tag{2.44}$$

and

$$\Delta(t,T) := \int_t^T e^{a(T-s)} \cdot \Sigma \cdot \Sigma' \cdot e^{a'(T-s)} ds, \tag{2.45}$$

with $I_n \in \mathbb{R}^{n \times n}$ denoting an identity matrix. Note that equations (2.43) and (2.44) assume that matrix a is non-singular. If this is not the case, functions $A(t,T)$ and $B(t,T)$ can always be obtained through the more general solutions described in Lund (1994, Appendix A). Moreover, function $\Delta(t,T)$ can be computed explicitly from Langetieg

(1980, Footnote 23), as long as matrix a is diagonalizable; otherwise, it is always possible to evaluate $\Delta(t,T)$ numerically using Padé approximations with scaling and squaring, based on Van Loan (1978, Theorem 1).

Using the diffusion process (2.41), applying Itô's lemma to equation (2.42), and considering the stochastic differential equation (2.1), it is easy to show that the $\mathbb{A}_0(n)$ specification adopted can be cast into the more general Gaussian HJM model presented in Section 2.2, as long as two conditions are met:

$$\sigma(t,T) = \Sigma' \cdot B(t,T); \tag{2.46}$$

and the discount function initially "observed" in the market must be replaced by equation (2.42). Adopting these two conditions, the pricing solutions proposed in Section 2.3 will be used under the nested $\mathbb{A}_0(n)$ specification, and function $l(t_0, T_a, T_b, T_c)$ will be computed explicitly—see Appendix A.

[Please insert Table 2.1 about here.]

Table 2.1 values at-the-money-forward (ATMF) European-style swaptions, using different valuation approaches, and under the three-factor Gaussian and affine model specified in Collin-Dufresne and Goldstein (2002, Exhibit 1) or Schrager and Pelsser (2006, Table 4.1), i.e. for $f = 6\%$, $G' = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $Y(t) = \begin{bmatrix} 1\% & 0.5\% & -2\% \end{bmatrix}$, $a = diag\{-1, -0.2, -0.5\}$, $b' = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, and $\Sigma = diag\{1\%, 0.5\%, 0.2\%\} \cdot L$, where L is the lower triangular matrix obtained from the Cholesky decomposition of a (3×3) correlation matrix R, with $R_{12} = -0.2$, $R_{13} = -0.1$, and $R_{23} = 0.3$. Four different option maturities (of 1, 2, 5 and 10 years) and eight different swap maturities (of 1, 2, 5, 10, 15, 20, 25 and 30 years) are considered, yielding a total of 32 swaptions.

Since there is no exact pricing solution in the literature for European-style swaptions under multifactor term structure models, the accuracy of each alternative valuation approach is measured by its percentage error with respect to a proxy of the exact swaption price. Such a proxy is obtained through Monte Carlo simulation, using the exact probability distribution of the state variables at the maturity date of the swaption contract—as described in Appendix B. All Monte Carlo experiments are run over 10^9 paths, and using standard antithetic variables. Consequently, for all the 32 swaptions valued, the ratio between the standard error and the Monte Carlo price estimate (labeled as the percentage standard error) is always below 0.3 basis points.

The Monte Carlo simulations were run under version 1.9.4.13 of GNU Pascal and on an Intel Xeon 3.33 GHz processor with 12 GB of RAM memory, whereas all the approximations tested (from the fourth to the last column of Table 2.1) were implemented through *Matlab* (R2010a). All non-linear equations involved in the implementation of the conditioning approach, the rank 1 approximation, the stochastic duration approach, or the hyperplane approximation were solved through the built-in function "fsolve" of *Matlab*.

The lower bound (2.25) provided by the conditioning approach proposed is the most accurate approximation tested: It yields a mean absolute percentage pricing error (MAPE, henceforth) of only 0.14 basis points, which is even smaller than the Monte Carlo percentage standard error. Moreover, the tight upper bound shown on the fifth column of Table 2.1 also provides a sharp forecast for the maximum approximation error attached to the conditioning approach: On average, the maximum absolute percentage pricing error associated to the lower bound approximation is equal to only 3.96 basis points (i.e. $0.0410\% - 0.0014\%$).

For ATMF swaptions, the low-variance martingale approach of Schrager and Pelsser (2006), the lognormal approximation, and the hyperplane approximation of Singleton and Umantsev (2002)—which is implemented at a 1% significance level—are almost as accurate as the conditioning approach.[2.11] Additionally, the low-variance martingale approach is also the most efficient approximation: The whole set of 32 swaptions contracts is priced

[2.11] Note that equation (A.3) of Schrager and Pelsser (2006, Page 692) contains a typo and has been replaced by

$$\sigma_{n,N} = \sqrt{\sum_{i=1}^{M} \widehat{\Sigma}^2_{(ii)} \left(\widetilde{C}^{(i)}_{n,N}\right)^2 \left[\frac{e^{2A_{(ii)}T_n} - 1}{2A_{(ii)}}\right] + 2\sum_{i=1}^{M}\sum_{j=i+1}^{M} \rho_{ij} \widehat{\Sigma}_{(ii)} \widehat{\Sigma}_{(jj)} \widetilde{C}^{(i)}_{n,N} \widetilde{C}^{(j)}_{n,N} \left[\frac{e^{\left[A_{(ii)}+A_{(jj)}\right]T_n} - 1}{A_{(ii)} + A_{(jj)}}\right]},$$

using the notation of the original paper.

under a CPU time of only 0.29 seconds.

The Edgeworth expansion method of Collin-Dufresne and Goldstein (2002) is the most time-consuming approximation tested: it takes more than $1,903$ seconds to price all the 32 swaptions. Since the m-th moment of the probability distribution of the model's factors $Y(T_0)$ requires the computation of $(N_0)^m$ terms (where N_0 represents the number of cash flow payment dates of the underlying swap), then the inefficiency of the Edgeworth expansion increases with the time to maturity of the underlying interest rate swap: For instance, the CPU times associated to the 10×1 and the 10×30 swaptions are equal to 0.033 and 239.37 seconds, respectively. Given the inefficiency of the Edgeworth expansion for long maturity swaps and following, for instance, Chu and Kwok (2007, Page 382), the Taylor series expansion of the log-characteristic function of $Y(T_0)$ is truncated only up to the third order.[2.12]

Overall, the less accurate methods are the rank 1 approximation, and the stochastic duration approach: Their average absolute percentage errors are about 200 and 118 times higher, respectively, than the MAPE associated to the lower bound of the conditioning approach. Moreover, these two approximations systematically overvalue the ATMF swaption contracts under analysis since their reported (positive) mean percentage errors are consistently identical to the corresponding MAPE. Nevertheless, and since swaptions are usually quoted in flat yield volatilities,[2.13] the penultimate line of Table 2.1 recomputes the mean absolute errors of each analytical approximation in terms of the Black (1976) flat yield volatility implicit to each swaption price. As expected, the error differences

[2.12]Note that equations (26) and (30) of Collin-Dufresne and Goldstein (2002, Page 16) contain a typo and have been replaced by

$$B_0(\tau) = -\delta\tau + \frac{1}{2}\sum_{i,j}\frac{\sigma_i\sigma_j\rho_{ij}}{\kappa_i\kappa_j}\left[\tau - B_{\kappa_i}(\tau) - B_{\kappa_j}(\tau) + B_{\kappa_i+\kappa_j}(\tau)\right],$$

and

$$M(\tau) = \sum_{i,j}\frac{\sigma_i\sigma_j\rho_{ij}}{\kappa_j}F_i\left[B_{\kappa_i}(\tau) - \frac{e^{-\kappa_j(W-T)} - e^{-\kappa_i\tau-\kappa_j(W-t)}}{\kappa_i+\kappa_j}\right] + \frac{1}{2}\sum_{i,j}\sigma_i\sigma_j\rho_{ij}F_iF_jB_{\kappa_i+\kappa_j}(\tau)$$

respectively, using the notation of the original paper.

[2.13]I.e. under the usual "market" assumption of lognormally distributed forward swap rates.

amongst all pricing methods are now less pronounced: All mean absolute volatility errors are lower than 1 basis point, and hence, are also clearly within the typical bid-ask spreads observed in the swaptions market.[2.14]

[Please insert Table 2.2 about here.]

The most challenging setup to test the accuracy of all the competing pricing approximations corresponds to the valuation of out-the-money-forward (OTMF) swaption contracts, i.e. option contracts with zero *intrinsic value*. For this purpose, Table 2.2 prices OTMF European-style swaptions under the three-factor Gaussian and affine model adopted in Table 2.1, and for two different strikes that are set at 85% and 90% of the current forward swap rate. Three different option maturities (of 1, 2 and 5 years) and six different swap maturities (of 1, 2, 5, 10, 20 and 30 years) are considered, yielding a total of 36 swaptions.

Percentage pricing errors are computed, under different analytical approximations, for only 27 out of the whole set of 36 swaption contracts. For nine of the swaption contracts considered, the Monte Carlo price estimate is so close to zero (and its standard error is so large) that the percentage pricing errors would not be meaningful for any of the analytical approximations tested. All the nine contracts with missing pricing errors in Table 2.2 possess Monte Carlo price estimates below 10^{-6} as well as percentage standard errors above 0.3%.

As before, the lower bound of the conditioning approach yields the most precise approximation for OTMF swaptions, with a MAPE of only 2.18 basis points. However, and specially for swaptions on long-dated swaps (with a time to maturity of 10 or more years), the upper bound is so loose that can no longer serve as an indicator for the error of the approximation.

Similarly to Table 2.1, the low-variance approximation of Schrager and Pelsser (2006) is still the fastest pricing methodology (with a CPU time of only 0.27 seconds), but now this is also the most inaccurate approximation tested—yielding an average percentage

[2.14]For instance, in 2011 the average bid-ask spread of US ATMF European-style swaptions (for the maturities and tenors described in the first column of Table 2.1) ranged between 30 and 67 basis points—data collected daily from Bloomberg between January 01, 2011 and December 31, 2011.

error above 9.56%. The sixth column of Table 2.2 shows that the significant upward bias of the low-variance approximation is mainly due to the large pricing errors attached to the swaption contracts on 20 and 30 years interest rate swaps, which were not considered in Schrager and Pelsser (2006, Table 6.2). However, and since these swaption contracts with long-term tenors possess small dollar values, the penultimate line of Table 2.2 shows that the corresponding mean absolute Black (1976) flat yield volatility errors are only around 3 basis points.

Again, the Edgeworth expansion technique is the most time-consuming approximation tested as well as one of the less accurate pricing methods: It yields a CPU time of $1,906.56$ seconds and a MAPE of 6.38%. Overall, the low-variance approximation and the Edgeworth expansion technique are both dominated by the hyperplane approach, and even by the simpler rank 1, lognormal and stochastic duration approximations: For instance, the stochastic duration approach combines a CPU time of only 0.57 seconds with a MAPE of 0.72%. Nevertheless, note that the mean absolute error of the Munk (1999) approach is still 33 times higher than the MAPE reported by the lower bound of the conditioning method.

[Please insert Table 2.3 about here.]

Table 2.3 shows that differences in accuracy among the alternative analytical approximations tested are much less pronounced for in-the-money-forward (ITMF) swaption contracts. Table 2.3 prices ITMF European-style swaptions under the same three-factor Gaussian and affine model already adopted in Tables 2.1 and 2.2, and for two different strikes that are set at 110% and 115% of the current forward swap rate. Similarly to Table 2.2, three different option maturities (of 1, 2 and 5 years) and six different swap maturities (of 1, 2, 5, 10, 20 and 30 years) are considered, yielding a total of 36 swaptions.

Once again, the lower bound of the conditioning approach is the most accurate valuation method with a MAPE of only 0.02 basis points. Moreover, the conditioning approach also offers tight error bounds since the upper bound on the swaption prices is even smaller than most of the price estimates produced by the alternative approximations tested: Its

MAPE of 0.52 basis points is only above the average pricing errors associated to the hyperplane, the lognormal, and the Edgeworth approximations.

As before, the low-variance approximation—that is still the fastest valuation method—is also the less accurate approach, yielding an average percentage error of -2.12 basis points (that corresponds to a mean absolute volatility error of 4.34 basis points). However, and in contrast with Table 2.2, the low-variance approximation consistently underprices ITMF swaptions since its mean percentage error and MAPE are exactly symmetrical.

Finally, a word of caution must be said about the accuracy of the proxy used for the exact price of the swaption contracts. Tables 2.1, 2.2, and 2.3 show that some of the percentage errors computed for the lower (upper) bound of the conditioning approach are slightly positive (negative), meaning that the Monte Carlo price estimate can be, for some contracts, slightly below or above the exact lower or upper price bound, respectively. Nevertheless, and given the large number of simulations run, the average pricing errors (over all swaption contacts in Tables 2.1, 2.2, and 2.3) are always non-positive(non-negative) for the lower (upper) bound of the conditioning approach.[2.15]

Note that this (small) bias occurs even though the Monte Carlo price estimates are produced using a large number of simulations (10^9 paths), antithetic variables, and the exact probability distribution of the state variables at the maturity date of the swaption contract, which is, to the authors knowledge, the most demanding setting already used in the literature on the pricing of swaption contracts. Therefore, one should expect this Monte Carlo bias to be even more relevant in the previous literature on the pricing of European-style swaptions; for instance, Schrager and Pelsser (2006, Page 689) test their low-variance approximation using only $500,000$ Monte Carlo simulations. The use, in this paper, of a more demanding Monte Carlo setting, and the inclusion of longer

[2.15] One way to avoid the noise introduced by the Monte Carlo estimates would be to consider a single-factor Gaussian term structure model, because an exact analytical pricing solution for European-style swaptions is provided by Jamshidian (1989). This approach is followed, for instance, by Schrager and Pelsser (2006, Table 4.2). However, such a test would be redundant in our case, because it is easy to show—following El Karoui and Rochet (1989, Page 22)—that the upper and lower bounds of the conditioning approach collapse into the exact swaption price under any single-factor Gauss-Markov and time-homogeneous term structure model.

swaption contracts (with 5 and 10 years of time to maturity) on long-term interest rate swaps (namely on 20 and 30 years' swaps) explains the novel findings with respect to the previous literature concerning the inaccuracy of the low-variance, and of the Edgeworth approximations for OTMF contracts.

2.5 Conclusions

This paper offers two contributions to the literature on swaptions pricing. First, this paper derives a new analytical approximation for European-style swaptions under a multifactor Gauss-Markov framework, and based on the *conditioning approach* proposed by Curran (1994), Rogers and Shi (1995), and Nielsen and Sandmann (2002). Second, a comprehensive and rigorous Monte Carlo study is run to compare, in terms of efficiency and accuracy, all the approximations already proposed in the literature for European-style swaptions under multifactor term structure models.

The numerical results obtained show that the exact lower bound of the swaption price provided by the conditioning approach is the most accurate pricing method for ATMF, OTMF and ITMF contracts. Moreover, the conditioning approach proposed in this paper also offers tight bounds for the approximation error, because the analytical lower and upper bounds proposed in Propositions 2.2 and 2.3 are usually very close to each other (except for some deep OTMF contracts).

By contrast, the low-variance martingale method of Schrager and Pelsser (2006)—which is the fastest pricing approach tested—and the Edgeworth expansion yield the highest pricing errors for OTMF swaption contracts. The latter approach proposed by Collin-Dufresne and Goldstein (2002) is also extremely time consuming for swaption contracts on long-term swaps. The hyperplane approximation of Singleton and Umantsev (2002) is more accurate and faster than the Edgeworth expansion technique, but still less accurate and slower than the proposed conditioning approach. Finally, the simpler rank 1, lognormal, and stochastic duration approximations are also very fast to implement but, nevertheless, still much less accurate than the lower bound of the conditioning approach.

A Function $l(t_0, T_a, T_b, T_c)$ under the $\mathbb{A}_0(n)$ specification

Under the nested $\mathbb{A}_0(n)$ specification, function $l(t_0, T_a, T_b, T_c)$ can be computed explicitly. For this purpose, equations (2.4) and (2.46) yield:

$$\begin{aligned} & l(t_0, T_a, T_b, T_c) && \text{(A-1)} \\ = \; & G' \cdot a^{-1} \cdot \int_{t_0}^{T_a} \left[e^{a(T_a-s)} - e^{a(T_b-s)}\right]' \cdot \Sigma \cdot \Sigma' \cdot \left[e^{a'(T_a-s)} - e^{a'(T_c-s)}\right] ds \cdot \left(a^{-1}\right)' \cdot G. \end{aligned}$$

Using definition (2.45), equation (A-1) can be restated as

$$\begin{aligned} l(t_0, T_a, T_b, T_c) &= G' \cdot a^{-1} \cdot \Big[\Delta(t_0, T_a) - \Delta(t_0, T_a) \cdot e^{a'(T_c-T_a)} - e^{a(T_b-T_a)} \cdot \Delta(t_0, T_a) \\ &\quad + e^{a(T_b-T_a)} \cdot \Delta(t_0, T_a) \cdot e^{a'(T_c-T_a)}\Big] \cdot \left(a^{-1}\right)' \cdot G \\ &= B(T_a, T_b)' \cdot \Delta(t_0, T_a) \cdot B(T_a, T_c), \qquad \text{(A-2)} \end{aligned}$$

where the last line follows from equation (2.43).

B Monte Carlo simulation

To implement equation (2.8) through Monte Carlo simulation, it is necessary to rewrite the stochastic differential equation (2.41) under the forward probability measure $\mathbb{Q}_0$. For this purpose, considering equation (2.46) and using, for instance, Nunes (2004, Equation (2.9)), it follows that

$$dW^{\mathbb{Q}_0}(t) = dW^{\mathbb{Q}}(t) - \Sigma' \cdot B(t, T_0)\, dt \qquad \text{(B-1)}$$

is also a vector of standard Brownian motion increments in $\mathbb{R}^n$ and under the forward measure $\mathbb{Q}_0$. Hence, equations (2.41) and (B-1) imply that

$$dY(t) = [a \cdot Y(t) + b + \Sigma \cdot \Sigma' \cdot B(t, T_0)]\, dt + \Sigma \cdot dW^{\mathbb{Q}_0}(t). \qquad \text{(B-2)}$$

Applying Itô's lemma to $e^{-at} \cdot Y(t)$ and using definition (2.43), the following strong solution is obtained for the stochastic differential equation (B-2):

$$Y(T_0) = e^{a(T_0-t_0)} \cdot Y(t_0) + \left[e^{a(T_0-t_0)} - I_n\right] \cdot a^{-1} \cdot \left[b + \Sigma \cdot \Sigma' \cdot \left(a^{-1}\right)' \cdot G\right] \qquad \text{(B-3)}$$
$$-\Delta(t_0, T_0) \cdot \left(a^{-1}\right)' \cdot G + \int_{t_0}^{T_0} e^{a(T_0-u)} \cdot \Sigma \cdot dW^{\mathbb{Q}_0}(u).$$

Equation (B-3) yields the exact probability distribution of the state vector $Y(T_0)$ after noting that Arnold (1992, Corollary 4.5.6) implies that the Itô integral $\int_{t_0}^{T_0} e^{a(T_0-u)} \cdot \Sigma \cdot dW^{\mathbb{Q}_0}(u)$ possesses a n-dimensional normal distribution with zero mean and variance-covariance matrix equal to $\Delta(t_0, T_0)$. The Itô integral is simulated by generating n normally distributed deviates with zero mean and unit variance—through routines "ran3" and "gasdev" of Press et al. (1994)—that are then correlated using the Cholesky decomposition of matrix $\Delta(t_0, T_0)$.

Table 2.1: Prices of ATMF European-style swaptions on plain-vanilla interest rate swaps with semiannual cash flows and under a three-factor Gauss-Markov HJM model

Swaption × Swap	Monte Carlo		Percentage Pricing Errors of Analytical Approximations							
			CA bounds		LVA	HA	EE	SD	LA	R1A
	price	% SE	lower	upper						
1×1	0.002082	0.0024%	0.0040%	0.0045%	0.0041%	0.0040%	0.0036%	0.0250%	0.0040%	0.0117%
1×2	0.003312	0.0024%	0.0001%	0.0035%	0.0004%	0.0001%	-0.0010%	0.0768%	0.0001%	0.0600%
1×5	0.005331	0.0024%	-0.0024%	0.0189%	-0.0014%	-0.0026%	-0.0054%	0.0690%	-0.0018%	0.3130%
1×10	0.006558	0.0024%	-0.0014%	0.0435%	0.0003%	-0.0025%	-0.0064%	0.0404%	0.0003%	0.5292%
1×15	0.006893	0.0024%	0.0004%	0.0550%	0.0023%	-0.0013%	-0.0054%	0.0382%	0.0026%	0.5954%
1×20	0.006984	0.0024%	0.0000%	0.0578%	0.0020%	-0.0018%	-0.0060%	0.0370%	0.0024%	0.6141%
1×25	0.007009	0.0024%	0.0000%	0.0587%	0.0021%	-0.0019%	-0.0061%	0.0368%	0.0024%	0.6195%
1×30	0.007016	0.0024%	-0.0003%	0.0587%	0.0018%	-0.0022%	-0.0064%	0.0364%	0.0022%	0.6208%
2×1	0.002355	0.0024%	-0.0019%	-0.0013%	-0.0017%	-0.0019%	-0.0024%	0.0376%	-0.0019%	0.0055%
2×2	0.003843	0.0024%	0.0044%	0.0084%	0.0049%	0.0044%	0.0028%	0.1387%	0.0045%	0.0566%
2×5	0.006369	0.0024%	-0.0007%	0.0233%	0.0009%	-0.0011%	-0.0055%	0.1618%	0.0004%	0.2444%
2×10	0.007907	0.0025%	-0.0035%	0.0463%	-0.0007%	-0.0057%	-0.0117%	0.1340%	-0.0005%	0.3931%
2×15	0.008327	0.0025%	0.0002%	0.0603%	0.0034%	-0.0025%	-0.0094%	0.1340%	0.0039%	0.4414%
2×20	0.008442	0.0025%	0.0002%	0.0638%	0.0035%	-0.0028%	-0.0098%	0.1332%	0.0043%	0.4545%
2×25	0.008474	0.0025%	-0.0006%	0.0639%	0.0028%	-0.0037%	-0.0107%	0.1321%	0.0035%	0.4573%
2×30	0.008483	0.0025%	-0.0009%	0.0639%	0.0024%	-0.0041%	-0.0111%	0.1317%	0.0032%	0.4579%
5×1	0.002321	0.0024%	0.0018%	0.0025%	0.0021%	0.0018%	0.0010%	0.0521%	0.0018%	0.0082%
5×2	0.003872	0.0024%	-0.0004%	0.0041%	0.0004%	-0.0004%	-0.0026%	0.1726%	-0.0001%	0.0411%
5×5	0.006568	0.0025%	-0.0017%	0.0237%	0.0008%	-0.0022%	-0.0090%	0.2652%	0.0002%	0.1749%
5×10	0.008216	0.0025%	-0.0015%	0.0501%	0.0028%	-0.0046%	-0.0143%	0.2812%	0.0032%	0.2750%
5×15	0.008667	0.0025%	-0.0027%	0.0595%	0.0023%	-0.0072%	-0.0176%	0.2858%	0.0033%	0.3029%
5×20	0.008792	0.0025%	-0.0033%	0.0623%	0.0019%	-0.0083%	-0.0189%	0.2865%	0.0031%	0.3106%
5×25	0.008826	0.0025%	0.0004%	0.0670%	0.0056%	-0.0048%	-0.0154%	0.2904%	0.0069%	0.3166%
5×30	0.008836	0.0025%	0.0007%	0.0676%	0.0059%	-0.0045%	-0.0151%	0.2907%	0.0072%	0.3175%

(continued)

Table 2.1—*Continued*

Swaption × Swap	Monte Carlo		Percentage Pricing Errors of Analytical Approximations							
			CA bounds		LVA	HA	EE	SD	LA	R1A
	price	% SE	lower	upper						
10 × 1	0.001800	0.0024%	-0.0006%	0.0001%	-0.0003%	-0.0006%	-0.0014%	0.0460%	-0.0005%	0.0054%
10 × 2	0.003020	0.0024%	0.0014%	0.0059%	0.0023%	0.0014%	-0.0011%	0.1740%	0.0017%	0.0395%
10 × 5	0.005153	0.0025%	-0.0011%	0.0243%	0.0016%	-0.0018%	-0.0094%	0.2200%	0.0009%	0.1573%
10 × 10	0.006459	0.0025%	-0.0009%	0.0507%	0.0039%	-0.0044%	-0.0154%	0.3214%	0.0044%	0.2458%
10 × 15	0.006816	0.0025%	-0.0017%	0.0605%	0.0039%	-0.0069%	-0.0186%	0.2602%	0.0050%	0.2707%
10 × 20	0.006914	0.0025%	-0.0018%	0.0639%	0.0040%	-0.0076%	-0.0194%	0.3596%	0.0055%	0.2782%
10 × 25	0.006941	0.0025%	-0.0006%	0.0662%	0.0053%	-0.0065%	-0.0184%	0.3617%	0.0068%	0.2815%
10 × 30	0.006948	0.0025%	0.0043%	0.0714%	0.0103%	-0.0016%	-0.0135%	0.3668%	0.0118%	0.2870%
MPE			-0.0003%	0.0409%	0.0025%	-0.0026%	-0.0087%	0.1687%	0.0028%	0.2871%
MAPE			0.0014%	0.0410%	0.0027%	0.0033%	0.0092%	0.1687%	0.0031%	0.2871%
MAE vol.			0.0001%	0.0010%	0.0001%	0.0001%	0.0002%	0.0046%	0.0001%	0.0077%
CPU (sec.)	180,821.50		5.91		0.29	16.21	1,903.50	0.55	3.17	0.59

This table values 32 at-the-money-forward (ATMF) European-style swaptions under the three-factor Gaussian and affine model specified in Collin-Dufresne and Goldstein (2002, Exhibit 1) or Schrager and Pelsser (2006, Table 4.1). The first column shows the maturity (in years) of the swaption contract and of its underlying swap. The second and third columns contain the Monte Carlo option price estimate and its percentage standard error (%SE)—i.e. the ratio between the standard error and the Monte Carlo price estimate—obtained using 10^9 paths, standard antithetic variables, and the exact probability distribution of the state variables at the maturity date of the swaption contract. The bounds provided by the conditioning approach (CA) in Propositions 2.2 and 2.3 are implemented in the fourth and fifth columns. The sixth column contains the low-variance martingale approximation (LVA) of Schrager and Pelsser (2006). The hyperplane approximation of Singleton and Umantsev (2002) is implemented at a 1% significance level in the seventh column. The eighth column presents the Edgeworth expansion (EE) method of Collin-Dufresne and Goldstein (2002) using a third-order approximation for the log-characteristic function of the terminal state vector. The last three columns contain the percentage pricing errors generated by the stochastic duration (SD) approach of Munk (1999), the lognormal approximation (LA) of Pang (1996), and the rank 1 approximation (R1A) of Brace and Musiela (1994). The last four lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE), mean absolute volatility errors (MAE vol.), and computation times (in seconds). All percentage errors are computed against the Monte Carlo price estimate.

