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BARRIER OPTION PRICING VIA HESTON MODEL

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Resumo

O objectivo desta tese é analisar a avaliação de opções de barreira no modelo de [Heston, S.L. (1993)]. Seguindo a abordagem presente em [Griebsch, S.A., and Pilz, K.F. (2012)] é estabelecido um preço para as opções de compra *up-and-out* via modelo de [Heston, S.L. (1993)]. Vários tópicos introdutórios são incluídos para evidenciar as semelhanças no método e nas técnicas aplicadas. A implementação numérica é feita em Matlab. O preço da opção barriera é calculado através de três métodos diferentes no caso mais simples. Para o caso geral, o algoritmo foi desenvolvido e implementado mas não produziu resultados satisfatórios.

Palavras-Chave: Modelo de Heston, opções barreira, transformada rápida de Fourier

Abstract

The thesis objective is to analyse the valuation of barrier options in the [Heston, S.L. (1993)] model. Following [Griebsch, S.A., and Pilz, K.F. (2012)] approach up-and-out calls are priced via Heston model. Several introductory topics are included to evidence the similarities in the method and the techniques applied. The numerical implementation is done in Matlab. The price for the option is calculated through three different methods in a simplest case. For the general case, the algorithm was developed and implemented but it did not present satisfactory results.

Key words: Heston model, barrier options, fast Fourier transform (FFT)

Contents

1 Introduction

Today, in the middle of the European debt crisis, the theme of complex financial instruments became an everyday public discussion. The fast developed option pricing methods, started with [Black, F., and Scholes, M. (1973)] model, have grown into complicated structures and became an inseparable part of the market. In this paradigm, the efficient price calculation techniques are highly valued. It is important to understand the difficulties presented by these instruments on both theoretical and implementation levels. Here, we will center our attention on the price calculation of one particular exotic option, the barrier option.

[Heston, S.L. (1993)] offers a closed form solution for the pricing of European-style standard options with stochastic volatility. The resulting model became quickly popular amongst the practitioners. In this thesis, mostly based on [Griebsch, S.A., and Pilz, K.F. (2012)], we look at the pricing of European-style barrier options under the [Heston, S.L. (1993)] model.

The thesis has the following structure.

Chapter 1 introduces the general concepts of the stochastic calculus and the Black-Sholes market model in particular. We focus in more detail on the first hitting time distribution, since it plays an important role in our approach to the pricing of barrier options. In the context of the Black-Sholes model we review how the barrier options are treated and what difficulties they present.

Chapter 2 presents the [Heston, S.L. (1993)] model and, in particular, the famous [Heston, S.L. (1993)] formula for standard call options.

Chapter 3 is dedicated to our main problem: to find the valuation formula for up-and-out calls under the [Heston, S.L. (1993)] model. We start by conditioning the options pay-off on variance paths and calculating the resulting joint distribution of the maximum to date and the generator of the process. The problem is analysed in three cases, because the model parameters influence the difficulty of the task. In the first case, we assume that there is no correlation between the two generator processes involved and the risk free rate is equal to the dividend yield. In the second case, we only maintain the zero correlation assumption and in the third case we study the general situation with no restrictions on the parameters. Next, we proceed to unconditioning. We analyse in detail only the cases 1 and 3. We derive the exact formula for the up-and-out call in the case 1 and an approximated one for the general case 3.

Finally, Chapter 4, is dedicated to the numerical analysis and to implementation issues. We describe some of the techniques used and develop the algorithms for the cases 1 and 3. The numerical tool used was Matlab. We compare the results obtained by the algorithms presented in the Appendix of the thesis.

1.1 Barrier options

The barrier option (BO) is a path dependent option which becomes either active or extinguished when the underlying asset reaches the "barrier" level. To illustrate this options a little better, we will define an European-style up-and-out call.

The *European up-and-out call* is an option to buy a certain asset S, at a strike price K and at a time T, if the price of the asset never reaches the barrier B until time T. The final pay-off of this option is

$$V_T = \begin{cases} (S_T - K)^+ & if \quad \sup\limits_{0 \le t \le T} S_t < B \\ 0 & otherwise \end{cases} \tag{1.1}$$

The previous definition includes the following concepts:

- *European* - the option that can only be exercised at the specific time T. Other possibilities could be Bermudian (can be exercised on several pre-chosen dates), American (can be exercised at any time), Asian (average price for a pre-chosen period of time), etc.
- *Up* - referring to an upper barrier B. Other possibilities would be down (down barrier) or both barriers (double barrier).
- *Out* - the BO becomes extinct when touching a barrier. Other possibility would be "in", if option is activated only upon reaching the barrier.

- *Call* - the option to buy an asset S, at a strike price K. Other possibility would be a put option, i.e. the right to sell an asset S, at a strike price K.

1.2 General assumptions

For the purpose of this thesis, we will consider certain theoretical assumptions regarding the market-trading universe, which are considered to be standard. We try to look at them from a BO point of view:

- Every variable has a continuos path, in particular the asset price S. In practice, of course, this is not true. Even if we consider a basis point to be our approximation to continuity, during times of "big news", market jumps are far from basis point ticks. An important consequence of the continuity assumption is that the probability of any particular price S_p (or $S_p = B$) is zero; hence, we can only work with intervals.

- The market is unique and liquid enough for any transactions, so in our model we know without ambiguity when the barrier is triggered. In practice these matters have to be carefully defined in the contract, namely: underlying asset, underlying market or markets, private or public trades, volume of the trade necessary to be considered a barrier breach, etc.

1.3 Brownian motion and the Black-Scholes model

With the previous assumptions in mind, we can start to develop a model for the market and later calculate the price for BOs. As ground work to obtain the price for BOs via [Heston, S.L. (1993)] model, we analyse the derivation via the simpler [Black, F., and Scholes, M. (1973)] model first. This will allow us to define some necessary notions and correspondent useful results. For this part we have [Shreve, S.E. (2000)] as our main reference.

1.3.1 First passage time distribution

Definition 1 (Brownian Motion) *Consider a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. For each* $\omega \in \Omega$*, the continuous function* $W(t)$*,* $t \geq 0$*, that satisfies* $W(0) = 0$*, depends on* ω *and for all* $t_0 < t_1 < ... < t_n$ *the increments*

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$$

are independent and normally distributed with:

$$\begin{aligned} \mathbb{E}[W(t_{i+1}) - W(t_i)] &= 0 \\ Var[W(t_{i+1}) - W(t_i)] &= t_{i+1} - t_i \end{aligned}$$

is called Brownian Motion (BM). [Shreve, S.E. (2000)] Definition 3.3.1.

The BM is the center of the stochastic calculus, developed first in physics to deal with the randomness of particle movements. One of the main concepts is the "differentiation" of the function whose variable is a BM (or more generally a stochastic process) through the Ito's Lemma, which is stated in the Appendix as Theorem 1.

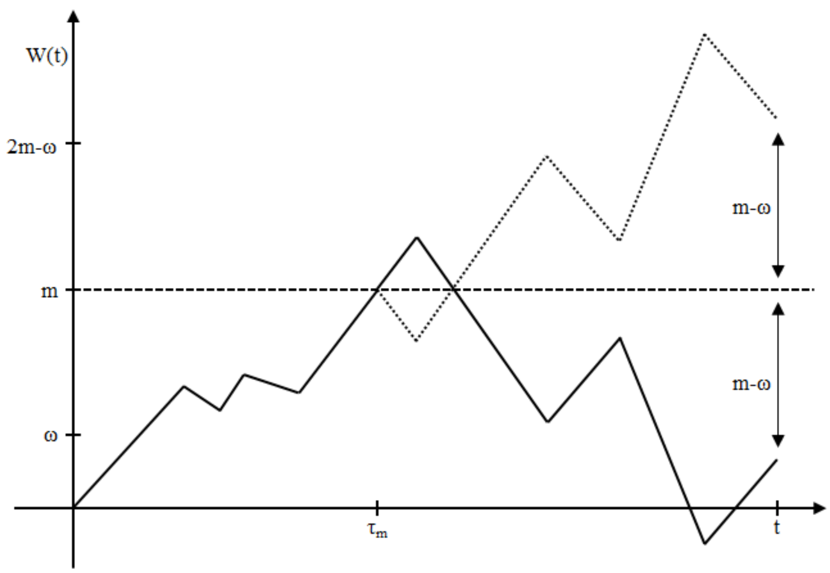

Figure 1.1: Illustration of Reflection Principle

The First passage time and maximum distributions

We start by deriving the distribution for the first passage time. Consider τ_m - the first passage time through level m for BM $W(t)$.

The Figure 1.1 illustrates the path of $W(t)$ and its reflected path

$$\widetilde{W}(t) = W_{\tau_m \wedge t} - (W_t - W_{\tau_m \wedge t}),$$

using the notation $a \wedge b = \inf(a, b)$.

The BM is symmetrical with respect to the time axis. We know that the path of $W(t)$ crosses level m at $\tau_m \leq t$. For every path that ends below level ω there exists another equally probable path that ends above

$2m-\omega = ((m-\omega)+m)$. This observation gives us the Reflection Principle, [Shreve, S.E. (2000)] Theorem 3.7.1.

$$\mathbb{P}(\tau_m \leq t, W(t) \leq \omega) = \mathbb{P}(W(t) \geq 2m-\omega). \tag{1.2}$$

We can use equation (1.2) to calculate the distribution of the first passage time . With choice $\omega = m$ and assuming $m > 0$, equation (1.2) becomes:

$$\mathbb{P}(\tau_m \leq t, W(t) \leq m) = \mathbb{P}(W(t) \geq m).$$

On the other hand, it is always true that:

$$\mathbb{P}(\tau_m \leq t, W(t) \geq m) = \mathbb{P}(W(t) \geq m).$$

Combining both equations:

$$\mathbb{P}(\tau_m \leq t) = 2\mathbb{P}(W(t) \geq m) = \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}} dx,$$

since, $W(t)$ has normal distribution with $\mathbb{E}[W(t)] = 0$ and $Var[W(t)] = t$.

Next, we introduce the concept of Maximum to Date*:*

$$M(t) = \sup_{0 \leq s \leq t} W(s). \tag{1.3}$$

We can rewrite the reflection principle (1.2) as:

$$\mathbb{P}(M(t) \geq m, W(t) \leq \omega) = \mathbb{P}(W(t) \geq 2m-\omega), \;\omega \leq m,\; m > 0. \tag{1.4}$$

From equation (1.4) we can obtain the joint density distribution of $(M(t), W(t))$, i.e. $f_{(M(t),W(t))}$ differentiating twice, first in order to m and then in order to ω:

$$\begin{aligned}
\int_m^\infty \int_{-\infty}^\omega f_{(M(t),W(t))}(x,y) dx dy &= \frac{1}{\sqrt{2\pi t}} \int_{2m-\omega}^\infty e^{-\frac{z^2}{2t}} dz \\
f_{(M(t),W(t))}(m,\omega) &= \frac{2(2m-\omega)}{t\sqrt{2\pi t}} e^{-\frac{(2m-\omega)^2}{2t}},
\end{aligned} \tag{1.5}$$

see [Shreve, S.E. (2000)] Theorem 3.7.3.

Consider the BM with a drift α constant

$$\hat{W}(t) = \alpha t + W(t),\ 0 \leq t \leq T,$$

where $W(t)$ is a BM that has zero drift (i.e. it is a martingale) under the original measure $\mathbb{P}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define maximum to date for $\hat{W}$ as

$$\hat{M}(t) = \sup_{0 \leq s \leq t} \hat{W}(s). \tag{1.6}$$

To calculate the joint density distribution of $(\hat{M}(T), \hat{W}(T))$ it is necessary to know how to change measures.

Theorem 2 (Girsanov, one-dimension) *Let $W(t),\ 0 \leq t \leq T$, be a BM on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{F}(t)$ be a filtration for this BM. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process. Define*

$$\begin{aligned} Z(t) &= \exp\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\}, \\ \widetilde{W}(t) &= W(t) + \int_0^t \Theta(u)du, \end{aligned}$$

and assume that

$$\mathbb{E}[\int_0^T \Theta^2(u)Z^2(u)du] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}} = \int_A Z(\omega)dP(\omega) \text{ for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t)$, $0 \leq t \leq T$, is a BM.

Proof. For a proof see, for instance, [Shreve, S.E. (2000)] Theorem 5.2.3 or [Øksendal, B. (2002)] Theorem 8.6.3 ■

With the help of Girsanov theorem and equation (1.5) we can calculate the joint density distribution of $(\hat{M}(T), \hat{W}(T))$ as

$$f_{(\hat{M}(T),\hat{W}(T))}(m,\omega) = \frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2}, \ \omega \le m, \ m \ge 0, \quad (1.7)$$

or zero for other values of m and ω, see [Shreve, S.E. (2000)] Theorem 7.2.1.

1.3.2 The Black and Scholes (1973) model

The Black-Sholes (BS) financial market model created a revolution in option pricing and, consequently, in option trading. It offered a clear pricing method for European-style standard options, making the option trading more transparent and understandable. In the BS model, the asset price $S(t)$ follows a geometric BM. The associated stochastic differential equation (SDE) under the physical measure $\mathbb{P}$ is

$$dS = \mu S dt + \sigma S dW^S, \quad (1.8)$$

where μ is the drift or expected rate return of the asset, σ is the standard deviation of the rate of return on the asset, and W^S is a standard BM. We can rewrite equation 1.8 using Girsanov theorem 2 with $\Theta = \frac{\mu-(r-q)}{\sigma}$, where r is the risk free rate and q is dividend yield we arrive at Black-Scholes-Merton equation [Merton, R (1973)]:

$$dS = (r-q)S dt + \sigma S d\widetilde{W}, \quad (1.9)$$

where $\widetilde{W}$ is a standard BM under the risk neutral measure $\mathbb{Q}$. The solution for the SDE (1.9) is

$$S = S_0 e^{\sigma \widetilde{W} + ((r-q) - \frac{1}{2}\sigma^2)t}. \tag{1.10}$$

Using Ito's Lemma, from the Theorem 1 in Appendix for an option $V(t,S)$ we derive:[1]

$$\begin{aligned} dV &= V_t dt + V_S dS + \frac{1}{2} V_{SS} \sigma^2 S^2 dt \\ &= (V_t + (r-q)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS})dt + V_S \sigma S d\widetilde{W}, \end{aligned} \tag{1.11}$$

Since $\widetilde{\mathbb{E}}[V_S \sigma S d\widetilde{W} | \mathcal{F}(t)] = 0$, then

$$\widetilde{\mathbb{E}}[dV|\mathcal{F}(t)] = (V_t + (r-q)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS})dt. \tag{1.12}$$

On the other hand, under (1.12) the risk neutral measure

$$\widetilde{\mathbb{E}}[dV|\mathcal{F}(t)] = rV dt. \tag{1.13}$$

Joining equations (1.12) and (1.13), we derive the Black-Scholes-Merton partial differential equation (PDE):

$$V_t + (r-q)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV. \tag{1.14}$$

1.3.3 Barrier options under Black-Scholes model

Here we follow [Shreve, S.E. (2000)], chapter 7.

[1] Here the subscript indicates the derivative $\frac{\partial V}{\partial t} := V_t$

Proposition 3 *The price of an up-and-out call at time t, satisfies the PDE equation (1.14) and the boundary conditions*

$$\begin{aligned} V(t,0) &= 0, \quad 0 \le t \le T, \\ V(t,B) &= 0, \quad 0 \le t \le T, \end{aligned}$$

and

$$V(T,S) = (S(t)-K)^+, \quad 0 \le S(t) < B,$$

assuming that the option was not knock-out before time t.

We will not proof Proposition 3 directly (i.e. solve the Cauchy problem) but we can verify it later. Instead, we will calculate the price of an up-and-out call by the direct application of the reflection principle.

The pay-off of the up-and-out call is given by equation (1.1). Since $S(t)$ is given by equation (1.10), defining $\hat{M}(T)$ as in (1.6) and

$$\hat{W}(T) = \widetilde{W}(T) + \frac{r-q-\frac{\sigma^2}{2}}{\sigma} T,$$

we have:

$$V(T) = \begin{cases} (S(T)-K)^+ = (S(0)e^{\sigma \hat{W}(T)} - K) & if \quad \hat{M}(T) < b \\ 0 & if \quad \hat{M}(T) \ge b \end{cases},$$

with $b = \frac{1}{\sigma}\log(\frac{B}{S(0)})$, or

$$\begin{aligned} V(T) &= (S(0)e^{\sigma \hat{W}(T)} - K)^+[\mathbf{1}_{\hat{M}_T < b}] \\ &= (S(0)e^{\sigma \hat{W}(T)} - K)[\mathbf{1}_{\hat{W}_T > k,\ \hat{M}_T < b}], \end{aligned}$$

with $k = \frac{1}{\sigma}\log(\frac{K}{S(0)})$. Hence,

$$V(0) = \int_k^b \int_{\omega^+}^b e^{-rT}(S(0)e^{\sigma w} - K) f_{(\hat{M}(T), \hat{W}(T))}(m, w) dm dw.$$

with $\omega^+ = \sup(\omega, 0)$. Using the density distribution of $(\hat{M}(T), \hat{W}(T))$ given by equation (1.7), with

$$\alpha = \frac{r - q - \frac{\sigma^2}{2}}{\sigma},$$

then

$$\begin{aligned} V(0) &= \int_k^b \int_{\omega^+}^b e^{-rT}(S(0)e^{\sigma\omega} - K)\frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2} dm d\omega \\ &= -\int_k^b (S(0)e^{\sigma\omega} - K)\frac{1}{\sqrt{2\pi T}} e^{-rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2}\Big|_{m=\omega^+}^{m=b} d\omega \\ &= S(0)I_1 - KI_2 + S(0)I_3 - KI_4, \end{aligned} \tag{1.15}$$

where I_j is the representation for the four integrals contained in equation 1.15, each of them is of the form

$$\frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta + \gamma\omega - \frac{1}{2\pi}\omega^2} dw = e^{\frac{1}{2}\gamma^2 T + \beta}[\mathcal{N}(\frac{b - \gamma T}{\sqrt{T}}) - \mathcal{N}(\frac{k - \gamma T}{\sqrt{T}})], \tag{1.16}$$

for some β and γ. Introducing

$$\delta_\pm(\tau, s) = \frac{1}{\sigma\sqrt{t}}[\log s + (r \pm \frac{1}{2}\sigma^2)\tau],$$

and using equation (1.16), I_j can be solved as

$$\begin{aligned}
I_1 &= \mathcal{N}(\delta_+(T, \frac{S(0)}{K})) - \mathcal{N}(\delta_+(T, \frac{S(0)}{B})) \\
I_2 &= e^{-rT}[\mathcal{N}(\delta_-(T, \frac{S(0)}{K})) - \mathcal{N}(\delta_-(T, \frac{S(0)}{B}))] \\
I_3 &= (\frac{S(0)}{B})^{-\frac{2r}{\sigma^2}-1}(\mathcal{N}(\delta_+(T, \frac{B^2}{KS(0)})) - \mathcal{N}(\delta_+(T, \frac{B}{S(0)})) \\
I_4 &= e^{-rT}(\frac{S(0)}{B})^{-\frac{2r}{\sigma^2}+1}(\mathcal{N}(\delta_-(T, \frac{B^2}{KS(0)})) - \mathcal{N}(\delta_-(T, \frac{B}{S(0)})).
\end{aligned}$$

We can check that under the assumptions $T > t$, $0 \leq S(0) = S \leq B$, our solution verifies the Black-Sholes-Merton PDE (1.14) with boundary conditions given by Proposition 3.

2 Heston (1993) model

2.1 Introduction

The stochastic volatility model, given by the equations

$$\begin{aligned}
dS_t &= (r-q)Sdt + \sqrt{v_t}SdW_t^S, \qquad (2.1) \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v, \qquad (2.2)
\end{aligned}$$

and

$$dW_t^S dW_t^v = \rho dt. \qquad (2.3)$$

is known as the Heston (1993) model. The typical introduction of any volatility model starts with the explanation of the limitations of the BS model.

One randomness source is simply not enough. Hence, to satisfy the customer (market), we need to introduce another element of randomness: the stochastic process for the variance given by equation (2.2)[2]. The goal is to make the new model "smile" on the volatility axis, in opposition to the rigid flat volatility from the BS model. The third equation establishes the correlation between the two BM's. The processes involved are: the spot price process S_t, the instantaneous volatility v_t of logarithmic spot price with initial variance v_0 fixed and the standard BM's W_t^S and W_t^v. Model parameters are: the dividend yield q, the risk free rate r, the mean reversion speed of variance κ, the mean reversion level for the variance θ, the volatility of the variance σ and the correlation ρ between two BM's. To ensure the strictly positive volatility we have to enforce the Feller condition on the parameters

$$2\kappa\theta > \sigma^2.$$

See, for instance, [Karatzas, I., and Shreve, S. (1991)].

Since the model has two BM's (random sources) and only one tradable asset S_t, our market model is incomplete. To complete it, we should have introduced another tradable asset, like an European call, for example. See [Hull, J. C., and White A. (1987)].

[2] The stochastic equation is known as a CIR (Cox, Ingersoll and Ross) process, see [Cox, J.C., J.E. Ingersoll and S.A. Ross (1985)].

2.2 Standard European call option price

The call price at time t under risk neutral measure $\mathbb{Q}$ is given by

$$c_t(S,K,T) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+].$$

where $\tau = T - t$ is the time to expiration and K is the strike price, i.e.:

$$c_t(S,K,T) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{S_T>K}] - e^{-r\tau}K\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{S_T>K}]. \tag{2.4}$$

Introducing the logarithmic spot price

$$x_t = \log S_t, \tag{2.5}$$

the probability of the call expiring in-the-money under risk neutral measure $\mathbb{Q}$ is the second term in equation (2.4):

$$\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{S_T>K}] = \mathbb{Q}(S_T > K) = \mathbb{Q}(\log S_T > \log K) = P_2(x,v,\tau). \tag{2.6}$$

The first term in equation (2.4) needs a measure change due to S_T term. We introduce the Radon-Nykodim derivative (see [Shreve, S.E. (2000)]) by

$$\mathbb{Z}_t = \frac{S_t e^{qt}}{S_T e^{qT}} e^{r\tau},$$

and define the probability measure

$$\mathbb{Q}_s = \int_A e^{r\tau}\mathbb{Z}_t d\mathbb{Q} \quad for\ all\ A \in \mathcal{F}(t).$$

The first term on the right-hand side of equation (2.4) becomes

$$\begin{aligned} e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{S_T>K}] &= \mathbb{E}^{\mathbb{Q}}[e^{-r\tau}S_T\mathbf{1}_{S_T>K}] \\ &= \mathbb{E}^{\mathbb{Q}_s}[e^{-r\tau}S_T\mathbf{1}_{S_T>K}\mathbb{Z}_t] \\ &= S_t e^{-q\tau}\mathbb{E}^{\mathbb{Q}_s}[\mathbf{1}_{S_T>K}] \\ &= e^{x_t - q\tau}\mathbb{Q}_s(S_T > K) = e^{x_t - q\tau}P_1(x,v,\tau). \end{aligned} \tag{2.7}$$

Considering equations (2.4), (2.6) and (2.7), then

$$c_t(S,K,T) = e^{x_t - q\tau} P_1(x,v,\tau) - e^{-r\tau} K P_2(x,v,\tau) \tag{2.8}$$

We can express the solution (2.8) through the characteristic function of x_t, method used in [Heston, S.L. (1993)]. Later on, we will use the same technique in our barrier option calculations and show it in more detail, so here, we will announce the idea and present the solution from [Heston, S.L. (1993)]. The both probabilities P_j have to satisfy the Heston PDE

$$V_t + \frac{1}{2} v S^2 V_{SS} + \rho \sigma_v S V_{vS} + \frac{1}{2} \sigma^2 v V_{vv} - rV + (r-q) S V_S + \kappa(\theta - v) V_v = 0.$$

We use the Feynman-Kac theorem described in the Appendix 5.2 to calculate the characteristic functions

$$f_j(x,v,T;u) = \mathbb{E}^{\mathbb{Q}_j}[\exp(iux_T)|\mathcal{F}_t],$$

with $\mathbb{Q}_1 = \mathbb{Q}_s$ and $\mathbb{Q}_2 = \mathbb{Q}$, using the guess

$$f_j(x,v,T;u) = \exp(C_j(\tau,u) + D_j(\tau,u)v + iux)$$

and obtaining the probabilities P_j through the Inversion Theorem 4 by

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}[\frac{e^{-iu \log K} f_j(x,v,u)}{iu}] du.$$

The solution for $j = 1, 2$ is

$$\begin{aligned} C_j &= (r-q) iu\tau + \frac{\kappa\theta}{\sigma^2}[(b_j - \rho\sigma iu + d_j)\tau - 2\log(\frac{1 - g_j e^{d_j \tau}}{1 - g_j})] \\ D_j &= \frac{b_j - \rho\sigma iu + d_j}{\sigma^2} (\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j \tau}}), \end{aligned}$$

with

$$
\begin{aligned}
g_j &= \frac{b_j - \rho\sigma iu + d_j}{b_j - \rho\sigma iu - d_j} \\
d_j &= \sqrt{(\rho\sigma iu - b_j)^2 - \sigma^2(2l_j iu - u^2)} \\
b_1 &= \kappa - \rho\sigma \\
b_2 &= \kappa \\
l_j &= (-1)^{1+j}\frac{1}{2}
\end{aligned}
$$

3 Barrier options via Heston (1993) model

Our objective is to compute the price of an up-and-out call assuming that the market is described by the Heston (1993) model. As in the case of the standard European call, a closed form solution is not known. Hence, we will use the same technique: to represent the solution in an integral form through the characteristic function. We will follow [Griebsch, S.A., and Pilz, K.F. (2012)], starting by the conditioning of the option expected payoff on variance paths, eliminating this way one source of randomness. Afterwords, we can calculate the arising conditioned expectation in closed form. At the end, we resolve the conditioning.

3.1 Conditioning on variance paths

The payoff of an up-and-out option is given by (1.1), and its discounted payoff is

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_T|\mathcal{F}_0]$$

where, $\mathcal{F}_t$ is the filtration associated with BM's W_s and W_v, or simply, $\mathcal{F}_0$ are initial conditions. We want to condition on a σ-algebra generated by variance paths up to time T, i.e.

$$\mathcal{G}_T^v = \sigma(\{v_s : 0 \le s \le T\}). \tag{3.1}$$

The conditioning on the variance paths will eliminate one source of randomness from the model. The tower property of conditional expectations and the payoff formula gives us[3]

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}[\mathbb{E}[V_T|\mathcal{G}_T^v]|\mathcal{F}_0] \\ &= e^{-rT}\mathbb{E}[\mathbb{E}[(S_T-K)^+\mathbb{I}_{\{\sup\limits_{0\le t\le T} S_t<B\}}|\mathcal{G}_T^v]|\mathcal{F}_0]. \end{aligned}$$

Introducing the notation

$$\mathcal{E}^v{=}\mathbb{E}[(S_T-K)^+\mathbb{I}_{\{\sup\limits_{0\le t\le T} S_t<B\}}|\mathcal{G}_T^v], \tag{3.2}$$

we rewrite

$$V_0 = e^{-rT}\mathbb{E}[\mathcal{E}^v|\mathcal{F}_0].$$

[3]Further the measure would be $\mathbb{Q}$ unless specified

Applying Ito's lemma to x_t, logarithmic spot price, given by (2.5) we have:

$$\begin{aligned} dx_t &= \frac{\partial \log S}{\partial t}dt + \frac{\partial \log S}{\partial S}dS - \frac{1}{2}\frac{\partial^2 \log S}{\partial S^2}dSdS \\ &= 0dt + \frac{1}{S}dS - \frac{1}{2}\frac{1}{S^2}dSdS \\ &= (r-q)dt + \sqrt{v}dW^S - \frac{1}{2}vdt, \end{aligned}$$

or in integral form

$$x_t = x_0 + (r-q)t + \int_0^t \sqrt{v}dW_s^S - \frac{1}{2}\int_0^t v_s ds. \tag{3.3}$$

We proceed by using the information from equations (2.2) and (2.3). Since W^S and W^v are correlated, we want to isolate W^v for more effective conditioning. Please consider the Cholesky decomposition $A = L \cdot L^*$ where L is a lower triangular matrix and L^* is its conjugate transpose:

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \\ L \cdot L^* &= \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \end{aligned}$$

Hence, there is an independent BM W that satisfies

$$dW^S = \rho dW^v + \sqrt{1-\rho^2}dW.$$

Then, equation (3.3) becomes

$$x_t = x_0 + (r-q)t - \frac{1}{2}\int_0^t v_s ds + \rho\int_0^t \sqrt{v_s}dW_s^v + \rho_2\int_0^t \sqrt{v_s}dW_s, \tag{3.4}$$

where $\rho_2 = \sqrt{1-\rho^2}$. The second equation of the model in the integral form is

$$v_t = v_0 + \kappa\theta t - \kappa \int_0^t v_s ds + \sigma \int_0^t \sqrt{v_s} dW_s^v.$$

Isolating the $\int_0^t \sqrt{v} dW_s^v$ term and inserting it in the equation (3.4), we obtain:

$$x_t = x_0 + (r-q)t - \frac{1}{2}\int_0^t v_s ds + \frac{\rho}{\sigma}(v_t - v_0 - \kappa\theta t + \kappa \int_0^t v_s ds) + \rho_2 \int_0^t \sqrt{v_s} dW_s. \quad (3.5)$$

Using the abbreviation

$$\alpha(t) = (r-q)t - \frac{1}{2}\int_0^t v_s ds + \frac{\rho}{\sigma}(v_t - v_0 - \kappa\theta t + \kappa \int_0^t v_s ds),$$

equation (3.5) becomes

$$x_t = x_0 + \alpha(t) + \rho_2 \int_0^t \sqrt{v_s} dW_s.$$

Logarithmic spot price is the sum of: its initial value, a time-dependent drift $\alpha(t)$ and an Itô integral $\rho_2 \int_0^t \sqrt{v_s} dW_s$. Now the conditioning on the variance paths ($\mathcal{G}_t^v$ defined in (3.1)) looks appealing and a new variable[4]

$$x_t^v = x_t | \mathcal{G}_t^v,$$

has a single random contribution, which arises from

$$Y_t = \rho_2 \int_0^t \sqrt{v_s} dW_s,$$

[4]Further, we will use the superscript v to identify variables subjected to conditioning on $\mathcal{G}_t^v$, or omit it when it creates no confusion

The variable v_t cannot be negative. The variable Y_t has normal distribution, because in $\mathcal{G}_t^v$, v_t is non-random, i.e.

$$Y_t \sim \mathcal{N}(0, \rho_2^2 \nu^2(t)),$$

where

$$\nu^2(t) = \int_0^t v_s ds.$$

The variable x_t^v is also normally distributed

$$x_t^v \sim \mathcal{N}(\mu^v(t), \rho_2^2 \nu^2(t)),$$

with mean

$$\mu^v(t) = x_0 + \alpha^v(t).$$

Note that $\alpha^v(t)$ is a deterministic continuos function, but nowhere differentiable (due to v_s term), unless $\rho = 0$.