Table 2.2: Prices of OTMF European-style receiver swaptions on plain-vanilla interest rate swaps with semiannual cash flows and under a three-factor Gauss-Markov HJM model

Swaption × Swap	Monte Carlo		Percentage Pricing Errors of Analytical Approximations							
			CA bounds		LVA	HA	EE	SD	LA	R1A
	price	% SE	lower	upper						
Panel A: Swaptions quoted at 85% of the forward swap rate										
1 × 1	0.000154	0.0111%	0.0197%	0.0255%	1.0749%	0.0199%	0.0075%	0.0913%	0.0117%	0.0488%
1 × 2	0.000103	0.0167%	-0.0009%	0.0941%	1.8874%	0.0070%	0.0603%	0.3109%	-0.0707%	0.2950%
1 × 5	0.000013	0.0543%	-0.1412%	7.3571%	6.9317%	0.3134%	4.3318%	-0.3886%	-0.9917%	2.6053%
1 × 10	0.000000	0.4928%	-	-	-	-	-	-	-	-
1 × 20	0.000000	72.5132%	-	-	-	-	-	-	-	-
1 × 30	0.000000	NA	-	-	-	-	-	-	-	-
2 × 1	0.000277	0.0089%	0.0009%	0.0047%	0.8982%	0.0012%	-0.0206%	0.1194%	-0.0065%	0.0242%
2 × 2	0.000254	0.0117%	0.0150%	0.0674%	1.4757%	0.0233%	-0.0242%	0.4851%	-0.0462%	0.2199%
2 × 5	0.000082	0.0252%	0.0115%	1.6128%	4.8410%	0.3809%	1.4967%	0.3586%	-0.6443%	1.5382%
2 × 10	0.000005	0.1010%	-0.0290%	65.5140%	21.0190%	3.0352%	24.1319%	-1.8662%	-3.2955%	4.2996%
2 × 20	0.000000	2.2576%	-	-	-	-	-	-	-	-
2 × 30	0.000000	23.4079%	-	-	-	-	-	-	-	-
5 × 1	0.000373	0.0076%	0.0057%	0.0091%	0.8107%	0.0060%	-0.0211%	0.1388%	-0.0016%	0.0235%
5 × 2	0.000422	0.0093%	-0.0062%	0.0286%	1.1962%	0.0025%	-0.0914%	0.5094%	-0.0625%	0.1300%
5 × 5	0.000240	0.0156%	0.0142%	0.6121%	3.5897%	0.3248%	0.2654%	0.7578%	-0.5167%	0.8544%
5 × 10	0.000040	0.0399%	-0.0153%	9.2530%	14.4558%	2.3050%	10.4057%	-0.2691%	-2.4444%	2.1306%
5 × 20	0.000001	0.3067%	-	-	-	-	-	-	-	-
5 × 30	0.000000	1.5701%	-	-	-	-	-	-	-	-
Panel B: Swaptions quoted at 90% of the forward swap rate										
1 × 1	0.000437	0.0067%	-0.0039%	-0.0019%	0.4571%	-0.0037%	-0.0150%	0.0476%	-0.0075%	0.0164%
1 × 2	0.000432	0.0085%	-0.0064%	0.0175%	0.7705%	-0.0030%	-0.0445%	0.2124%	-0.0366%	0.1872%
1 × 5	0.000185	0.0159%	-0.0124%	0.5469%	2.5783%	0.1643%	0.1077%	0.0149%	-0.3427%	1.5981%
1 × 10	0.000023	0.0460%	-0.0519%	11.6615%	10.4281%	1.4425%	5.3939%	-1.3531%	-1.6902%	4.5859%
1 × 20	0.000000	0.4229%	-	-	-	-	-	-	-	-
1 × 30	0.000000	2.4067%	-	-	-	-	-	-	-	-

(continued)

Table 2.2—*Continued*

Swaption × Swap	Monte Carlo		Percentage Pricing Errors of Analytical Approximations							
	price	% SE	CA bounds lower	CA bounds upper	LVA	HA	EE	SD	LA	R1A
2 × 1	0.000642	0.0058%	-0.0020%	-0.0001%	0.4063%	-0.0018%	-0.0140%	0.0854%	-0.0055%	0.0149%
2 × 2	0.000758	0.0069%	0.0074%	0.0259%	0.6397%	0.0112%	-0.0387%	0.3438%	-0.0204%	0.1483%
2 × 5	0.000523	0.0106%	0.0165%	0.2826%	1.9030%	0.1684%	-0.1359%	0.3738%	-0.2524%	0.9671%
2 × 10	0.000143	0.0216%	-0.0183%	2.4887%	7.1171%	1.1153%	1.4861%	-0.4124%	-1.2304%	2.4083%
2 × 20	0.000007	0.0890%	0.0265%	67.5232%	32.4355%	4.4250%	24.2704%	-2.6396%	-4.0460%	5.0962%
2 × 30	0.000001	0.2661%	0.0630%	700.0488%	69.2668%	7.3914%	59.6131%	-4.7587%	-6.6443%	7.0860%
5 × 1	0.000756	0.0053%	-0.0038%	-0.0020%	0.3744%	-0.0036%	-0.0168%	0.0973%	-0.0074%	0.0095%
5 × 2	0.001010	0.0060%	0.0190%	0.0344%	0.5624%	0.0232%	-0.0314%	0.4012%	-0.0076%	0.1167%
5 × 5	0.000938	0.0082%	-0.0100%	0.1511%	1.4690%	0.1244%	-0.2617%	0.5970%	-0.2387%	0.5424%
5 × 10	0.000408	0.0136%	-0.0114%	0.9370%	5.2659%	0.8892%	-0.0271%	0.3407%	-0.9567%	1.2640%
5 × 20	0.000053	0.0360%	-0.0265%	9.9383%	22.0731%	3.2570%	9.9860%	-0.7644%	-3.0194%	2.4497%
5 × 30	0.000011	0.0753%	0.0511%	49.4814%	44.4240%	5.4328%	30.0070%	-1.8024%	-4.7930%	3.3877%
MPE			-0.0033%	34.3597%	9.5686%	1.1427%	6.3267%	-0.3322%	-1.1617%	1.5573%
MAPE			0.0218%	34.3600%	9.5686%	1.1436%	6.3817%	0.7237%	1.1626%	1.5573%
MAE vol.			0.0002%	0.0319%	0.0311%	0.0030%	0.0131%	0.0041%	0.0037%	0.0067%
CPU (sec.)	175,024.62		5.48		0.27	15.40	1,906.56	0.57	2.93	0.60

This table values 36 out-the-money-forward (OTMF) European-style swaptions under the three-factor Gaussian and affine model specified in Table 2.1. The first column shows the maturity (in years) of the swaption contract and of its underlying swap. The second and third columns contain the Monte Carlo option price estimate and its percentage standard error (%SE) obtained using 10^9 paths, standard antithetic variables, and the exact probability distribution of the state variables at the maturity date of the swaption contract. The %SE that are not available (NA) correspond to Monte Carlo price estimates that are equal to zero. The bounds provided by the conditioning approach (CA) in Propositions 2.2 and 2.3 are implemented in the fourth and fifth columns. The sixth column contains the low-variance martingale approximation (LVA) of Schrager and Pelsser (2006). The hyperplane approximation of Singleton and Umantsev (2002) is implemented at a 1% significance level in the seventh column. The eighth column presents the Edgeworth expansion (EE) method of Collin-Dufresne and Goldstein (2002) using a third-order approximation for the log-characteristic function of the terminal state vector. The last three columns contain the percentage pricing errors generated by the stochastic duration (SD) approach of Munk (1999), the lognormal approximation (LA) of Pang (1996), and the rank 1 approximation (R1A) of Brace and Musiela (1994). The last four lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE), mean absolute volatility errors (MAE vol.), and computation times (in seconds).All percentage errors are computed against the Monte Carlo price estimate, except for the nine contracts with percentage standard errors above 0.3%.

Table 2.3: Prices of ITMF European-style receiver swaptions on plain-vanilla interest rate swaps with semiannual cash flows and under a three-factor Gauss-Markov HJM model

Swaption × Swap	Monte Carlo		Percentage Pricing Errors of Analytical Approximations							
			CA bounds		LVA	HA	EE	SD	LA	R1A
	price	% SE	lower	upper						
Panel A: Swaptions quoted at 110% of the forward swap rate										
1 × 1	0.005634	0.0005%	0.0008%	0.0010%	-0.0349%	0.0008%	-0.0001%	0.0063%	0.0011%	0.0027%
1 × 2	0.010674	0.0004%	0.0001%	0.0012%	-0.0313%	-0.0001%	-0.0015%	0.0137%	0.0016%	0.0096%
1 × 5	0.024284	0.0001%	-0.0002%	0.0049%	-0.0201%	-0.0018%	0.0005%	0.0063%	0.0029%	0.0150%
1 × 10	0.042493	0.0000%	0.0000%	0.0075%	-0.0061%	-0.0011%	0.0032%	0.0014%	0.0011%	0.0034%
1 × 20	0.065979	0.0000%	0.0000%	0.0066%	-0.0002%	0.0000%	0.0006%	0.0001%	0.0000%	0.0001%
1 × 30	0.078861	0.0000%	0.0000%	0.0057%	0.0000%	0.0000%	0.0001%	0.0000%	0.0000%	0.0000%
2 × 1	0.005689	0.0007%	0.0002%	0.0004%	-0.0458%	0.0001%	-0.0012%	0.0131%	0.0006%	0.0025%
2 × 2	0.010659	0.0005%	0.0001%	0.0017%	-0.0447%	-0.0002%	-0.0031%	0.0338%	0.0026%	0.0123%
2 × 5	0.023572	0.0003%	0.0001%	0.0072%	-0.0418%	-0.0039%	-0.0036%	0.0243%	0.0073%	0.0259%
2 × 10	0.040481	0.0001%	0.0001%	0.0107%	-0.0259%	-0.0047%	0.0044%	0.0093%	0.0055%	0.0113%
2 × 20	0.062458	0.0000%	0.0000%	0.0093%	-0.0045%	-0.0008%	0.0048%	0.0011%	0.0008%	0.0011%
2 × 30	0.074583	0.0000%	0.0000%	0.0080%	-0.0010%	-0.0002%	0.0021%	0.0002%	0.0001%	0.0002%
5 × 1	0.005142	0.0008%	0.0002%	0.0005%	-0.0552%	0.0002%	-0.0017%	0.0196%	0.0009%	0.0026%
5 × 2	0.009537	0.0007%	-0.0005%	0.0015%	-0.0577%	-0.0010%	-0.0058%	0.0552%	0.0030%	0.0121%
5 × 5	0.020483	0.0004%	0.0003%	0.0092%	-0.0669%	-0.0071%	-0.0112%	0.0592%	0.0132%	0.0312%
5 × 10	0.034347	0.0002%	-0.0002%	0.0132%	-0.0636%	-0.0129%	-0.0027%	0.0321%	0.0137%	0.0190%
5 × 20	0.052436	0.0001%	0.0000%	0.0119%	-0.0248%	-0.0046%	0.0104%	0.0079%	0.0044%	0.0039%
5 × 30	0.062523	0.0000%	0.0000%	0.0104%	-0.0096%	-0.0016%	0.0090%	0.0027%	0.0016%	0.0012%
Panel B: Swaptions quoted at 115% of the forward swap rate										
1 × 1	0.007948	0.0002%	-0.0002%	-0.0001%	-0.0207%	-0.0002%	-0.0005%	0.0021%	0.0000%	0.0006%
1 × 2	0.015460	0.0001%	-0.0001%	0.0008%	-0.0127%	-0.0001%	0.0003%	0.0040%	0.0006%	0.0026%
1 × 5	0.036149	0.0000%	0.0000%	0.0036%	-0.0027%	-0.0002%	0.0018%	0.0007%	0.0004%	0.0014%
1 × 10	0.063698	0.0000%	0.0000%	0.0052%	-0.0001%	0.0000%	0.0004%	0.0000%	0.0000%	0.0000%
1 × 20	0.098967	0.0000%	0.0000%	0.0046%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
1 × 30	0.118291	0.0000%	0.0000%	0.0040%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%

(continued)

Table 2.3—*Continued*

Swaption	Monte Carlo		Percentage Pricing Errors of Analytical Approximations							
×			CA bounds		LVA	HA	EE	SD	LA	R1A
Swap	price	% SE	lower	upper						
2 × 1	0.007845	0.0003%	-0.0001%	0.0001%	-0.0319%	-0.0001%	-0.0009%	0.0062%	0.0002%	0.0010%
2 × 2	0.015098	0.0002%	0.0002%	0.0014%	-0.0245%	0.0000%	-0.0006%	0.0135%	0.0016%	0.0049%
2 × 5	0.034636	0.0001%	0.0001%	0.0051%	-0.0117%	-0.0011%	0.0033%	0.0056%	0.0022%	0.0052%
2 × 10	0.060483	0.0000%	0.0000%	0.0073%	-0.0021%	-0.0004%	0.0034%	0.0007%	0.0004%	0.0006%
2 × 20	0.093669	0.0000%	0.0000%	0.0065%	0.0000%	0.0000%	0.0002%	0.0000%	0.0000%	0.0000%
2 × 30	0.111871	0.0000%	0.0000%	0.0056%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
5 × 1	0.006950	0.0004%	-0.0005%	-0.0002%	-0.0437%	0.0005%	-0.0019%	0.0102%	0.0000%	0.0008%
5 × 2	0.013205	0.0003%	0.0010%	0.0025%	-0.0374%	0.0007%	-0.0017%	0.0278%	0.0035%	0.0069%
5 × 5	0.029533	0.0002%	-0.0002%	0.0062%	-0.0295%	0.0036%	0.0008%	0.0196%	0.0056%	0.0092%
5 × 10	0.050896	0.0001%	0.0000%	0.0093%	-0.0122%	0.0026%	0.0087%	0.0053%	0.0028%	0.0026%
5 × 20	0.078540	0.0000%	0.0000%	0.0083%	-0.0009%	0.0002%	0.0031%	0.0002%	0.0001%	0.0001%
5 × 30	0.093753	0.0000%	0.0000%	0.0071%	-0.0001%	0.0001%	0.0008%	0.0000%	0.0000%	0.0000%
MPE			0.0000%	0.0052%	-0.0212%	-0.0013%	0.0006%	0.0106%	0.0022%	0.0053%
MAPE			0.0002%	0.0052%	0.0212%	0.0014%	0.0026%	0.0106%	0.0022%	0.0053%
MAE vol.			0.0076%	0.1455%	0.0434%	0.0124%	0.0387%	0.0138%	0.0093%	0.0129%
CPU (sec.)	174,856.55		5.46		0.28	15.47	1,903.70	0.56	2.96	0.60

This table values 36 in-the-money-forward (ITMF) European-style swaptions under the three-factor Gaussian and affine model specified in Tables 2.1 and 2.2. The first column shows the maturity (in years) of the swaption contract and of its underlying swap. The second and third columns contain the Monte Carlo option price estimate and its percentage standard error (%SE) obtained using 10^9 paths, standard antithetic variables, and the exact probability distribution of the state variables at the maturity date of the swaption contract. The bounds provided by the conditioning approach (CA) in Propositions 2.2 and 2.3 are implemented in the fourth and fifth columns. The sixth column contains the low-variance martingale approximation (LVA) of Schrager and Pelsser (2006). The hyperplane approximation of Singleton and Umantsev (2002) is implemented at a 1% significance level in the seventh column. The eighth column presents the Edgeworth expansion (EE) method of Collin-Dufresne and Goldstein (2002) using a third-order approximation for the log-characteristic function of the terminal state vector. The last three columns contain the percentage pricing errors generated by the stochastic duration (SD) approach of Munk (1999), the lognormal approximation (LA) of Pang (1996), and the rank 1 approximation (R1A) of Brace and Musiela (1994). The last four lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE), mean absolute volatility errors (MAE vol.), and computation times (in seconds). All percentage errors are computed against the Monte Carlo price estimate.

3 Pricing European-style Barrier Options under Stochastic Interest Rates

Abstract

This paper offers an extremely fast and accurate novel methodology for the pricing of (long-term) European-style single barrier options on underlying spot prices driven by a geometric Brownian motion and under the stochastic interest rates framework of Vasiček (1977). The proposed valuation methodology extends the *stopping time approach* of Kuan and Webber (2003) to a more general setting, and expresses the price of a European-style barrier option in terms of the first passage time density of the underlying asset price to the barrier level. Using several model parameter constellations and option maturities, our numerical results show that the proposed pricing approach is much more accurate and faster than the two-dimensional *extended Fortet method* of Bernard et al. (2008).

JEL Classification: G13

Keywords: European-style barrier options; geometric Brownian motion; Vasiček model; first passage time.

3.1 Introduction

The main objective of this paper is to offer a fast and accurate novel methodology for the pricing of European-style single barrier options on asset prices driven by a geometric Brownian motion and under the stochastic interest rates framework of Vasiček (1977). This paper generalizes the *stopping time approach* (ST approach, hereafter) first proposed by Kuan and Webber (2003) for options on pure discount bonds, under single-factor term structure models, and later extended by Dias et al. (2014) to the pricing of European-style single and double barrier options under the *jump to default extended constant elasticity of variance* (JDCEV) framework of Carr and Linetsky (2006).

European-style barrier options are path-dependent contingent claims, which are characterized by a strike price and an upper or lower barrier level. These contracts become standard European-style options if the barrier level is—for *knock-in options*—or is not—for *knock-out* options—breached by the underlying spot price, during the option lifetime. If not, the option expires worthless, in which case a cash rebate may be received by the option holder. The existence of a barrier makes these contracts cheaper than their standard counterparts, and allow investors to better express their views about the future evolution of the underlying spot price.

The pricing of European-style single barrier options has already been extensively analyzed over the last decades—see, for instance, Merton (1973), Goldman et al. (1979), Rubinstein and Reiner (1991) and Rich (1994)—but these studies are confined to a constant interest rates setup. However, and as argued, for example, by Amin and Bodurtha (1995), the underlying assumption regarding the evolution of interest rates has a significant impact on the option price, namely when valuing long-maturity contracts.

To the authors' knowledge, the pricing of European-style barrier options in the context of a stochastic interest rate framework has only been pursued through the *extended Fortet method* of Bernard et al. (2008). These authors extend the Fortet (1943) method and offer a two-dimensional Markovian pricing approximation. In different contexts, the Fortet

(1943) method has also been adopted, for instance, by Longstaff and Schwartz (1995), for the pricing of risky debt, and by Collin-Dufresne and Goldstein (2001), for credit risk modelling.

Our pricing methodology extends the ST approach of Kuan and Webber (2003) to a two-factor option pricing model with stochastic interest rates, and expresses the European-style barrier option price in terms of the density function of the first passage time of the underlying asset price to the barrier level. Using the standard partition method of Park and Schuurmann (1976), we are able to recover this hitting density as the implicit solution of a non-linear integral equation. However, and since we are working under a two-factor model, our valuation approach involves a double integral, in both time and interest rate dimensions. In Section 3.3, we will show that our proposed pricing solution can be simplified to require only one integration with respect to time, because the probability density function of the short-term interest rate, conditional on the knock-in or knock-out event, will be obtained in closed-form. Therefore, the ST approach will be shown to be much more accurate and efficient than the extended Fortet method of Bernard et al. (2008).

The remainder of this work is organized as follows. Section 3.2 succinctly describes the model framework and the main features of the contracts under analysis. Sections 3.3 and 3.4 describe the novel pricing methodology proposed. Section 3.5 reviews the extended Fortet method of Bernard et al. (2008). Section 3.6 implements the ST and the extended Fortet valuation approaches, and compares both methodologies in terms of efficiency and accuracy. Section 3.7 summarizes the main conclusions. All auxiliary results are relegated to the Appendix.

3.2 Modelling architecture and contractual features

This section summarizes the pricing model adopted as well as the contractual features of the barrier option contracts under analysis.

3.2.1 Model set-up

The valuation of European-style barrier options will be explored in the context of an arbitrage-free and frictionless financial market with continuous trading on the interval $\mathcal{T} := [t_0, T]$, for some fixed time $T > 0$. As usual, uncertainty will be represented by a complete probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathbb{Q}$, taken as given, will denote the equivalent martingale measure obtained when the numéraire of the economy is a money market account.

The underlying asset price S is specified through the geometric Brownian motion

$$\frac{dS_t}{S_t} = (r_t - q)\, dt + \sigma_S dW_S^{\mathbb{Q}}(t), \tag{3.1}$$

where $q \in \mathbb{R}$ represents the dividend yield, r_t denotes the time-t risk-free short-term interest rate, $\sigma_S \in \mathbb{R}_+$ corresponds to the instantaneous volatility of asset returns, and $\{W_S^{\mathbb{Q}}(t), t \geq t_0\}$ is a standard Brownian motion, defined under measure $\mathbb{Q}$, and initialized at zero. Furthermore, the short-term interest rate dynamics are described by the Ornstein-Uhlenbeck process adopted by Vasiček (1977):

$$dr_t = \alpha(\gamma - r_t)\, dt + \sigma_r dW_r^{\mathbb{Q}}(t), \tag{3.2}$$

where $\gamma \in \mathbb{R}$ is the risk-adjusted long-term mean of r, $\alpha \in \mathbb{R}_+$ is the speed of mean reversion, $\sigma_r \in \mathbb{R}_+$ is the instantaneous volatility of r, and $\{W_r^{\mathbb{Q}}(t), t \geq t_0\}$ is also a standard Brownian motion, defined under measure $\mathbb{Q}$, and initialized at zero. The Brownian motions $\{W_S^{\mathbb{Q}}(t), t \geq t_0\}$ and $\{W_r^{\mathbb{Q}}(t), t \geq t_0\}$ generate the augmented, right continuous, and completed filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$, and are assumed to be correlated:

$$d\langle W_S^{\mathbb{Q}}, W_r^{\mathbb{Q}}\rangle(t) = \rho dt, \tag{3.3}$$

where $|\rho| < 1$.

In order to simplify the exposition, the correlated Brownian motions $\{W_S^{\mathbb{Q}}(t), t \geq$

$t_0\}$ and $\{W_r^{\mathbb{Q}}(t), t \geq t_0\}$ can be decomposed into two independent Brownian motions $\{Z_S^{\mathbb{Q}}(t), t \geq t_0\}$ and $\{Z_r^{\mathbb{Q}}(t), t \geq t_0\}$, such that

$$dW_S^{\mathbb{Q}}(t) = dZ_S^{\mathbb{Q}}(t), \tag{3.4}$$

and

$$dW_r^{\mathbb{Q}}(t) = \rho dZ_S^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dZ_r^{\mathbb{Q}}(t). \tag{3.5}$$

Moreover, and using, for instance, Musiela and Rutkowski (2005, Equation (9.33)), it follows that

$$dZ_S^{\mathbb{Q}^T}(t) = dZ_S^{\mathbb{Q}}(t) + \rho\sigma_r B_\alpha(t,T)\, dt \tag{3.6}$$

and

$$dZ_r^{\mathbb{Q}^T}(t) = dZ_r^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}\sigma_r B_\alpha(t,T)\, dt, \tag{3.7}$$

where $dZ_S^{\mathbb{Q}^T}(t)$ and $dZ_r^{\mathbb{Q}^T}(t)$ are now standard Brownian motion increments in $\mathbb{R}$ and under the forward martingale measure $\mathbb{Q}^T$, which is obtained when the numéraire is taken to be a default-free pure discount bond with maturity at time $T\,(\geq t)$, whose time-t price is defined as

$$P(r_t, T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\int_t^T r_u du\right]\middle| \mathcal{F}_t\right], \tag{3.8}$$

where $\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$ denotes the expected value of a random variable X, conditional on the σ-algebra $\mathcal{F}_t$, and computed under the equivalent martingale measure $\mathbb{Q}$.

Under the Vasiček (1977) model, and following, for instance, Brigo and Mercurio (2006, Section 3.2.1), it is well known that $P(r_t, T)$ is an exponential-affine function of the state variable r:

$$P(r_t, T) = \exp\left[A(t,T) - B_\alpha(t,T)\, r_t\right], \tag{3.9}$$

where

$$A(t,T) := \left[B_\alpha(t,T) - (T-t)\right]\left(\gamma - \frac{\sigma_r^2}{2\alpha^2}\right) - \frac{\sigma_r^2}{4\alpha} B_\alpha^2(t,T), \tag{3.10}$$

and

$$B_\alpha(t,T) := \frac{1 - e^{-\alpha(T-t)}}{\alpha}. \tag{3.11}$$

Hereafter, the analysis will be focused on the time-t value of European-style up barrier put options on the asset price S, with strike price K, constant barrier level $U \in \mathbb{R}_+$ and maturity at date $T\,(\geq t)$, under a stochastic interest rate framework.[3.1] For this purpose, under the framework described by equations (3.1)-(3.3), and following, for instance, Nunes (2011, Equations (29)-(31)), it is straightforward to show that the time-t price of a standard European-style put on the asset price S, with strike price K, with maturity date $T\,(\geq t)$, and defined by

$$p_t(S, r, K, T) := P(r_t, T)\, \mathbb{E}_{\mathbb{Q}^T}\left[\left.(K - S_T)^+\right| \mathcal{F}_t\right], \tag{3.12}$$

can be computed through the following closed-form solution:

$$p_t(S, r, K, T) = KP(r_t, T)\, \Phi\left[-d(S, r_t) + v_x(t,T)\right] - S_t e^{-q(T-t)} \Phi\left[-d(S, r_t)\right], \tag{3.13}$$

with

$$d(S, r_t) := \frac{\ln\left[\frac{S_t}{KP(r_t,T)}\right] - q(T-t) + \frac{v_x^2(t,T)}{2}}{v_x(t,T)}, \tag{3.14}$$

[3.1] The corresponding down barrier, as well as European-style barrier call options, with similar technical features, can be valued in a similar fashion, but will be omitted to contain the paper length.

$$\begin{aligned} v_x^2(t,T) \;&:=\; \sigma_S^2(T-t) + \frac{2\rho\sigma_S\sigma_r}{\alpha}\left[T-t-B_\alpha(t,T)\right] \\ &\quad + \frac{\sigma_r^2}{\alpha^2}\left[T-t-2B_\alpha(t,T)+B_{2\alpha}(t,T)\right], \end{aligned} \tag{3.15}$$

and where $P(r_t,T)$ is given by equation (3.9), $B_\alpha(t,T)$ is given by definition (3.11),

$$B_{2\alpha}(t,T) := \frac{1-e^{-2\alpha(T-t)}}{2\alpha}, \tag{3.16}$$

and $\Phi(.)$ represents the cumulative density function of the univariate standard normal law.

3.2.2 European-style barrier options

The holder of an up-and-out put owns a standard put if and only if the barrier is never breached by the underlying spot price, during the option lifetime. However, if at any time between the contract inception and its expiration date, the barrier is breached, then the option contract is canceled (i.e. it is *knocked out*) and expires worthless, in which case a cash rebate may be received, either when the barrier is breached, or at the maturity date of the option. The next definition summarizes the contractual features of an up-and-out put, with no rebate and with a constant barrier level.[3.2]

Definition 3.1. *The time-T price of a unit face value and zero-rebate European-style up-and-out put on the asset price S, with strike K, upper barrier level $U \in \mathbb{R}_+$, inception at time t_0, and maturity at time $T\,(\geq t_0)$ is equal to:*

$$EKO_T(S,r,U,K,T) = (K-S_T)^+\,\mathbb{I}_{\{\tau_U>T\}}, \tag{3.17}$$

where

$$\tau_U := \inf\{u > t_0 : S_u = U\} \tag{3.18}$$

[3.2] Similarly to Bernard et al. (2008), we will ignore the existence of rebates.

is the first passage time (from below) of the underlying asset price to the upper barrier level, while $\mathbb{I}_{\{A\}}$ *denotes the indicator function of the set* A.

On the other hand, the holder of an up-and-in barrier put owns a standard put if and only if the barrier is breached by the underlying spot price during the option lifetime (i.e. if the option is *knocked in*). However, if the barrier is never breached between the contract inception and its expiration date, then the option contract is canceled and expires worthless, in which case a cash rebate may be received at the maturity date of the option. The next definition summarizes the contractual features of an up-and-in put, with no rebate and with a constant barrier level.

Definition 3.2. *The time-*T *price of a unit face value and zero-rebate European-style up-and-in put on the asset price* S*, with strike* K*, upper barrier level* $U \in \mathbb{R}_+$*, inception at time* t_0*, and maturity at time* $T\,(\geq t_0)$ *is equal to:*

$$EKI_T\left(S, r, U, K, T\right) = \left(K - S_T\right)^+ \mathbb{I}_{\{\tau_U \leq T\}}, \tag{3.19}$$

where τ_U *is still defined by equation* (3.18).

3.3 The stopping time approach

This section offers a new approach for pricing European-style barrier options under the financial model specified by equations (3.1) through (3.3). Assuming that the up-and-out put option is still alive at the valuation date, i.e. $\tau_U > t_0$, equation (3.17) implies that its time-t_0 price is

$$EKO_{t_0}\left(S, r, U, K, T\right) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\int_{t_0}^{T} r_u du\right]\left(K - S_T\right)^+ \mathbb{I}_{\{\tau_U > T\}}\middle|\,\mathcal{F}_{t_0}\right]. \tag{3.20}$$

Applying the change of measure technique of Geman et al. (1995, Corollary 2), equation (3.20) can be restated as

$$EKO_{t_0}\left(S, r, U, K, T\right) = P\left(r_{t_0}, T\right)\mathbb{E}_{\mathbb{Q}^T}\left[\left(K - S_T\right)^+ \mathbb{I}_{\{\tau_U > T\}}\middle|\,\mathcal{F}_{t_0}\right]. \tag{3.21}$$

Since $\mathbb{I}_{\{\tau_U > T\}} = 1 - \mathbb{I}_{\{\tau_U \leq T\}}$, equations (3.12), (3.19) and (3.21) yield the usual in-out parity relation

$$EKO_{t_0}(S, r, U, K, T) = p_{t_0}(S, r, K, T) - EKI_{t_0}(S, r, U, K, T), \tag{3.22}$$

where $p_{t_0}(S, r, K, T)$ and

$$EKI_{t_0}(S, r, U, K, T) = P(r_{t_0}, T)\, \mathbb{E}_{\mathbb{Q}^T}\left[(K - S_T)^+ \mathbb{I}_{\{\tau_U \leq T\}} \middle| \mathcal{F}_{t_0} \right] \tag{3.23}$$

are, respectively, the time-t_0 price of a standard European-style put and the time-t_0 price of a European-style up-and-in put, both contracts with technical features identical to those of the up-and-out put under analysis.

The next proposition provides a quasi-analytical solution for the up-and-in put defined by equation (3.23). Although this term involves three different random variables—namely, the terminal asset price S_T, the short-term interest rate r, and the first passage time τ_U—, the Markov property inherent to equations (3.1) and (3.2) enables the decomposition of their joint density via the convolution of their marginal probability laws.

Proposition 3.1. *Under the financial model described by equations* (3.1)-(3.3)*, the time-t_0 value of a unit face value and zero-rebate European-style up-and-in put on the asset price S, with strike K, upper barrier level $U \in \mathbb{R}_+$ and maturity at time $T\,(\geq t_0)$ is equal to*

$$\begin{aligned} & EKI_{t_0}(S, r, U, K, T) \\ = \quad & P(r_{t_0}, T) \int_{t_0}^{T} \left\{ \int_{\mathbb{R}} \frac{p_u(U, r, K, T)}{P(r, T)} \mathbb{Q}^T(r_u \in dr |\, S_u = U, r_{t_0}) \right\} \mathbb{Q}^T(\tau_U \in du |\, \mathcal{F}_{t_0}), \end{aligned} \tag{3.24}$$

where $\mathbb{Q}^T(\tau_U \in du |\, \mathcal{F}_{t_0})$ is the density function of the first passage time (3.18) *under the forward martingale measure $\mathbb{Q}^T$.*

Proof. Using identity (3.18), the expectation of the right-hand side of equation (3.23) can be written as a function of the joint density of only two random variables: r and τ_U. That is,

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}^T}\left[\left.(K-S_T)^+\,\mathbb{I}_{\{\tau_U\leq T\}}\right|\mathcal{F}_{t_0}\right] \\
= & \int_{\mathbb{R}}\int_{t_0}^{T}\mathbb{E}_{\mathbb{Q}^T}\left[\left.(K-S_T)^+\right|S_u=U,r_u\right]\mathbb{Q}^T\left(r_u\in dr,\tau_U\in du\right|\mathcal{F}_{t_0}\left.\right).
\end{aligned} \tag{3.25}$$

Because both state variables (S and r) follow a Markov process, the joint density contained in the right-hand side of equation (3.25) is simply the convolution between the density of the first passage time τ_U, and the transition probability density function of r (conditional on the asset price crossing the barrier U):

$$\mathbb{Q}^T\left(r_u\in dr,\tau_U\in du\right|\mathcal{F}_{t_0}\left.\right)=\mathbb{Q}^T\left(r_u\in dr\right|S_u=U,r_{t_0}\left.\right)\mathbb{Q}^T\left(\tau_U\in du\right|\mathcal{F}_{t_0}\left.\right). \tag{3.26}$$

Combining equations (3.23), (3.25) and (3.26), then

$$\begin{aligned}
& EKI_{t_0}\left(S,r,U,K,T\right) \\
= & P\left(r_{t_0},T\right)\int_{t_0}^{T}\left\{\int_{\mathbb{R}}\mathbb{E}_{\mathbb{Q}^T}\left[\left.(K-S_T)^+\right|S_u=U\right]\mathbb{Q}^T\left(r_u\in dr\right|S_u=U,r_{t_0}\left.\right)\right\} \\
& \times\mathbb{Q}^T\left(\tau_U\in du\right|\mathcal{F}_{t_0}\left.\right),
\end{aligned} \tag{3.27}$$

and equation (3.24) follows from equation (3.12). □

To solve explicitly the inner integration on the right-side of equation (3.24), it is essential to obtain a closed-form solution for the probability density of the short-term interest rate, under the forward measure, and conditional on the asset price crossing the barrier level U. Such result is provided in the following proposition.

Proposition 3.2. *For the pricing model described through equations* (3.1)-(3.3)*, and for* $t_0\leq t\leq u\leq T$*,*

$$\mathbb{Q}^T\left(r_u\in dr\right|S_u=U,r_{t_0}\left.\right)=\phi\left[r;\mu_{r|x}\left(x_{t_0},r_{t_0},t_0,\ln U,u\right),v^2_{r|x}\left(t_0,u\right)\right]dr, \tag{3.28}$$

with

$$\mu_{r|x}\left(x_t, r_t, t, x_u, u\right) := \mu_r\left(t,u\right) + \frac{v_{r,x}\left(t,u\right)}{v_x^2\left(t,u\right)}\left(x_u - \mu_x\left(x_t, r_t, t, u\right)\right), \tag{3.29}$$

$$v_{r|x}^2\left(t,u\right) := v_r^2\left(t,u\right) - \frac{v_{r,x}^2\left(t,u\right)}{v_x^2\left(t,u\right)}, \tag{3.30}$$

$$\mu_r\left(t,u\right) := e^{-\alpha(u-t)} r_t + \bar{\mu}_r\left(t,u\right), \tag{3.31}$$

$$\bar{\mu}_r\left(t,u\right) := \left(\alpha\gamma - \frac{\sigma_r^2}{\alpha}\right) B_\alpha\left(t,u\right) + \frac{\sigma_r^2}{\alpha} e^{-\alpha(T-u)} B_{2\alpha}\left(t,u\right), \tag{3.32}$$

$$v_r^2\left(t,u\right) := \sigma_r^2 B_{2\alpha}\left(t,u\right), \tag{3.33}$$

$$\mu_x\left(x_t, r_t, t, u\right) := x_t + B_\alpha\left(t,u\right) r_t + \bar{\mu}_x\left(t,u\right), \tag{3.34}$$

$$\begin{aligned}\bar{\mu}_x\left(t,u\right) \quad :=& \quad \left(\gamma - \frac{\sigma_r^2}{\alpha^2}\right)\left[u - t - B_\alpha\left(t,u\right)\right] \\ &+ \left[\left(\frac{\sigma_r^2}{2\alpha^2} + \frac{\rho\sigma_S\sigma_r}{\alpha}\right) e^{-\alpha(T-u)} - \frac{\sigma_r^2}{2\alpha^2} e^{-\alpha(T-t)}\right] B_\alpha\left(t,u\right) \\ &- \left(\frac{\rho\sigma_S\sigma_r}{\alpha} + q + \frac{\sigma_S^2}{2}\right)\left(u-t\right),\end{aligned} \tag{3.35}$$

$$\begin{aligned}v_x^2\left(t,u\right) \quad :=& \quad \sigma_S^2\left(u-t\right) + \frac{2\rho\sigma_S\sigma_r}{\alpha}\left[u - t - B_\alpha\left(t,u\right)\right] \\ &+ \frac{\sigma_r^2}{\alpha^2}\left[u - t - 2B_\alpha\left(t,u\right) + B_{2\alpha}\left(t,u\right)\right],\end{aligned} \tag{3.36}$$

$$v_{r,x}\left(t,u\right) := \left(\rho\sigma_S\sigma_r + \frac{\sigma_S^2}{\alpha}\right) B_\alpha\left(t,u\right) - \frac{\sigma_r^2}{\alpha} B_{2\alpha}\left(t,u\right), \tag{3.37}$$

and where

$$x_t := \ln S_t, \tag{3.38}$$

while $\phi(z; \mu, \sigma^2)$ represents the probability density function of a normally distributed univariate random variable z, with mean μ and standard deviation σ.

Proof. Please see Appendix C. □

Combining Propositions 3.1 and 3.2, as well as equations (3.9) and (3.13), the next proposition offers a closed-form solution for the fair value of a European-style up-and-in put that involves only one time-integral; i.e. the integration with respect to the short-term interest rate state variable is solved explicitly.

Proposition 3.3. *Under the financial model described by equations* (3.1)-(3.3)*, the time-t_0 price of a unit face value and zero-rebate European-style up-and-in put on the asset price S, with strike price K, upper barrier level $U \in \mathbb{R}_+$, and maturity at time $T (\geq t_0)$ is equal to:*

$$\begin{aligned}
& EKI_{t_0}(S, r, U, K, T) \\
= & \int_{t_0}^{T} \mathbb{Q}^T\left(\tau_U \in du \middle| \mathcal{F}_{t_0}\right) \left\{ K\Phi\left[\eta\left(x_{t_0}, r_{t_0}, t_0, u, T\right)\right] \right. \\
& \left. -\varphi\left(x_{t_0}, r_{t_0}, t_0, u, T\right) \Phi\left[\eta\left(x_{t_0}, r_{t_0}, t_0, u, T\right) - \sqrt{v_x^2(u, T) + B_\alpha^2(u, T)\, v_{r|x}^2(t_0, u)}\right] \right\},
\end{aligned} \tag{3.39}$$

with

$$\begin{aligned}
& \varphi\left(x_t, r_t, t, u, T\right) \\
:= & \exp\left\{ \ln U - q(T-u) - A(u, T) + B_\alpha(u, T)\, \mu_{r|x}\left(x_t, r_t, t, \ln U, u\right) \right. \\
& \left. + \frac{1}{2} B_\alpha^2(u, T)\, v_{r|x}^2(t, u) \right\},
\end{aligned} \tag{3.40}$$

and

$$\eta\left(x_t, r_t, t, u, T\right) \tag{3.41}$$
$$:= \frac{\ln K - \ln U + A\left(u,T\right) + q\left(T-u\right) + \frac{v_x^2(u,T)}{2} - B_\alpha\left(u,T\right)\mu_{r|x}\left(x_t, r_t, t, \ln U, u\right)}{\left[v_x^2\left(u,T\right) + B_\alpha^2\left(u,T\right) v_{r|x}^2\left(t,u\right)\right]^{\frac{1}{2}}}.$$

Proof. Please see Appendix D. □

3.4 First passage time density

To implement the pricing solution offered by Proposition 3.3, it is necessary to compute the first passage time density of the underlying asset price through the barrier level U. The next proposition offers a Volterra integral equation of the second kind for this density.

Proposition 3.4. *Under the pricing model described by equations* (3.1)-(3.3)*, the first passage time density of the underlying asset price to the barrier level U is the implicit solution of the following nonlinear integral equation:*

$$\Phi\left[f\left(t_0, u\right)\right] = \int_{t_0}^{u} \Phi\left[g\left(t_0, l, u\right)\right] \mathbb{Q}^T\left(\tau_U \in dl \middle| \mathcal{F}_{t_0}\right), \tag{3.42}$$

for $u \in [t_0, T]$, with

$$f\left(t_0, u\right) := \frac{\mu_x\left(x_{t_0}, r_{t_0}, t_0, u\right) - \ln U}{v_x\left(t_0, u\right)}, \tag{3.43}$$

and

$$g\left(t_0, l, u\right) := \frac{\bar{\mu}_x\left(l,u\right) + B_\alpha\left(l,u\right)\mu_{r|x}\left(x_{t_0}, r_{t_0}, t_0, \ln U, l\right)}{\sqrt{v_x^2\left(l,u\right) + B_\alpha^2\left(l,u\right) v_{r|x}^2\left(t_0, l\right)}}. \tag{3.44}$$

Proof. By the law of total probability, and since $S_{t_0} < U$,

$$\mathbb{Q}^T \left(\tau_U \leq u \middle| \mathcal{F}_{t_0} \right) = \mathbb{Q}^T \left\{ \sup_{t_0 \leq l \leq u} (S_l) \geq U, S_u \geq U \middle| \mathcal{F}_{t_0} \right\} + \mathbb{Q}^T \left\{ \sup_{t_0 \leq l \leq u} (S_l) \geq U, S_u < U \middle| \mathcal{F}_{t_0} \right\}. \tag{3.45}$$

Given that

$$\mathbb{Q}^T \left\{ \sup_{t_0 \leq l \leq u} (S_l) \geq U, S_u \geq U \middle| \mathcal{F}_{t_0} \right\} = \mathbb{Q}^T \left(S_u \geq U \middle| \mathcal{F}_{t_0} \right), \tag{3.46}$$

and because both the underlying price process and the short-term interest rate are Markovian, then equations (3.18), (3.45) and (3.46) yield

$$\begin{aligned} & \mathbb{Q}^T \left(\tau_U \leq u \middle| \mathcal{F}_{t_0} \right) \\ = & \mathbb{Q}^T \left(S_u \geq U \middle| \mathcal{F}_{t_0} \right) + \mathbb{Q}^T \left(\tau_U \leq u, S_u < U \middle| \mathcal{F}_{t_0} \right) \\ = & \mathbb{Q}^T \left(S_u \geq U \middle| \mathcal{F}_{t_0} \right) + \mathbb{Q}^T \left(\tau_U \leq u \middle| \mathcal{F}_{t_0} \right) \\ & - \int_{t_0}^{u} \int_{\mathbb{R}} \mathbb{Q}^T \left(S_u \geq U \middle| S_l = U, r \right) \mathbb{Q}^T \left(r_l \in dr, \tau_U \in dl \middle| \mathcal{F}_{t_0} \right). \end{aligned} \tag{3.47}$$

Furthermore, equation (3.26) allows equation (3.47) to be rewritten as

$$\begin{aligned} & \mathbb{Q}^T \left(S_u \geq U \middle| \mathcal{F}_{t_0} \right) \\ = & \int_{t_0}^{u} \left[\int_{\mathbb{R}} \mathbb{Q}^T \left(S_u \geq U \middle| S_l = U, r \right) \mathbb{Q}^T \left(r_l \in dr \middle| S_l = U, r_{t_0} \right) \right] \mathbb{Q}^T \left(\tau_U \in dl \middle| \mathcal{F}_{t_0} \right). \end{aligned} \tag{3.48}$$

Using the marginal density function (C-12), the left-hand side of equation (3.48) becomes

$$\begin{aligned} \mathbb{Q}^T \left(S_u \geq U \middle| \mathcal{F}_{t_0} \right) & = \mathbb{Q}^T \left(x_u \geq \ln U \middle| \mathcal{F}_{t_0} \right) \\ & = \Phi \left[f \left(t_0, u \right) \right], \end{aligned} \tag{3.49}$$

where function $f(.)$ is defined by equation (3.43). Furthermore, equations (3.34) and

(C-12) imply that

$$\begin{aligned}
& \int_{\mathbb{R}} \mathbb{Q}^T \left(S_u \geq U \middle| S_l = U, r \right) \mathbb{Q}^T \left(r_l \in dr \middle| S_l = U, r_{t_0} \right) \\
= & \int_{\mathbb{R}} \mathbb{Q}^T \left(x_u \geq \ln U \middle| x_l = \ln U, r \right) \mathbb{Q}^T \left(r_l \in dr \middle| x_l = \ln U, r_{t_0} \right) \\
= & \int_{\mathbb{R}} \Phi \left[\frac{\bar{\mu}_x (l,u) + B_\alpha (l,u) r}{v_x (l,u)} \right] \mathbb{Q}^T \left(r_l \in dr \middle| x_l = \ln U, r_{t_0} \right). \qquad (3.50)
\end{aligned}$$

Through Lemma D.1, the previous equation can be rewritten more concisely as

$$\int_{\mathbb{R}} \mathbb{Q}^T \left(S_u \geq U \middle| S_l = U, r \right) \mathbb{Q}^T \left(r_l \in dr \middle| S_l = U, r_{t_0} \right) = \Phi \left[g \left(t_0, l, u \right) \right], \qquad (3.51)$$

where function $g(.)$ is defined by equation (3.44). Combining equations (3.48), (3.49) and (3.51), equation (3.42) follows immediately. □

Following, for instance, Kuan and Webber (2003) or Nunes (2009, Proposition 6), the first passage time density can be efficiently computed through the standard partition method proposed by Park and Schuurmann (1976). Thus, dividing the time interval $[t_0, T]$ into N_{PS} subintervals of size $h := (T - t_0)/N_{PS}$, the probabilities $\mathbb{Q}\left(\tau_U = t_0 + ih \middle| \mathcal{F}_{t_0} \right)$ are obtained from the following recurrence relation:

$$\begin{aligned}
& \mathbb{Q}\left(\tau_U = t_0 + ih \middle| \mathcal{F}_{t_0} \right) \\
= & \mathbb{Q}\left(\tau_U = t_0 + (i-1) h \middle| \mathcal{F}_{t_0} \right) + \left[g \left(t_0, t_0 + \frac{(2i-1) h}{2}, t_0 + ih \right) \right]^{-1} \qquad (3.52) \\
& \times \left\{ f \left(t_0, t_0 + ih \right) - \sum_{j=1}^{i-1} g \left(t_0, t_0 + \frac{(2j-1) h}{2}, t_0 + ih \right) \right. \\
& \left. \times \left[\mathbb{Q}\left(\tau_U = t_0 + jh \middle| \mathcal{F}_{t_0} \right) - \mathbb{Q}\left(\tau_U = t_0 + (j-1) h \middle| \mathcal{F}_{t_0} \right) \right] \right\},
\end{aligned}$$

for $i = 1, ..., N_{PS}$ and where $\mathbb{Q}\left(\tau_U = t_0 \middle| \mathcal{F}_{t_0} \right) = 0$.

3.5 Review of the extended Fortet method

To benchmark the accuracy and efficiency of the pricing approach proposed in Sections 3.3 and 3.4, this section briefly summarizes the extended Fortet method proposed by Bernard et al. (2008) for the pricing of European-style barrier put options, under the financial model defined by equations (3.1)-(3.3).

Following, for instance, Bernard et al. (2008) or Collin-Dufresne and Goldstein (2001, Appendix B), the extended Fortet method starts by dividing the interval $[t_0, T]$ into n_T subintervals of length $\delta_t = (T - t_0)/n_T$, and the interest rate domain $[r_{min}, r_{max}]$ into n_r subintervals of length $\delta_r = (r_{max} - r_{min})/n_r$, where r_{min} and r_{max} are arbitrarily defined bounds that induce a *truncation error*. In this setting, the time-t_0 price of a European-style up-and-out put is given by the in-out parity relation (3.22), where the plain-vanilla European-style put price is still given by equation (3.13), while the corresponding European-style up-and-in put price is equal to

$$EKI_{t_0}(S, r, U, K, T) = P(r_{t_0}, T)(K\mathcal{B} - \mathcal{A}). \tag{3.53}$$

Denoting by $t_j = t_0 + j\delta_t$ (for $j = 1, ..., n_T$) and $r_i = r_{min} + i\delta_r$ (for $i = 1, ..., n_r$) the discretized time-periods and short-term interest rate values, it can be shown that $\mathcal{A}$ and $\mathcal{B}$ can be computed through the following quasi-analytical formulas:

$$\mathcal{A} \approx \sum_{j=1}^{n_T}\sum_{i=1}^{n_r} \kappa\left(\mu_x(\ln U, r_i, t_j, T), v_x(t_j, T), K\right) q(i, j) \tag{3.54}$$

and

$$\mathcal{B} \approx \sum_{j=1}^{n_T}\sum_{i=1}^{n_r} \Phi\left[\frac{\ln K - \mu_x(\ln U, r_i, t_j, T)}{v_x(t_j, T)}\right] q(i, j), \tag{3.55}$$

with

$$\kappa(m, \sigma, a) := e^{m+\frac{\sigma^2}{2}}\Phi\left[\frac{\ln a - m - \sigma^2}{\sigma}\right], \tag{3.56}$$

and where

$$q(i,j) := \mathbb{Q}^T \left\{ r_{\tau_U} \in [r_i, r_{i+1}], \tau_U \in [t_j, t_{j+1}] \middle| \mathcal{F}_{t_0} \right\} \tag{3.57}$$

represents the discretized version of the joint density of the short-term interest rate and the first passage time (3.18), while $\mu_x(.)$ and $v_x^2(.)$ are computed through equations (3.34) and (3.36), respectively. The densities $q(i,j)$ can be computed through

$$q(i,1) = \delta_r \Psi(x_{t_0}, r_{t_0}, t_0, r_i, t_1; U), \tag{3.58}$$

for $j = 1$, and

$$q(i,j) = \delta_r \left[\Psi(x_{t_0}, r_{t_0}, t_0, r_i, t_j; U) - \sum_{k=1}^{j-1} \sum_{p=1}^{n_r} q(p,k) \Psi(\ln U, r_p, t_k, r_i, t_j; U) \right], \tag{3.59}$$

for $j = 2, ..., n_T$, with

$$\Psi(x_t, r_t, t, r_u, u; U) := f(r_t, t, r_u, u) \Phi \left[\frac{\mu_{x|r}(x_t, r_t, t, r_u, u) - \ln U}{v_{x|r}(t,u)} \right], \tag{3.60}$$

$$f(r_t, t, r_u, u) := \phi \left[r_u; \mu_r(t,u), v_r^2(t,u) \right], \tag{3.61}$$

$$\begin{aligned} \mu_{x|r}(x_t, r_t, t, r_u, u) &:= \mathbb{E}_{\mathbb{Q}^T}\left[x_u | r_u, x_t \right] \\ &= \mu_x(x_t, r_t, t, u) + \frac{v_{r,x}(t,u)}{v_r^2(t,u)} (r_u - \mu_r(t,u)), \end{aligned} \tag{3.62}$$

and

$$\begin{aligned} v_{x|r}^2(t,u) &:= \mathbb{E}_{\mathbb{Q}^T}^2 \left[(x_u - \mathbb{E}_{\mathbb{Q}^T}[x_u | r_u, x_t])^2 \middle| r_u, x_t \right] \\ &= v_x^2(t,u) - \frac{v_{r,x}^2(t,u)}{v_r^2(t,u)}, \end{aligned} \tag{3.63}$$

where $\mu_r(.)$ and $v_r^2(.)$ are given by equations (3.31) and (3.33), respectively.

Equations (3.54), (3.55) and (3.59) show that the extended Fortet method is a two-dimensional pricing approach since a double summation is run over both the time and interest rate dimensions. This is similar, in nature, to the double integral that is presented in Proposition 3.1, which constitutes a general result that can be applied to any Markovian single factor interest rate model. However, under the Gaussian interest rate setup provided equations (3.1)-(3.3), Proposition 3.3 shows that the integration with respect to the short-term interest rate process can be solved explicitly. Therefore, one should expect the one-dimensional ST approach to be more efficient than the two-dimensional extended Fortet method.

3.6 Numerical results

This section implements the methods described in Sections 3.3, 3.4 and 3.5, by pricing European-style up-and-out options, under the financial model described by equations (3.1)-(3.3). The efficiency and accuracy of the novel ST approach is compared against the methodology proposed by Bernard et al. (2008). Under the same framework, and in order to obtain a benchmark to evaluate the accuracy of both alternative pricing methods, a proxy for the exact European-style up-and-out put price is obtained through the numerical solution of the following partial differential equation:

$$\begin{aligned} 0 = {} & \frac{1}{2}\sigma_S^2 \frac{\partial^2 EKO_t(S,r,U,K,T)}{\partial x^2} + \rho\sigma_S\sigma_r \frac{\partial^2 EKO_t(S,r,U,K,T)}{\partial x \partial r} \\ & + \frac{1}{2}\sigma_r^2 \frac{\partial^2 EKO_t(S,r,U,K,T)}{\partial r^2} + \left(r_t - q - \frac{\sigma_S^2}{2}\right)\frac{\partial EKO_t(S,r,U,K,T)}{\partial x} \\ & + \alpha(\gamma - r_t)\frac{\partial EKO_t(S,r,U,K,T)}{\partial r} + \frac{\partial EKO_t(S,r,U,K,T)}{\partial t} \\ & - r_t EKO_t(S,r,U,K,T), \end{aligned} \tag{3.64}$$

subject to the boundary conditions

$$\begin{cases} \lim_{x\to\infty} \frac{\partial EKO_t(S,r,U,K,T)}{dx} = 0 \\ \lim_{x\to-\infty} \frac{\partial EKO_t(S,r,U,K,T)}{dx} = -1 \end{cases} \tag{3.65}$$

and

$$\begin{cases} \lim_{r\to\infty} \frac{\partial EKO_t(S,r,U,K,T)}{dr} = 0 \\ \lim_{r\to-\infty} \frac{\partial EKO_t(S,r,U,K,T)}{dr} = 0 \end{cases}. \tag{3.66}$$

Table 3.1 reports at-the-money one-year European-style up-and-out option prices, for different initial asset price values, and under various model specifications retrieved from Menkveld and Vorst (2000) or Nunes (2011, Table 2). The ST approach proposed is implemented through Propositions 3.3 and 3.4, and equation (3.52) is applied with different numbers of discretization time steps. The extended Fortet method proposed by Bernard et al. (2008) is also implemented with different numbers of discretization steps for both the time and interest rate dimensions.

In order to perceive the magnitude of the corresponding European-style up-and-in put, the sixth column of Table 3.1 reports the value of the corresponding standard European-style put, computed through equation (3.13). The benchmark for the true European-style up-and-out price (seventh column of Table 3.1) is obtained by solving the partial differential equation (3.64) via an explicit finite difference method with $20,000$ time intervals (using a forward difference approximation) and 200 space steps (using a central difference approximation, in both x and r dimensions).[3.3] The ST and the extended Fortet valuation approaches are implemented through *Matlab* (R2013a), while the benchmark prices are obtained through Python 3.4.4 (all computations are made on an Intel Core i7 PC).

[Please insert Table 3.1 about here]

Table 3.1 shows that the ST approach is the most accurate pricing approximation, yielding a mean absolute percentage error (MAPE, hereafter) of only 11.2 basis points,

[3.3] It is well known that the convergence of numerical methods based on trees for pricing barrier options can be very slow, since the barrier level assumed by the tree is different from the true barrier level. Therefore, and in order to improve the accuracy of the benchmark price, we follow a simple procedure described in Hull (2008, Section 26.6). First, we compute inner and outer barrier levels, corresponding to the lattice nodes immediately before and after the true barrier level, respectively. Second, we calculate the price of the barrier option, assuming that the inner barrier is the true barrier, and then repeat the same procedure, assuming that the outer barrier is the true barrier. The benchmark price is then obtained via interpolation between the two previously computed prices.

even for a discretization involving only 256 time steps. The novel approach demonstrates an excellent speed of convergence, since prices are virtually indistinguishable using 256, 512 or 1024 time steps. This applies across all contract specifications, including the ones in which the initial underlying asset price is closer to the barrier level, thus suggesting that our proposed approach does not suffer from the well known *near barrier problem*. The ST approach is also the most efficient approximation, since the whole set of 36 contracts is priced under a CPU time of only 2.7 seconds.

In contrast, the extended Fortet method of Bernard et al. (2008) is considerable less accurate, yielding a MAPE of 57.4 basis points for a discretization with as much as 200 space steps. The accuracy of this approach considerably deteriorates as the instantaneous volatility of asset returns increases, and the distance between the initial asset price and the barrier level decreases. The extended Fortet method is also very time-consuming, as it takes $16,881.9$ seconds to price all the 36 contracts. This is mainly the consequence of the double summations involved in equations (3.54), (3.55) and (3.59).

[Please insert Table 3.2 about here]

Table 3.2 repeats the previous exercise, but for contracts with a time-to-maturity of two years, yielding similar results: The ST approach yields a MAPE of 9.1 basis points with a CPU time of only 6.1 seconds, using 512 time steps, while the extended Fortet method is less accurate and much slower to convergence, reporting a MAPE of 68.3 basis points and a CPU time of $16,992.8$ seconds.

[Please insert Table 3.3 about here]

The most challenging setup to test the efficiency and accuracy of both pricing methods corresponds to the valuation of long-dated contracts. For this purpose, Table 3.3 reports at-the-money European-style up-and-out option prices with a time-to-maturity of five years. As before, the performance ST approach offers the best accuracy/efficiency trade-off: It yields a MAPE of 12.4 basis points for a CPU time of only 6.1 seconds, using 512 time steps. On the other hand, the performance of the extended Fortet method substantially worsens with the increased contract maturity, reporting a MAPE of 98 basis points—more than 8 times higher than the MAPE offered by the ST approach—and a CPU time of $17,035.7$ seconds.

3.7 Conclusions

This paper extends the ST approach originally proposed by Kuan and Webber (2003) and offers a novel approach for pricing European-style barrier options on asset prices driven by a geometric Brownian motion and under the stochastic interest rates setup specified by the Vasiček (1977) model.