Next objective is to rewrite the equation (3.2) for $\mathcal{E}^v$, through the new variables

$$\begin{aligned} \hat{Y}_t &= \alpha(t) + Y_t \\ \hat{M}_t &= \sup\{\hat{Y}_s : 0 \leq s \leq t\}, \end{aligned}$$

and using the observations

$$x_t = \log S_t = x_0 + \hat{Y}_t,$$

$$S_t = e^{x_0} e^{\hat{Y}_t} = S_0 e^{\hat{Y}_t},$$

$$\sup S_t < B \Longrightarrow \sup S_0 e^{\hat{Y}_t} < B \Longrightarrow \hat{M}_t < b = \log(\frac{B}{S_0}),$$

$$S_T - K > 0 \Longrightarrow \hat{Y}_T > k = \log(\frac{K}{S_0}),$$

we can rewrite (3.2) as

$$\begin{aligned}
\mathcal{E}^v &= \mathbb{E}[(S_0 e^{\hat{Y}_t} - K)\mathbb{I}_{\{\hat{M}_t < b, \hat{Y}_t > k\}} | \mathcal{G}_T^v] \\
&= \mathbb{E}^v[(S_0 e^{\hat{Y}_t} - K)\mathbb{I}_{\{\hat{M}_t < b, \hat{Y}_t > k\}}]. \qquad (3.6)
\end{aligned}$$

To calculate the expectation $\mathcal{E}^v$ we need the joint distribution of $(\hat{Y}, \hat{M})$ under the probability measure $\mathbb{Q}^v$.

3.1.1 Derivation of the joint density distribution for $(M(T), Y(T))$

Proposition 1 (Reflection Principle) *Let $\{Y_t\}_{t\geq 0}$ be an Itô process of the form*

$$Y_t = \int_0^t \beta(s) dW_s \qquad (3.7)$$

with deterministic β and $M_t = \sup_{s\leq t} Y_s$ for $t \geq 0$. Then the reflection principle holds,

$$\mathbb{Q}(M_t \geq x, Y_t < y) = \mathbb{Q}(Y_t > 2x - y) \qquad for\ all\ t \geq 0,\ x \geq y \vee 0.$$

Therefore,

$$\mathbb{Q}^v(M_t \geq x, Y_t \leq y) = \mathbb{Q}^v(Y_t \geq 2x - y). \qquad (3.8)$$

Proof. For a proof see [Griebsch, S.A., and Pilz, K.F. (2012)], theorem 1.
■

We can use the previous proposition to calculate the density distribution of $(M(T), Y(T))$. Consider Y_t defined by (3.7). Then,

Proposition 2 *For $m \neq 0$ the random variable $\tau_m = \inf\{t \geq 0 : Y_t \geq m\}$ has the following distribution*

$$\mathbb{Q}^v(M_t \geq m) = \mathbb{Q}^v(\tau_m \leq t) = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{\rho_2^2 \nu^2(t)}}}^{+\infty} e^{-\frac{1}{2}y^2} dy.$$

Proof. Following [Griebsch, S.A., and Pilz, K.F. (2012)], proposition 1. The proof is a straightforward application of the reflection principle. Setting $m > 0$, $x = y = m$ and using equation (3.8), then

$$\mathbb{Q}^v(M_t \geq m, Y_t \leq m) = \mathbb{Q}^v(\tau_m \leq t, Y_t \leq m) = \mathbb{Q}^v(Y_t \geq m)$$

Since $Y_0 = 0$, if $Y_t \geq m$, then $\tau_m \leq t$, i.e.

$$\mathbb{Q}^v(\tau_m \leq t, Y_t \geq m) = \mathbb{Q}^v(Y_t \geq m)$$

Adding the previous two quantities, we get the cumulative distribution of τ_m:

$$\begin{aligned}
\mathbb{Q}^v(\tau_m &\leq t) = \mathbb{Q}^v(\tau_m \leq t, Y_t \geq m) + \mathbb{Q}^v(\tau_m \leq t, Y_t \leq m) \\
&= 2\mathbb{Q}^v(Y_t \geq m) \\
&= \frac{2}{\sqrt{2\pi}\rho_2\nu(t)} \int_m^{\infty} e^{-\frac{1}{2}\frac{x^2}{\rho_2^2\nu^2(t)}} dx
\end{aligned}$$

If $m < 0$, then $\tau_m \overset{d}{=} \tau_{|m|}$. ■

Proposition 3 *For $t > 0$ the joint distribution of (M_t, Y_t) is given by:*

$$f_{M,Y}(m, w) = \frac{2(2m - w)}{\sqrt{2\pi}\rho_2^3\nu^3(t)} \exp\left(-\frac{1}{2}\frac{(2m - w)^2}{\rho_2^2\nu^2(t)}\right), \quad \textit{for } w \leq m, m > 0. \quad (3.9)$$

Proof. Following [Griebsch, S.A., and Pilz, K.F. (2012)], proposition 2. Since,

$$\mathbb{Q}^v(M_t \geq m, Y_t \leq w) = \mathbb{Q}^v(Y_t \geq 2m - w),$$

and

$$\mathbb{Q}^v(M_t \geq m, Y_t \leq w) = \int_m^{\infty} \int_{-\infty}^{w} f_{M,Y}(u,s) du ds,$$

by the reflection principle and by definition, and because

$$\mathbb{Q}^v(Y_t \geq 2m - w) = \frac{1}{\sqrt{2\pi}\rho_2 \nu(t)} \int_{2m-w}^{\infty} e^{-\frac{1}{2}\frac{y^2}{\rho_2^2 \nu^2(t)}} dy,$$

then

$$\int_m^{\infty} \int_{-\infty}^{w} f_{M,Y}(u,s) du ds = \frac{1}{\sqrt{2\pi}\rho_2 \nu(t)} \int_{2m-w}^{\infty} e^{-\frac{1}{2}\frac{y^2}{\rho_2^2 \nu^2(t)}} dy.$$

Differentiating twice, first with respect to m and then with respect to w leads to equation (3.9). ■

3.1.2 Case 1 - interest rate equal to the dividend yield and no correlation ($r = q$ and $\rho = 0$)

For case 1, where $r = q$ and $\rho = 0$, we define:

$$\begin{aligned} Y_t &= \rho_2 \int_0^t \sqrt{v_s} dW_s = \int_0^t \sqrt{v_s} dW_s, && (3.10) \\ \alpha(t) &= -\frac{1}{2}\int_0^t v_s ds, && (3.11) \\ \hat{Y}_t &= \alpha(t) + Y_t = -\frac{1}{2}\int_0^t v_s ds + Y_t, && (3.12) \\ \hat{M}_t &= \sup\{\hat{Y}_s : 0 \leq s \leq t\}. && (3.13) \end{aligned}$$

We want to calculate $f_{\hat{M},\hat{Y}}(m, w)$ under the $\mathbb{Q}^v$ measure.

Proposition 4 *The joint density of $(\hat{M}_T, \hat{Y}_T)$ under $\mathbb{Q}^v$ is given by*

$$f_{\hat{M},\hat{Y}}(m, w) = \exp(-\frac{1}{2}w - \frac{1}{8}\nu^2(T))\frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)}\exp(-\frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}), \quad (3.14)$$

$$for\ w \leq m, m > 0.$$

Proof. We will proceed as following: 1) define measure for which $(\hat{M}_T, \hat{Y}_T)$ has no drift, 2) use Girsanov theorem and Proposition 3, 3) change back to the measure $\mathbb{Q}^v$. Since $\hat{Y}_0 = 0$ and $\hat{M}_t \geq \hat{Y}_t$, $(\hat{M}, \hat{Y})$ take values on a set $\{(m,w) : m \geq 0, w \leq m\}$. Considering equations (3.10)-(3.12), then

$$\begin{aligned}
dY_t &= \sqrt{v_t} dW_t, \\
d\hat{Y}_t &= d\alpha(t) + \sqrt{v_t} dW_t = \sqrt{v_t} d\hat{W}_t, \\
d\hat{W}_t &= dW_t + \frac{d\alpha(t)}{\sqrt{v_t}} = dW_t + \gamma(t)dt \quad with \quad \gamma(t) = -\frac{1}{2}\sqrt{v_t}, \quad (3.15) \\
\hat{W}_t &= \int_0^t \gamma(s)ds + W_t.
\end{aligned}$$

Hence, $\hat{W}_t$ is a BM under $\mathbb{Q}^v$ with drift $\gamma(t)$. Next, we introduce an exponential martingale:

$$\begin{aligned}
\hat{H}_t &= \exp(-\int_0^t \gamma(s)dW_s - \frac{1}{2}\int_0^t \gamma^2(s)ds) \qquad (3.16) \\
&= \exp(-\int_0^t \gamma(s)d\hat{W}_s + \frac{1}{2}\int_0^t \gamma^2(s)ds), \qquad (3.17)
\end{aligned}$$

where the last equality is due to equation (3.15). We define a new measure

$$\hat{\mathbb{Q}}(A) = \int_A \hat{H}_T d\mathbb{Q}^v \qquad for\ all\ A \in \mathcal{G}_T^v \qquad (3.18)$$

conditioned on a σ-algebra generated by variance paths. $\hat{\mathbb{Q}}(A)$ is well defined, satisfying Novikov's condition (appendix Theorem 2), in our case:

$$\mathbb{E}^{\mathbb{Q}}[\exp\{\frac{1}{2}\int_0^T (-\frac{1}{2}v_s)^2 ds\} \mid \mathcal{G}_T^v] < \infty,$$

since $+\infty$ is a natural boundary, i.e. cannot be reached. Now we are able to use Girsanov's theorem 2 to conclude that $\hat{W}$ is a BM with zero drift under

$\hat{\mathbb{Q}}$ and through proposition 3 we know its density distribution. To finish the proof we have to calculate it in the $\mathbb{Q}^v$ measure.

$$\begin{aligned}
\mathbb{Q}^v(\hat{M}_T &\leq m, \widehat{Y}_T \leq w) = \mathbb{E}^v[\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}] \\
&= \widehat{\mathbb{E}}\left[\frac{1}{\hat{H}}\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \qquad (3.19) \\
&= \widehat{\mathbb{E}}\left[\exp(\int_0^T \gamma(s) d\hat{W}_s - \frac{1}{2}\int_0^T \gamma^2(s) ds)\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \widehat{\mathbb{E}}\left[\exp(-\frac{1}{2}\int_0^T \sqrt{v_s} d\hat{W}_s - \frac{1}{8}\int_0^T v_s ds)\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \int_{-\infty}^{w}\int_{-\infty}^{m} \exp(-\frac{1}{2}y - \frac{1}{8}\nu^2(T))\hat{f}_{\hat{M},\hat{Y}}(x,y) dx dy.
\end{aligned}$$

To obtain the density distribution, for set $\{(m,w) : m \geq 0, w \leq m\}$ and zero for other values, we just have to compute:

$$\begin{aligned}
\frac{\partial^2}{\partial m \partial w}\mathbb{Q}^v(\hat{M}_T &\leq m, \widehat{Y}_t \leq w) = \\
&= \exp(-\frac{1}{2}w - \frac{1}{8}\nu^2(T))\frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)}\exp(-\frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}).
\end{aligned}$$

■

3.1.3 Case 2 - no correlation ($\rho = 0$) and arbitrary rates r and q

The analysis of this case will be similar to the previous one, (re)introducing

$$Y_t = \rho_2 \int_0^t \sqrt{v_s} dW_s = \int_0^t \sqrt{v_s} dW_s, \qquad (3.20)$$

$$\alpha(t) = (r-q)t - \frac{1}{2}\int_0^t v_s ds \qquad (3.21)$$

and also

$$\begin{aligned}\hat{Y}_t &= Y_t + \alpha(t),\\ d\hat{Y}_t &= \sqrt{v_t}d\hat{W}_t,\\ d\hat{W}_t &= dW_t + \gamma(t)dt,\end{aligned}$$

with a new drift $\gamma(t)$ defined by

$$\gamma(s) = \frac{\frac{\partial}{\partial s}\alpha(s)}{\sqrt{v_s}} = \frac{r-q}{\sqrt{v_s}} - \frac{1}{2}\sqrt{v_s}. \tag{3.22}$$

The new $\alpha(t)$ from equation (3.21) is still differentiable, but the newly defined drift $\gamma(t)$ (3.22) is not, so we can not use the same technique as in the previous case. We still rely on the change of measure (3.18), but with a different $\gamma(t)$. We have

$$\begin{aligned}\mathbb{Q}^v(\hat{M}_T &\leq m, \hat{Y}_T \leq w) = \widehat{\mathbb{E}}[\frac{1}{\hat{H}}\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}]\\ &= \widehat{\mathbb{E}}[\exp(\int_0^T \gamma(s)d\hat{W}_s - \frac{1}{2}\int_0^T \gamma^2(s)ds)\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}]\\ &= \widehat{\mathbb{E}}[\exp((r-q)\int_0^T \frac{1}{\sqrt{v_s}}d\hat{W}_s - \frac{1}{2}\int_0^T \sqrt{v_s}d\hat{W}_t - \qquad (3.23)\\ &\quad -\frac{1}{2}\int_0^T \gamma^2(s)ds)\mathbb{I}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}].\end{aligned}$$

Introducing the random variable $\hat{X}_T$ and the function $b_I(t)$:

$$\begin{aligned}\hat{X}_T &= \int_0^T \frac{1}{\sqrt{v_s}}d\hat{W}_s\\ b_I(t) &= \frac{1}{2}\int_0^t \gamma^2(s)ds = \frac{1}{2}\int_0^t (\frac{(r-q)^2}{v_s} - (r-q) + \frac{1}{4}v_s)ds\end{aligned}$$

equation (3.23) becomes

$$\mathbb{Q}^v(M_T \le m, Y_T \le w) = \widehat{\mathbb{E}}[\exp((r-q)\hat{X}_T - \frac{1}{2}\hat{Y}_T - b_I(T))\mathbb{I}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}]. \tag{3.24}$$

Proceeding similarly to the previous case, we need to calculate the joint distribution $\hat{f}_{\hat{X},\hat{Y},\hat{M}}$. Instead of looking for it, we will linearly approximate $\hat{X}$ by $\hat{Y}$ and rewrite (3.24) as

$$\begin{aligned} &\mathbb{Q}^v(M_T \le m, Y_T \le w) = \\ &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_0^{\infty} \exp((r-q)x - \frac{1}{2}y - b_I(T))\mathbb{I}_{\{z\le m, y \le w\}}\hat{f}_{\hat{X},\hat{Y},\hat{M}}(x,y,z)dzdydx \\ &= \int\limits_{\mathbb{R}}\int\limits_0^{\infty} \exp(-\frac{1}{2}y - b_I(T))\mathbb{I}_{\{z\le m, y \le w\}}\hat{f}_{\hat{Y},\hat{M}}(y,z)\times \\ &\times\widehat{\mathbb{E}}[e^{(r-q)\hat{X}_T}|\hat{Y}_T = y, \hat{M}_T = z]dzdy. \end{aligned} \tag{3.25}$$

With this approach we can use the density distribution from Proposition 3 for $(\hat{M}_T, \hat{Y}_T)$.