Similarly to Kuan and Webber (2003), Proposition 3.1 writes the barrier option price in terms of the first passage time density of the underlying asset price through the barrier level. Again, this density is recovered—in Proposition 3.4—as the implicit solution of a non-linear integral equation. However, and since we are dealing with a two-factor model, our pricing solution involves a double integral, in both time and interest rate dimensions.

Given the Gaussian specification adopted for the short-term interest rate, and following Nunes (2011), we are able to obtain an explicit solution—in Proposition 3.2—for the probability density of the short-term interest rate, conditional on the knock-in or knock-out event, and, therefore, we are left with a pricing solution—in Proposition 3.3—that only involves an integration with respect to time. Moreover, and as shown by Nunes (2011), our one-dimensional pricing solutions can be easily extended from the single-factor Vasiček (1977) model to a multifactor Gaussian Heath, Jarrow, and Morton (1992) framework, without increasing the dimensionality of the pricing problem. In contrast, the extended Fortet method adopted by Bernard et al. (2008) is a two-dimensional pricing approach.

The accuracy and efficiency of the ST approach is compared against the extended Fortet method of Bernard et al. (2008) using several model parameter constellations and option maturities. The numerical results obtained show that the ST approach is the most accurate and efficient pricing method, considering both short-term and long-term contracts.

C Proof of Proposition 3.2

Starting with the short-term interest rate, applying Itô's lemma to $y_t = e^{\alpha t} r_t$, and using equations (3.2) as well as (3.4)-(3.7), yields

$$\begin{aligned} dy_t &= \alpha e^{\alpha t} r_t dt + e^{\alpha t} dr_t \\ &= e^{\alpha t}\left[\alpha r_t + \alpha(\gamma - r_t) - \sigma_r^2 B_\alpha(t,T)\right] dt + e^{\alpha t}\left[\rho\sigma_r dZ_S^{\mathbb{Q}^T}(t) + \sqrt{1-\rho^2}\sigma_r dZ_r^{\mathbb{Q}^T}(t)\right] \\ &= e^{\alpha t}\left[\alpha\gamma - \sigma_r^2 B_\alpha(t,T)\right] dt + e^{\alpha t}\left[\rho\sigma_r dZ_S^{\mathbb{Q}^T}(t) + \sqrt{1-\rho^2}\sigma_r dZ_r^{\mathbb{Q}^T}(t)\right]. \end{aligned} \tag{C-1}$$

Integrating both sides of equation (C-1) between t and $u\,(\geq t)$, and using definition (3.11), yields

$$\begin{aligned} y_u &= y_t + \int_t^u e^{\alpha l}\left[\alpha\gamma - \sigma_r^2 B_\alpha(l,T)\right] dl \\ &\quad + \rho\sigma_r \int_t^u e^{\alpha l} dZ_S^{\mathbb{Q}^T}(l) + \sqrt{1-\rho^2}\sigma_r \int_t^u e^{\alpha l} dZ_r^{\mathbb{Q}^T}(l) \\ &= y_t + \gamma\left(e^{\alpha u} - e^{\alpha t}\right) - \frac{\sigma_r^2}{\alpha^2}\left(e^{\alpha u} - e^{\alpha t}\right) + \frac{\sigma_r^2}{2\alpha^2}\left[e^{-\alpha(T-2u)} - e^{-\alpha(T-2t)}\right] \\ &\quad + \rho\sigma_r \int_t^u e^{\alpha l} dZ_S^{\mathbb{Q}^T}(l) + \sqrt{1-\rho^2}\sigma_r \int_t^u e^{\alpha l} dZ_r^{\mathbb{Q}^T}(l). \end{aligned} \tag{C-2}$$

Attending to the definition of y_t, and using definitions (3.11) and (3.16), the previous equation yields the following strong solution for the stochastic differential equation (3.2):

$$\begin{aligned} r_u &= e^{-\alpha(u-t)} r_t + \left(\alpha\gamma - \frac{\sigma_r^2}{\alpha}\right) B_\alpha(t,u) + \frac{\sigma_r^2}{\alpha} e^{-\alpha(T-u)} B_{2\alpha}(t,u) \\ &\quad + \rho\sigma_r \int_t^u e^{-\alpha(u-l)} dZ_S^{\mathbb{Q}^T}(l) + \sqrt{1-\rho^2}\sigma_r \int_t^u e^{-\alpha(u-l)} dZ_r^{\mathbb{Q}^T}(l). \end{aligned} \tag{C-3}$$

Since any Itô's integral with a deterministic integrand possesses a normal distribution with zero mean and variance equal to its quadratic variation, equation (C-3) implies that

$$\mathbb{Q}^T\left(r_u \in dr \middle| \mathcal{F}_t\right) = \phi\left[r; \mu_r(t,u), v_r^2(t,u)\right] dr, \tag{C-4}$$

where $\mu_r(.)$ is given by equation (3.31), and the variance $v_r^2(.)$ is equal to

$$\begin{aligned} v_r^2(t,u) &= \mathbb{E}_{\mathbb{Q}^T}\left\{\left[r_u - \mathbb{E}_{\mathbb{Q}^T}\left(r_u \middle| \mathcal{F}_t\right)\right]^2 \middle| \mathcal{F}_t\right\} \\ &= \mathbb{E}_{\mathbb{Q}^T}\left\{\left[\rho\sigma_r \int_t^u e^{-\alpha(u-l)} dZ_S^{\mathbb{Q}^T}(l) + \sqrt{1-\rho^2}\sigma_r \int_t^u e^{-\alpha(u-l)} dZ_r^{\mathbb{Q}^T}(l)\right]^2 \middle| \mathcal{F}_t\right\} \\ &= \sigma_r^2 \int_t^u e^{-2\alpha(u-l)} dl, \qquad \text{(C-5)} \end{aligned}$$

where the last line follows from Itô's isometry. Equation (3.33) follows directly from equations (3.16) and (C-5).

Concerning the underlying asset price, combining equations (3.1) and (3.4)-(3.7), while applying Itô's lemma to $x_t = \ln S_t$, yields

$$dx_t = \left[r_t - q - \frac{\sigma_S^2}{2} - \rho\sigma_S\sigma_r B_\alpha(t,T)\right] dt + \sigma_S dZ_S^{\mathbb{Q}^T}(t). \qquad \text{(C-6)}$$

Integrating both sides of equation (C-6) between t and $u\,(\geq t)$, while using equation (C-3), yields

$$\begin{aligned} x_u &= x_t + B_\alpha(t,u) r_t + \bar{\mu}_x(t,u) \qquad \text{(C-7)} \\ &\quad + \rho\sigma_r \int_t^u \int_t^l e^{-\alpha(l-s)} dZ_S^{\mathbb{Q}^T}(s)\, dl + \sqrt{1-\rho^2}\sigma_r \int_t^u \int_t^l e^{-\alpha(l-s)} dZ_r^{\mathbb{Q}^T}(s)\, dl \\ &\quad + \sigma_S \int_t^u dZ_S^{\mathbb{Q}^T}(l), \end{aligned}$$

where

$$\begin{aligned} &\bar{\mu}_x(t,u) \qquad \text{(C-8)} \\ = &\int_t^u \left[\left(\alpha\gamma - \frac{\sigma_r^2}{\alpha}\right) B_\alpha(t,l) + \frac{\sigma_r^2}{\alpha} e^{-\alpha(T-l)} B_{2\alpha}(t,l) - q - \frac{\sigma_S^2}{2} - \rho\sigma_S\sigma_r B_\alpha(l,T)\right] dl \end{aligned}$$

yields equation (3.35) with the help of definitions (3.11) and (3.16). Changing the order of integration, the second and third lines on the right-hand side of equation (C-7) can be restated as

$$
\begin{aligned}
& \rho\sigma_r \int_t^u \int_t^l e^{-\alpha(l-s)} dZ_S^{\mathbb{Q}^T}(s)\, dl + \sqrt{1-\rho^2}\sigma_r \int_t^u \int_t^l e^{-\alpha(l-s)} dZ_r^{\mathbb{Q}^T}(s)\, dl \\
& +\sigma_S \int_t^u dZ_S^{\mathbb{Q}^T}(l) \\
= \; & \rho\sigma_r \int_t^u \int_s^u e^{-\alpha(l-s)} dl\, dZ_S^{\mathbb{Q}^T}(s) + \sqrt{1-\rho^2}\sigma_r \int_t^u \int_s^u e^{-\alpha(l-s)} dl\, dZ_r^{\mathbb{Q}^T}(s) \\
& +\sigma_S \int_t^u dZ_S^{\mathbb{Q}^T}(l) \\
= \; & \int_t^u \left[\sigma_S + \rho\sigma_r B_\alpha(s,u)\right] dZ_S^{\mathbb{Q}^T}(s) + \sqrt{1-\rho^2}\sigma_r \int_t^u B_\alpha(s,u)\, dZ_r^{\mathbb{Q}^T}(s), \qquad \text{(C-9)}
\end{aligned}
$$

and, therefore, equation (C-7) can be rewritten as

$$
\begin{aligned}
x_u \; = \; & x_t + B_\alpha(t,u)\, r_t + \bar{\mu}_x(t,u) \qquad \text{(C-10)} \\
& + \int_t^u \left[\sigma_S + \rho\sigma_r B_\alpha(s,u)\right] dZ_S^{\mathbb{Q}^T}(s) + \sqrt{1-\rho^2}\sigma_r \int_t^u B_\alpha(s,u)\, dZ_r^{\mathbb{Q}^T}(s).
\end{aligned}
$$

Using equation (C-10),

$$
\begin{aligned}
v_x^2(t,u) \; := \; & \mathbb{E}_{\mathbb{Q}^T}\left\{ \left[x_u - \mathbb{E}_{\mathbb{Q}^T}\left(x_u \middle| \mathcal{F}_{t_0}\right)\right]^2 \middle| \mathcal{F}_{t_0} \right\} \\
= \; & \int_t^u \left[\sigma_S + \rho\sigma_r B_\alpha(l,u)\right]^2 dl + \left(1-\rho^2\right)\sigma_r^2 \int_t^u B_\alpha^2(l,u)\, dl \\
= \; & \int_t^u \left[\sigma_S^2 + 2\rho\sigma_S\sigma_r B_\alpha(l,u) + \sigma_r^2 B_\alpha^2(l,u)\right] dl, \qquad \text{(C-11)}
\end{aligned}
$$

where the second line follows from Itô's isometry, and equation (3.36) is obtained from definitions (3.11) and (3.16). Again using the fact that any Itô's integral with a deterministic integrand possesses a normal distribution with zero mean and variance equal to its quadratic variation, then equations (C-10) and (C-11) imply that

$$
\mathbb{Q}^T\left(x_u \in dx \middle| \mathcal{F}_t\right) = \phi\left[x; \mu_x(x_t, r_t, t, u), v_x^2(t,u)\right] dx, \qquad \text{(C-12)}
$$

where $\mu_x(.)$ is given by equation (3.34).

Finally, in order to obtain the conditional density function (3.28), and following Mood et al. (1974, Page 167), it is necessary to show that the random variables r and x possess a bivariate normal distribution. For this purpose, consider a linear combination of both random variables,

$$y = ar + bx, \tag{C-13}$$

for $a, b \in \mathbb{R}$. Given the distribution functions (C-4) and (C-12), it follows directly that such linear combination also possesses a univariate normal density function of the following form:

$$\begin{aligned} & \mathbb{Q}^T\left(y_u \in dy \middle| \mathcal{F}_t\right) && \text{(C-14)} \\ = \; & \phi\left[y; a\mu_r\left(t,u\right) + b\mu_x\left(x_t, r_t, t, u\right), a^2 v_r^2\left(t,u\right) + b^2 v_x^2\left(t,u\right) + 2ab v_{x,y}\left(t,u\right)\right] dy, \end{aligned}$$

where the covariance $v_{x,y}\left(.\right)$ is given by

$$\begin{aligned} v_{x,y}\left(t,u\right) &:= \mathbb{E}_{\mathbb{Q}^T}\left\{\left[r_u - \mathbb{E}_{\mathbb{Q}^T}\left(r_u \middle| \mathcal{F}_t\right)\right]\left[x_u - \mathbb{E}_{\mathbb{Q}^T}\left(x_u \middle| \mathcal{F}_t\right)\right] \middle| \mathcal{F}_t\right\} \\ &= \int_t^u \left[\sigma_S + \rho\sigma_r B_\alpha\left(l,u\right)\right]\left[\rho\sigma_r e^{-\alpha(u-l)}\right] dl \\ &\quad + \int_t^u \left[\sqrt{1-\rho^2}\sigma_r B_\alpha\left(l,u\right)\right]\left[\sqrt{1-\rho^2}\sigma_r e^{-\alpha(u-l)}\right] dl \\ &= \int_t^u \left[\rho\sigma_S\sigma_r e^{-\alpha(u-l)} + \sigma_r^2 e^{-\alpha(u-l)} B_\alpha\left(l,u\right)\right] dl, \end{aligned} \tag{C-15}$$

from which equation (3.37) is obtained. Consequently, the moment generating function of y is equal to

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^T}\left\{\exp\left[s\left(ar_u + bx_u\right)\right] \middle| \mathcal{F}_t\right\} && \text{(C-16)} \\ = \; & \exp\Big\{s\left[a\mu_r\left(t,u\right) + b\mu_x\left(x_t, r_t, t, u\right)\right] \\ & + \frac{s^2}{2}\left[a^2 v_r^2\left(t,u\right) + b^2 v_x^2\left(t,u\right) + 2ab v_{x,y}\left(t,u\right)\right]\Big\} \end{aligned}$$

for any $s \in \mathbb{R}$. Taking $s = 1$, and following Mood et al. (1974, Page 164), the right-hand side of equation (C-16) can be understood as the joint moment generating function of r

and x. The condition density function (3.28) follows directly. □

D Proof of Proposition 3.3

Using equations (3.9) and (3.13), the inner integral on the right-hand side of equation (3.24) can be rewritten as

$$\begin{aligned} & \int_{\mathbb{R}} \frac{p_u(U,r,K,T)}{P(r,T)} \mathbb{Q}^T\left(r_u \in dr \middle| S_u = U, r_{t_0}\right) \\ = \; & K \int_{\mathbb{R}} \Phi\left[-d(U,r) + v_x(u,T)\right] \mathbb{Q}^T\left(r_u \in dr \middle| S_u = U, r_{t_0}\right) \\ & - \exp\left[\ln U - q(T-u) - A(u,T)\right] \\ & \int_{\mathbb{R}} \exp\left[B_\alpha(u,T) r\right] \Phi\left[-d(U,r)\right] \mathbb{Q}^T\left(r_u \in dr \middle| S_u = U, r_{t_0}\right). \end{aligned} \tag{D-1}$$

Through equation (3.14), the first integral on the right-hand side of equation (D-1) becomes

$$\begin{aligned} & K \int_{\mathbb{R}} \Phi\left[-d(U,r) + v_x(u,T)\right] \mathbb{Q}^T\left(r_u \in dr \middle| S_u = U, r_{t_0}\right) \\ = \; & K \int_{\mathbb{R}} \Phi\left[\frac{\ln K - \ln U + A(u,T) + q(T-u) + \frac{v_x^2(u,T)}{2} - B_\alpha(u,T) r}{v_x(u,T)}\right] \\ & \times \mathbb{Q}^T\left(r_u \in dr \middle| S_u = U, r_{t_0}\right), \end{aligned} \tag{D-2}$$

where $v_x^2(.)$ is given by equation (3.15), and following Nunes (2011, Appendix B), the integral contained in equation (D-2) can be explicitly computed through the following lemma.

Lemma D.1. *For a, b, c, θ, $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$,*

$$\int_{\mathbb{R}} \exp(\theta z)\, \phi\left(z; \mu, \sigma^2\right) \Phi\left(\frac{a - bz}{c}\right) dz = \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right) \Phi\left[\frac{a - b\left(\mu + \sigma^2\theta\right)}{\sqrt{c^2 + b^2\sigma^2}}\right]. \tag{D-3}$$

Applying Lemma D.1 to equation (D-2), while using Proposition 3.2, then

$$K\int_{\mathbb{R}}\Phi\left[-d\left(U,r\right)+v_x\left(u,T\right)\right]\mathbb{Q}^T\left(\left.r_u\in dr\right|S_u=U,r_{t_0}\right)=K\Phi\left[\eta\left(x_{t_0},r_{t_0},t_0,u,T\right)\right], \quad \text{(D-4)}$$

where $\eta(.)$ is given by equation (3.41).

Furthermore, and using again equation (3.14), the second integral on the right-hand side of equation (D-1) can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}}\exp\left[B_\alpha\left(u,T\right)r\right]\Phi\left[-d\left(U,r\right)\right]\mathbb{Q}^T\left(\left.r_u\in dr\right|S_u=U,r_{t_0}\right)\\
=&\int_{\mathbb{R}}\exp\left[B_\alpha\left(u,T\right)r\right]\Phi\left[\frac{\ln K-\ln U+A\left(u,T\right)+q\left(T-u\right)-\frac{v_x^2\left(u,T\right)}{2}-B_\alpha\left(u,T\right)r}{v_x\left(u,T\right)}\right]\\
&\times\mathbb{Q}^T\left(\left.r_u\in dr\right|S_u=U,r_{t_0}\right)\\
=&\exp\left[\mu_{r|x}\left(x_{t_0},r_{t_0},t_0,\ln U,u\right)B_\alpha\left(u,T\right)+\frac{1}{2}v_{r|x}^2\left(t_0,u\right)B_\alpha^2\left(u,T\right)\right] \qquad \text{(D-5)}\\
&\times\Phi\left\{\left[\ln K-\ln U+A\left(u,T\right)+q\left(T-u\right)-\frac{v_x^2\left(u,T\right)}{2}\right.\right.\\
&\left.-B_\alpha\left(u,T\right)\left(\mu_{r|x}\left(x_{t_0},r_{t_0},t_0,\ln U,u\right)+v_{r|x}^2\left(t_0,u\right)B_\alpha\left(u,T\right)\right)\right]\\
&\left.\times\left[v_x^2\left(u,T\right)+B_\alpha^2\left(u,T\right)v_{r|x}^2\left(t_0,u\right)\right]^{-\frac{1}{2}}\right\}
\end{aligned}$$

where the second equality follows from Proposition 3.2 and Lemma D.1. Using definitions (3.40) and (3.41), equation (D-5) implies that

$$\begin{aligned}
&\exp\left[\ln U-q\left(T-u\right)-A\left(u,T\right)\right]\int_{\mathbb{R}}\exp\left[B_\alpha\left(u,T\right)r\right]\Phi\left[-d\left(U,r\right)\right] \qquad \text{(D-6)}\\
&\times\mathbb{Q}^T\left(\left.r_u\in dr\right|S_u=U,r_{t_0}\right)\\
=&\ \varphi\left(x_{t_0},r_{t_0},t_0,u,T\right)\Phi\left[\eta\left(x_{t_0},r_{t_0},t_0,u,T\right)-\sqrt{v_x^2\left(u,T\right)+B_\alpha^2\left(u,T\right)v_{r|x}^2\left(t_0,u\right)}\right].
\end{aligned}$$

Proposition 3.3 arises after combining equations (D-1), (D-4) and (D-6). □

Table 3.1: Prices of European-style up-and-out puts under the financial framework defined by equations (3.1)-(3.3), with a time-to-maturity of one year

$T-t$	S_{t_0}	σ_S	r_{t_0}	γ	EPut	FD	ST Approach			Ext. Fortet method		
(Yrs.)		(%)	(%)				256	512	$1,024$	50	100	200
1	100	20	4	0.04332	5.91	5.90	5.90	5.90	5.90	5.90	5.90	5.90
1	110	20	4	0.04332	6.50	6.35	6.36	6.36	6.36	6.38	6.37	6.36
1	120	20	4	0.04332	7.09	5.94	5.95	5.95	5.95	6.03	5.98	5.96
1	130	20	4	0.04332	7.68	2.93	2.94	2.94	2.94	3.09	2.98	2.95
1	100	20	8	0.08332	4.33	4.32	4.32	4.32	4.32	4.33	4.32	4.32
1	110	20	8	0.08332	4.76	4.65	4.65	4.65	4.65	4.66	4.66	4.65
1	120	20	8	0.08332	5.20	4.30	4.30	4.30	4.30	4.36	4.32	4.31
1	130	20	8	0.08332	5.63	2.06	2.06	2.06	2.06	2.11	2.08	2.07
1	100	20	12	0.12332	3.09	3.09	3.09	3.09	3.09	3.09	3.09	3.09
1	110	20	12	0.12332	3.40	3.31	3.31	3.31	3.31	3.32	3.32	3.31
1	120	20	12	0.12332	3.71	3.03	3.03	3.03	3.03	3.06	3.04	3.03
1	130	20	12	0.12332	4.02	1.41	1.41	1.41	1.41	1.43	1.41	1.41
1	100	30	4	0.04332	9.73	9.50	9.51	9.51	9.51	9.54	9.52	9.51
1	110	30	4	0.04332	10.71	9.56	9.58	9.58	9.58	9.68	9.61	9.59
1	120	30	4	0.04332	11.68	7.90	7.92	7.92	7.92	8.13	7.99	7.95
1	130	30	4	0.04332	12.65	3.45	3.46	3.46	3.46	3.81	3.54	3.50
1	100	30	8	0.08332	7.93	7.73	7.73	7.73	7.73	7.76	7.74	7.74
1	110	30	8	0.08332	8.72	7.74	7.75	7.75	7.75	7.83	7.78	7.76
1	120	30	8	0.08332	9.52	6.33	6.34	6.34	6.34	6.49	6.39	6.36
1	130	30	8	0.08332	10.31	2.71	2.72	2.72	2.72	2.87	2.78	2.74
1	100	30	12	0.12332	6.39	6.22	6.22	6.22	6.22	6.24	6.23	6.22
1	110	30	12	0.12332	7.03	6.20	6.20	6.20	6.20	6.27	6.22	6.21
1	120	30	12	0.12332	7.67	5.02	5.02	5.02	5.02	5.13	5.05	5.03
1	130	30	12	0.12332	8.31	2.11	2.12	2.12	2.12	2.21	2.15	2.13
1	100	50	4	0.04332	17.36	14.83	14.85	14.85	14.85	15.07	14.93	14.88
1	110	50	4	0.04332	19.09	13.40	13.42	13.42	13.42	13.79	13.55	13.47
1	120	50	4	0.04332	20.83	9.88	9.89	9.89	9.89	10.46	10.07	9.97
1	130	50	4	0.04332	22.56	3.92	3.93	3.93	3.93	4.81	4.14	4.03
1	100	50	8	0.08332	15.28	12.99	13.00	13.00	13.00	13.20	13.07	13.03
1	110	50	8	0.08332	16.81	11.68	11.70	11.70	11.70	12.00	11.80	11.74
1	120	50	8	0.08332	18.34	8.56	8.57	8.57	8.57	8.95	8.71	8.63
1	130	50	8	0.08332	19.86	3.38	3.38	3.38	3.38	3.81	3.57	3.45
1	100	50	12	0.12332	13.40	11.34	11.34	11.34	11.34	11.51	11.40	11.37
1	110	50	12	0.12332	14.74	10.15	10.16	10.16	10.16	10.42	10.25	10.20
1	120	50	12	0.12332	16.08	7.39	7.40	7.40	7.40	7.74	7.51	7.45
1	130	50	12	0.12332	17.42	2.89	2.90	2.90	2.90	3.21	3.02	2.95
MPE							0.112%	0.112%	0.112%	3.408%	1.187%	0.574%
MAPE							0.112%	0.112%	0.112%	3.408%	1.187%	0.574%
CPU						25,136.4	2.7	6.1	14.3	222.6	1,831.6	16,881.9

This table reports at-the-money European-style up-and-out put option prices under the financial model described by equations (3.1)-(3.3), with a time to maturity of one year. The initial asset value (S_{t_0}), the instantaneous volatility of asset returns (σ_S), the short-term interest rate (r_{t_0}), and the risk-adjusted long term mean of r_t (γ) are given in the second to fifth columns. It is assumed that the barrier level is \$135, the dividend yield (q) is 0%, the speed of mean reversion (α) is 0.1, the instantaneous volatility of the short-term interest rate (σ_r) is 1%, and the correlation coefficient between asset returns and the short-term interest rate (ρ) is equal to -0.5. The sixth column contains the price provided by equation (3.13) for the corresponding standard European-style put contracts, while the seventh column reports benchmark prices computed through finite differences, with $20,000$ time intervals and 200 space steps. The next three columns report option prices obtained via the ST approach implemented through Propositions 3.3 and 3.4, and using different numbers of time steps. Finally, the last three columns report prices obtained via the extended Fortet method, as described in Section 3.5, also using different numbers of steps in the time and interest rate dimensions. The last three lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE) and computation times (in seconds), for the whole set of contracts under analysis.

Table 3.2: Prices of European-style up-and-out puts under the financial framework defined by equations (3.1)-(3.3), with a time-to-maturity of two years

$T-t$	S_{t_0}	σ_S	r_{t_0}	γ	EPut	FD	ST Approach			Ext. Fortet method		
(Yrs.)		(%)	(%)				256	512	1,024	50	100	200
2	100	20	4	0.04332	7.15	7.01	7.01	7.01	7.01	7.03	7.02	7.02
2	110	20	4	0.04332	7.86	7.10	7.11	7.11	7.11	7.15	7.13	7.12
2	120	20	4	0.04332	8.58	5.88	5.88	5.88	5.88	5.97	5.92	5.90
2	130	20	4	0.04332	9.29	2.53	2.54	2.54	2.54	2.63	2.57	2.56
2	100	20	8	0.08332	4.42	4.33	4.33	4.33	4.33	4.34	4.33	4.33
2	110	20	8	0.08332	4.86	4.34	4.34	4.34	4.34	4.37	4.35	4.35
2	120	20	8	0.08332	5.31	3.52	3.52	3.52	3.52	3.55	3.53	3.52
2	130	20	8	0.08332	5.75	1.46	1.46	1.46	1.46	1.47	1.47	1.46
2	100	20	12	0.12332	2.59	2.52	2.53	2.53	2.53	2.53	2.53	2.53
2	110	20	12	0.12332	2.85	2.51	2.51	2.51	2.51	2.52	2.52	2.51
2	120	20	12	0.12332	3.11	1.99	1.99	1.99	1.99	2.00	1.99	1.99
2	130	20	12	0.12332	3.36	0.80	0.80	0.80	0.80	0.78	0.79	0.79
2	100	30	4	0.04332	12.33	11.11	11.12	11.12	11.12	11.20	11.16	11.14
2	110	30	4	0.04332	13.57	10.30	10.31	10.31	10.31	10.46	10.38	10.34
2	120	30	4	0.04332	14.80	7.75	7.76	7.76	7.76	7.96	7.84	7.80
2	130	30	4	0.04332	16.03	3.11	3.12	3.12	3.12	3.37	3.21	3.16
2	100	30	8	0.08332	9.01	8.05	8.05	8.05	8.05	8.11	8.08	8.06
2	110	30	8	0.08332	9.91	7.39	7.39	7.39	7.39	7.48	7.43	7.41
2	120	30	8	0.08332	10.81	5.47	5.48	5.48	5.48	5.58	5.52	5.50
2	130	30	8	0.08332	11.71	2.16	2.16	2.16	2.16	2.24	2.20	2.18
2	100	30	12	0.12332	6.43	5.69	5.70	5.70	5.70	5.73	5.71	5.70
2	110	30	12	0.12332	7.07	5.17	5.18	5.18	5.18	5.23	5.20	5.19
2	120	30	12	0.12332	7.71	3.78	3.78	3.78	3.78	3.83	3.80	3.79
2	130	30	12	0.12332	8.35	1.46	1.46	1.46	1.46	1.48	1.47	1.47
2	100	50	4	0.04332	22.63	16.18	16.19	16.19	16.19	16.51	16.33	16.26
2	110	50	4	0.04332	24.89	13.77	13.78	13.78	13.78	14.20	13.96	13.87
2	120	50	4	0.04332	27.15	9.62	9.63	9.63	9.63	10.17	9.85	9.74
2	130	50	4	0.04332	29.42	3.65	3.66	3.66	3.66	4.41	3.93	3.79
2	100	50	8	0.08332	18.58	13.11	13.12	13.12	13.12	13.35	13.22	13.17
2	110	50	8	0.08332	20.44	11.08	11.09	11.09	11.09	11.39	11.22	11.16
2	120	50	8	0.08332	22.29	7.69	7.69	7.69	7.69	8.03	7.85	7.77
2	130	50	8	0.08332	24.15	2.90	2.90	2.90	2.90	3.29	3.08	2.99
2	100	50	12	0.12332	15.13	10.54	10.54	10.54	10.54	10.72	10.62	10.58
2	110	50	12	0.12332	16.65	8.85	8.86	8.86	8.86	9.08	8.95	8.90
2	120	50	12	0.12332	18.16	6.10	6.10	6.10	6.10	6.34	6.20	6.15
2	130	50	12	0.12332	19.68	2.28	2.28	2.28	2.28	2.53	2.39	2.33
MPE							0.095%	0.091%	0.091%	2.956%	1.277%	0.658%
MAPE							0.095%	0.091%	0.091%	3.085%	1.331%	0.683%
CPU						26,165.4	2.7	6.1	14.6	217.3	1,832.7	16,992.8

This table reports at-the-money European-style up-and-out put option prices under the financial model described by equations (3.1)-(3.3), with a time to maturity of two years. The initial asset value (S_{t_0}), the instantaneous volatility of asset returns (σ_S), the short-term interest rate (r_{t_0}), and the risk-adjusted long term mean of r_t (γ) are given in the second to fifth columns. It is assumed that the barrier level is \$135, the dividend yield (q) is 0%, the speed of mean reversion (α) is 0.1, the instantaneous volatility of the short-term interest rate (σ_r) is 1%, and the correlation coefficient between asset returns and the short-term interest rate (ρ) is equal to -0.5. The sixth column contains the price provided by equation (3.13) for the corresponding standard European-style put contracts, while the seventh column reports benchmark prices computed through finite differences, with $20,000$ time intervals and 200 space steps. The next three columns report option prices obtained via the ST approach implemented through Propositions 3.3 and 3.4, and using different numbers of time steps. Finally, the last three columns report prices obtained via the extended Fortet method, as described in Section 3.5, also using different numbers of steps in the time and interest rate dimensions. The last three lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE) and computation times (in seconds), for the whole set of contracts under analysis.