Proposition 5 *The random variable $(\hat{X}_T, \hat{Y}_T)$ is normally distributed with* 0 *mean and covariance matrix given by*

$$\Sigma = \begin{pmatrix} \nu_{inv}^2(T) & T \\ T & \nu^2(T) \end{pmatrix}, \tag{3.26}$$

where

$$\begin{aligned} \nu_{inv}^2(T) &= \int\limits_0^T \frac{1}{v_s}ds, \\ \nu^2(T) &= \int\limits_0^T v_s ds. \end{aligned}$$

Proof. We are going to calculate the characteristic function of $(\hat{X}_T, \hat{Y}_T)$ and compare it with the bivariate normal distribution:

$$\widehat{\mathbb{E}}[\exp(u_1\hat{X}_T + u_2\hat{Y}_T)] = \widehat{\mathbb{E}}[\exp(\int_0^T (u_1\frac{1}{\sqrt{v_s}} + u_2\sqrt{v_s})d\hat{W}_s)].$$

We define

$$Z(t) = \exp(-\int_0^t (u_1\frac{1}{\sqrt{v_s}} + u_2\sqrt{v_s})d\hat{W}_s) - \frac{1}{2}\int_0^t (u_1\frac{1}{\sqrt{v_s}} + u_2\sqrt{v_s})^2 ds.$$

By Girsanov multi-dimensional theorem (see [Shreve, S.E. (2000)]), $\mathbb{E}^v[Z(T)] = 1$. Under the probability measure defined by equation (3.18), the characteristic function of $(\hat{X}_T, \hat{Y}_T)$ is

$$\begin{aligned} & \widehat{\mathbb{E}}[\exp(\int_0^T (u_1\frac{1}{\sqrt{v_s}} + u_2\sqrt{v_s})d\hat{W}_s)] \\ = \ & \widehat{\mathbb{E}}[\exp(\frac{1}{2}\int_0^T (u_1\frac{1}{\sqrt{v_s}} + u_2\sqrt{v_s})^2 ds)] \\ = \ & \exp(-\frac{1}{2}(u_1^2\int_0^T \frac{1}{v_s}ds + 2u_1u_2T + u_2^2\int_0^T v_s ds), \end{aligned}$$

which is equal to the characteristic function of the normal distribution with the covariate matrix given by equation (3.26), for further reference see [Øksendal, B. (2002)]. ■

In order to linearly approximate $\hat{X}_T$ by $\hat{Y}_T$, we take

$$\hat{X}_T = k_0 + k_1\hat{Y}_T + \varepsilon,$$

for some normal distribution ε. We have to find the constants k_0 and k_1 such that $\widehat{\mathbb{E}}[\varepsilon] = 0$ and $\widehat{\mathbb{E}}[\varepsilon^2]$ is minimal. Hence,

$$k_0 = \widehat{\mathbb{E}}[\hat{X}_T] - k_1\widehat{\mathbb{E}}[\hat{Y}_T] - \widehat{\mathbb{E}}[\varepsilon] = 0,$$

and

$$\begin{aligned}\widehat{\mathbb{E}}[\varepsilon^2] &= \widehat{\mathbb{E}}[(\hat{X}_T - k_1\hat{Y}_T)^2] = \widehat{\mathbb{E}}[\hat{X}_T^2] + k_1^2\widehat{\mathbb{E}}[\hat{Y}_T^2] - 2Cov(\hat{X}_T, k_1\hat{Y}_T) \\ &= \nu_{inv}^2(T) + k_1^2\nu^2(T) - 2Tk_1 \end{aligned} \tag{3.27}$$

by the Proposition 5. The expected value $\widehat{\mathbb{E}}[\varepsilon^2]$ is minimal for

$$\begin{aligned} 2k_1\nu^2(T) - 2T &= 0 \\ k_1 &= \frac{T}{\nu^2(T)}. \end{aligned} \tag{3.28}$$

Since, the variable $(\hat{X}_T, \hat{Y}_T)$ is normally distributed, any linear combination of $\hat{X}_T$ and $\hat{Y}_T$ and, in particular,

$$\varepsilon = \hat{X}_T - k_1\hat{Y}_T$$

is still normally distributed. Moreover, the variable $(\hat{Y}_T, \varepsilon)$ is normally distributed, since all the linear combination of $(\hat{Y}_T, \varepsilon)$ are the same as of $(\hat{X}_T,\hat{Y}_T)$ [Gut, A. (2009)]. The variance of ε is given by (3.27):

$$\widehat{\mathbb{E}}[\varepsilon^2] = \nu_{inv}^2(T) - \frac{T}{\nu^2(T)}. \tag{3.29}$$

also, ε is uncorrelated to $\hat{Y}_T$:

$$\begin{aligned} Cov(\hat{Y}_T, \varepsilon) &= Cov(\hat{Y}_T, \hat{X}_T - k_1\hat{Y}_T) = \\ &= \widehat{\mathbb{E}}[\hat{Y}_T(\hat{X}_T - k_1\hat{Y}_T)] - \widehat{\mathbb{E}}[\hat{Y}_T]\widehat{\mathbb{E}}[\hat{X}_T - k_1\hat{Y}_T] \\ &= \widehat{\mathbb{E}}[\hat{Y}_T\hat{X}_T] - k_1\widehat{\mathbb{E}}[\hat{Y}_T^2] - 0 \\ &= T - T = 0, \end{aligned}$$

and hence, independent.

We introduce a standard normal variable U, such that

$$\begin{aligned} \varepsilon &= k_2 U \\ k_2 &= \sqrt{\nu_{inv}^2(T) - \frac{T}{\nu^2(T)}}, \end{aligned}$$

and U and $\hat{Y}_T$ are independent. Now, the decomposition of $\hat{X}_T$ is complete:

$$\hat{X}_T = k_1 \hat{Y}_T + k_2 U.$$

The independence between U and $\hat{Y}_T$ was verified, but to completely solve the inner expected value from equation (3.25), we need the independence of U from $\hat{M}$. We will ignore this dependence, since U is the "rest" of the decomposition of $\hat{X}_T$ after the $\hat{Y}_T$ contribution. The variable $\hat{M}_T$ is the supremum of $\hat{Y}_T$ and should not contribute much. We will proceed as if U was independent from $\hat{M}_T$ and consider it as an approximation.

The expected value from equation (3.25) becomes

$$\begin{aligned} \widehat{\mathbb{E}}[e^{(r-q)\hat{X}_T}|\hat{Y}_T &= y, \hat{M}_T = z] = \widehat{\mathbb{E}}[e^{(r-q)(k_1\hat{Y}_T + k_2 U)}|\hat{Y}_T = y, \hat{M}_T = z] \\ &= e^{(r-q)k_1 y} \times \widehat{\mathbb{E}}[e^{(r-q)k_2 U}|\hat{Y}_T = y, \hat{M}_T = z] \\ &\approx e^{(r-q)k_1 y} \times \widehat{\mathbb{E}}[e^{(r-q)k_2 U}]. \end{aligned}$$

The distribution of $e^{(r-q)k_2 U}$, is log-normal and we have:

$$\widehat{\mathbb{E}}[e^{(r-q)k_2 U}] = e^{\frac{1}{2}(r-q)^2 k_2^2}.$$

We can now rewrite equation (3.25) as

$$\begin{aligned} \mathbb{Q}^v(\hat{M}_T &\leq m, \hat{Y}_T \leq w) \approx \\ &\approx e^{\frac{1}{2}k_2^2(r-q)^2} \int_{-\infty}^{w} \int_{0}^{m} \exp(((r-q)k_1 - \frac{1}{2})y - b_I(T)) \hat{f}_{\hat{Y},\hat{M}}(y,z) dz dy. \end{aligned}$$

The application of Proposition 3 for the density distribution $\hat{f}_{\hat{Y},\hat{M}}(y,z)$ and the differentiation lead to the following result for the case 2 ($\rho = 0$ and $r \neq q$):

Proposition 6 *In case 2, the density of* $(\hat{M}_T, \hat{Y}_T)$ *under* $\mathbb{Q}^v$ *is*

$$\begin{aligned}\frac{\partial}{\partial w \partial m}\mathbb{Q}^v(\hat{M}_T &\leq m, \hat{Y}_T \leq w) \approx \\ &\approx \frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)}\exp(((r-q)k_1 - \frac{1}{2})w - b_I(T) - \frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}),\end{aligned}$$

where $b_I(T) = \frac{1}{2}[(r-q)^2\frac{T^2}{\nu^2(T)} - (r-q)T + \frac{1}{4}\nu^2(T))]$.

3.1.4 Case 3 - the general case, arbitrary r, q and ρ

We now consider the general case: an arbitrary rates r, q and correlation ρ. In this case,

$$\alpha(t) = (r-q)t - \frac{1}{2}\int_0^t v_s ds + \frac{\rho}{\sigma}(v_t - v_0 - k\theta t + k\int_0^t v_s ds), \tag{3.30}$$

and hence, $\gamma(t)$ should be

$$\gamma(t)dt = \frac{d\alpha(t)}{\rho_2\sqrt{v_t}}. \tag{3.31}$$

The new $\alpha(t)$ is not differentiable, due to term v_t, which is actually a realization of the Heston (1993) model. As we will show in the next proposition, choosing an approximation of v_t, $\bar{v}_t$, such that $\bar{v}_t$ is differentiable

$$v_0 = \bar{v}_0 \textit{ and } v_T = \bar{v}_T \tag{3.32}$$

will lead to the same result. The final formula (for $\hat{f}_{\hat{Y},\hat{M}}$ under $\mathbb{Q}^v$) in the next proposition only depends on the path of v_s and boundary values v_0, and v_T. The following theorem, gives us the density function conditioned on variance paths, under the general case.

Theorem 7 *Consider, the general model with arbitrary r, q and ρ. Let $\bar{v}_t$ be a differentiable approximation of v_t satisfying conditions (3.32). Then, the density function of $(\hat{M},\hat{Y})$ under $\mathbb{Q}^v$ is given by*

$$\begin{aligned}\hat{f}_{\hat{M},\hat{Y}}(m,w) \approx & \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(T)}\exp\left(-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}\right)\times \\ & \times\exp\left(\frac{c_1k_1+c_2+c_3k_3}{\rho_2^2}w-b_2(T)\right),\end{aligned} \tag{3.33}$$

with

$$\begin{aligned} b_2(T) &= \frac{1}{2\rho_2^2}[c_1^2\frac{T^2}{\nu^2(T)}+c_3^2\frac{(v_T-v_0)^2}{\nu^2(T)}+2c_1c_3\frac{(v_T-v_0)T}{\nu^2(T)}+c_2^2\nu^2(T)+ \\ &\quad +2c_1c_2T+2c_2c_3(v_T-v_0)] \\ c_1 &= (r-q)-\frac{\kappa\theta\rho}{\sigma} && (3.34) \\ c_2 &= (\frac{\kappa\rho}{\sigma}-\frac{1}{2}) && (3.35) \\ c_3 &= \frac{\rho}{\sigma}. && (3.36)\end{aligned}$$

Proof. Following [Griebsch, S.A., and Pilz, K.F. (2012)], theorem 2. Approximating v_t by $\bar{v}_t$, and taking $\bar{v}'_t=\frac{d\bar{v}_t}{dt}$, equation (3.31) yields:

$$\begin{aligned}\gamma(t) &= \frac{1}{\rho_2\sqrt{v_t}}\left((r-q)-\frac{1}{2}v_t+\frac{\rho}{\sigma}(\bar{v}'_t-\kappa\theta+\kappa v_t)\right) \\ &= \frac{1}{\rho_2}((r-q)-\frac{\kappa\theta\rho}{\sigma})\frac{1}{\sqrt{v_t}}+\frac{1}{\rho_2}(\frac{\kappa\rho}{\sigma}-\frac{1}{2})\sqrt{v_t}+\frac{1}{\rho_2}\frac{\rho}{\sigma}\frac{\bar{v}'_t}{\sqrt{v_t}}.\end{aligned}$$

Function $\gamma(t)$ is now composed of 3 types of terms, two we encountered already and the new $\frac{\bar{v}'_t}{\sqrt{v_t}}$. As usual, we define the new measure given by (3.18) where $\hat{H}_t$ is given by equation (3.16). Under this measure $\hat{Y}_t$, has no drift. The first term under expected value on the right-hand side of equation (3.19) is

$$\begin{aligned}\int_0^T \gamma(s)d\hat{W}_s &= \exp(\frac{1}{\rho_2}((r-q)-\frac{\kappa\theta\rho}{\sigma})\int_0^T \frac{1}{\sqrt{v_s}}d\hat{W}_s + \\ &+\frac{1}{\rho_2}(\frac{\kappa\rho}{\sigma}-\frac{1}{2})\int_0^T \sqrt{v_t}d\hat{W}_s + \frac{1}{\rho_2}\frac{\rho}{\sigma}\int_0^T \frac{\bar{v}'_t}{\sqrt{v_t}}d\hat{W}_s).\end{aligned} \tag{3.37}$$

If we (re)introduce

$$\begin{aligned}\hat{Y}_t &= \rho_2\int_0^t \sqrt{v_s}d\hat{W}_s, \\ \hat{X}_t &= \rho_2\int_0^t \frac{1}{\sqrt{v_s}}d\hat{W}_s, \\ \hat{Z}_t &= \rho_2\int_0^t \frac{\bar{v}'_s}{\sqrt{v_s}}d\hat{W}_s,\end{aligned}$$

equation (3.37) can be rewritten as

$$\begin{aligned}\int_0^T \gamma(s)d\hat{W}_s &= \frac{1}{\rho_2^2}((r-q)-\frac{\kappa\theta\rho}{\sigma})\hat{X}_T + \frac{1}{\rho_2^2}(\frac{\kappa\rho}{\sigma}-\frac{1}{2})\hat{Y}_T + \\ &+\frac{1}{\rho_2^2}\frac{\rho}{\sigma}\hat{Z}_T \\ &= \frac{1}{\rho_2^2}(c_1\hat{X}_T + c_2\hat{Y}_T + c_3\hat{Z}_T),\end{aligned} \tag{3.38}$$

with c_1, c_2 and c_3 defined by equations (3.34)-(3.36). As in the previous case, we want to approximate $\hat{X}_T$ and $\hat{Z}_T$ by $\hat{Y}_T$ and some independent normal variables, i.e.

$$\begin{aligned}\hat{X}_T &= k_0 + k_1\hat{Y}_T + \varepsilon^X \\ \hat{Z}_T &= k_2 + k_3\hat{Y}_T + \varepsilon^Z\end{aligned}$$

such that

$$\mathbb{E}[\varepsilon^X] = \mathbb{E}[\varepsilon^Z] = 0$$
$$\mathbb{E}[(\varepsilon^X)^2] \text{ and } \mathbb{E}[(\varepsilon^Z)^2] \text{ are minimal.}$$

■

Proposition 8 *1) The random variable $(\hat{X}_T, \hat{Y}_T, \hat{Z}_T)$ is normally distributed with zero mean and a covariance matrix*

$$\Sigma = \rho_2^2 \times \begin{bmatrix} \nu_{inv}^2(T) & T & \nu_{II}^2(T) \\ T & \nu^2(T) & (\nu'(T))^2 \\ \nu_{II}^2(T) & (\nu'(T))^2 & \nu_I^2(T) \end{bmatrix}$$

where

$$\begin{aligned}\nu_I(T) &= \sqrt{\int_0^T \frac{(\bar{v}_s')^2}{v_s} ds}, \\ \nu_{II}(T) &= \sqrt{\int_0^T \frac{\bar{v}_s'}{v_s} ds}, \\ \nu'(T) &= \sqrt{\int_0^T \bar{v}_s' ds},\end{aligned}$$

provided that all these integrals exist.

2) The variable $(\varepsilon^X, \varepsilon^Z)$ is normally distributed with zero mean and covariance matrix

$$\Sigma = \rho_2^2 \times \begin{bmatrix} \nu_{inv}^2(T) - k_1 T & \nu_{II}^2(T) - k_3 T \\ \nu_{II}^2(T) - k_3 T & \nu_I^2(T) - k_3(\nu'(T))^2 \end{bmatrix}$$

and is independent of $\hat{Y}_T$.

Proof. For a proof see [Griebsch, S.A., and Pilz, K.F. (2012)], p 29.

Using the first result from Proposition 8, we have:

$$\begin{aligned}
\mathbb{E}[\varepsilon^X] &= 0 = \mathbb{E}[[\hat{X}_T - k_0 + k_1\hat{Y}_T] \Rightarrow k_0 = 0, \\
\mathbb{E}[\varepsilon^Z] &= 0 = \mathbb{E}[\hat{Z}_T - k_2 + k_3\hat{Y}_T] \Rightarrow k_2 = 0, \\
k_0 &= k_2 = 0, \\
k_1 &= \frac{T}{\nu^2(T)} \text{ and } k_3 = \frac{(\nu'(T))^2}{\nu^2(T)} = \frac{v_T - v_0}{\nu^2(T)}.
\end{aligned}$$

For k_1 and k_3 we minimize the variances as

$$\begin{aligned}
\mathbb{E}[(\varepsilon^X)^2] &= \mathbb{E}[(\hat{X}_T)^2] + k_1^2\mathbb{E}[\hat{Y}_T] - 2k_1 Cov(\hat{X}_T, \hat{Y}_T) = \rho_2^2(\nu_{inv}^2 + k_1^2\nu^2 - 2k_1 T) \\
&\Rightarrow \; k_1 = \frac{T}{\nu^2(T)} \\
\mathbb{E}[(\varepsilon^Z)^2] &= \mathbb{E}[(\hat{Z}_T)^2] + k_3^2\mathbb{E}[\hat{Y}_T] - 2k_3 Cov(\hat{Z}_T, \hat{Y}_T) = \rho_2^2(\nu_I^2 + k_3^2\nu^2 - 2k_3(\nu')^2) \\
k_3 &= \frac{(\nu'(T))^2}{\nu^2(T)} = \frac{v_T - v_0}{\nu^2(T)}.
\end{aligned}$$

Please note that k_3 is well defined even without the differentiability of v_s.

Equation (3.38) now becomes

$$\int_0^T \gamma(s) d\hat{W}_s = \frac{1}{\rho_2^2}(c_1(k_1\hat{Y}_T + \varepsilon^X) + c_2\hat{Y}_T + c_3(k_3\hat{Y}_T + \varepsilon^Z)).$$

We introduce $b_I(T)$ as the second term under the expected value on the right-hand side of equation (3.19):

$$\begin{aligned} b_I(T) &= -\frac{1}{2}\int_0^T \gamma^2(s)ds \\ &= -\frac{1}{2\rho_2^2}[c_1^2\nu_{inv}^2(T) + c_2^2\nu^2(T) + c_3^2\nu_I^2(T) + 2c_1c_2T + 2c_1c_3\nu_{II}^2(T) + \\ &\quad +2c_2c_3(v_T - v_0)]. \end{aligned}$$

The $(\hat{M}_T, \hat{Y}_T)$ probability is

$$\begin{aligned} &\mathbb{Q}^v(\hat{M}_T \le m, \hat{Y}_T \le w) = \\ &\quad = \widehat{\mathbb{E}}[(\exp(b_I(T) + \frac{1}{\rho_2^2}(c_1\hat{X}_T + c_2\hat{Y}_T + c_3\hat{Z}_T))\mathbb{I}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}] \\ &\quad = e^{b_I(T)}\widehat{\mathbb{E}}[(\exp\{\frac{1}{\rho_2^2}(c_1(k_1\hat{Y}_T + \varepsilon^X) + c_2\hat{Y}_T + \\ &\quad + c_3(k_3\hat{Y}_T + \varepsilon^Z)\})\mathbb{I}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}] \\ &\quad = e^{b_I(T)}\widehat{\mathbb{E}}[e^{\frac{c_1\varepsilon^X + c_2\varepsilon^Z}{\rho_2^2}}]\times \qquad (3.39) \\ &\quad \times \widehat{\mathbb{E}}[(\exp\{\frac{1}{\rho_2^2}(c_1k_1 + c_2 + c_3k_3)\hat{Y}_T\})\mathbb{I}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}], \end{aligned}$$

where the first passage arises because b_I is deterministic in $\mathcal{G}_T^v$, the second follows from Proposition 8, due to the independence of $(\varepsilon^X, \varepsilon^Z)$ from $\hat{Y}_T$, and, as in the previous case, with a similar argument, we assume that $(\varepsilon^X, \varepsilon^Z)$ are independent from $\hat{M}_T$. Introducing

$$\begin{aligned} b_{II} &= \widehat{\mathbb{E}}[e^{\frac{c_1\varepsilon^X + c_2\varepsilon^Z}{\rho_2^2}}] \\ &= \frac{1}{2\rho_2^2}[c_1^2(\nu_{inv}^2(T) - \frac{T^2}{\nu^2(T)}) + c_3^2(\nu_I^2(T) - \frac{(v_T - v_0)^2}{\nu^2(T)}) + \\ &\quad +2c_1c_3(\nu_{II}^2 - \frac{(v_T - v_0)}{\nu^2(T)})] \end{aligned}$$

and

$$-b_2 = b_I(T) + b_{II}(T),$$

equation (3.39) becomes

$$\begin{aligned}\mathbb{Q}^v(\hat{M}_T &\leq m, \hat{Y}_T \leq w) = e^{-b_2} \int_{-\infty}^{w} \int_{-\infty}^{m} e^{\frac{y}{\rho_2^2}(c_1 k_1 + c_2 + c_3 k_3)} \hat{f}_{\hat{M}_T, \hat{Y}_T}(x, y) dx dy \\ &= e^{-b_2} \int_{-\infty}^{w} \int_{-\infty}^{m} e^{\frac{y}{\rho_2^2}(c_1 k_1 + c_2 + c_3 k_3)} \frac{2(2x - y)}{\sqrt{2\pi} \rho_2^3 \nu^3(T)} \exp(-\frac{1}{2} \frac{(2x - y)^2}{\rho_2^2 \nu^2(T)}) dx dy,\end{aligned}$$

after another application of Proposition 3. The differentiation will lead to equation (3.33). ■

3.2 Up-and-out call formula

Having computed the density distributions for the three cases, our next objective is to solve the inner expectation from equation (3.6), i.e. the value of up-and-out call, conditioned on variance paths,

$$\mathcal{E}^v = \mathbb{E}^v[(S_0 e^{\hat{Y}_t} - K)\mathbb{I}_{\{\hat{M}_t < b, \hat{Y}_t > k\}}].$$

The density distributions for all three cases can be represented as

$$f_{\hat{Y}, \hat{M}}(m, w) \approx \frac{2(2m - w)}{\sqrt{2\pi} \rho_2^3 v^2(T)} e^{-\frac{1}{2} \frac{(2m-w)^2}{\rho_2^2 v^2(T)} + Fw + G}$$

where

$$F = \begin{cases} -\frac{1}{2} & case\ 1 \\ (r - q)k_1 - \frac{1}{2} & case\ 2 \\ (c_1 k_1 + c_2 + c_3 k_3)/\rho_2^2 & case\ 3 \end{cases} \qquad G = \begin{cases} -\frac{1}{8} v^2(T) & case\ 1 \\ -b_1(T) & case\ 2 \\ -b_2(T) & case\ 3 \end{cases}.$$

Note that the quantities F and G are completely deterministic in $\mathcal{G}_T^v$, r, q and c_i depend only on model parameters, but k_i and b_i are additionally functions of the time-integrated variance v and v_T. The formula for case 1 is exact, but for cases 2 and 3 is approximated.

The limits of integration for the inner expectation $\mathcal{E}^v$ are

$$(\hat{Y}_t > k,\ \hat{M}_t < b) \Longrightarrow \{k \leq w \leq b,\ w^+ \leq m \leq b\}.$$

Hence,

$$\begin{aligned}
\mathcal{E}^v &= \int_k^b \int_{w^+}^b (S_0 e^w - K) f_{\hat{Y},\hat{M}}(m,w)\, dm dw \\
&= \int_k^b \int_{w^+}^b (S_0 e^w - K) \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3 \nu^2(T)} e^{-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)} + Fw + G} dm dw.
\end{aligned}$$

We can integrate with respect to m using the substitution

$$y = \frac{(2m-w)^2}{2\rho_2^2\nu^2(T)},$$

i.e.

$$dy = \frac{2(2m-w)}{\rho_2^2\nu^2(T)} dm.$$

Yielding,

$$\begin{aligned}
\mathcal{E}^v &= -\frac{1}{\sqrt{2\pi}\rho_2\nu^2} \int_k^b (S_0 e^w - K) \exp(Fw + G) \int_{-\frac{(2w^+-w)^2}{2\rho_2^2\nu^2(T)}}^{-\frac{(2b-w)^2}{2\rho_2^2\nu^2(T)}} e^y dy dw \\
&= \frac{1}{\sqrt{2\pi}\rho_2\nu(T)} \int_k^b (S_0 e^w - K) \exp(Fw + G - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)})\Big|_{m=w^+}^{m=b} dw.
\end{aligned}$$

The term e^G is independent from w, so we can take it out and separate the inner bracket as

$$\begin{aligned}
\mathcal{E}^v &= \frac{1}{\sqrt{2\pi}\rho_2\nu(T)} e^G \int\limits_k^b (S_0 e^w - K)\exp(Fw - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)})\Big|_{m=w^+}^{m=b} dw \\
&= \frac{S_0}{\sqrt{2\pi}\rho_2\nu(T)} e^G \int\limits_k^b \exp((F+1)w - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)})\Big|_{m=w^+}^{m=b} dw \qquad (3.40)\\
&\quad - \frac{K}{\sqrt{2\pi}\rho_2\nu(T)} e^G \int\limits_k^b \exp(Fw - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)})\Big|_{m=w^+}^{m=b} dw.
\end{aligned}$$

Both integrals on the right-hand side of equation (3.40) have the form of a normal distribution (after completing the square), and can be expressed as:

$$\begin{aligned}
I_{1,x} &= \exp(\frac{1}{2}(F+x)^2\rho_2^2\nu^2(T) + G) \times \\
&\quad \times \left[\mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\
I_{2,x} &= \exp(\frac{1}{2}(F+x)^2\rho_2^2\nu^2(T) + G + 2b(F+x)) \times \\
&\quad \times \left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0K}) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right],
\end{aligned}$$

for $x \in \{0,1\}$. Finally, the inner expectation can be expressed as

$$\mathcal{E}^v = S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x).$$

The separated formulas are presented in Appendix 5.1. The value of the up-and-out call option in the [Heston, S.L. (1993)] model at time 0 is given by

$$V_0 \approx e^{-rT}\mathbb{E}[S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x)]. \qquad (3.41)$$

The formula is exact for the simple case 1 and an approximation for the cases 2 and 3. It is important to note, for the purposes of unconditioning, that the only path dependent variables "to solve" are $\nu^2(T)$ for the case 1 and $(\nu^2(T), v_T)$ for the general case 3. Hence, for the unconditioning we will need the density distribution of $\nu^2(T)$ and $(\nu^2(T), v_T)$. We will not show here the unconditioning for case 2, since it is similar in idea and less complicated than the case 3.

3.3 Resolving the conditioning

Next, we have to solve the conditioning on the variance paths i.e.:

$$V_0 = e^{-rT}\mathbb{E}[\mathcal{E}^v|\mathcal{F}_0]. \tag{3.42}$$

For this purpose we need the distribution of $\nu^2(T)$ for case 1 and the distribution $(\nu^2(T), v(T))$ for case 3. We will proceed in the following steps:

1. Find the characteristic function PDE
2. Solve PDE (get the characteristic function)
3. Use the inversion theorem to calculate the distribution of $\nu^2(T)$ or $(\nu^2(T), v(T))$, depending on the case under analysis
4. Use the obtained distribution in equation (3.42)

3.3.1 Case 1

We will start with the simpler case 1, with $r = q$ and $\rho = 0$. We are looking for the characteristic function of $\nu^2(T)$.

Proposition 9 *The characteristic function of $\nu^2(T)$ in case 1 is given by*

$$\phi_{\nu^2(T)}(u) = \mathbb{E}[e^{iu\nu^2(T)}] = \exp[A(u)v_0 + B(u)],$$

where

$$\begin{aligned} A(u) &= \frac{2iue^-}{de^+ + \kappa e^-}, \\ B(u) &= \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2}\log(\frac{2d}{de^+ + \kappa e^-}), \end{aligned}$$

with $d = \sqrt{\kappa^2 - 2\sigma^2 iu}$ and $e^\pm = 1 \pm \exp(-dT)$.