Table 3.3: Prices of European-style up-and-out puts under the financial framework defined by equations (3.1)-(3.3), with a time-to-maturity of five years

$T-t$	S_{t_0}	σ_S	r_{t_0}	γ	EPut	FD	ST Approach			Ext. Fortet method		
(Yrs.)		(%)	(%)				256	512	$1,024$	50	100	200
5	100	20	4	0.04332	7.84	6.79	6.80	6.80	6.80	6.84	6.82	6.81
5	110	20	4	0.04332	8.62	6.09	6.09	6.09	6.09	6.15	6.12	6.11
5	120	20	4	0.04332	9.41	4.39	4.39	4.39	4.39	4.44	4.41	4.40
5	130	20	4	0.04332	10.19	1.67	1.68	1.68	1.68	1.68	1.68	1.68
5	100	20	8	0.08332	3.18	2.69	2.69	2.69	2.69	2.70	2.70	2.70
5	110	20	8	0.08332	3.50	2.35	2.35	2.35	2.35	2.36	2.36	2.36
5	120	20	8	0.08332	3.82	1.64	1.64	1.64	1.64	1.63	1.63	1.64
5	130	20	8	0.08332	4.14	0.60	0.60	0.60	0.60	0.56	0.58	0.59
5	100	20	12	0.12332	1.11	0.91	0.91	0.91	0.91	0.91	0.91	0.91
5	110	20	12	0.12332	1.22	0.78	0.78	0.78	0.78	0.78	0.78	0.78
5	120	20	12	0.12332	1.33	0.53	0.53	0.53	0.53	0.51	0.52	0.52
5	130	20	12	0.12332	1.44	0.18	0.18	0.18	0.18	0.16	0.17	0.18
5	100	30	4	0.04332	15.03	10.73	10.74	10.74	10.74	10.88	10.81	10.77
5	110	30	4	0.04332	16.53	9.05	9.06	9.06	9.06	9.22	9.14	9.10
5	120	30	4	0.04332	18.04	6.24	6.25	6.25	6.25	6.42	6.33	6.28
5	130	30	4	0.04332	19.54	2.34	2.33	2.33	2.33	2.50	2.41	2.37
5	100	30	8	0.08332	8.42	5.81	5.82	5.82	5.82	5.87	5.84	5.83
5	110	30	8	0.08332	9.27	4.82	4.83	4.83	4.83	4.87	4.85	4.83
5	120	30	8	0.08332	10.11	3.26	3.26	3.26	3.26	3.29	3.27	3.27
5	130	30	8	0.08332	10.95	1.19	1.19	1.19	1.19	1.19	1.19	1.19
5	100	30	12	0.12332	4.43	2.96	2.96	2.96	2.96	2.98	2.97	2.96
5	110	30	12	0.12332	4.88	2.42	2.42	2.42	2.42	2.42	2.42	2.42
5	120	30	12	0.12332	5.32	1.60	1.61	1.61	1.61	1.59	1.60	1.60
5	130	30	12	0.12332	5.76	0.57	0.57	0.57	0.57	0.53	0.55	0.56
5	100	50	4	0.04332	29.23	15.14	15.15	15.15	15.15	15.61	15.38	15.26
5	110	50	4	0.04332	32.15	12.22	12.23	12.23	12.23	12.77	12.50	12.36
5	120	50	4	0.04332	35.07	8.16	8.17	8.17	8.17	8.81	8.47	8.32
5	130	50	4	0.04332	38.00	2.99	3.00	2.99	2.99	3.80	3.37	3.17
5	100	50	8	0.08332	20.22	10.12	10.13	10.13	10.13	10.38	10.25	10.19
5	110	50	8	0.08332	22.24	8.10	8.11	8.11	8.11	8.39	8.25	8.17
5	120	50	8	0.08332	24.27	5.36	5.37	5.37	5.37	5.68	5.52	5.44
5	130	50	8	0.08332	26.29	1.95	1.96	1.95	1.95	2.31	2.12	2.03
5	100	50	12	0.12332	13.71	6.64	6.64	6.64	6.64	6.77	6.70	6.67
5	110	50	12	0.12332	15.09	5.27	5.28	5.28	5.28	5.41	5.34	5.30
5	120	50	12	0.12332	16.46	3.46	3.46	3.46	3.46	3.59	3.52	3.49
5	130	50	12	0.12332	17.83	1.24	1.25	1.25	1.25	1.38	1.31	1.28
MPE							0.140%	0.124%	0.120%	2.224%	1.090%	0.554%
MAPE							0.140%	0.124%	0.120%	4.127%	1.985%	0.980%
CPU						25,137.2	2.7	6.1	14.5	217.9	1,848.3	17,035.7

This table reports at-the-money European-style up-and-out put option prices under the financial model described by equations (3.1)-(3.3), with a time to maturity of five years. The initial asset value (S_{t_0}), the instantaneous volatility of asset returns (σ_S), the short-term interest rate (r_{t_0}), and the risk-adjusted long term mean of r_t (γ) are given in the second to fifth columns. It is assumed that the barrier level is \$135, the dividend yield (q) is 0%, the speed of mean reversion (α) is 0.1, the instantaneous volatility of the short-term interest rate (σ_r) is 1%, and the correlation coefficient between asset returns and the short-term interest rate (ρ) is equal to -0.5. The sixth column contains the price provided by equation (3.13) for the corresponding standard European-style put contracts, while the seventh column reports benchmark prices computed through finite differences, with $20,000$ time intervals and 200 space steps. The next three columns report option prices obtained via the ST approach implemented through Propositions 3.3 and 3.4, and using different numbers of time steps. Finally, the last three columns report prices obtained via the extended Fortet method, as described in Section 3.5, also using different numbers of steps in the time and interest rate dimensions. The last three lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE) and computation times (in seconds), for the whole set of contracts under analysis.

4 Pricing Credit and Equity Default Swaps under the Jump to Default Extended CEV Model

Abstract

This paper offers a novel methodology for the pricing of credit and equity default swaps under the *jump to default extended constant elasticity of variance* (JDCEV) model of Carr and Linetsky (2006). The proposed method extends the *stopping time approach* of Kuan and Webber (2003), and expresses the value of the building blocks of both contracts in terms of the first passage time density of the underlying asset price to the contract triggering level. The numerical results show that the proposed pricing methodology is extremely accurate and much faster than the Laplace transform approach of Mendoza-Arriaga and Linetsky (2011).

JEL Classification: G13

Keywords: credit risk; default; credit default swaps; equity default swaps; CEV model; JDCEV model; first passage time.

4.1 Introduction

The main purpose of the present paper is to offer a novel valuation methodology for credit default swaps (CDSs) and equity default swaps (EDSs) under the *jump to default extended constant elasticity of variance* (hereafter, JDCEV) model proposed by Carr and Linetsky (2006). This paper generalizes the *stopping time approach* (from now on, ST approach) first proposed by Kuan and Webber (2003) for options on pure discount bonds, under single-factor term structure models, and later extended by Dias et al. (2014) to the pricing of European-style single and double barrier options under the JDCEV framework.

With the global financial crisis of 2007-09, CDSs became the most widely traded credit derivative in financial markets.[4.1] These securities can be thought of as an insurance contract, which provides its buyer compensation in the case of a credit event of a reference entity. A credit event can encompass, but is not limited to, bankruptcy of the reference entity, failure to pay, or a debt restructuring. In return, the credit default swap (CDS) seller receives a series of periodic payments, up to the credit event or the contract maturity, whichever occurs first. EDSs are hybrid credit-equity securities, which combine characteristics of CDS contracts and equity barrier derivatives. These instruments, originally launched around fifteen years ago, allow investors to simultaneously hedge the equity and credit risk associated with a reference entity. Similarly to CDSs, the equity default swap (EDS) pays its buyer a pre-determined amount in the case of a triggering event, which in this case is defined as a sharp decrease (typically of 50% to 70%) in the underlying stock of the reference entity. Conversely, the EDS seller also receives regular payments through the life of the contract, up to the triggering event, if it occurs. Hence, a CDS can be understood as an EDS with a triggering level equal to zero, since it is expected that, in the event of a default, the stock price trades near zero.

There are two important consequences stemming from the differences between the structures of CDSs and EDSs. First, the EDS swap rate is always greater than the CDS

[4.1] According to summary statistics reported by the Bank of International Settlements, the notional amount of CDS contracts outstanding rose from \$6.4 trillion in 2004, to \$58.2 trillion in 2007, decreasing thereafter to \$12.3 trillion, at the end of 2015.

swap rate. This is because the probability of a sharp stock decrease that triggers the EDS protection payment is larger than the probability of default. Second, EDSs can be considered more transparent and unambiguous than CDSs, since these contracts are triggered when the stock price, an observable market variable, reaches a new low, whereas CDSs are only triggered when a credit event occurs, which in many cases is not simple to determine.

The pricing of EDS contracts has already been the subject of analysis in the academic literature. Albanese and Chen (2005), based on the results of Davydov and Linetsky (2001, 2003) for European-style barrier options with rebates, price EDS contracts under the *constant elasticity of variance* (CEV, hereafter) model; Campi and Sbuelz (2009) employ a Laplace transform approach to price EDS contracts also under the CEV framework; Atlan and Leblanc (2005, 2006) also work under the CEV assumption, but analyze other specifications for the underlying price evolution, such as the *constant elasticity of stochastic variance* (CESV) process—i.e. the Heston (1993) stochastic volatility model, coupled with an uncorrelated CEV diffusion for the underlying price process; Baaquie et al. (2011) study this issue from an empirical perspective, performing a calibration exercise of a CEV process to market observed CDS and EDS spreads, using a data sample between 2004 and 2005.

The aforementioned models are consistent with two well-known empirical facts documented in the literature: the existence of a negative correlation between asset returns and historical volatility—the leverage effect, shown, for instance, by Bekaert and Wu (2000)—, and the inverse relation between option strike prices and implied volatility—the implied volatility skew, observed, for example, by Dennis and Mayhew (2002). However, they do not comply with the evidence of a positive link between default probabilities and stock volatility, as described, for instance, by Campbell and Taksler (2003). To circumvent this issue, Carr and Linetsky (2006) propose the JDCEV model, a unified framework to price equity and credit derivatives, in which the stock price is modeled as a CEV process, prior to default. Under this model, a default event can occur by diffusion of the underlying price process to zero, or by a jump to default, with an intensity specified as an affine function of the instantaneous stock variance. Thus, this framework is able to link the underlying stock price, the volatility of stock returns and the default intensity function.

To the authors' knowledge, the valuation of EDSs under the JDCEV framework has only been pursued by Mendoza-Arriaga and Linetsky (2011). In their paper, the authors offer pricing formulae for the building blocks of an EDS contract (protection leg, premium leg and accrued interest) via the inversion of the Laplace transform of several expectations containing the first passage time of the underlying price process through the contract triggering level. These authors are also able to price CDS contracts, by considering the limit when the triggering level tends to zero.

Our faster pricing methodology extends the ST approach of Kuan and Webber (2003), and expresses the value of the building blocks of CDS and EDS contracts in terms of the density function of the first passage time of the underlying asset price to the contract triggering level. Through the standard partition method of Park and Schuurmann (1976), this hitting density is recovered as the implicit solution of a non-linear integral equation. We note that the ST approach is able to accommodate the valuation of CDS and EDS contracts under the *constant elasticity of variance* (CEV) model of Cox (1975), as a special case. Moreover, we show that when the contract triggering level is set to zero, our ST approach nests the CDS pricing solutions already offered, under the JDCEV model, by Carr and Linetsky (2006), which do not depend on the first passage time density.

The remainder of this paper is organized as follows. Section 4.2 briefly describes the JDCEV model framework and the main features underlying CDS and EDS contracts. Sections 4.3, 4.4 and 4.5 generalize the ST approach for the valuation of the building blocks of an EDS contract under the CEV and JDCEV models, thus nesting pricing formulas for CDSs as a particular case. Section 4.6 reviews the Laplace transform approach of Mendoza-Arriaga and Linetsky (2011). Section 4.7 implements the ST and the Laplace transform approach, and compares both methodologies in terms of efficiency and accuracy. Finally, Section 4.8 summarizes the main conclusions.

4.2 Model setup

The valuation of CDSs and EDSs will be explored in the context of an arbitrage-free and frictionless financial market, with continuous trading on the time interval $\mathcal{T} := [t_0, T]$, for some fixed time $T > 0$. As usual, uncertainty will be represented by a complete probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathbb{Q}$, taken as a given, will denote the equivalent martingale measure obtained when the numéraire of the economy is a money market account.

In the JDCEV framework of Carr and Linetsky (2006), the (pre-default) price of the defaultable stock is modeled as a time-inhomogeneous diffusion process solving the stochastic differential equation

$$\frac{dS_t}{S_t} = [r(t) - q(t) + \lambda(t, S)]dt + \sigma(t, S)dW_t^{\mathbb{Q}}, \tag{4.1}$$

with $S_{t_0} > 0$, and where the time-t risk-free short-term interest rate $r(t) \in \mathbb{R}$ and the time-t dividend yield $q(t) \in \mathbb{R}$ are both modeled as deterministic functions of time. Furthermore, the hazard rate $\lambda(t, S) \geq 0$ (which compensates equityholders for default with no recovery, thus ensuring an expected rate of return equal to the risk-free interest rate, under measure $\mathbb{Q}$) and the instantaneous volatility of returns $\sigma(t, S) \in \mathbb{R}_+$ can also be state-dependent. Finally, $\{W_t^{\mathbb{Q}}, t \geq t_0\}$ denotes a standard Brownian motion, defined under measure $\mathbb{Q}$, initialized at 0, and generating the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.

Following Carr and Linetsky (2006, Page 311), and in order to ensure consistency with the well-known leverage effect and the implied volatility skew, the instantaneous stock volatility is specified as a power function of the stock price:

$$\sigma(t, S) = a(t) S_t^{\bar{\beta}}, \tag{4.2}$$

where $\bar{\beta} < 0$ is the elasticity of volatility, and $a(t) > 0$ is a deterministic volatility scale function. Also, to be coherent with the empirical evidence of a positive relationship between default probabilities and equity volatility, the default intensity is modeled as an affine function of the instantaneous stock variance:

$$\begin{aligned}\lambda(t,S) &= b(t) + c\sigma(t,S)^2 \\ &= b(t) + ca(t)^2 S_t^{2\bar{\beta}},\end{aligned} \tag{4.3}$$

where $b(t) \geq 0$, and $c \geq 0$ measures the sensitivity of λ to σ^2.

In the JDCEV model, a default event is formally modeled as the stock price dropping to zero, which can happen by diffusion of the underlying price process, or by a jump to default. In the first case, default occurs at the first passage time of the stock price to 0:

$$\tau_0 := \inf\{t > t_0 : S_t = 0\}. \tag{4.4}$$

Alternatively, the stock price can jump to an absorbing *cemetery state* (given that the stock price has not yet reached zero by diffusion) whenever the integrated hazard process

$$\Lambda_t = \int_{t_0}^{t} \lambda(u,S)du \tag{4.5}$$

is greater or equal to the level drawn from an exponential random variable Θ independent of $\{W_t^{\mathbb{Q}}, t \geq t_0\}$ and with unit mean, i.e. at the first jump time

$$\tilde{\zeta} := \inf\{t > t_0 : \Lambda_t \geq \Theta\} \tag{4.6}$$

of a doubly stochastic Poisson process with intensity $\lambda(t,S)$. The default time is therefore decomposable into a predictable component, τ_0, whenever the diffusion process hits zero via diffusion, and a totally inaccessible part, $\tilde{\zeta}$, associated to the jump to default, i.e.:

$$\zeta = \tau_0 \wedge \tilde{\zeta}. \tag{4.7}$$

Moreover, $\mathbb{D} = \{\mathcal{D}_t : t \geq t_0\}$ is the filtration generated by the default indicator process $\mathcal{D}_t = \mathbb{I}_{\{t>\zeta\}}$, and $\mathbb{G} = \{\mathcal{G}_t : t \geq t_0\}$ defines the enlarged filtration, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.

This general framework encompasses several other option pricing models as special cases: if r, q and σ are assumed to be constant, and $b = c = 0$, then the standard geometric Brownian motion (GBM) arises; also, if r and q are constant, $b = c = 0$, and $\sigma(t, S) = \delta S_t^{\frac{\beta}{2}-1}$, with $\delta \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, then the CEV model of Cox (1975) is obtained.

4.2.1 Contractual features

An EDS is a financial contract between two parties, through which the EDS buyer (also known as the *protection buyer* or the holder of the *protection leg*) receives a protection payment if and when the stock price drops below a pre-specified *triggering level* (defined as a percentage of the stock price at the contract initiation[4.2]) until the expiry date of the contract. The protection payment corresponds to the *loss given default*, calculated as a percentage of the EDS nominal amount deducted from the recovery value. Conversely, the EDS seller (the *protection seller* or the holder of the *premium leg*) receives periodic premium payments at the EDS swap rate, up to the triggering event or the contract maturity, whichever occurs first. Additionally, if the triggering event occurs between premium payment dates, the EDS seller also receives accrued interest up to that time.

Therefore, and since no cash flows are exchanged at the inception of the contract, the time-t_0 arbitrage-free EDS swap rate ϱ must be such that the present value of the protection leg is equal to the present value of the premium leg, plus accrued interest. The following definitions summarize the contractual features of these three building blocks. Hereafter, and without loss of generality, a nominal amount of 1 is assumed.

Definition 4.1. *The time-t_0 value of the protection leg of an EDS contract written on the asset price S, with triggering level $L\,(>0)$, recovery rate $R \in (0,1]$ and maturity at time $T\,(\geq t_0)$ is equal to*

$$PROT_{t_0}(S, L, R, T) \;=\; \mathbb{I}_{\{\tau_D > t_0\}}(1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\,\mathbb{I}_{\{\tau_D \leq T\}} \middle| \mathcal{G}_{t_0} \right], \qquad (4.8)$$

[4.2]If the triggering level is set to 0%, then the problem boils down to the pricing of a CDS.

where

$$\tau_D := \tilde{\zeta} \wedge \tau_L \tag{4.9}$$

is the contract triggering event time, and

$$\tau_L := \inf\{t > t_0 : S_t = L\} \tag{4.10}$$

is the first time the stock price drops below the triggering level $L\,(\geq 0)$.

Definition 4.2. *The time-t_0 value of the stream of periodic premium payments of an EDS contract on the asset price S, with an EDS swap rate ϱ, and at times $t_i = t_0 + i\Delta$, for $i = 1, 2, ..., N$, where $\Delta := (T - t_0)/N$ represents the length of each time interval, and N is the maximum number of premium payments, is equal to*[4.3]

$$PREM_{t_0}(S, L, \varrho, N, T) \;=\; \mathbb{I}_{\{\tau_D > t_0\}} \varrho \Delta \sum_{i=1}^{N} e^{-\int_{t_0}^{t_i} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau_D \geq t_i\}} \middle| \mathcal{G}_{t_0} \right]. \tag{4.11}$$

Definition 4.3. *The time-t_0 value of the accrued interest of an EDS contract on the asset price S, with an EDS swap rate ϱ, payable if the triggering event occurs between premium*

[4.3]To ease the notation, we have assumed that Δ is constant for all time intervals. In practice, the length of the time intervals $[t_i, t_{i+1}]$ must be adjusted attending to the daycount convention adopted.

payment dates, is equal to

$$
\begin{aligned}
& ACCINT_{t_0}\left(S,L,\varrho,N,T\right) \\
= \; & \mathbb{I}_{\{\tau_D>t_0\}}\varrho\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\left(\tau_D-\Delta\left\lfloor\frac{\tau_D}{\Delta}\right\rfloor\right)\mathbb{I}_{\{\tau_D\le T\}}\middle|\,\mathcal{G}_{t_0}\right] \\
= \; & \mathbb{I}_{\{\tau_D>t_0\}}\varrho\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\left(\tau_D-i\Delta\right)\mathbb{I}_{\{t_i\le\tau_D\le t_{i+1}\}}\middle|\,G_{t_0}\right] \\
= \; & \mathbb{I}_{\{\tau_D>t_0\}}\varrho\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\tau_D\left(\mathbb{I}_{\{\tau_D\le t_{i+1}\}}-\mathbb{I}_{\{\tau_D<t_i\}}\right)\middle|\,\mathcal{G}_{t_0}\right] \\
& -\mathbb{I}_{\{\tau_D>t_0\}}\varrho\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}i\Delta\left(\mathbb{I}_{\{\tau_D\le t_{i+1}\}}-\mathbb{I}_{\{\tau_D<t_i\}}\right)\middle|\,\mathcal{G}_{t_0}\right] \\
= \; & \mathbb{I}_{\{\tau_D>t_0\}}\varrho\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\tau_D\mathbb{I}_{\{\tau_D\le T\}}\middle|\,\mathcal{G}_{t_0}\right] \\
& -\mathbb{I}_{\{\tau_D>t_0\}}\varrho\sum_{i=0}^{N-1}i\Delta\left(\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\mathbb{I}_{\{\tau_D\le t_{i+1}\}}\middle|\,\mathcal{G}_{t_0}\right]-\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl}\mathbb{I}_{\{\tau_D<t_i\}}\middle|\,\mathcal{G}_{t_0}\right]\right).
\end{aligned}
\tag{4.12}
$$

where the $\lfloor . \rfloor$ *denotes the floor function.*

From now on, and to lighten the notation, we will assume that neither the triggering event nor the jump to default have occurred yet (i.e., $\tau_D > t_0$).

4.3 The stopping time approach

This section offers a new method for pricing EDSs under the time-inhomogeneous JDCEV model. The pricing formulas for CDS contracts are also obtained as a particular case.

Before presenting our main results, the following proposition, taken as a special case of Nunes et al. (2015, Proposition A.1), provides a general result for further reference throughout the paper.

Proposition 4.1. *Under the JDCEV framework, and for τ_L defined through equation (4.10),*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L}\lambda(l,S)dl}\mathbb{I}_{\{\inf_{t_0<l<\tau_L}(S_l)>0\}}\gamma\left(\tau_L\right)\mathbb{I}_{\{\tau_L\leq T\}}\middle|\,\mathcal{F}_{t_0}\right] \\ = & \int_{t_0}^{T}\gamma\left(u\right)SP\left(S_{t_0},t_0;u\right)\mathbb{Q}\left(\tau_L\in du|\,\mathcal{F}_{t_0}\right), \end{aligned} \tag{4.13}$$

where $\gamma:\mathcal{T}\to\mathbb{R}_+$ is any real-valued deterministic function of time, $\mathbb{Q}\left(\tau_L\in dv|\,\mathcal{F}_{t_0}\right)$ denotes the $\mathbb{Q}$-measured density function of the first passage time τ_L, and

$$\begin{aligned} SP\left(S_{t_0},t_0;T\right) & := \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\zeta>T\}}\middle|\,\mathcal{G}_{t_0}\right] \\ & = \mathbb{I}_{\{\zeta>t_0\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T}\lambda(l,S)dl}\mathbb{I}_{\{\tau_0>T\}}\middle|\,\mathcal{F}_{t_0}\right] \end{aligned} \tag{4.14}$$

represents the risk-neutral survival probability beyond time $T\,(>t_0)$, as defined in Carr and Linetsky (2006, Equation 3.1).

Proof. Since S_t follows a pure Markovian process with respect to $\mathcal{F}_{t_0}$, the left-hand side of equation (4.13) can be restated in terms of the convolution between the densities of the first passage time τ_L and of the random vector (S,τ_0):

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L}\lambda(l,S)dl}\mathbb{I}_{\{\inf_{t_0<l<\tau_L}(S_l)>0\}}\gamma\left(\tau_L\right)\mathbb{I}_{\{\tau_L\leq T\}}\middle|\,\mathcal{F}_{t_0}\right] \\ = & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L}\lambda(l,S)dl}\gamma\left(\tau_L\right)\mathbb{I}_{\{\tau_0\geq\tau_L\}}\mathbb{I}_{\{\tau_L\leq T\}}\middle|\,\mathcal{F}_{t_0}\right] \\ = & \int_{t_0}^{T}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\gamma\left(u\right)\mathbb{I}_{\{\tau_0\geq u\}}\middle|\,\sigma\left(\mathcal{F}_{t_0}\cup\{\tau_L=u\}\right)\right]\mathbb{Q}\left(\tau_L\in du|\,\mathcal{F}_{t_0}\right) \\ = & \int_{t_0}^{T}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\gamma\left(u\right)|\,S_u=L\right]e^{-\int_{t_0}^{u}\lambda(l,S)dl}\mathbb{I}_{\{\tau_0\geq u\}}\middle|\,\mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_L\in du|\,\mathcal{F}_{t_0}\right), \end{aligned} \tag{4.15}$$

where the last line follows from the tower law of conditional expectations. Taking into account that $\gamma\left(u\right)$ is nonrandom, equation (4.15) becomes:

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L}\lambda(l,S)dl}\mathbb{I}_{\{\inf_{t_0<l<\tau_L}(S_l)>0\}}\gamma\left(\tau_L\right)\mathbb{I}_{\{\tau_L\leq T\}}\middle|\,\mathcal{F}_{t_0}\right] \\ = & \int_{t_0}^{T}\gamma\left(u\right)\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\mathbb{I}_{\{\tau_0\geq u\}}\middle|\,\mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_L\in du|\,\mathcal{F}_{t_0}\right), \end{aligned} \tag{4.16}$$

and equation (4.13) follows immediately from definition (4.14). $\square$

Remark 4.1. *To use equation* (4.13) *under the CEV model, it is only necessary to take* $b(t) = c = 0$*, for all* $t \in \mathcal{T}$*.*

Under the JDCEV framework and following Carr and Linetsky (2006, Equation 5.14), the survival probability given by equation (4.14) can be obtained as

$$SP\left(S_{t_0}, t_0; T\right) = e^{-\int_{t_0}^{T} b(l)dl} \left(\frac{x^2\left(S_{t_0}\right)}{\theta\left(t_0, T\right)}\right)^{\frac{1}{2\left|\bar{\beta}\right|}} \mathcal{M}\left(-\frac{1}{2\left|\bar{\beta}\right|}; 2\left(1 + v_+\right), \frac{x^2\left(S_{t_0}\right)}{\theta\left(t_0, T\right)}\right), \quad (4.17)$$

where

$$x(S) := \frac{1}{\left|\bar{\beta}\right|} S^{\left|\bar{\beta}\right|}, \quad (4.18)$$

$$v_+ := \frac{c + \frac{1}{2}}{\left|\bar{\beta}\right|}, \quad (4.19)$$

$$\theta\left(t_0, T\right) := \int_{t_0}^{T} a(u)^2 e^{-2\left|\bar{\beta}\right| \int_{t_0}^{u} [r(l) - q(l) + b(l)]dl} du, \quad (4.20)$$

and $\mathcal{M}(p; v, \lambda) := \mathbb{E}^{\chi^2(v,\lambda)}\left[X^p\right]$ is the p-th raw moment of a non-central chi-square random variable X with v degrees of freedom and non-centrality parameter λ, as defined in Carr and Linetsky (2006, Equation 5.10). As in Ruas et al. (2013), Dias et al. (2014) or Nunes et al. (2015), the algorithm proposed by Dias and Nunes (2014) will be used for valuing the truncated p-th moments

$$\Phi_\xi\left(p, y; v, \lambda\right) := \mathbb{E}^{\chi^2(v,\lambda)}\left[X^p \mathbb{I}_{\{\xi X \geq \xi y\}}\right], \quad (4.21)$$

with $\xi \in \{-1, 1\}$. The raw moments $\mathcal{M}(p; v, \lambda)$ are then computed via the following identity, provided by Carr and Linetsky (2006, Equation 5.13):

$$\mathcal{M}(p;v,\lambda) = \Phi_{-1}(p,y;v,\lambda) + \Phi_{+1}(p,y;v,\lambda), \tag{4.22}$$

for any $y \in \mathbb{R}$.