Proof. To compute the characteristic function, we will find the martingale (time invariant) and the PDE associated with it. Afterwords, we solve the resulting PDE through guessing the from of the solution. We define

$$F(u,t,v) = \mathbb{E}[\exp(iu\int_0^t v(s)ds|\mathcal{F}_0],$$

and a candidate for martingale

$$M(t) = \exp(iu\int_0^t v(s)ds)F(u,\tau,v_t),$$

with $\tau = T - t$. The $M(t)$ is martingale

$$\mathbb{E}[M(T)|\mathcal{F}_0] = \mathbb{E}[\exp(iu\int_0^T v(s)ds)F(0,v_T)|\mathcal{F}_0] = F(u,T,v_0) = M_0.$$

Applying Ito's Lemma to $M(t)$ and using the subscript index as a derivative, i.e. $\frac{\partial A}{\partial x} = A_x$, we have

$$dM = M_t dt + M_v dv + \frac{1}{2} M_{vv} dv dv. \tag{3.43}$$

To further simplify notation we will use $\varpi = iu \int_0^t v(s) ds$. Therefore,

$$\begin{aligned} M_t &= F_t e^{\varpi} + iuve^{\varpi} F, \\ M_v &= F_v e^{\varpi} + F(e^{\varpi})_v = F_v e^{\varpi}, \\ M_{vv} &= F_{vv} e^{\varpi}, \\ dvdv &= \sigma^2 v dt. \end{aligned}$$

It is important to note that all F derivatives and F itself are taken at a point (u, τ, v). Combining the previous results with equation (3.43),

$$dM = (F_t e^{\varpi} + iuve^{\varpi} F + \kappa(\theta - v) F_v e^{\varpi} + \frac{1}{2}\sigma^2 v F_{vv} e^{\varpi}) dt + \sigma \sqrt{v} F_v e^{\varpi} dW^v.$$

Since, M is a martingale, the dt term must be 0. So, our function F satisfies the following PDE:

$$\begin{aligned} F_t e^{\varpi} + iuve^{\varpi} F + \kappa(\theta - v) F_v e^{\varpi} + \frac{1}{2}\sigma^2 v F_{vv} e^{\varpi} &= 0 \\ F_t + iuvF + \kappa(\theta - v) F_v + \frac{1}{2}\sigma^2 v F_{vv} &= 0. \end{aligned} \tag{3.44}$$

We guess that the solution can be represented as

$$F(u, \tau, v) = \exp[A(u, \tau) v_t + B(u, \tau)].$$

For this trial function the derivatives are

$$\begin{aligned} F_t &= -(A_t v + B_t) F \\ F_v &= AF \\ F_{vv} &= A^2 F, \end{aligned}$$

and the equation (3.44) is

$$-(A_t v + B_t) + iuv + \kappa(\theta - v)A + \frac{1}{2}\sigma^2 v A^2 = 0.$$

Combining v-terms and non-v-terms, since it has to be valid for all v's, we get the system of equations

$$A_t + iu - \kappa v A + \frac{1}{2}\sigma^2 A^2 = 0 \tag{3.45}$$

$$-B_t + \kappa\theta A = 0. \tag{3.46}$$

We can solve the first equation and through it solve the second one. The first equation (3.45) is a Riccati ordinary differential equation

$$y_t = P + Qy + Ry^2,$$

which is solved by considering the second order auxiliary differential equation for $\omega(t)$

$$\omega_{tt} - (\frac{P_t}{P} + Q)\omega_t + PR\omega = 0,$$

or

$$\omega_{tt} - p\omega_t + q\omega = 0, \tag{3.47}$$

where solution $y(t)$ is given by

$$y(t) = -\frac{\omega_t}{\omega}\frac{1}{R}.$$

The solution for the equation (3.47) has the form

$$\omega(t) = ae^{\alpha t} + be^{\beta t},$$

where the roots α and β are obtained from the quadratic equation $x^2 + px + q = 0$:

$$\alpha = \frac{-p + \sqrt{p^2 - 4q}}{2},$$

$$\beta = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

For our case

$$\begin{aligned} P &= iu, \\ Q &= -\kappa, \\ R &= \frac{1}{2}\sigma^2, \end{aligned}$$

and, consequently,

$$\begin{aligned} p &= -(\frac{P_t}{P} + Q) = -Q = \kappa \\ q &= PR = \frac{1}{2}iu\sigma^2, \end{aligned}$$

and, introducing, $d = \sqrt{\kappa^2 - 2iu\sigma^2}$,

$$\begin{aligned} \alpha &= \frac{1}{2}(-\kappa + d), \\ \beta &= \frac{1}{2}(-\kappa - d). \end{aligned}$$

Hence,

$$A(u,\tau) = -\frac{1}{R}(\frac{K\alpha e^{\alpha\tau} + \beta e^{\beta\tau}}{Ke^{\alpha\tau} + e^{\beta\tau}}), \tag{3.48}$$

with $K = \frac{a}{b}$. Up until now, we did not mention the boundary conditions on our function F. They are:

$$F(u,0,v) = 1 \Rightarrow A(u,0)v_0 + B(u,0) = 0,$$

for every v_0, hence for the trial function, the boundary conditions are

$$A(u,0) = 0, \tag{3.49}$$

$$B(u,0) = 0. \tag{3.50}$$

Combining equations (3.49) and (3.48), then

$$K\alpha + \beta = 0 \Rightarrow K = -\frac{\beta}{\alpha}.$$

Hence,

$$\begin{aligned}
A(u,\tau) &= -\frac{1}{R}\frac{-\frac{\beta}{\alpha}\alpha e^{\alpha\tau}+\beta e^{\beta\tau}}{-\frac{\beta}{\alpha}e^{\alpha\tau}+e^{\beta\tau}} \\
&= -\frac{\beta}{R}\frac{-e^{\alpha\tau}+e^{\beta\tau}}{-\frac{\beta}{\alpha}e^{\alpha\tau}+e^{\beta\tau}} \\
&= -\frac{\beta}{R}\frac{e^{\alpha\tau}}{e^{\alpha\tau}}\frac{-1+e^{(\beta-\alpha)\tau}}{-\frac{\beta}{\alpha}e^{\alpha\tau}+e^{(\beta-\alpha)\tau}} \\
&= -\frac{\beta}{R}\frac{-1+e^{-d\tau}}{-\frac{\beta}{\alpha}e^{\alpha\tau}+e^{-d\tau}} \\
&= \frac{\beta}{R}\frac{e^{-}}{-\frac{\beta}{\alpha}e^{\alpha\tau}+e^{-d\tau}} \\
&= \frac{\beta}{R}\frac{e^{-}}{-\frac{d+\kappa}{d-\kappa}e^{\alpha\tau}+e^{-d\tau}} \\
&= \frac{\beta(d-\kappa)}{R}\frac{e^{-}}{(\kappa e^{-}+de^{+})} \\
&= \frac{2uie^{-}}{\kappa e^{-}+de^{+}},
\end{aligned}$$

with $e^{\pm}=1\pm e^{-d\tau}$. The equation for the function $B(u,\tau)$ follows from (3.46):

$$B_\tau = \frac{2uik\theta e^{-}}{\kappa e^{-}+de^{+}}.$$

Introducing $\lambda = 2\kappa\theta ui$ and integrating, then

$$B(u,\tau) = \lambda\int_0^\tau \frac{e^{-}}{\kappa e^{-}+de^{+}} + C,$$

with a constant C. After using the boundary condition (3.50), we find that $C=0$. To solve the integral we want to rewrite it as

$$\int_a^b \frac{1-x}{(1-\beta x)x}dx = [\log x + \frac{1-\beta}{\beta}\log(1-\beta x)]\Big|_a^b. \tag{3.51}$$

With a change of coordinates $x = e^{-d\tau}$, then

$$
\begin{aligned}
B(u,\tau) &= \lambda \int_0^{\tau} \frac{e^-}{\kappa e^- + de^+} \\
&= \lambda \int_1^{e^{-d\tau}} \frac{1-x}{d(1-x)+d(1+x)} \frac{1}{x(-d)} dx \\
&= \frac{\lambda}{(-d)(\kappa+d)} \int_1^{e^{-d\tau}} \frac{1-x}{(1+\frac{d-\kappa}{d+\kappa}x)x} dx \\
&= -\frac{\alpha}{d(d+\kappa)} \int_1^{e^{-d\tau}} \frac{1-x}{(1-\beta x)x} dx,
\end{aligned}
$$

with $\beta = -\frac{d-\kappa}{d+\kappa}$. Applying equation (3.51),

$$
\begin{aligned}
B(u,\tau) &= -\frac{\lambda}{d(d+\kappa)} [\log x + \frac{1-\beta}{\beta} \log(1-\beta x)]|_1^{e^{-d\tau}} \\
&= -\frac{\lambda}{d(d+\kappa)} [(-d\tau) - 0 + \frac{1-\beta}{\beta} \log(\frac{1-\beta e^{-d\tau}}{1-\beta})] \\
&= \frac{\kappa\theta}{\sigma^2}(\kappa - d)\tau - \frac{2\kappa\theta}{\sigma^2} \log(\frac{de^+ + \kappa e^-}{2d}) \\
&= \frac{\kappa\theta}{\sigma^2}(\kappa - d)\tau + \frac{2\kappa\theta}{\sigma^2} \log(\frac{2d}{de^+ + \kappa e^-}).
\end{aligned}
$$

■

We have calculated the characteristic function of $\nu^2(T)$. Next, we use the Inversion theorem 4 from the Appendix to calculate the density distribution of $\nu^2(T)$:

$$
d_{\nu^2(T)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} \phi_{\nu^2(T)}(u) du = \frac{1}{\pi} \int_{\mathbb{R}^+} e^{-ixu} \phi_{\nu^2(T)}(u) du, \qquad (3.52)
$$

since the last integral is symmetric. We combine the previous results with equation (3.41), the value for our up-and-out call is

$$V_0 = e^{-rT} \int_{\mathbb{R}^+} (S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x)) \times d_{\nu^2(T)}(x) dx. \quad (3.53)$$

3.3.2 Case 3

The approach to the general case is similar to the first simpler case. We start by looking for the characteristic function of $(\nu^2(T), v_T)$.

Proposition 10 *The characteristic function in case 3 is given by*

$$\phi_{\nu^2(T),v(T)}(u, w, v_0) = \mathbb{E}[e^{-iu\nu^2(T)-iwv(T)}] = \exp[A(u, w, T)v_0 + B(u, w, T)],$$

where

$$\begin{aligned} A(u, w, \tau) &= \frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}, \\ B(u, w, \tau) &= \frac{\kappa\theta}{\sigma^2}(\kappa - d)\tau + \frac{2\theta}{\sigma^2}\log(\frac{2d}{\gamma(w)}), \end{aligned}$$

with $d = \sqrt{\kappa^2 - 2\sigma^2 iu}$, $e^{\pm} = 1 \pm \exp(-d\tau)$ *and* $\gamma(w) = 2d\exp(-d\tau) + (\kappa + d - \sigma^2 iw)e^-$

Proof. *(Sketch)* We follow [Nunes, J.P.] and [Lamberton, D., and Lapeyre, P. (1996)]. As in previous the case, there is only one stochastic process (v) involved, and we (re)introduce

$$F(u, w, t, v) = \mathbb{E}[\exp(iwv)\exp(iu\int_0^t v(s)ds)]$$

and the martingale

$$M(t) = \exp(iu\int_0^t v(s)ds)F(u, v, \tau, v).$$

The PDE is derived in a similar way to equation (3.44) as

$$F_t + iuvF + \kappa(\theta - v)F_v + \frac{1}{2}\sigma^2 vF_{vv} = 0, \tag{3.54}$$

for the new function F. To complete our Cauchy problem, in this case, the terminal conditions are

$$F(u, w, 0, v) = e^{iwv}. \tag{3.55}$$

The trial function is

$$F(u, w, \tau, v) = \exp[A(u, w, \tau)v + B(u, w, \tau)]. \tag{3.56}$$

As in case 1, we can solve the PDE (3.54) with boundary conditions (3.55) for the trial function (3.56). The techniques in this case are similar to the previous one. ■

The density distribution can be obtained by the Inversion theorem 4

$$d_{v^2,v_T}(x, y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{-ixu-i\omega y}\phi_{\nu^2(T),v(T)}(u, \omega)d\omega du. \tag{3.57}$$

Following [Griebsch, S.A., and Pilz, K.F. (2012)], our aim is to solve analytically one of the integrals above. Applying the solution from the Proposition 10 to (3.57),

$$\begin{aligned} d_{\nu^2,v_T}(x,y) &= \frac{1}{(2\pi)^2}\iint\limits_{\mathbb{R}^2} e^{-ixu-i\omega y}\exp[A(u,w,T)v_0+B(u,w,T)]dwdu \\ &= \frac{1}{(2\pi)^2}\iint\limits_{\mathbb{R}^2} e^{-ixu-i\omega y} e^{\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0+\frac{\kappa\theta}{\sigma^2}(\kappa-d)T+\frac{2\theta}{\sigma^2}\log(\frac{2d}{\gamma(w)})}dwdu \\ &= \frac{1}{(2\pi)^2}\int\limits_{\mathbb{R}} e^{-ixu}e^{\frac{\kappa\theta}{\sigma^2}(\kappa-d)T}\int\limits_{\mathbb{R}} e^{-iwy}(\frac{\gamma(w)}{2d})^{-\frac{2\kappa\theta}{\sigma^2}}\times \\ &\quad \times e^{\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0}dwdu. \end{aligned} \tag{3.58}$$

We want to solve the inner integral relative to w, i.e.

$$\chi = \int\limits_{\mathbb{R}} e^{-iwy}(\frac{\gamma(w)}{2d})^{-\frac{2\kappa\theta}{\sigma^2}} e^{\frac{2iue^- - \kappa iwe^- + diwe^+}{\gamma(w)}v_0}dw. \tag{3.59}$$

For the next transformation, we should have in mind the following inverse Laplace transform:

$$\frac{1}{2\pi i}\int\limits_{\xi-i\infty}^{\xi+i\infty} e^{cz}z^{-2L}e^{kz^{-1}}dz = \mathcal{L}_c^{-1}(z^{-2L}e^{kz^{-1}}), \tag{3.60}$$

with L and k constants. The inverse Laplace transform of the complex function $f(s)$ is the operator defined as

$$\mathcal{L}_t^{-1}(f(s)) = \frac{1}{2\pi i}\int\limits_{\xi-i\infty}^{\xi+i\infty} e^{ts}f(s)ds,$$

for some $\xi \in \mathbb{R}$. For $f(s) = e^{ks^{-1}}s^{-\mu}$, we have

$$\mathcal{L}_t^{-1}(f(s)) = \frac{1}{2\pi i}\int\limits_{\xi-i\infty}^{\xi+i\infty} e^{ts}e^{ks^{-1}}s^{-\mu}ds = (\frac{t}{k})^{\frac{\mu-1}{2}}I_{\mu-1}(2\sqrt{kt}),$$

where $I_{\mu-1}$ is modified Bessel function for complex variables of the first kind, defined as

$$I_\nu(z) = (\frac{z}{2})^\nu \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n}}{\Gamma(n+1)\Gamma(\nu+n+1)},$$

where $\Gamma(z)$ is Gamma function as defined by [Abramowitz, M., and Stegun, I.A.] and [Lamberton, D., and Lapeyre, P. (1996)]. Defining

$$z = \frac{\gamma(w)}{2d} = \frac{de^+ + \kappa e^-}{2d} - iw\frac{\sigma^2 e^-}{2d} := n - iwn$$

and noting that $dw = -\frac{1}{in}dz$, the integral (3.59) becomes

$$\chi = -\frac{1}{2\pi in} e^{-\frac{y}{n}m} \int_{m-in\infty}^{m+in\infty} e^{\frac{y}{n}z} e^{\frac{\kappa e^- - de^+}{2dn}v_0} e^{\frac{2iue^- - m(\kappa e^- - de^+)}{2dn}v_0 z^{-1}} z^{-\frac{2\kappa\theta}{\sigma^2}} dz. \quad (3.61)$$