4.3.1 CEV model

The next proposition offers a new approach for valuing the building blocks of an EDS contract, under the CEV model.

Proposition 4.2. *Under the CEV model, equations* (4.8), (4.11) *and* (4.12) *can be restated as*

$$PROT_{t_0}(S,L,R,T) = (1-R)\int_{t_0}^{T} e^{-r(u-t_0)}\mathbb{Q}\left(\tau_L \in du \middle| \mathcal{F}_{t_0}\right), \tag{4.23}$$

$$PREM_{t_0}(S,L,\varrho,N,T) = \varrho\Delta\sum_{i=1}^{N}\left[e^{-r(t_i-t_0)} - e^{-r(t_i-t_0)}\int_{t_0}^{t_i}\mathbb{Q}\left(\tau_L \in du \middle| \mathcal{F}_{t_0}\right)\right], \tag{4.24}$$

and

$$\begin{aligned} & ACCINT_{t_0}(S,L,\varrho,N,T) \\ = \; & \varrho\int_{t_0}^{T} e^{-r(u-t_0)}u\mathbb{Q}\left(\tau_L \in du \middle| \mathcal{F}_{t_0}\right) - \varrho\sum_{i=1}^{N-1} i\Delta\int_{t_i}^{t_{i+1}} e^{-r(u-t_0)}\mathbb{Q}\left(\tau_L \in du \middle| \mathcal{F}_{t_0}\right). \end{aligned} \tag{4.25}$$

Proof. In the CEV model there is no jump to default, i.e., $\tau_D = \tau_L$, and the interest rate r is constant. Moreover, $\tau_0 > \tau_L$, i.e., the default time occurs always after the first passage time of the price process through L. Therefore, the protection leg value (4.8) can be rewritten as

$$PROT_{t_0}(S,L,R,T) = (1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_L-t_0)}\mathbb{I}_{\{\tau_L \le T\}} \middle| \mathcal{F}_{t_0}\right]. \tag{4.26}$$

Using Proposition 4.1, with $\gamma(\tau_L) = e^{-r(\tau_L - t_0)}$ and $b(t) = c = 0$, equation (4.23) follows from equation (4.26). Similarly, the premium leg value (4.11) becomes

$$\begin{aligned} PREM_{t_0}(S, L, \varrho, N, T) &= \varrho\Delta \sum_{i=1}^{N} e^{-r(t_i - t_0)} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau_L \geq t_i\}} \middle| \mathcal{F}_{t_0} \right] \\ &= \varrho\Delta \sum_{i=1}^{N} \left\{ e^{-r(t_i - t_0)} - e^{-r(t_i - t_0)} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau_L < t_i\}} \middle| \mathcal{F}_{t_0} \right] \right\}. \end{aligned} \tag{4.27}$$

Using Proposition 4.1, with $\gamma(\tau_L) = 1$ and $b(t) = c = 0$, equation (4.24) follows immediately. Finally, the accrued interest component (4.12) becomes

$$\begin{aligned} & ACCINT_{t_0}(S, L, \varrho, N, T) \\ = \; & \varrho \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_L - t_0)} \tau_L \mathbb{I}_{\{\tau_L \leq T\}} \middle| \mathcal{G}_{t_0} \right] \\ & -\varrho \sum_{i=1}^{N-1} i\Delta \left\{ \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_L - t_0)} \mathbb{I}_{\{\tau_L \leq t_{i+1}\}} \middle| \mathcal{G}_{t_0} \right] - \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_L - t_0)} \mathbb{I}_{\{\tau_L < t_i\}} \middle| \mathcal{G}_{t_0} \right] \right\}. \end{aligned} \tag{4.28}$$

Using Proposition 4.1 with $\gamma(\tau_L) = e^{-r(\tau_L - t_0)} \tau_L$ and $b(t) = c = 0$, for the first expectation on the right-hand side of equation (4.28), and with $\gamma(\tau_L) = e^{-r(\tau_L - t_0)}$ and $b(t) = c = 0$, for the second and third expectations, equation (4.25) is obtained. □

Remark 4.2. *In order to compute the time-t_0 value of the three building blocks for a CDS contract, Proposition 4.2 can be used while taking $L = 0$.*

4.3.2 JDCEV model

Next proposition contains our main result and generalizes the previous analysis to the context of the JDCEV framework. Again, we are able to price the three building blocks of an EDS in terms of the first passage time density through the triggering level L.

Proposition 4.3. *Under the JDCEV model, equations* (4.8), (4.11) *and* (4.12) *can be rewritten as*

$$\begin{aligned} & PROT_{t_0}(S,L,R,T) \\ = \; & (1-R)\left\{\int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} SP(S_{t_0},t_0;u)\,\mathbb{Q}(\tau_L \in du|\,\mathcal{F}_{t_0}) \right. \qquad (4.29) \\ & \left. + \int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl}\left[H_{t_0}^{D}(S_{t_0};u) - \int_{t_0}^{u} H_v^D(L;u)\,SP(S_{t_0},t_0;v)\,\mathbb{Q}(\tau_L \in dv|\,\mathcal{F}_{t_0})\right] du\right\}, \end{aligned}$$

$$\begin{aligned} & PREM_{t_0}(S,L,\varrho,N,T) \qquad (4.30) \\ = \; & \varrho\Delta\sum_{i=1}^{N} e^{-\int_{t_0}^{t_i} r(l)dl}\left[SP(S_{t_0},t_0;t_i) - \int_{t_0}^{t_i} SP(L,u;t_i)\,SP(S_{t_0},t_0;u)\,\mathbb{Q}(\tau_L \in du|\,\mathcal{F}_{t_0})\right], \end{aligned}$$

and

$$\begin{aligned} & ACCINT_{t_0}(S,L,\varrho,N,T) \\ = \; & \varrho\int_{t_0}^{T} u e^{-\int_{t_0}^{u} r(l)dl} \qquad (4.31) \\ & \times\left[H_{t_0}^{D}(S_{t_0};u) - \int_{t_0}^{u} H_v^D(L;u)\,SP(S_{t_0},t_0;v)\,\mathbb{Q}(\tau_L \in dv|\,\mathcal{F}_{t_0})\right] du \\ & +\varrho\int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} u SP(S_{t_0},t_0;u)\,\mathbb{Q}(\tau_L \in du|\,\mathcal{F}_{t_0}) - \varrho\sum_{i=1}^{N-1} i\Delta\left\{\int_{t_i}^{t_{i+1}} e^{-\int_{t_0}^{u} r(l)dl}\right. \\ & \times\left[H_{t_0}^{D}(S_{t_0};u) - \int_{t_0}^{u} H_v^D(L;u)\,SP(S_{t_0},t_0;v)\,\mathbb{Q}(\tau_L \in dv|\,\mathcal{F}_{t_0})\right] du \\ & \left. + \int_{t_i}^{t_{i+1}} e^{-\int_{t_0}^{u} r(l)dl} SP(S_{t_0},t_0;u)\,\mathbb{Q}(\tau_L \in du|\,\mathcal{F}_{t_0})\right\}, \end{aligned}$$

where

$$\begin{aligned} & H_t^D(S_t;T) \\ := \; & b(T)\,e^{-\int_t^T b(l)dl}\left[\frac{x^2(S_t)}{\theta(t,T)}\right]^{\frac{1}{2|\bar\beta|}} \mathcal{M}\left(-\frac{1}{2|\bar\beta|};2(1+v_+),\frac{x^2(S_t)}{\theta(t,T)}\right) \qquad (4.32) \\ & +ca(T)^2\,e^{\int_t^T\left[2\bar\beta(r(l)-q(l)+b(l))-b(l)\right]dl} \\ & \times[x(S_t)]^{\frac{1}{|\bar\beta|}}\,|\bar\beta|^{-2}\,[\theta(t,T)]^{-1-\frac{1}{2|\bar\beta|}}\,\mathcal{M}\left(\frac{2\bar\beta-1}{2|\bar\beta|};2(1+v_+)\,\frac{x^2(S_t)}{\theta(t,T)}\right). \end{aligned}$$

Proof. Concerning the protection leg, and using definition (4.9), equation (4.8) can be restated as

$$\begin{aligned} PROT_{t_0}(S,L,R,T) &= (1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tilde{\zeta}\wedge\tau_L} r(l)dl}\mathbb{I}_{\{\tilde{\zeta}\wedge\tau_L\leq T\}} \middle| \mathcal{G}_{t_0}\right] \\ &= (1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L} r(l)dl}\mathbb{I}_{\{\tilde{\zeta}\geq\tau_L,\tau_L\leq T\}} \middle| \mathcal{G}_{t_0}\right] \\ &\quad +(1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tilde{\zeta}} r(l)dl}\mathbb{I}_{\{\tilde{\zeta}<\tau_L,\tilde{\zeta}\leq T\}} \middle| \mathcal{G}_{t_0}\right]. \end{aligned} \tag{4.33}$$

For the first term on the right-hand side of equation (4.33), we get

$$\begin{aligned} & (1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L} r(l)dl}\mathbb{I}_{\{\tilde{\zeta}\geq\tau_L,\tau_L\leq T\}} \middle| \mathcal{G}_{t_0}\right] \\ =\; & (1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L} (r(l)+\lambda(l,S))dl}\mathbb{I}_{\{\tau_0\geq\tau_L,\tau_L\leq T\}} \middle| \mathcal{F}_{t_0}\right] \\ =\; & (1-R)\int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} SP\left(S_0,t_0;u\right)\mathbb{Q}\left(\tau_L\in du|\,\mathcal{F}_{t_0}\right), \end{aligned} \tag{4.34}$$

where the last line follows from Proposition 4.1, with $\gamma\left(\tau_L\right)=e^{-\int_{t_0}^{\tau_L} r(l)dl}$. For the second term on the right-hand side of equation (4.33), and using, for instance, Carr and Linetsky (2006, Equation 3.4), then

$$\begin{aligned} & (1-R)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tilde{\zeta}} r(l)dl}\mathbb{I}_{\{\tilde{\zeta}<\tau_L,\tilde{\zeta}\leq T\}} \middle| \mathcal{G}_{t_0}\right] \\ =\; & (1-R)\int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_L>u\}} \middle| \mathcal{F}_{t_0}\right] du. \end{aligned} \tag{4.35}$$

Since $\tau_0\geq\tau_L$ and $\mathbb{I}_{\{\tau_0\geq\tau_L,\tau_L>u\}}=\mathbb{I}_{\{\tau_0>u,\tau_L>u\}}$, the expectation on the right-hand side of equation (4.35) becomes

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u,\tau_L>u\}} \middle| \mathcal{F}_{t_0}\right] \\ =\; & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u\}} \middle| \mathcal{F}_{t_0}\right] \\ & -\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u,\tau_L\leq u\}} \middle| \mathcal{F}_{t_0}\right]. \end{aligned} \tag{4.36}$$

Since the pre-default stock price S follows a Markovian process with respect to $\mathcal{F}_{t_0}$, the second expectation on the right-hand side of equation (4.36) can be rewritten in terms of the convolution between the densities of the first passage time τ_L and of the random vector (S_u, τ_0). Therefore, equation (4.36) yields

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u,\tau_L>u\}}\middle|\,\mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u\}}\middle|\,\mathcal{F}_{t_0}\right] \\
& -\int_{t_0}^{u}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u\}}\middle|\,\sigma\left(\mathcal{F}_{t_0}\cup\{\tau_L=v\}\right)\right]\mathbb{Q}\left(\tau_L\in dv|\,\mathcal{F}_{t_0}\right) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\tau_0>u\}}\middle|\,\mathcal{F}_{t_0}\right] \\
& -\int_{t_0}^{u}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{v}^{u}\lambda(l,S)dl}\lambda\left(u,S\right)\mathbb{I}_{\{\inf_{v\le l\le u}(S_l)>0\}}\middle|\,S_v=L\right]\right. \\
& \left. e^{-\int_{t_0}^{v}\lambda(l,S)dl}\mathbb{I}_{\{\inf_{t_0<l\le v}(S_l)>0\}}\middle|\,\mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_L\in dv|\,\mathcal{F}_{t_0}\right) \\
= \; & H_{t_0}^{D}\left(S_{t_0};u\right)-\int_{t_0}^{u}H_{v}^{D}\left(L;u\right)SP\left(S_{t_0},t_0;v\right)\mathbb{Q}\left(\tau_L\in dv|\,\mathcal{F}_{t_0}\right),
\end{aligned} \tag{4.37}$$

where the second equality follows from the tower law of conditional expectations, and the last line uses equation (4.14) and the following definition:

$$H_t^D\left(S;T\right) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T\lambda(l,S)dl}\lambda\left(T,S\right)\mathbb{I}_{\{\inf_{t\le l\le T}(S_l)>0\}}\middle|\,S_t=S\right]. \tag{4.38}$$

Following Carr and Linetsky (2006, Proposition 5.4), the right-hand side of equation (4.38) can be restated as an expectation over a function of a time-changed Bessel process

$\{R_{\theta(t,u)}; u \geq t\}$ of index v_+, and started at $R_{\theta(t,t)} = x\left(S_t\right)$:

$$
\begin{aligned}
& H_t^D\left(S_t; T\right) \\
= \; & e^{-\int_t^T b(l)dl} \mathbb{E}_{x(S_t)}^{(v_+)}\left[\left(\frac{R_{\theta(t,T)}}{x(S_t)}\right)^{-\frac{1}{|\bar{\beta}|}}\left(b\left(T\right) + ca\left(T\right)^2 e^{2\bar{\beta}\int_t^T [r(l)-q(l)+b(l)]dl}\left(|\bar{\beta}| R_{\theta(t,T)}\right)^{\frac{2\bar{\beta}}{|\bar{\beta}|}}\right)\right] \\
= \; & e^{-\int_t^T b(l)dl}\Bigg\{ b\left(T\right)\left(\frac{1}{x(S_t)}\right)^{-\frac{1}{|\bar{\beta}|}} \mathbb{E}_{x(S_t)}^{(v_+)}\left[\left(R_{\theta(t,T)}\right)^{-\frac{1}{|\bar{\beta}|}}\right] \\
& + ca\left(T\right)^2 e^{2\bar{\beta}\int_t^T [r(l)-q(l)+b(l)]dl}\left(\frac{1}{x(S_t)}\right)^{-\frac{1}{|\bar{\beta}|}}\left(|\bar{\beta}|\right)^{\frac{2\bar{\beta}}{|\bar{\beta}|}} \mathbb{E}_{x(S_t)}^{(v_+)}\left[\left(R_{\theta(t,T)}\right)^{\frac{2\bar{\beta}-1}{|\bar{\beta}|}}\right]\Bigg\} \\
= \; & b\left(T\right) e^{-\int_t^T b(l)dl}\left[\frac{\theta\left(t,T\right)}{x^2(S_t)}\right]^{-\frac{1}{2|\bar{\beta}|}} \mathbb{E}_{x(S_t)}^{(v_+)}\left[\left(\frac{R^2_{\theta(t,T)}}{\theta\left(t,T\right)}\right)^{-\frac{1}{2|\bar{\beta}|}}\right] + ca\left(T\right)^2 \qquad (4.39) \\
& \times e^{\int_t^T\left[2\bar{\beta}(r(l)-q(l)+b(l))-b(l)\right]dl}\left[x(S_t)\right]^{\frac{1}{|\bar{\beta}|}}\left(|\bar{\beta}|\right)^{\frac{2\bar{\beta}}{|\bar{\beta}|}}\left[\theta\left(t,T\right)\right]^{\frac{2\bar{\beta}-1}{2|\bar{\beta}|}} \mathbb{E}_{x(S_t)}^{(v_+)}\left[\left(\frac{R^2_{\theta(t,T)}}{\theta\left(t,T\right)}\right)^{\frac{2\bar{\beta}-1}{2|\bar{\beta}|}}\right],
\end{aligned}
$$

where the expectation is taken with respect to a Bessel process $\{R_{\theta(t,u)}; u \geq t\}$ of index v_+ and started at $R_{\theta(t,t)} = x\left(S_t\right)$. Equation (4.39) yields equation (4.32) because $\frac{R^2_{\theta(t,T)}}{\theta(t,T)}$ follows a non-central chi-square law with $2\left(1 + v_+\right)$ degrees of freedom and non-centrality parameter $\frac{x^2(S_t)}{\theta(t,T)}$. Finally, combining equations (4.33)-(4.35) and (4.37)-(4.39), equation (4.29) follows immediately.

For the premium leg, using again definition (4.9), the expectation on the right-hand side of equation (4.11) can be written as

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau_D \geq t_i\}}\middle|\, \mathcal{G}_{t_0}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta}\wedge\tau_L \geq t_i\}}\middle|\, \mathcal{G}_{t_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta}\geq t_i, \tilde{\zeta}\leq \tau_L\}} + \mathbb{I}_{\{\tau_L \geq t_i, \tilde{\zeta} > \tau_L\}}\middle|\, \mathcal{G}_{t_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{t_i \leq \tilde{\zeta} \leq \tau_L\}} + \mathbb{I}_{\{t_i \leq \tau_L < \tilde{\zeta}\}}\middle|\, \mathcal{G}_{t_0}\right]. \qquad (4.40)
\end{aligned}
$$

Since

$$
\mathbb{I}_{\{t_i \leq \tau_L\}} = \mathbb{I}_{\{\tilde{\zeta} < t_i \leq \tau_L\}} + \mathbb{I}_{\{t_i \leq \tilde{\zeta} \leq \tau_L\}} + \mathbb{I}_{\{t_i \leq \tau_L < \tilde{\zeta}\}},
$$

then equation (4.40) becomes

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau_D \geq t_i\}}\middle|\,\mathcal{G}_{t_0}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{t_i \leq \tau_L\}} - \mathbb{I}_{\{\tilde{\zeta} < t_i \leq \tau_L\}}\middle|\,\mathcal{G}_{t_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{t_i \leq \tau_L\}} - \mathbb{I}_{\{\tilde{\zeta} < t_i, t_i \leq \tau_L\}}\middle|\,\mathcal{G}_{t_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta} \geq t_i, t_i \leq \tau_L\}}\middle|\,\mathcal{G}_{t_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta} \geq t_i\}}\middle|\,\mathcal{G}_{t}\right] - \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta} \geq t_i, \tau_L < t_i\}}\middle|\,\mathcal{G}_{t_0}\right].
\end{aligned} \tag{4.41}$$

Using definition (4.7) and equation (4.14), the first term on the right-hand side of equation (4.41) becomes

$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta} \geq t_i\}}\middle|\,\mathcal{G}_{t_0}\right] = SP\left(S_{t_0}, t_0; t_i\right). \tag{4.42}$$

Similarly, the second term on the right-hand side of equation (4.41) yields

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tilde{\zeta} \geq t_i, \tau_L < t_i\}}\middle|\,\mathcal{G}_{t_0}\right] \\
=\ & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_i}\lambda(l,S)dl}\mathbb{I}_{\{\tau_0 \geq t_i, \tau_L < t_i\}}\middle|\,\mathcal{F}_{t_0}\right] \\
=\ & \int_{t_0}^{t_i}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_i}\lambda(l,S)dl}\mathbb{I}_{\{\tau_0 \geq t_i\}}\middle|\,\sigma\left(\mathcal{F}_{t_0}\cup\{\tau_L = u\}\right)\right]\mathbb{Q}\left(\tau_L \in du\middle|\,\mathcal{F}_{t_0}\right) \\
=\ & \int_{t_0}^{t_i}\mathbb{E}_{\mathbb{Q}}\Bigg\{\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{u}^{t_i}\lambda(l,S)dl}\mathbb{I}_{\{\inf_{u \leq l < t_i}(S_l) > 0\}}\middle|\,S_u = L\right] \\
& e^{-\int_{t_0}^{u}\lambda(l,S)dl}\mathbb{I}_{\{\inf_{t_0 < l \leq u}(S_l) > 0\}}\Bigg|\,\mathcal{F}_{t_0}\Bigg\}\mathbb{Q}\left(\tau_L \in du\middle|\,\mathcal{F}_{t_0}\right) \\
=\ & \int_{t_0}^{t_i} SP\left(L, u; t_i\right)SP\left(S_{t_0}, t_0; u\right)\mathbb{Q}\left(\tau_L \in du\middle|\,\mathcal{F}_{t_0}\right),
\end{aligned} \tag{4.43}$$

where the last line follows from equation (4.14). Combining equations (4.11), (4.41), (4.42) and (4.43), equation (4.30) is obtained.

Finally, for the accrued interest component, and using again definition (4.7), the first expectation on the right-hand side of equation (4.12) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl} \tau_D \mathbb{I}_{\{\tau_D \leq T\}} \middle| \mathcal{G}_{t_0} \right] \\
= & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tilde{\zeta}\wedge\tau_L} r(l)dl} \left(\tilde{\zeta} \wedge \tau_L \right) \mathbb{I}_{\{\tilde{\zeta}\wedge\tau_L \leq T\}} \middle| \mathcal{G}_{t_0} \right] \\
= & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tilde{\zeta}} r(l)dl} \tilde{\zeta} \mathbb{I}_{\{\tilde{\zeta}<\tau_L, \tilde{\zeta}\leq T\}} \middle| \mathcal{G}_{t_0} \right] + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L} r(l)dl} \tau_L \mathbb{I}_{\{\tau_L\leq\tilde{\zeta}, \tau_L\leq T\}} \middle| \mathcal{G}_{t_0} \right] \\
= & \int_{t_0}^{T} u e^{-\int_{t_0}^{u} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl} \lambda(u,S) \mathbb{I}_{\{\tau_L > u\}} \middle| \mathcal{F}_{t_0} \right] du \qquad (4.44) \\
& + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L} [r(l)+\lambda(l,S)]dl} \tau_L \mathbb{I}_{\{\tau_0 \geq \tau_L, \tau_L \leq T\}} \middle| \mathcal{F}_{t_0} \right],
\end{aligned}$$

where the last line uses, for instance, Carr and Linetsky (2006, Equation 3.4). The inner expectation on the first term of the right-hand side of equation (4.44) is exactly given by equation (4.37). For the second term on the right-hand side of equation (4.44), Proposition 4.1, with $\gamma(\tau_L) = \tau_L e^{-\int_{t_0}^{\tau_L} r(l)dl}$ implies that

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_L} [r(l)+\lambda(l,S)]dl} \tau_L \mathbb{I}_{\{\tau_0 \geq \tau_L, \tau_L \leq T\}} \middle| \mathcal{F}_{t_0} \right] \qquad (4.45) \\
= & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} u SP(S_{t_0}, t_0; u) \mathbb{Q}(\tau_L \in du | \mathcal{F}_{t_0}).
\end{aligned}$$

The other two expectations on the right-hand side of (4.12) are similar to the one contained in equation (4.8) and, hence, can be obtained from equation (4.29) with T replaced by t_i or t_{i+1}:

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl} \mathbb{I}_{\{\tau_D \leq t_{i+1}\}} \middle| \mathcal{G}_{t_0} \right] - \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl} \mathbb{I}_{\{\tau_D < t_i\}} \middle| \mathcal{G}_{t_0} \right] \\
= & \int_{t_i}^{t_{i+1}} e^{-\int_{t_0}^{u} r(l)dl} SP(S_{t_0}, t_0; u) \mathbb{Q}(\tau_L \in du | \mathcal{F}_{t_0}) \qquad (4.46) \\
& + \int_{t_i}^{t_{i+1}} e^{-\int_{t_0}^{u} r(l)dl} \left[H_{t_0}^D(S_{t_0}; u) - \int_{t_0}^{u} H_v^D(L; u) SP(S_{t_0}, t_0; v) \mathbb{Q}(\tau_L \in dv | \mathcal{F}_{t_0}) \right] du.
\end{aligned}$$

In summary, combining equations (4.12), (4.37), (4.44), (4.45) and (4.46), then equation (4.31) arises immediately. □

Remark 4.3. *For the pricing of CDS contracts, the above formulae can be further simplified since we can take $\tau_L = \tau_0$, and, hence, $\tau_D = \tilde{\zeta}$.*

Taking $\tau_D = \tilde{\zeta}$ in equation (4.8), and using Carr and Linetsky (2006, Equation 5.15), the time-t_0 value of the protection leg is now given by

$$PROT_{t_0}(S, 0, R, T) = (1 - R) \int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} H_{t_0}^{D}(S_{t_0}; u)\, du. \tag{4.47}$$

Likewise, the expectation contained in the right-hand side of equation (4.11) is now the survival probability of Carr and Linetsky (2006, Equation 5.14), and, therefore, the present value of premium payments is equal to

$$PREM_{t_0}(S, 0, \varrho, R, T) = \varrho \Delta \sum_{i=1}^{N} e^{-\int_{t_0}^{t_i} r(l)dl} SP(S_{t_0}, t_0; t_i). \tag{4.48}$$

Finally, and concerning the accrued interest value, equation (4.44) can be rewritten as

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_D} r(l)dl} \tau_D \mathbb{I}_{\{\tau_D \leq T\}} \middle| \mathcal{G}_{t_0} \right] &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tilde{\zeta}} r_l dl} \tilde{\zeta} \mathbb{I}_{\{\tilde{\zeta} \leq T\}} \middle| \mathcal{G}_{t_0} \right] \\ &= \int_{t_0}^{T} u e^{-\int_{t_0}^{u} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl} \lambda(u, S)\, \mathbb{I}_{\{\tau_0 > u\}} \middle| \mathcal{F}_{t_0} \right] du \\ &= \int_{t_0}^{T} u e^{-\int_{t_0}^{u} r(l)dl} H_{t_0}^{D}(S_{t_0}; u)\, du, \end{aligned} \tag{4.49}$$

where the last line follows from definition (4.38). Since the last two expectations on the right-hand side of equation (4.12) are now given by Carr and Linetsky (2006, Equation 5.15), equations (4.12) and (4.49) yield

$$\begin{aligned} & ACCINT_{t_0}(S, 0, \varrho, N, T) \\ = \; & \varrho \left\{ \int_{t_0}^{T} u e^{-\int_{t_0}^{u} r(l)dl} H_{t_0}^{D}(S_{t_0}; u)\, du - \sum_{i=1}^{N-1} i \Delta \int_{t_i}^{t_{i+1}} e^{-\int_{t_0}^{u} r(l)dl} H_{t_0}^{D}(S_{t_0}; u)\, du \right\}. \end{aligned} \tag{4.50}$$

Equations (4.47) and (4.48) correspond exactly to Carr and Linetsky (2006, Equations 5.15 and 5.14). Hence, our formulae for EDS contracts yield the pricing solutions for CDS contracts already offered by Carr and Linetsky (2006) when $L = 0$. Furthermore, *equations* (4.47), (4.48) *and* (4.50) *do not depend on the density of* τ_L *and, therefore, should be very fast to implement.*

4.4 First passage time density

To implement the pricing solutions offered by Propositions 4.2 and 4.3, it is necessary to compute the first passage time density of the underlying asset price through the triggering level L.

4.4.1 CEV model

We recall that under the CEV model of Cox (1975) the risk free interest rate (r) and the dividend yield (q) are both assumed to be constant. Moreover, the elasticity parameter (β) is related to the one of the JDCEV model through the relation $\beta = 2\left(\bar{\beta}+1\right)$.

Proposition 4.4. *Assuming that the underlying asset price S follows a CEV process, the first passage time density of the underlying asset price through the triggering level L is the implicit solution of the following nonlinear integral equation:*

$$G_{-1}\left(t_0, S_{t_0}; u, L\right) = \int_{t_0}^{u} G_{-1}\left(v, L; u, L\right) \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right), \tag{4.51}$$

for $u \in [t_0, T]$, with

$$G_{-1}\left(v, S_v; u, S_u\right) = \begin{cases} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa_{v,u} S_u^{2-\beta}\right)}\left(2\kappa_{v,u} S_v^{2-\beta} e^{(2-\beta)(r-q)(u-v)}\right) & \Leftarrow \quad \beta < 2 \\ Q_{\chi^2\left(2+\frac{2}{\beta-2}, 2\kappa_{v,u} S_v^{2-\beta} e^{(2-\beta)(r-q)(u-v)}\right)}\left(2\kappa_{v,u} S_u^{2-\beta}\right) & \Leftarrow \quad \beta > 2 \end{cases}, \tag{4.52}$$

$$\kappa_{v,u} := \frac{2\left(r-q\right)}{\left(2-\beta\right)\delta^2\left[e^{(2-\beta)(r-q)(u-v)} - 1\right]}, \tag{4.53}$$

and where $Q_{\chi^2(v,\lambda)}\left(x\right)$ represents the complementary distribution function of a non-central chi-square law with v degrees of freedom, and non-centrality parameter λ.

Proof. By the law of total probability,

$$\begin{aligned}\mathbb{Q}\left(S_u \leq L \middle| \mathcal{F}_{t_0}\right) &= \mathbb{Q}\left(S_u \leq L, \tau_L \leq u \middle| \mathcal{F}_{t_0}\right) + \mathbb{Q}\left(S_u \leq L, \tau_L > u \middle| \mathcal{F}_{t_0}\right) \\ &= \mathbb{Q}\left(S_u \leq L, \tau_L \leq u \middle| \mathcal{F}_{t_0}\right). \qquad (4.54)\end{aligned}$$

The left-hand side of equation (4.54), which will be denoted by $G_{-1}(t_0, S_{t_0}; u, L)$, represents the cumulative probability distribution function of the time-u ($\geq t_0$) value for the CEV process, conditional on its value at time-t_0, and can be computed using equations (4.52) and (4.53)—see, for instance, Schroder (1989, Equation 1) for $\beta < 2$, or Emanuel and MacBeth (1982, Equation 7) for $\beta > 2$. Equation (4.52) can be rewritten in terms of the function $\Phi_{+1}(.)$ defined in equation (4.21), with zero as its first argument, and it will be also computed using the algorithm proposed by Dias and Nunes (2014).

Concerning the right-hand side of equation (4.54), and since S follows a Markovian process, such probability can be written in terms of the convolution between the densities of the first passage time τ_L and of the random vector (S_u, τ_L):

$$\begin{aligned}\mathbb{Q}\left(S_u \leq L, \tau_L \leq u \middle| \mathcal{F}_{t_0}\right) &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{S_u \leq L, \tau_L \leq u\}} \middle| \mathcal{F}_{t_0}\right] \\ &= \int_{t_0}^{u} \mathbb{E}_{Q}\left[\mathbb{I}_{\{S_u \leq L\}} \middle| S_v = L\right] \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right) \\ &= \int_{t_0}^{u} \mathbb{Q}\left(S_u \leq L \middle| S_v = L\right) \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right). \qquad (4.55)\end{aligned}$$

Since $\mathbb{Q}\left(S_u \leq L \middle| S_v = L\right) = G_{-1}(v, L; u, L)$, equation (4.51) follows immediately from equations (4.54) and (4.55). □

Following, for instance, Kuan and Webber (2003) or Nunes (2009, Proposition 6), the first passage time density can be efficiently computed through the standard partition method proposed by Park and Schuurmann (1976). Thus, dividing the time interval $[t_0, T]$ into N_{PS} subintervals of size $h := (T - t_0)/N_{PS}$, the probabilities $\mathbb{Q}\left(\tau_L = t_0 + ih \middle| \mathcal{F}_{t_0}\right)$

are obtained from the following recurrence relation:

$$
\begin{aligned}
& \mathbb{Q}\left(\tau_L = t_0 + ih \middle| \mathcal{F}_{t_0}\right) \\
= \; & \mathbb{Q}\left(\tau_L = t_0 + (i-1)\,h \middle| \mathcal{F}_{t_0}\right) + \left[G_{-1}\left(t_0 + \frac{(2i-1)\,h}{2}, L; t_0 + ih, L\right)\right]^{-1} \\
& \left\{G_{-1}\left(t_0, S_{t_0}; t_0 + ih, L\right) - \sum_{j=1}^{i-1} G_{-1}\left(t_0 + \frac{(2j-1)\,h}{2}, L; t_0 + ih, L\right)\right. \\
& \left.\left[\mathbb{Q}\left(\tau_L = t_0 + jh \middle| \mathcal{F}_{t_0}\right) - \mathbb{Q}\left(\tau_L = t_0 + (j-1)\,h \middle| \mathcal{F}_{t_0}\right)\right]\right\},
\end{aligned} \tag{4.56}
$$

for $i = 1, ..., N_{PS}$, and where $\mathbb{Q}\left(\tau_L = t_0 \middle| \mathcal{F}_{t_0}\right) = 0$.

4.4.2 JDCEV model

Under the JDCEV model, the first passage time density of τ_L is still recovered through the same type of Volterra integral equation as in Proposition 4.4. However, we now have to consider also the possibility of a default event.

Proposition 4.5. *Assuming that the underlying asset price S follows a JDCEV process, the first passage time density of the underlying asset price through the triggering level L is the implicit solution of the following nonlinear integral equation:*

$$
G_{-1}\left(t_0, S_{t_0}; u, L\right) = \int_{t_0}^{u} G_{-1}\left(v, L; u, L\right) SP\left(S_{t_0}, t_0; v\right) \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right), \tag{4.57}
$$

for $u \in [t_0, T]$, where

$$
\begin{aligned}
& G_{-1}\left(v, S_v; u, S_u\right) \\
= \; & e^{-\int_v^u b(l)dl} \left[\frac{k^2\left(v, v; S_v\right)}{\theta\left(v, u\right)}\right]^{\frac{1}{2\left|\bar{\beta}\right|}} \Phi_{-1}\left[-\frac{1}{2\left|\bar{\beta}\right|}, \frac{k^2\left(v, u; S_u\right)}{\theta\left(v, u\right)}; 2\left(1 + v_+\right), \frac{k^2\left(v, v; S_v\right)}{\theta\left(v, u\right)}\right],
\end{aligned} \tag{4.58}
$$

and

$$
k\left(v, u; S\right) := \frac{1}{\left|\bar{\beta}\right|} S^{\left|\bar{\beta}\right|} e^{-\left|\bar{\beta}\right| \int_v^u [r(l) - q(l) + b(l)]dl}, \tag{4.59}
$$

while v_+ *and* $\theta(.)$ *are still given by equations* (4.19) *and* (4.20), *respectively.*

Proof. By the law of total probability,

$$
\begin{aligned}
& \mathbb{Q}\left(S_u \leq L, \zeta > u \middle| \mathcal{G}_{t_0}\right) \\
= \; & \mathbb{Q}\left(S_u \leq L, \zeta > u, \tau_L \leq u \middle| \mathcal{G}_{t_0}\right) + \mathbb{Q}\left(S_u \leq L, \zeta > u, \tau_L > u \middle| \mathcal{G}_{t_0}\right) \\
= \; & \mathbb{Q}\left(S_u \leq L, \zeta > u, \tau_L \leq u \middle| \mathcal{G}_{t_0}\right). \qquad (4.60)
\end{aligned}
$$

Using Carr and Linetsky (2006, Equation 3.2), the left-hand side of equation (4.60) becomes

$$
\begin{aligned}
\mathbb{Q}\left(S_u \leq L, \zeta > u \middle| \mathcal{G}_{t_0}\right) &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{S_u \leq L, \zeta > u\}} \middle| \mathcal{G}_{t_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl} \mathbb{I}_{\{S_u \leq L, \tau_0 > u\}} \middle| \mathcal{F}_{t_0}\right]. \qquad (4.61)
\end{aligned}
$$

Furthermore, applying Carr and Linetsky (2006, Proposition 5.4) to equation (4.61), then

$$
\begin{aligned}
& \mathbb{Q}\left(S_u \leq L, \zeta > u \middle| \mathcal{G}_{t_0}\right) \\
= \; & e^{-\int_{t_0}^{u} b(l)dl} \mathbb{E}^{(v_+)}_{k\left(t_0,t_0;S_{t_0}\right)} \left[\left(\frac{R_{\theta(t_0,u)}}{k\left(t_0,t_0;S_{t_0}\right)} \right)^{-\frac{1}{|\bar{\beta}|}} \mathbb{I}_{\left\{ e^{\int_{t_0}^{u}[r(l)-q(l)+b(l)]dl} \left(\left|\bar{\beta}\right| R_{\theta(t_0,u)}\right)^{\frac{1}{|\bar{\beta}|}} \leq L \right\}} \right] \\
= \; & e^{-\int_{t_0}^{u} b(l)dl} \frac{ \mathbb{E}^{(v_+)}_{k\left(t_0,t_0;S_{t_0}\right)} \left[\left(\frac{R^2_{\theta(t_0,u)}}{\theta(t_0,u)} \right)^{-\frac{1}{2|\bar{\beta}|}} \mathbb{I}_{\left\{ \frac{R^2_{\theta(t_0,u)}}{\theta(t_0,u)} \leq \frac{L^{2|\bar{\beta}|} e^{-2|\bar{\beta}| \int_{t_0}^{u}[r(l)-q(l)+b(l)]dl}}{|\bar{\beta}|^2 \theta(t_0,u)} \right\}} \right] }{ \left[\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\theta(t_0,u)} \right]^{-\frac{1}{2|\bar{\beta}|}} }, \qquad (4.62)
\end{aligned}
$$

where the expectation is taken with respect to the law of a Bessel process $\{R_{\theta(t_0,u)}; u \geq t_0\}$ of index v_+ and started at $R_{\theta(t_0,t_0)} = k\left(t_0, t_0; S_{t_0}\right) = x\left(S_{t_0}\right)$. Finally, and since $\frac{R^2_{\theta(t_0,u)}}{\theta(t_0,u)}$ follows a non-central chi-square law with $2\left(1 + v_+\right)$ degrees of freedom and non-centrality parameter $\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\theta(t_0,u)}$, equation (4.62) yields

$$\begin{aligned}\mathbb{Q}\left(S_u\leq L,\zeta>u\right|\mathcal{G}_{t_0}) &= e^{-\int_{t_0}^u b(l)dl}\frac{\Phi_{-1}\left[-\frac{1}{2\left|\bar{\beta}\right|},\frac{k^2(t_0,u;L)}{\theta(t_0,u)};2\left(1+v_+\right),\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\theta(t_0,u)}\right]}{\left[\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\theta(t_0,u)}\right]^{-\frac{1}{2\left|\bar{\beta}\right|}}}\\ &= G_{-1}\left(t_0,S_{t_0};u,L\right),\end{aligned}\tag{4.63}$$

where the last line follows from definition (4.58).

Concerning the right-hand side of equation (4.60),

$$\begin{aligned}\mathbb{Q}\left(S_u\leq L,\zeta>u,\tau_L\leq u\right|\mathcal{G}_{t_0}) &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{S_u\leq L,\zeta>u,\tau_L\leq u\}}\middle|\mathcal{G}_{t_0}\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^u\lambda(l,S)dl}\mathbb{I}_{\{S_u\leq L,\tau_0>u,\tau_L\leq u\}}\middle|\mathcal{F}_{t_0}\right],\end{aligned}\tag{4.64}$$

where the last line follows again from Carr and Linetsky (2006, Equation 3.2). Since S is a Markovian process with respect to the filtration $\mathbb{F}$, equation (4.64) can be rewritten in terms of the convolution between the densities of the first passage time τ_L and of the random vector (S_u,τ_0):

$$\begin{aligned}&\mathbb{Q}\left(S_u\leq L,\zeta>u,\tau_L\leq u\right|\mathcal{G}_{t_0})\\ =&\int_{t_0}^u\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^u\lambda(l,S)dl}\mathbb{I}_{\{S_u\leq L,\tau_0>u\}}\middle|\sigma\left(\mathcal{F}_{t_0}\cup\{\tau_L=v\}\right)\right]\mathbb{Q}\left(\tau_L\in dv\right|\mathcal{F}_{t_0})\\ =&\int_{t_0}^u\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_v^u\lambda(l,S)dl}\mathbb{I}_{\{S_u\leq L,\inf_{v\leq l\leq u}(S_l)>0\}}\middle|S_v=L\right]\right.\\ &\left.e^{-\int_{t_0}^v\lambda(l,S)dl}\mathbb{I}_{\{\inf_{t_0<l\leq v}(S_l)>0\}}\middle|\mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_L\in dv\right|\mathcal{F}_{t_0}),\end{aligned}\tag{4.65}$$

where the last line follows from the tower law for conditional expectations. Comparing the inner expectation on the right-hand side of equation (4.65) with equation (4.61), and using equation (4.63), then

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_v^u\lambda(l,S)dl}\mathbb{I}_{\{S_u\leq L,\inf_{v\leq l\leq u}(S_l)>0\}}\middle|S_v=L\right]=G_{-1}\left(v,L;u,L\right),\tag{4.66}$$

and, hence, equation (4.65) can be rewritten as

$$
\begin{aligned}
& \mathbb{Q}\left(S_u \leq L, \zeta > u, \tau_L \leq u \middle| \mathcal{G}_{t_0}\right) \\
= & \int_{t_0}^{u} G_{-1}\left(v, L; u, L\right) \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{v} \lambda(l,S)dl} \mathbb{I}_{\{\inf_{t_0 < l \leq v}(S_l) > 0\}} \middle| \mathcal{F}_{t_0}\right] \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{u} G_{-1}\left(v, L; u, L\right) SP\left(S_{t_0}, t_0; v\right) \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right), \qquad (4.67)
\end{aligned}
$$

where the last line follows from the definition of the survival probability, as given by equation (4.14). Finally, combining equations (4.60), (4.63) and (4.67), equation (4.57) follows immediately. □

To compute the first passage time density, under the JDCEV model, equation (4.56) can be easily adapted by replacing the term $G_{-1}\left(v, L; u, L\right)$ with $G_{-1}\left(v, L; u, L\right) SP\left(S_{t_0}, t_0; v\right)$:

$$
\begin{aligned}
& \mathbb{Q}\left(\tau_L = t_0 + ih \middle| \mathcal{F}_{t_0}\right) \\
= & \mathbb{Q}\left(\tau_L = t_0 + (i-1)h \middle| \mathcal{F}_{t_0}\right) \qquad (4.68) \\
& + \left[G_{-1}\left(t_0 + \frac{(2i-1)h}{2}, L; t_0 + ih, L\right) SP\left(t_0, S_{t_0}; t_0 + \frac{(2i-1)h}{2}\right)\right]^{-1} \\
& \Bigg\{G_{-1}\left(t_0, S_{t_0}; t_0 + ih, L\right) \\
& - \sum_{j=1}^{i-1} G_{-1}\left(t_0 + \frac{(2j-1)h}{2}, L; t_0 + ih, L\right) SP\left(t_0, S_{t_0}; t_0 + \frac{(2j-1)h}{2}\right) \\
& \left[\mathbb{Q}\left(\tau_L = t_0 + jh \middle| \mathcal{F}_{t_0}\right) - \mathbb{Q}\left(\tau_L = t_0 + (j-1)h \middle| \mathcal{F}_{t_0}\right)\right]\Bigg\},
\end{aligned}
$$

for $i = 1, ..., N_{PS}$, and where $\mathbb{Q}\left(\tau_L = t_0 \middle| \mathcal{F}_{t_0}\right) = 0$.

4.5 Review of the Laplace transform pricing methodology

To benchmark the accuracy and efficiency of the approach proposed in Sections 4.3 and 4.4, this section briefly summarizes the Laplace transform approach of Mendoza-Arriaga and Linetsky (2011). The Laplace transform methodology is only applicable when working under the time-homogeneous version of the JDCEV model (as well as under the CEV model), and expresses the present value of the building blocks of CDS and EDS contracts in terms of infinite sums.

In this setting, the time-t_0 value of the protection payment of an EDS contract, with $L > 0$ and $r - q + b > 0$,[4.4] is equal to

$$\begin{aligned} & PROT_{t_0}(S, L, R, T) \\ = & (1-R)\left(\frac{S^{\frac{1}{2}-c+\bar{\beta}}\Gamma\left(\frac{c}{|\bar{\beta}|}+1\right)}{\Gamma(1+v)} e^{-\frac{A}{2}S^{-2\bar{\beta}}}\right. \\ & \times\sum_{n=1}^{+\infty}\left\{\frac{b\mathcal{L}_+(n-1,0)}{r+b+\omega(n-1)}\left(1-e^{-(r+b+\omega(n-1))T}\right)\right. \\ & +\frac{|\bar{\beta}|a^2\mathcal{L}_+(n-1,2\bar{\beta})}{r+b+\omega n}\left(1-e^{-(r+b+\omega n)T}\right) \\ & \left.+\frac{b\mathcal{M}_+(n,0)+|\bar{\beta}|a^2\mathcal{M}_+(n,2\bar{\beta})}{r+\omega\left(x_n-\frac{v-1}{2}\right)+\xi}\left(1-e^{-\left(r+\omega\left(x_n-\frac{v-1}{2}\right)+\xi\right)T}\right)\right\} \\ & +\left(\frac{S}{L}\right)^{\bar{\beta}-c+\frac{1}{2}} e^{-\frac{A}{2}\left(S^{-2\bar{\beta}}-L^{-2\bar{\beta}}\right)}\left\{\frac{W_{\frac{v-1}{2}-\frac{r+\xi}{\omega},\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)}{W_{\frac{v-1}{2}-\frac{r+\xi}{\omega},\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)}\right. \\ & \left.\left.+\sum_{n=1}^{+\infty}\frac{\omega e^{-\left(\omega\left(x_n-\frac{v-1}{2}\right)+r+\xi\right)T}}{\omega\left(x_n-\frac{v-1}{2}\right)+r+\xi}\frac{W_{x_n,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)}{\left[\frac{\partial}{\partial x}W_{x,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)\right]\Big|_{x=x_n}}\right\}\right), \end{aligned} \tag{4.69}$$

with

$$\begin{aligned} & \mathcal{L}_+(n,p) \\ := & \frac{A^{\frac{1-2c}{4|\bar{\beta}|}+\frac{1}{2}-\delta_p}\left(1+\frac{1}{2|\bar{\beta}|}-\delta_p\right)_n}{n!}\times\left\{M_{\frac{1-2c}{4|\bar{\beta}|}+\frac{2n+1}{2}-\delta_p,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)\right. \\ & \left.-\frac{M_{\frac{1-2c}{4|\bar{\beta}|}+\frac{2n+1}{2}-\delta_p,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)}{W_{\frac{1-2c}{4|\bar{\beta}|}+\frac{2n+1}{2}-\delta_p,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)}W_{\frac{1-2c}{4|\bar{\beta}|}+\frac{2n+1}{2}-\delta_p,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)\right\}, \end{aligned} \tag{4.70}$$

[4.4] Mendoza-Arriaga and Linetsky (2011, Appendix B) also offer pricing solutions for the case $r-q+b < 0$. Nevertheless, the analysis carried out in Section 4.6 will be limited to the case $r - q + b > 0$.

$$
\begin{aligned}
& \mathcal{M}_{+}(n,p) \\
:= \ & \frac{M_{x_n,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right) W_{x_n,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)}{\left[\frac{\partial}{\partial x} W_{x,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)\right]\Big|_{x=x_n}} \\
& \times \left\{ \frac{\Gamma(v)\, {}_2F_2\begin{pmatrix} \delta_p - \frac{1}{2|\bar{\beta}|}, & \frac{1-v-2x_n}{2} & ; AL^{-2\bar{\beta}} \\ 1+\delta_p - \frac{1}{2|\bar{\beta}|}, & 1-v & \end{pmatrix}}{A^{\frac{v-1}{2}} L^{2\bar{\beta}\delta_p+1} \Gamma\left(\frac{c}{|\bar{\beta}|}+\delta_p\right)\left(\frac{1}{2|\bar{\beta}|}-\delta_p\right)} \right. \\
& + A^{\frac{1-2c}{4|\bar{\beta}|}+\frac{1}{2}-\delta_p} \frac{\Gamma\left(\delta_p - \frac{1}{2|\bar{\beta}|}\right)\Gamma\left(\frac{1-2c}{4|\bar{\beta}|}+\frac{1}{2}-\delta_p - x_n\right)}{\Gamma\left(\frac{1-v}{2}-x_n\right)} \\
& \left. - \frac{\Gamma(-v)\,\Gamma\left(\frac{1+v-2x_n}{2}\right) {}_2F_2\begin{pmatrix} \delta_p + \frac{c}{|\bar{\beta}|}, & \frac{1+v-2x_n}{2} & ; AL^{-2\bar{\beta}} \\ 1+\delta_p + \frac{c}{|\bar{\beta}|}, & 1+v & \end{pmatrix}}{A^{-\frac{1+v}{2}} L^{2\bar{\beta}\delta_p - 2c} \Gamma\left(\frac{c}{|\bar{\beta}|}+\delta_p+1\right)\Gamma\left(\frac{1-v}{2}-x_n\right)} \right\},
\end{aligned}
\tag{4.71}
$$

$$
v := \frac{1+2c}{2\left|\bar{\beta}\right|}, \tag{4.72}
$$

$$
A := \frac{|r-q+b|}{a^2\left|\bar{\beta}\right|}, \tag{4.73}
$$

$$
\omega := 2\left|\bar{\beta}\,(r-q+b)\right|, \tag{4.74}
$$

$$
\xi := 2c\,(r-q+b)+b, \tag{4.75}
$$

$$
\{x_n, n=1,2,...\} := \left\{ x_n \middle| \, W_{x,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right) = 0 \right\}, \tag{4.76}
$$

$$\delta_p := \begin{cases} 1 & \Leftarrow \quad p = 0 \\ 0 & \Leftarrow \quad p = 2\left|\bar{\beta}\right| \end{cases}, \tag{4.77}$$

and where $M_{x,m}(z)$ and $W_{x,m}(z)$ denote the first and second Whittaker functions, respectively, $\Gamma(z)$ is the Euler Gamma function, ${}_2F_2$ is the generalized hypergeometric function, $(z)_n = \Gamma(z+n)/\Gamma(z)$ is the Pochhammer symbol, and $n!$ is the factorial of n.

Furthermore, the present value of the EDS periodic premium payments, also with $L > 0$ and $r - q + b > 0$, is given by

$$\begin{aligned} & PREM_{t_0}(S, L, \varrho, N, T) \\ = \; & \varrho\Delta \frac{S^{\frac{1}{2}-c+\bar{\beta}}\Gamma\left(\frac{c}{|\bar{\beta}|}+1\right)}{\Gamma(1+v)} e^{-\frac{A}{2}S^{-2\bar{\beta}}} \\ & \times \sum_{n=1}^{+\infty} \left\{ \mathcal{L}_+(n-1,0)\left(\frac{1-e^{-(r+b+\omega(n-1))\Delta N}}{e^{(r+b+\omega(n-1))\Delta}-1}\right) \right. \\ & \left. + \mathcal{M}_+(n,0)\left(\frac{1-e^{-\left(r+\omega\left(x_n-\frac{v-1}{2}\right)+\xi\right)\Delta N}}{e^{\left(r+\omega\left(x_n-\frac{v-1}{2}\right)+\xi\right)\Delta}-1}\right)\right\}, \end{aligned} \tag{4.78}$$

whereas the present value of the accrued interest component is

$$
\begin{aligned}
& ACCINT_{t_0}\left(S, L, \varrho, N, T\right) \\
= \ & \varrho\Delta\frac{S^{\frac{1}{2}-c+\bar{\beta}}\Gamma\left(\frac{c}{\left|\bar{\beta}\right|}+1\right)}{\Gamma\left(1+v\right)}e^{-\frac{A}{2}S^{-2\bar{\beta}}} \\
& \times\sum_{n=1}^{\infty}\left\{\frac{b\mathcal{L}_{+}\left(n-1,0\right)\left(1-e^{-\left(r+b+\omega(n-1)\right)T}\right)}{r+b+\omega\left(n-1\right)}\right. \\
& \times\left(\frac{1}{r+b+\omega\left(n-1\right)}+\frac{\Delta}{1-e^{\left(r+b+\omega(n-1)\right)\Delta}}\right) \\
& +\frac{\mathcal{L}_{+}\left(n-1,2\bar{\beta}\right)\left(1-e^{-\left(r+b+\omega n\right)T}\right)}{\left(r+b+\omega n\right)/\left(\left|\bar{\beta}\right|a^{2}\right)}\left(\frac{1}{r+b+\omega n}+\frac{\Delta}{1-e^{\left(r+b+\omega n\right)\Delta}}\right) \\
& +\frac{\left(b+\mathcal{L}_{+}\left(n,0\right)+\left|\bar{\beta}\right|a^{2}\mathcal{M}_{+}\left(n,2\bar{\beta}\right)\right)\left(1-e^{-\left(r+\omega\left(x_{n}-\frac{v-1}{2}+\xi\right)T\right)}\right)}{r+b+\left(x_{n}-\frac{v-1}{2}\right)+\xi} \\
& \left.\times\left(\frac{1}{r+\omega\left(x_{n}-\frac{v-1}{2}\right)+\xi}+\frac{\Delta}{1-e^{\left(r+\omega\left(x_{n}-\frac{v-1}{2}\right)+\xi\right)\Delta}}\right)\right\} \\
& +\varrho\left(\frac{S}{L}\right)^{\bar{\beta}-c+\frac{1}{2}}+\frac{e^{-\frac{AS^{-2\bar{\beta}}}{2}}}{e^{-\frac{AL^{-2\bar{\beta}}}{2}}}\left\{\sum_{n=1}^{\infty}\frac{W_{x_{n},\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)}{\left(\frac{2x_{n}-v+1}{2}+\frac{r+\xi}{\omega}\right)\left[\frac{\partial}{\partial x}W_{x,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)\right]\Big|_{x=x_{n}}}\right. \\
& \times\left(\frac{e^{-\left(\omega\left(x_{n}-\frac{v-1}{2}\right)+r+\xi\right)T}}{\omega\left(x_{n}-\frac{v-1}{2}\right)+r+\xi}-\frac{\Delta\left(1-e^{-\left(\omega\left(x_{n}-\frac{v-1}{2}\right)+r+\xi\right)T}\right)}{1-e^{\left(\omega\left(x_{n}-\frac{v-1}{2}\right)+r+\xi\right)\Delta}}\right) \\
& \left.+\left(\frac{\left[\frac{\partial}{\partial\rho}W_{\rho,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)\right]}{\omega\left[\frac{\partial}{\partial\rho}W_{\rho,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)\right]}-\frac{W_{\rho,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right)\left[\frac{\partial}{\partial\rho}W_{\rho,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)\right]}{\omega\left[\frac{\partial}{\partial\rho}W_{\rho,\frac{v}{2}}\left(AL^{-2\bar{\beta}}\right)\right]^{2}}\right)\Bigg|_{\rho=\frac{v-1}{2}-\frac{r+\xi}{\omega}}\right\}.
\end{aligned}
\tag{4.79}
$$

For the pricing of CDS contracts, since $L = 0$, equations (4.69), (4.78) and (4.79) can be simplified into:

$$\begin{aligned} & PROT_{t_0}(S,0,R,T) \\ = \; & (1-R)\frac{S^{\frac{1}{2}-c+\bar{\beta}}\Gamma\left(\frac{c}{|\bar{\beta}|}+1\right)}{\Gamma(1+v)}e^{-\frac{A}{2}S^{-2\bar{\beta}}} \\ & \times\sum_{n=1}^{+\infty}\left\{\frac{b\mathcal{D}_+(n-1,0)}{r+b+\omega(n-1)}\left(1-e^{-(r+b+\omega(n-1))T}\right)\right. \\ & \left.+\frac{|\bar{\beta}|\,a^2\mathcal{D}_+\left(n-1,2\bar{\beta}\right)}{r+b+\omega n}\left(1-e^{-(r+b+\omega n)T}\right)\right\}, \end{aligned} \tag{4.80}$$

$$\begin{aligned} & PREM_{t_0}(S,0,\varrho,N,T) \\ = \; & \varrho\Delta\frac{S^{\frac{1}{2}-c+\bar{\beta}}\Gamma\left(\frac{c}{|\bar{\beta}|}+1\right)}{\Gamma(1+v)}e^{-\frac{A}{2}S^{-2\bar{\beta}}}\times\sum_{n=1}^{+\infty}\mathcal{D}_+(n-1,0)\left(\frac{1-e^{-(r+b+\omega(n-1))\Delta N}}{e^{(r+b+\omega(n-1))\Delta}-1}\right), \end{aligned} \tag{4.81}$$

and

$$\begin{aligned} & ACCINT_{t_0}(S,0,\varrho,N,T) \\ = \; & \varrho\Delta\frac{S^{\frac{1}{2}-c+\bar{\beta}}\Gamma\left(\frac{c}{|\bar{\beta}|}+1\right)}{\Gamma(1+v)}e^{-\frac{A}{2}S^{-2\bar{\beta}}} \\ & \times\sum_{n=1}^{+\infty}\left\{\frac{b\mathcal{D}_+(n-1,0)\left(1-e^{-(r+b+\omega(n-1))T}\right)}{r+b+\omega(n-1)}\right. \\ & \times\left(\frac{1}{r+b+\omega(n-1)}+\frac{\Delta}{1-e^{(r+b+\omega(n-1))\Delta}}\right) \\ & \left.+\frac{\mathcal{D}_+\left(n-1,2\bar{\beta}\right)\left(1-e^{-(r+b+\omega n)T}\right)}{(r+b+\omega n)/\left(|\bar{\beta}|\,a^2\right)}\left(\frac{1}{r+b+\omega n}+\frac{\Delta}{1-e^{(r+b+\omega n)\Delta}}\right)\right\}, \end{aligned} \tag{4.82}$$

where

$$\mathcal{D}_+(n,p):=A^{\frac{1-2c}{|\bar{\beta}|}+\frac{1}{2}-\delta_p}\frac{\left(1+\frac{1}{2|\bar{\beta}|}-\delta_p\right)_n}{n!}M_{\frac{1-2c}{4|\bar{\beta}|}+\frac{2n+1}{2}-\delta_p,\frac{v}{2}}\left(AS^{-2\bar{\beta}}\right). \tag{4.83}$$

4.6 Numerical results

This section implements the methods described in Sections 4.3, 4.4 and 4.5, by pricing CDS and EDS contracts written on CEV and JDCEV processes. The efficiency and accuracy of the novel ST approach is compared against the Laplace transform approach proposed by Mendoza-Arriaga and Linetsky (2011). Both valuation approaches are implemented through *Mathematica 9*, running on an Intel Core i7 PC.