Introducing

$$\begin{aligned} k &= \frac{2iue^- - m(\kappa e^- - de^{+)}}{2dn}v_0, \\ L &= \frac{\kappa\theta}{\sigma^2}, \\ c &= \frac{y}{n}, \end{aligned}$$

we can rewrite (3.61) and solve it using (3.60), see [Griebsch, S.A., and Pilz, K.F. (2012)]:

$$-\frac{1}{2\pi in} e^{-\frac{y}{n}m} e^{\frac{\kappa e^- - de^+}{2dn}v_0} \int_{m-in\infty}^{m+in\infty} e^{cz} e^{kz^{-1}} z^{-2L} dz$$

$$= \frac{2d}{\sigma^2 e^-} \exp(-y\frac{de^+ + \kappa e^-}{\sigma^2 e^-}) \exp(v_0 \frac{\kappa e^- - de^+}{\sigma^2 e^-})(\frac{y}{v_0}e^{d\tau})^{L-\frac{1}{2}} I_{2L-1}(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}).$$

Combining the previous result with equation (3.58),

$$\begin{aligned} d_{v^2,v_T}(x,y) &= \frac{1}{2\pi} e^{L\kappa\tau + (v_0 - y)\frac{\kappa}{\sigma^2}} \times \quad (3.62) \\ &\times \int_{\mathbb{R}} \frac{2d}{\sigma^2 e^-} e^{-iux - Ld\tau - (v_0+y)\frac{de^+}{\sigma^2 e^-}} (\frac{ye^{d\tau}}{v_0})^{L-\frac{1}{2}} I_{2L-1}(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}) du. \end{aligned}$$

The value for up-and-out call in the [Heston, S.L. (1993)] model for the general case (case 3) is

$$\begin{aligned} V_0 &\approx e^{-rT} \iint\limits_{\mathbb{R}_+^2} (S_0 I_{1,1}(x,y) - K I_{1,0}(x,y) - S_0 I_{2,1}(x,y) + \\ &\quad + K I_{2,0}(x,y)) d_{\nu^2, v_T}(x,y) dx dy, \end{aligned} \tag{3.63}$$

where $d_{v^2,v_T}(x,y)$ is given by the equation (3.62) and $I_{j,k}$ are obtained from Appendix 5.1, case 3.

4 Numerical analysis

4.1 Matlab implementation

The big part of the problem with exotic options pricing is the numerical implementation. We will spend some time announcing the methods and techniques adopted. The implementation was done in Matlab.

4.1.1 The (Log-)Euler discretization and the Monte-Carlo Simulation

One of the most frequently used techniques for the pricing of exotic options and, in general, for the BM modeling is the discretization of the equations followed by the Monte-Carlo simulation.

To simulate the BM we divide time into intervals, and for each interval we assume a normally distributed "jump"

$$W(t+h) - W(t).$$

A complete path (possibility) is a sequence of "jumps" from the time t_0 to the final destination at time T. Using the pseudo-random routines, we can run a large number of simulations (possible paths) and take the average of all the results. This average will be the approximation for the expected value.

We start by discreticising the model equations (2.1), (2.2) and (2.3). Beginning with (2.2), for every jump $\Delta t = T/N$, where N is the number of jumps, we have

$$v(t+\Delta t) = v(t) - \kappa(\theta - v(t))\Delta t + \sigma\sqrt{v(t)}Z_v\sqrt{\Delta t},$$

where Z_v has a standard normal distribution and the last term is the approximation of the stochastic integral

$$\begin{aligned}
\int_t^{t+\Delta t} \sigma\sqrt{v(t)}dW^v &\approx \sigma\sqrt{v(t)} \int_t^{t+\Delta t} dW^v \\
&= \sigma\sqrt{v(t)}(W(t+\Delta t) - W(t)) \sim \sigma\sqrt{v(t)}\mathcal{N}(0, \sqrt{t}) \\
&= \sigma\sqrt{v(t)}Z_v\sqrt{\Delta t}.
\end{aligned}$$

The values for $v(t)$ have to be real and positive, which is not guaranteed. The negative values have the probability

$$\begin{aligned}
\mathbb{Q}(v(t) &< 0) = \mathbb{Q}^v(Z_v < \frac{-v(s) + \kappa(\theta - v(s))\Delta t}{\sigma\sqrt{v(s)}\sqrt{\Delta t}}) \\
&= \Phi(\frac{-v(s) + \kappa(\theta - v(s))\Delta t}{\sigma\sqrt{v(s)}\sqrt{\Delta t}}),
\end{aligned}$$

which tends to zero as $\Delta t \to 0$ but is still always positive. The easiest way to deal with this problem is to truncate $v(t)$ to $v^+(t) = \sup(v(t), 0)$ and hope the bias introduced is not very significant. This is the most popular approach in practical situations [Andersen, L. (2008)]. The full truncation scheme for the variance equation is

$$v(t + \Delta t) = v(t) - \kappa(\theta - v^+(t))\Delta t + \sigma\sqrt{v^+(t)}Z_v\sqrt{\Delta t}.$$

The direct discretization of the stock price equation (2.1) leads to

$$S(t + \Delta t) = S(t)(1 + (r - q)\Delta t + \sqrt{v(t)}Z_S\sqrt{\Delta t}),$$

where Z_S is a normal random variable correlated by ρ with Z_v. For the practical implementation in Matlab simulation, we used the Cholesky decomposition:

$$\begin{aligned} Z_1 &= Z_v, \\ Z_2 &= \rho Z_v + \sqrt{1-\rho^2}Z_2. \end{aligned}$$

Alternatively, we can use the exact solution

$$S(t + \Delta t) = S(t)\exp[\int_t^{t+\Delta t}((r - q) - \frac{1}{2}v(u))du + \int_t^{t+\Delta t}\sqrt{v(t)}dW^S(u)],$$

and, as stated by [Haastrecht, A., and Pelsser, A. (2008)], "...taking logarithms and discreticising in an Eulerly fashion, one obtains the following log-Euler scheme":

$$\log S(t + \Delta t) = \log(S(t)) + ((r - q) - \frac{1}{2}v^+(t))\Delta t + \sqrt{v^+(t)}Z_S\sqrt{\Delta t},$$

with the same choice of truncation for the negative $v(t)$ values.

4.1.2 The Fast Fourier Transform (FFT)

The FFT is the algorithm invented by Gauss to efficiently compute the sums of the form

$$FFT(X)_l = Y[l] = \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N} jl} X[j] \tag{4.1}$$

for any complex input $\{X[j] \in \mathbb{C} : j = 0, .., N-1\}$. So why do we need it? The technique is very commonly used for option pricing, due to its connection to the option pricing schemes when the characteristic function approach is used, see [Carr, P., and Madan, D. (1999)]. During the efforts to calculate the oscillatory integrals, it was decided to apply the FFT as the numerical technique. The struggle with Matlab was enlightening, after trying several solutions for oscillatory integrals, this was the method that proved to be robust, and last but not least, it is fun.

4.1.3 The trapezoidal rule and 2D integration

The objective is to calculate numerically $\iint\limits_{\mathbb{R}^2} f(x,y)dxdy$.

Since, we are talking about a numerical calculation, the infinite limits have to be changed. The integrals appearing in our formulas are fast converging, so, we will not be losing much on a limit truncation. Fortunately, the 2D integration or any multi-dimensional integration scheme is much more interesting then a simple integration on a lane. It is possible to choose different shapes for the contour, weights, etc., besides the alternatives in the point selection (random, equally spaced, etc.). Said that, we will pick the simplest of them all, the extension of the trapezoidal rule to two dimensions. We will be integrating over the rectangles $R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$, with

$$\begin{aligned} x_i &= x_0 + ih, \quad h = \frac{b-a}{m}, \quad i = 0, .., m \\ y_j &= y_0 + jk, \quad k = \frac{d-c}{n}, \quad j = 0, .., n, \end{aligned}$$

so the interval R will be divided in the same length rectangles $\{(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]\}$. Besides border points, all other points have the same weights.

$$\begin{aligned} \iint\limits_{\mathbb{R}^2} f(x,y)dxdy &= Trap2D(f,h,k) = \frac{1}{4}hk(f(a,c) + f(b,c) + f(a,b) \\ &\quad + f(b,d) + 2\sum_{i=1}^{m-1} f(x_i, c) + 2\sum_{i=1}^{m-1} f(x_i, d) + \\ &\quad +2\sum_{j=1}^{n-1} f(a, y_j) + 2\sum_{j=1}^{n-1} f(b, y_j) + 4\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} f(x_i, y_j)). \end{aligned}$$

The error in this approximation is of magnitude $O(h^2) + O(k^2)$, i.e.

$$Err(Trap2D(f,h,k)) = \iint\limits_{\mathbb{R}^2} f(x,y)dxdy - Trap2D(f,h,k) = O(h^2) + O(k^2).$$

For the further reference see [Atkinson, K. (1989)].

4.2 Case 1

For case 1 we have to calculate (3.53), with $d_{\nu^2(T)}$ given by (3.52). This simple case provides a good illustration for the oscillatory behavior of the integral calculated through FFT. We begin with $d_{\nu^2(T)}(x)$ and afterwords V_0. To make the notation a little bit easier we will change

$$\begin{aligned} d_{\nu^2(T)}(x) &: = d_1(x) \\ \phi_{\nu^2(T)}(u) &: = \phi_1(u) \end{aligned}$$

and define

$$f(x) = [S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x)].$$

The steps for our algorithm will be:

1. A definition of the limits of integration
2. A discretization of the integrals and of the FFT application for $d_1(x)$
3. A calculation of V_0

We have to limit the indefinite integrals to finite intervals and do it in a way that does not affect the result. Since, well defined limits will drastically change the performance of an algorithm, it is worth spending time trying different inputs. The variables in this case are u and x. We simply define the functions in Matlab and by intelligent observation verify where we need to stop. It is possible to create a routine for this step, however for this case it is not necessary. The $d_1(x)$ is an oscillatory integral. Hence, for large u or x, as oscillations become more frequent, they cancel themselves. The function $\phi_1(u)$ converges to 0 quickly. All the above will influence our choice of the interval $u \in [-a, a]$. Table 4.1 contains the model parameters adopted.

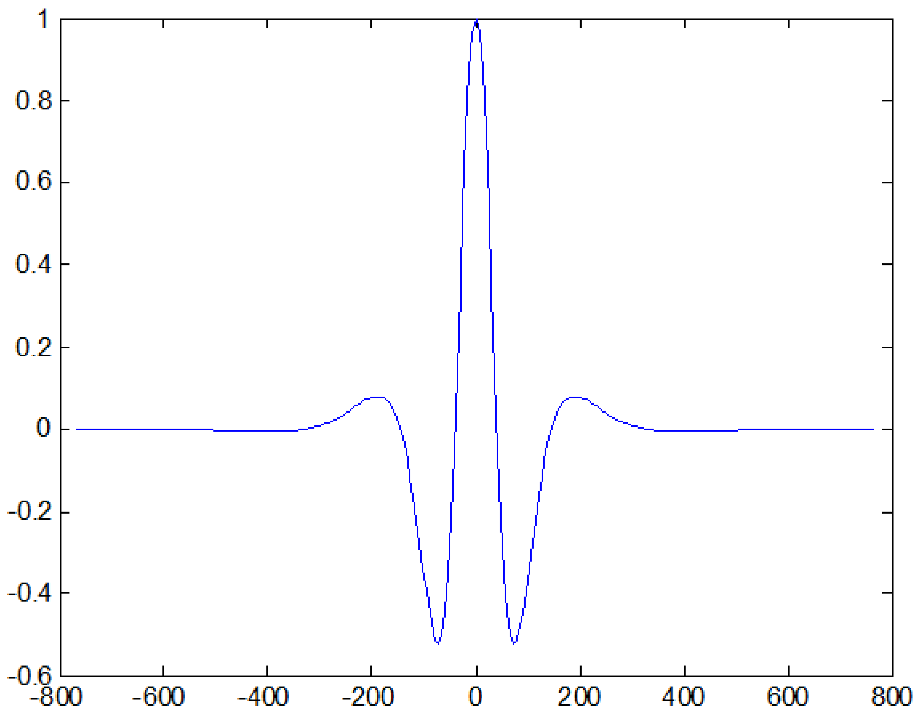

Figure 4.1: $\phi(u)$ for the tested parameters

Table 4.1: The model parameters adopted

Parameter	Value(s)	Parameter	Value(s)
κ	2	S_0	80-100
ρ	0	K	100
θ	0.04	r	0.03
σ	0.25	q	0.03
v_0	0.04	B	105-145
		T	1

For tested parameters $\phi(u)$ near 800 is almost zero (see Figure 4.1), so the choice $a = 800$ is adequate. For $x \in [-b, b]$, we have to look at $f(x)$. It is also a well behaved fast converging function. We do not need the negative x values for the calculation of V_0, but it is easier to work with them in the FFT algorithm. From the analysis above we can observe that it would be beneficial to discreticise the functions unevenly. Unfortunately, this is not compatible with the FFT algorithm. The d_1 integral in a discrete form is

$$\begin{aligned} d_1(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu}\phi_1(u)du \approx \frac{1}{2\pi}\int_{-a}^{a} e^{-ixu}\phi_1(u)du \\ &\approx \frac{1}{2\pi}\sum_{n=0}^{N-1} e^{-ixu_n}\phi_1(u)\Delta_u, \end{aligned} \tag{4.2}$$

where Δ_u is a u step, i.e. $u_n = (n - \frac{N}{2})\Delta_u$ for $n = 0, .., N$ with N intervals for $[-a, a]$. The sum (4.2) is almost the same as equation (4.1). All we need to do is to choose wisely the x points:

$$x_p = (p - \frac{N}{2})\Delta_x \quad \text{and} \quad \Delta_x\Delta_u = \frac{2\pi}{N}, \tag{4.3}$$

for $p = 0, .., N-1$. With these choices,

$$\begin{aligned} X[p] &= \frac{1}{2\pi}\sum_{n=0}^{N-1} e^{-i(n-\frac{N}{2})\Delta_u(p-\frac{N}{2})\frac{2\pi}{N\Delta_u}}\phi_1(u)\Delta_u \\ &= \frac{1}{2\pi}\sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}(np+\frac{N^2}{4}-\frac{Np}{2}-\frac{Nn}{2})}\phi_1(u)\Delta_u. \end{aligned} \tag{4.4}$$

Note that the exponential parcels are:

1. $e^{-\frac{2\pi i}{N}np} \rightarrow$ ready for the FFT
2. $e^{-\frac{2\pi i N}{4}} = 1 \rightarrow$ if we choose $N = 2^{2k}$
3. $e^{+\pi i p} = (-1)^p \rightarrow$ independent of n
4. $e^{+\pi i n} = (-1)^n$

Combining all the terms, equation (4.4) becomes

$$X[p] = \frac{1}{2\pi}(-1)^p\sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}np}[(-1)^n\phi_1(u_n)].$$

Hence, the application of the FFT to

$$U[n] = (-1)^n\phi_1(u_n)$$

will give us the sum (integral estimation) at point x_p. Note that the second condition from (4.3) is the trade-off between the number of points N (speed) and the values of Δ_x and Δ_u (accuracy). These choices have to be well made and verified or we will get some "impressive" results. For our tested parameters, we choose $N = 2^8$ and $\Delta_u = 6$.