For the ST approach, all numerical integrations are performed through a 5-point Gauss-Legendre quadrature, as described in Press et al. (1994, Page 184). For the Laplace transform approach, and following Mendoza-Arriaga and Linetsky (2011, Footnote 2), the first and second Whittaker functions are computed via the algorithm offered by Abad and Sesma (1995, Page 76). Moreover, the zeros of the second Whittaker function, defined by equation (4.76), are computed using the built-in *Mathematica* function *FindRoot*. In order to obtain a benchmark to evaluate the accuracy of both alternative pricing methods, we also present the results reported by Mendoza-Arriaga and Linetsky (2011, Tables 2 and 3).

Table 4.1 reports CDS and EDS swap rates under the standard CEV model, for a set of 8 contract maturities specified in the first column, and using the same parameter constellation as in Mendoza-Arriaga and Linetsky (2011, Table 2), i.e. $S_{t_0} = 50$, $r = 5\%$, $q = 0\%$, $\delta = 20$, $\beta = 0$, $R = 50\%$ and $\Delta = 0.25$. Under this framework, the ST approach proposed is implemented through Propositions 4.2 and 4.4, and equation (4.56) is applied with different numbers of discretization time steps. The Laplace transform approach of Mendoza-Arriaga and Linetsky (2011) is also implemented with different numbers of summation terms.

[Insert Table 4.1 about here]

Table 4.1 shows that the ST approach yields extremely accurate swap rates, identical to the ones reported by Mendoza-Arriaga and Linetsky (2011, Table 2), even for a crude discretization involving as few as 128 time steps. The computational efficiency of the novel ST approach is also remarkable: It takes only 16.9 seconds for CDS rates, and 31.7 (38.9) seconds for EDS rates with $L = 30\%$ ($L = 50\%$, respectively), when using 128 time steps.

In contrast, the Laplace transform approach of Mendoza-Arriaga and Linetsky (2011) involves a larger computational burden: It takes 116.4 seconds for computing the whole set of CDS rates, with 250 summation terms, and 232.9 (271.2) seconds for EDS rates, with $L = 30\%$ ($L = 50\%$, respectively), when using 500 summation terms. This methodology is substantially slower than the ST approach we propose, mainly because the computation of EDS swap rates involves the time-consuming search for the roots of the second Whittaker function.

Table 4.2 repeats the previous analysis, but for the time-homogeneous version of the JDCEV model. Since we test the accuracy of our results against the Laplace transform approach of Mendoza-Arriaga and Linetsky (2011), the analysis will be focused on the time-independent parameterization, even though it is straightforward to extend the ST approach proposed for the time-inhomogeneous case.

Remark 4.4. *Under the time-homogeneous JDCEV model, equations* (4.20) *and* (4.59) *are reduced to*

$$\theta\left(v,u\right) := \begin{cases} \frac{a^2}{2\left|\bar{\beta}\right|(r-q+b)}\left(1 - e^{-2\left|\bar{\beta}\right|(r-q+b)(u-v)}\right) & \Leftarrow \quad r-q+b \neq 0 \\ a^2\left(u-v\right) & \Leftarrow \quad r-q+b = 0 \end{cases}, \tag{4.84}$$

and

$$k\left(v,u;S\right) := \frac{1}{\left|\bar{\beta}\right|} S^{\left|\bar{\beta}\right|} e^{-\left|\bar{\beta}\right|(r-q+b)(u-v)}. \tag{4.85}$$

The JDCEV model parameters considered are also borrowed from Mendoza-Arriaga and Linetsky (2011, Table 3): $S_{t_0} = 50$, $r = 5\%$, $q = 0\%$, $a = 20$, $b = 0.02$, $c = 1$, $\bar{\beta} = -1$, $R = 50\%$ and $\Delta = 0.25$. The ST approach is now implemented using Remark 4.2 for CDSs, and Propositions 4.3 and 4.5, for EDSs, while equation (4.68) is applied considering different numbers of discretization time steps.

[Insert Table 4.2 about here]

Table 4.2 further highlights the superiority of the ST approach, in terms of accuracy and efficiency: For CDS rates, it takes only 0.3 seconds to price all the 8 contracts, which is substantially faster than under the CEV model (see Table 4.1), since, in this setting, equations (4.47), (4.48) and (4.50) do not depend on the density of τ_L, and hence on the number of time steps used. For EDS rates with $L = 30\%$ ($L = 50\%$), the CPU time corresponds to 853.4 (863.7, respectively) seconds, when using 2^{11} time steps.

The methodology proposed by Mendoza-Arriaga and Linetsky (2011) is considerably less efficient: It takes 619.1 seconds to compute the whole set of CDS rates (using $9,000$ summation terms), and $1,702,8$ ($1,493.3$) seconds for EDS rates with $L = 30\%$ ($L = 50\%$, respectively) and $4,000$ ($3,000$, respectively) summation terms. As before, the efficiency of the Laplace transform approach is negatively impacted by the need to find the zeros of the second Whittaker function for the computation of EDS swap rates.

4.7 Conclusions

This paper extends the ST approach of Kuan and Webber (2003) and offers a novel approach for valuing CDS and EDS contracts under the CEV model of Cox (1975) and the JDCEV model of Carr and Linetsky (2006). Under the CEV model, the triggering event may occur only by diffusion of the underlying price process; under the JDCEV framework, the triggering event may also occur via a jump-to-default of the price process.

Propositions 4.2 (under the CEV model) and 4.3 (under the JDCEV framework) offer pricing solutions for each one of the building blocks of CDS and EDS contracts, which involve only one integration with respect to the density function of the first passage time of the underlying asset price to the contract triggering level. Furthermore, for the pricing of CDS contracts under the JDCEV model, Remark 4.3 shows that the ST approach nests the pricing solutions already offered by Carr and Linetsky (2006) when the contract triggering level is set to zero. Again, under both models, the hitting time density is recovered—in Propositions 4.4 and 4.5—as the implicit solution of a non-linear equation.

The accuracy and efficiency of ST approach is compared against the Laplace transform valuation methodology proposed by Mendoza-Arriaga and Linetsky (2011). The numerical results show that the ST approach is the most efficient pricing method. Moreover, and in opposition to the Laplace transform methodology, the ST approach has been formulated under the most general time-inhomogeneous formulation of the JDCEV model.

Table 4.1: Time-t_0 value of the protection leg, premium leg and accrued interest of CDS and EDS contracts, and the corresponding premium rates ϱ (in basis points per annum), under the CEV model

Panel A - CDS spreads																
Time	Reported				Stopping time approach								Laplace transform approach			
(Yrs.)	MA-L (2011)				$N_{PS} = 64$				$N_{PS} = 128$				$n = 250$				$n = 500$			
	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ
0.25	0.0000	0.2469	0.0000	0	0.0000	0.2469	0.0000	0	0.0000	0.2469	0.0000	0	0.0000	0.2469	0.0000	0	0.0000	0.2469	0.0000	0
0.5	0.0002	0.4906	0.0001	3	0.0002	0.4906	0.0001	3	0.0002	0.4906	0.0001	3	0.0002	0.4906	0.0001	3	0.0002	0.4906	0.0001	3
1	0.0050	0.9660	0.0014	52	0.0050	0.9660	0.0014	52	0.0050	0.9660	0.0014	52	0.0050	0.9660	0.0014	52	0.0050	0.9660	0.0014	52
2	0.0295	1.8497	0.0077	159	0.0295	1.8497	0.0077	159	0.0295	1.8497	0.0077	159	0.0295	1.8497	0.0077	159	0.0295	1.8497	0.0077	159
3	0.0547	2.6397	0.0139	206	0.0547	2.6397	0.0139	206	0.0547	2.6397	0.0139	206	0.0547	2.6397	0.0139	206	0.0547	2.6397	0.0139	206
5	0.0905	3.9880	0.0228	226	0.0905	3.9880	0.0228	226	0.0905	3.9880	0.0228	226	0.0905	3.9880	0.0228	226	0.0905	3.9880	0.0228	226
7	0.1119	5.1055	0.0281	218	0.1119	5.1055	0.0280	218	0.1119	5.1055	0.0281	218	0.1119	5.1055	0.0281	218	0.1119	5.1055	0.0281	218
10	0.1300	6.4778	0.0325	200	0.1300	6.4778	0.0325	200	0.1300	6.4778	0.0325	200	0.1300	6.4778	0.0325	200	0.1300	6.4778	0.0325	200
MPE	–	–	–	–	0.000%	0.000%	-0.044%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
MAPE	–	–	–	–	0.000%	0.000%	0.044%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
CPU	–	–	–	–	7.9				16.9				116.4				154.6			
Panel B - EDS spreads (with $L = 30\%$)																				
Time	Reported				Stopping time approach								Laplace transform approach							
(Yrs.)	MA-L (2011)				$N_{PS} = 64$				$N_{PS} = 128$				$n = 250$				$n = 500$			
	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ
0.25	0.0002	0.2468	0.0001	8	0.0002	0.2468	0.0001	8	0.0002	0.2468	0.0001	8	0.0002	0.2468	0.0001	9	0.0002	0.2468	0.0001	8
0.5	0.0056	0.4878	0.0018	114	0.0056	0.4878	0.0018	114	0.0056	0.4878	0.0018	114	0.0056	0.4878	0.0018	114	0.0056	0.4878	0.0018	114
1	0.0329	0.9414	0.0088	346	0.0329	0.9414	0.0088	346	0.0329	0.9414	0.0088	346	0.0329	0.9413	0.0088	346	0.0329	0.9414	0.0088	346
2	0.0852	1.7335	0.0217	485	0.0852	1.7335	0.0217	485	0.0852	1.7335	0.0217	485	0.0852	1.7335	0.0217	485	0.0852	1.7335	0.0217	485
3	0.1194	2.4078	0.0301	490	0.1194	2.4078	0.0301	490	0.1194	2.4078	0.0301	490	0.1194	2.4078	0.0301	490	0.1194	2.4078	0.0301	490
5	0.1577	3.5237	0.0395	443	0.1577	3.5237	0.0395	443	0.1577	3.5237	0.0395	443	0.1577	3.5237	0.0396	443	0.1577	3.5237	0.0396	443
7	0.1773	4.4332	0.0444	396	0.1773	4.4332	0.0444	396	0.1773	4.4332	0.0444	396	0.1773	4.4332	0.0444	396	0.1773	4.4332	0.0444	396
10	0.1925	5.5443	0.0481	344	0.1925	5.5443	0.0481	344	0.1925	5.5443	0.0481	344	0.1925	5.5443	0.0482	344	0.1925	5.5443	0.0482	344
MPE	–	–	–	–	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	-0.001%	0.058%	1.563%	0.000%	0.000%	0.058%	0.000%
MAPE	–	–	–	–	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.001%	0.058%	1.563%	0.000%	0.000%	0.058%	0.000%
CPU	–	–	–	–	11.3				31.7				144.5				232.9			

Table 4.1—*Continued*

Panel C - EDS spreads (with $L = 50\%$)																				
Time	Reported				Stopping time approach								Laplace transform approach							
(Yrs.)	MA-L (2011)				$N_{PS} = 64$				$N_{PS} = 128$				$n = 250$				$n = 500$			
	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ
0.25	0.0054	0.2442	0.0022	221	0.0054	0.2442	0.0022	221	0.0054	0.2442	0.0022	221	0.0054	0.2442	0.0022	221	0.0054	0.2442	0.0022	221
0.5	0.0333	0.4715	0.0097	693	0.0333	0.4715	0.0097	693	0.0333	0.4715	0.0097	693	0.0333	0.4715	0.0097	693	0.0333	0.4715	0.0097	693
1	0.0896	0.8748	0.0234	997	0.0896	0.8748	0.0234	997	0.0896	0.8748	0.0234	997	0.0896	0.8748	0.0234	997	0.0896	0.8748	0.0234	997
2	0.1550	1.5392	0.0392	982	0.1550	1.5392	0.0392	982	0.1550	1.5392	0.0392	982	0.1550	1.5392	0.0392	982	0.1550	1.5392	0.0392	982
3	0.1890	2.0866	0.0475	885	0.1890	2.0866	0.0475	885	0.1890	2.0866	0.0475	885	0.1890	2.0866	0.0475	885	0.1890	2.0866	0.0475	885
5	0.2228	2.9778	0.0558	734	0.2228	2.9778	0.0558	735	0.2228	2.9778	0.0558	734	0.2228	2.9778	0.0558	734	0.2228	2.9778	0.0558	734
7	0.2389	3.6994	0.0598	636	0.2389	3.6994	0.0597	636	0.2389	3.6994	0.0598	636	0.2389	3.6994	0.0598	636	0.2389	3.6994	0.0598	636
10	0.2508	4.5804	0.0627	540	0.2508	4.5804	0.0626	540	0.2508	4.5804	0.0627	540	0.2508	4.5804	0.0627	540	0.2508	4.5804	0.0627	540
MPE	–	–	–	–	0.000%	0.000%	-0.041%	0.017%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
MAPE	–	–	–	–	0.000%	0.000%	0.041%	0.017%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
CPU	–	–	–	–	13.1				38.9				155.2				271.2			

This table reports the time-t_0 value of the building blocks of CDS and EDS contracts, and the corresponding premium rates (in basis points per annum) under the CEV model, using the same model parameters as in Mendoza-Arriaga and Linetsky (2011, Table 2), i.e. $S_{t_0} = 50$, $r = 5\%$, $q = 0\%$, $\delta = 20$, $\beta = 0$, $R = 50\%$ and $\Delta = 0.25$, for the contract maturities specified in the first column. Panel A reports results for CDS contracts, while Panels B and C report results for EDS contracts, with $L = 30\%$ and $L = 50\%$, respectively. In the three panels, columns 2 to 5 reproduce the values reported in Mendoza-Arriaga and Linetsky (2011, Table 2). Columns 6 to 13 are obtained via the ST approach of Propositions 4.2 and 4.4, and using different numbers of time steps. Columns 14 to 21 describe the results generated by the MA-L valuation approach of Mendoza-Arriaga and Linetsky (2011), as described in Section 4.6, with different numbers of summation terms. The last three lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE) and computation times (in seconds), for the whole set of maturities under analysis.

Table 4.2: Time-t_0 value of the protection leg, premium leg and accrued interest of CDS and EDS contracts, and the corresponding premium rates ϱ (in basis points per annum), under the JDCEV model

Panel A - CDS spreads																				
Time	Reported				Stopping time approach								Laplace transform approach							
(Yrs.)	MA-L (2011)				$N_{PS} = 1,024$				$N_{PS} = 2,048$				$n = 8,000$				$n = 9,000$			
	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ
0.25	0.0219	0.2360	0.0054	908	0.0219	0.2360	0.0054	908	0.0219	0.2360	0.0054	908	0.0219	0.2360	0.0054	909	0.0219	0.2360	0.0054	908
0.5	0.0427	0.4588	0.0106	909	0.0427	0.4588	0.0106	909	0.0427	0.4588	0.0106	909	0.0427	0.4588	0.0106	910	0.0427	0.4588	0.0106	909
1	0.0799	0.8680	0.0197	900	0.0799	0.8680	0.0197	900	0.0799	0.8680	0.0197	900	0.0799	0.8680	0.0197	900	0.0799	0.8680	0.0197	900
2	0.1341	1.5689	0.0330	837	0.1341	1.5689	0.0330	837	0.1341	1.5689	0.0330	837	0.1341	1.5689	0.0330	837	0.1341	1.5689	0.0330	837
3	0.1689	2.1553	0.0416	769	0.1689	2.1553	0.0416	769	0.1689	2.1553	0.0416	769	0.1690	2.1553	0.0416	769	0.1689	2.1553	0.0416	769
5	0.2097	3.1019	0.0516	665	0.2097	3.1019	0.0516	665	0.2097	3.1019	0.0516	665	0.2097	3.1019	0.0516	665	0.2097	3.1019	0.0516	665
7	0.2324	3.8472	0.0572	595	0.2324	3.8472	0.0572	595	0.2324	3.8472	0.0572	595	0.2324	3.8472	0.0572	595	0.2324	3.8472	0.0572	595
10	0.2521	4.7191	0.0621	527	0.2521	4.7191	0.0621	527	0.2521	4.7191	0.0621	527	0.2521	4.7191	0.0621	527	0.2521	4.7191	0.0621	527
MPE	–	–	–	–	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.007%	0.000%	0.000%	0.028%	0.000%	0.000%	0.000%	0.000%
MAPE	–	–	–	–	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.007%	0.000%	0.000%	0.028%	0.000%	0.000%	0.000%	0.000%
CPU	–	–	–	–	0.3				0.3				527.3				619.1			

Panel B - EDS spreads (with $L = 30\%$)																				
Time	Reported				Stopping time approach								Laplace transform approach							
(Yrs.)	MA-L (2011)				$N_{PS} = 1,024$				$N_{PS} = 2,048$				$n = 3,000$				$n = 4,000$			
	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ
0.25	0.0220	0.2360	0.0055	910	0.0220	0.2360	0.0055	910	0.0220	0.2360	0.0055	910	0.0220	0.2360	0.0055	909	0.0220	0.2360	0.0055	910
0.5	0.0437	0.4582	0.0109	932	0.0437	0.4582	0.0109	932	0.0437	0.4582	0.0109	932	0.0437	0.4582	0.0109	932	0.0437	0.4582	0.0109	932
1	0.0841	0.8640	0.0208	950	0.0841	0.8640	0.0208	950	0.0841	0.8640	0.0208	950	0.0841	0.8640	0.0208	950	0.0841	0.8640	0.0208	950
2	0.1405	1.5536	0.0346	884	0.1405	1.5536	0.0346	884	0.1405	1.5536	0.0346	884	0.1405	1.5536	0.0346	884	0.1405	1.5536	0.0346	884
3	0.1752	2.1282	0.0431	807	0.1752	2.1282	0.0431	807	0.1752	2.1282	0.0431	807	0.1752	2.1282	0.0431	807	0.1752	2.1282	0.0431	807
5	0.2150	3.0554	0.0530	692	0.2150	3.0554	0.0530	692	0.2150	3.0554	0.0530	692	0.2150	3.0554	0.0530	692	0.2150	3.0554	0.0530	692
7	0.2371	3.7861	0.0584	617	0.2371	3.7861	0.0584	617	0.2371	3.7861	0.0584	617	0.2371	3.7861	0.0584	617	0.2371	3.7861	0.0584	617
10	0.2563	4.6418	0.0632	545	0.2563	4.6418	0.0631	545	0.2563	4.6418	0.0632	545	0.2563	4.6418	0.0632	545	0.2563	4.6418	0.0632	545
MPE	–	–	–	–	0.000%	0.000%	-0.020%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	-0.014%	0.000%	0.000%	0.000%	0.000%
MAPE	–	–	–	–	0.000%	0.000%	0.020%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.014%	0.000%	0.000%	0.000%	0.000%
CPU	–	–	–	–	373.8				853.4				1,303.0				1,702.8			

Table 4.2—*Continued*

Panel C - EDS spreads (with $L = 50\%$)																				
Time	Reported				Stopping time approach								Laplace transform approach							
(Yrs.)	MA-L (2011)				$N_{PS} = 1,024$				$N_{PS} = 2,048$				$n = 2,000$				$n = 3,000$			
	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ	$PROT$	$PREM$	$ACCINT$	ϱ
0.25	0.0242	0.2349	0.0064	1004	0.0242	0.2349	0.0064	1004	0.0242	0.2349	0.0064	1004	0.0242	0.2349	0.0064	1005	0.0242	0.2349	0.0064	1004
0.5	0.0552	0.4514	0.0142	1187	0.0552	0.4514	0.0142	1187	0.0552	0.4514	0.0142	1187	0.0552	0.4514	0.0142	1187	0.0552	0.4514	0.0142	1187
1	0.1065	0.8371	0.0266	1233	0.1065	0.8371	0.0266	1233	0.1065	0.8371	0.0266	1233	0.1065	0.8371	0.0266	1233	0.1065	0.8371	0.0266	1233
2	0.1661	1.4786	0.0410	1093	0.1661	1.4786	0.0410	1093	0.1661	1.4786	0.0410	1093	0.1661	1.4786	0.0410	1093	0.1661	1.4786	0.0410	1093
3	0.1993	2.0086	0.0492	969	0.1993	2.0086	0.0492	969	0.1993	2.0086	0.0492	969	0.1993	2.0086	0.0492	969	0.1993	2.0086	0.0492	969
5	0.2359	2.8628	0.0582	808	0.2359	2.8628	0.0582	808	0.2359	2.8628	0.0582	808	0.2359	2.8628	0.0582	808	0.2359	2.8628	0.0582	808
7	0.2558	3.5375	0.0631	711	0.2558	3.5375	0.0631	711	0.2558	3.5375	0.0631	711	0.2558	3.5375	0.0631	711	0.2558	3.5375	0.0631	711
10	0.2732	4.3297	0.0674	621	0.2732	4.3297	0.0673	621	0.2732	4.3297	0.0674	621	0.2732	4.3297	0.0674	621	0.2732	4.3297	0.0674	621
MPE	–	–	–	–	0.000%	0.000%	-0.019%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.012%	0.000%	0.000%	0.000%	0.000%
MAPE	–	–	–	–	0.000%	0.000%	0.019%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.012%	0.000%	0.000%	0.000%	0.000%
CPU	–	–	–	–	392.0				863.7				1,014.3				1,493.3			

This table reports the time-t_0 value of the building blocks of CDS and EDS contracts, and the corresponding premium rates (in basis points per annum) under the time-homogeneous JDCEV model, using the same model parameters as in Mendoza-Arriaga and Linetsky (2011, Table 2), i.e. $S_{t_0} = 50$, $r = 5\%$, $q = 0\%$, $a = 20$, $b = 0.02$, $c = 1$, $\bar{\beta} = -1$, $R = 50\%$ and $\Delta = 0.25$, for the contract maturities specified in the first column. Panel A reports results for CDS contracts, while Panels B and C report results for EDS contracts, with $L = 30\%$ and $L = 50\%$, respectively. In the three panels, columns 2 to 5 reproduce the values reported in Mendoza-Arriaga and Linetsky (2011, Table 2). Columns 6 to 13 are obtained via the ST approach of Propositions 4.3 and 4.5 (or Remark 4.3, for Panel A), and using different numbers of time steps. Columns 14 to 21 describe the results generated by the MA-L valuation approach of Mendoza-Arriaga and Linetsky (2011), as described in Section 4.6, with different numbers of summation terms. The last three lines report mean percentage pricing errors (MPE), mean absolute percentage pricing errors (MAPE) and computation times (in seconds), for the whole set of maturities under analysis.

5 Conclusions

This thesis presents three essays on option pricing, providing important results with applications on interest rates, equity and credit derivatives.

The first paper offers two contributions to the literature on swaptions pricing. First, this paper derives a new analytical approximation for European-style swaptions under a multifactor Gauss-Markov framework, and based on the *conditioning approach* proposed by Curran (1994), Rogers and Shi (1995), and Nielsen and Sandmann (2002). Second, a comprehensive and rigorous Monte Carlo study is run to compare, in terms of efficiency and accuracy, all the approximations already proposed in the literature for European-style swaptions under multifactor term structure models.

The numerical results obtained show that the exact lower bound of the swaption price provided by the conditioning approach is the most accurate pricing method for ATMF, OTMF and ITMF contracts. Moreover, the conditioning approach proposed in this paper also offers tight bounds for the approximation error, because the analytical lower and upper bounds proposed are usually very close to each other (except for some deep OTMF contracts).

The second paper extends the ST approach originally proposed by Kuan and Webber (2003) and offers a novel approach for pricing European-style barrier options on asset prices driven by a geometric Brownian motion and under the stochastic interest rates setup specified by the Vasiček (1977) model.

Similarly to Kuan and Webber (2003), we are able to write the barrier option price in terms of the first passage time density of the underlying asset price through the barrier level. Again, this density is recovered as the implicit solution of a non-linear integral equation. However, and since we are dealing with a two-factor model, our pricing solution involves a double integral, in both time and interest rate dimensions.

Given the Gaussian specification adopted for the short-term interest rate, and following

Nunes (2011), we are able to obtain an explicit solution for the probability density of the short-term interest rate, conditional on the knock-in or knock-out event, and, therefore, we are left with a pricing solution that only involves an integration with respect to time. Moreover, and as shown by Nunes (2011), our one-dimensional pricing solutions can be easily extended from the single-factor Vasiček (1977) model to a multifactor Gaussian Heath, Jarrow, and Morton (1992) framework, without increasing the dimensionality of the pricing problem. In contrast, the extended Fortet method adopted by Bernard et al. (2008) is a two-dimensional pricing approach.

The accuracy and efficiency of the ST approach is compared against the extended Fortet method of Bernard et al. (2008) using several model parameter constellations and option maturities. The numerical results obtained show that the ST approach is the most accurate and efficient pricing method, considering both short-term and long-term contracts.

The third paper extends the ST approach of Kuan and Webber (2003) and offers a novel approach for valuing CDS and EDS contracts under the CEV model of Cox (1975) and the JDCEV model of Carr and Linetsky (2006). Under the CEV model, the triggering event may occur only by diffusion of the underlying price process; under the JDCEV framework, the triggering event may also occur via a jump-to-default of the price process.

We offer pricing solutions for each one of the building blocks of CDS and EDS contracts, which involve only one integration with respect to the density function of the first passage time of the underlying asset price to the contract triggering level. Furthermore, for the pricing of CDS contracts under the JDCEV model, we show that the ST approach nests the pricing solutions already offered by Carr and Linetsky (2006) when the contract triggering level is set to zero. Again, under both models, the hitting time density is recovered as the implicit solution of a non-linear equation.

The accuracy and efficiency of ST approach is compared against the Laplace transform valuation methodology proposed by Mendoza-Arriaga and Linetsky (2011). The numerical results show that the ST approach is the most efficient pricing method. Moreover, and in opposition to the Laplace transform methodology, the ST approach has been formulated under the most general time-inhomogeneous formulation of the JDCEV model.

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