The double barrier option price in this case can be computed by the Liptons eigenfunction expansion method from [Lipton, A. (2001)] presented in Appendix 5.4, with the lower barrier close to zero ($L = 0.001$) and the upper barrier $U = B$. The error level was established at 10^{-13}. The results are presented in the Table 4.2.

Table 4.2: The barrier option price via Liptons eigenfunction expansion, (Spot 80-100, Barrier 105-145)

	80	85	90	95	100
105	1.27	0.70	0.31	0.09	0.01
110	3.32	2.05	1.10	0.48	0.14
115	5.81	3.90	2.36	1.25	0.53
120	8.39	5.94	3.89	2.31	1.20
125	10.81	7.93	5.48	3.51	2.04
130	12.91	9.73	6.97	4.70	2.95
135	14.67	11.25	8.27	5.78	3.81
140	16.08	12.50	9.35	6.71	4.58
145	17.18	13.49	10.23	7.47	5.24

The differences between equation (3.53) and our algorithm in Appendix 5.5 are presented in table 4.3.

In comparison, the Monte-Carlo simulation from Appendix 5.5 with $N = 10^5$ and the jump interval $\Delta t = 1/300$ is presented in Table 4.4.

As expected, due to the full truncation algorithm, the Monte-Carlo simulation is less accurate.

Table 4.3: The difference between case 1 formula and Liptons eigenfunction expansion, (Spot 80-100, Barrier 105-145)

	80	85	90	95	100
105	0.002%	0.003%	0.006%	0.015%	0.304%
110	-0.001%	-0.002%	-0.002%	-0.003%	0.237%
115	-0.001%	-0.001%	-0.001%	-0.001%	0.163%
120	-0.001%	-0.001%	-0.001%	-0.001%	0.105%
125	0.000%	0.000%	0.000%	0.000%	0.067%
130	0.000%	0.000%	0.000%	0.000%	0.044%
135	0.000%	0.000%	0.000%	0.000%	0.031%
140	0.000%	0.000%	0.000%	0.000%	0.024%
145	0.000%	0.000%	0.000%	0.000%	0.020%

4.3 Case 3

For the general case, we will be dealing with equations (3.63) and (3.62). In more comfortable notation

$$V_0 \approx e^{-rT} \iint_{\mathbb{R}_+^2} f_3(x,y) \times d_3(x,y) dx dy, \tag{4.5}$$

with

$$\begin{aligned} f_3(x,y) &= S_0 I_{1,1}(x,y) - K I_{1,0}(x,y) - S_0 I_{2,1}(x,y) + K I_{2,0}(x,y) \\ d_3 &= d_{v^2,v_T}(x,y) = g(y) \int_{\mathbb{R}} e^{-ixu} h(y,u) du, \end{aligned} \tag{4.6}$$

Table 4.4: The difference between the Monte-Carlo simulation and Liptons eigenfunction expansion, (Spot 80-100, Barrier 105-145)

	80	85	90	95	100
105	-19.492%	-23.811%	-24.158%	-13.294%	-47.534%
110	-6.644%	-11.961%	-10.848%	-14.311%	-18.082%
115	-6.811%	-9.093%	-13.712%	-9.089%	-16.118%
120	-4.585%	-7.851%	-4.287%	-6.667%	-4.882%
125	-2.179%	-4.003%	-5.056%	-5.605%	-3.238%
130	-3.108%	-2.391%	-3.944%	-3.357%	-2.804%
135	-3.231%	-2.630%	-2.811%	-2.896%	-5.419%
140	-1.221%	-2.268%	-3.145%	-2.423%	-2.123%
145	-2.280%	-1.163%	-3.112%	-1.879%	-0.541%

where $I_{j,k}$ is given for case 3, in Appendix 5.1, and

$$\begin{aligned} g(y) &= \frac{1}{2\pi} e^{L\kappa\tau + (v_0 - y)\frac{\kappa}{\sigma^2}}, \\ h(y,u) &= e^{-Ld\tau - (v_0+y)\frac{de^+}{\sigma^2 e^-}} (\frac{ye^{d\tau}}{v_0})^{L-\frac{1}{2}} I_{2L-1}(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}). \end{aligned}$$

From equation (4.6) for the fixed y we have the same problem as in case 1 for the function $d_1(x)$. Hence, we will apply the FFT for each y and create a (x, y) grid for the integral in (4.5). We can summarize our algorithm as:

1. Fix y_p
2. Calculate $d_3(x_i, y_p)$ through the FFT
3. Repeat the steps 1 and 2 for the entire y grid

4. Combine all the $d_3(x_i, y_p)$ values with $f_3(x_i, y_p)$ and use the 2D trapezoidal rule

5. Hope that everything goes well, as planned

Unfortunately the final step was very hard to accomplish. The algorithm is hard to calibrate.

5 Appendix

Theorem 1 *Ito's Lemma or Ito's Formula. Let*

$$dX(t) = \alpha dt + \beta dB(t)$$

be an n-dimensional Itô process. Let $g : [0, \infty[\times \mathbb{R}^n \to \mathbb{R}^p$ be a C^2 map. Then the process

$$Y(t, w) = g(t, X(t))$$

is again the Itô process whose component Y_k, is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_i \frac{\partial g_k}{\partial X_i}(t, X)dX_i + \frac{1}{2}\sum_{ij} \frac{\partial g_k}{\partial X_i \partial X_j},$$

where $dB_i dB_j = \delta_{ij} dt$ and $dB_i dt = dt dB_i = 0$.

For reference see [Øksendal, B. (2002)].

5.1 The inner expectation formulas

Case 1: $r = q$ and $\rho = 0$

$$\begin{aligned}
I_{1,1} &= \mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) \\
I_{1,0} &= \mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) \\
I_{2,1} &= \frac{B}{S_0}\left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0 K}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right] \\
I_{2,0} &= \frac{S_0}{B}\left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0 K}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right].
\end{aligned}$$

Case 2: $r \neq q$ and $\rho = 0$

$$
\begin{aligned}
I_{1,1} &= \exp((r-q)T) \times \left[\mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + (r-q)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + (r-q)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right] \\
I_{1,0} &= \left[\mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + (r-q)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + (r-q)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right] \\
I_{2,1} &= \exp((r-q)T(\frac{2}{\nu^2(T)}\log(\frac{B}{S_0}) + 1)) \times \\
&\quad \times \frac{B}{S_0}\left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0 K}) + (r-q)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + (r-q)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right] \\
I_{2,0} &= \exp((r-q)T(\frac{2}{\nu^2(T)}\log(\frac{B}{S_0}) + 1)) \times \\
&\quad \times \frac{S_0}{B}\left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0 K}) + (r-q)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + (r-q)T - \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right].
\end{aligned}
$$

Case 3: $r \neq q$ and ρ arbitrary

$$
\begin{aligned}
I_{1,1} &= \exp(\frac{1}{2}\rho_2^2\nu^2(T) + \overline{q}) \times \\
&\quad \times \left[\mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + \overline{q} + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + \overline{q} + \rho_2^2\nu^2(T))}{\rho_2\nu(T)}\right)\right] \\
I_{1,0} &= \left[\mathcal{N}\left(\frac{\log(\frac{S_0}{K}) + \overline{q}}{\rho_2\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{S_0}{B}) + \overline{q}}{\rho_2\nu(T)}\right)\right] \\
I_{2,1} &= \exp(\frac{1}{2}\rho_2^2\nu^2(T) + \overline{q})(\frac{B}{S_0})^{\frac{2\overline{q}}{\rho_2^2\nu^2(T)}+2} \times \\
&\quad \times \left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0 K}) + \overline{q} + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + \overline{q} + \rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\
I_{2,0} &= (\frac{B}{S_0})^{\frac{2\overline{q}}{\rho_2^2\nu^2(T)}}\left[\mathcal{N}\left(\frac{\log(\frac{B^2}{S_0 K}) + \overline{q}}{\rho_2\nu(T)}\right) - \mathcal{N}\left(\frac{\log(\frac{B}{S_0}) + \overline{q}}{\rho_2\nu(T)}\right)\right],
\end{aligned}
$$

where

$$\overline{q} = c_1 T + c_2 \nu^2(T) + c_3(v_T + v_0).$$

Proposition 2 (Novikov's Condition) *For X_t adapted process, on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ and BM W_t, if*

$$\mathbb{E}\left[e^{\int_0^T |X_t|^2 dt}\right] < \infty$$

then

$$Z_t = e^{-\int\limits_0^t X_s dW_s - \frac{1}{2}\int\limits_0^t X_s^2 ds}$$

is a Martingale. See e.g. [Øksendal, B. (2002)], and the reference therein.

5.2 Feynman-Kac Theorem

Theorem 3 (Feyman-Kac) *Consider the stochastic differential equation*

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t)$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$g(t, x) = \mathbb{E}^{t,x}[h(X(T)] = [\mathbb{E}[h(X(T)|\mathcal{F}(t)],$$

(We assume that $\mathbb{E}^{t,x}[h(X(T)] < \infty$ for all t and x). Then $g(t, x)$ satisfies the PDE

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0$$

and the terminal condition

$$g(T, x) = h(x) \quad \forall x$$

See e.g. [Shreve, S.E. (2000)].

5.3 Inversion Theorem

From the characteristic function we can retrieve the probability density through the following theorem.

Theorem 4 (Inversion Theorem) *If X has a characteristic function $\varphi_X(t)$ than for any interval (a, b)*

$$P[a < x < b] + \frac{P[x = a] + P[x = b]}{2} = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-iua} - e^{-iub}}{iu} \varphi_X(u) du.$$

If φ_X is integrable, then P_X is absolutely continuous and X has a probability density function given by:

$$f_X(x) = P'(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \varphi_X(u) du.$$

For a proof and further detail see [Billingsley, P. (1995)].

5.4 Lipton eigenfunction expansion method

$$V_0^D = 2e^{-qT}\sqrt{SK} \sum_{n=1}^{\infty} \sin(k_n \ln \frac{S}{L}) e^{A(k_n)v_0 + B(k_n)} \frac{(-1)^{n+1} k_n (\sqrt{\frac{U}{K}} - \sqrt{\frac{K}{U}}) + \sin(k_n \ln \frac{L}{K})}{(k_n^2 + \frac{1}{4}) \ln(\frac{U}{L})},$$

with

$$
\begin{aligned}
e^{\pm}(k) &= 1 \pm e^{-\zeta(k)T} \\
k_n &= \frac{\pi n}{\ln(\frac{U}{L})} \\
\zeta(k) &= \sqrt{\kappa^2 + \sigma^2(k^2 + \frac{1}{4})} \\
A(k) &= \frac{-(k^2 + \frac{1}{4})e^{-}(k)}{\zeta(k)e^{+}(k) + \kappa e^{-}(k)} \\
B(k) &= \frac{\kappa\theta}{\sigma^2}(\kappa - \zeta(k))T + \frac{2\kappa\theta}{\sigma^2}\log(\frac{2\zeta(k)}{\zeta(k)e^{+}(k) - \kappa e^{-}(k)}).
\end{aligned}
$$

See e.g. [Lipton, A. (2001)]

5.5 Matlab code

Monte-Carlo simulation

```
syms u x

tic

v0=Data(1); % initial variance

k=Data(2); % rate of mean inversion

theta=Data(3); % long term variance

ro=Data(4); % Correlation between two processes Z1 e Z2

T=Data(5); % time to maturity (T)

rd=Data(6); % domestic rate
```

```
rf=Data(7); % foreign rate

LIVRE=Data(8);

sigma=Data(9); % volatility of the variance

K=Data(10); % strike price

L=Data(11); % lower barrier

U=Data(12); % upper barrier

C=Data(13); % cap

S0=Data(14); % spot

rr=0;

bb=1;

for Strike=80:5:100

for Barrier=105:5:145

K=Strike;

U=Barrier;

d=sqrt(k^2-2*sigma^2*1i*u);

e_plus=1+exp(-d*T);

e_minus=1-exp(-d*T);

A=(2*1i*u*e_minus)/(d*e_plus+k*e_minus);
```

```
B=k*theta/sigma^2*(k-d)*T+2*k*theta/sigma^2*log(2*d/(d*e_plus+k*e_minus));

Char_Nu_squared=exp(A*v0+B); % symetrica

ff_f=matlabFunction(Char_Nu_squared);

I_1_1= @(x) normcdf((log(S0/K)+0.5.*x)./sqrt(x))-normcdf((log(S0/U)+0.5.*x)./sqrt(x));
% N(z)=1-N(-z)

I_1_0= @(x) normcdf((log(S0/K)-0.5.*x)/sqrt(x))-normcdf((log(S0/U)-
0.5.*x)/sqrt(x));

I_2_1= @(x) (U/S0)*(normcdf((log(U^2/(S0*K))+0.5.*x)/sqrt(x))-normcdf((log(U/S0)+0.5.*x)/sqrt

I_2_0= @(x) (S0/U)* (normcdf((log(U^2/(S0*K))-0.5.*x)/sqrt(x))-normcdf((log(U/S0)-
0.5.*x)/sqrt(x)));

f_Int_Case_A=@(x) S0*I_1_1(x)-K*I_1_0(x)-S0*I_2_1(x)+K*I_2_0(x);

H = @(x) (x >= 0);

f_f=ff_f;

N=2^8; % 16384

d_step=6;

lambda_step=2*pi/(N*d_step);

mm=0:1:N;

pp=0:1:N;

uu=(mm-N/2)*d_step;
```

```
uu_max=(N-N/2)*d_step;
uu_min=(0-N/2)*d_step;
xx=(pp-N/2)*lambda_step;
xx0=N/2+1;
phi_uu=f_f(uu);
four=zeros(1,N);
for m=1:N
    four(m)=((-1)^(m-1))*phi_uu(m);
end
GAMMA=fft(four);
real_GAMMA=real(GAMMA);
INV_GAMMA=ifft(real_GAMMA);
for p=1:N
    I(p)=GAMMA(p)*d_step*((-1)^(p-1));
end
real_I=real(I);
x_plot=5000;
Integral_2=quadgk(f_f,uu_min,uu_max);
```

```
Integral_3=I(xx0);

jj=1;

for ii=1:N

    if abs(I(ii))>1/10^6 && xx(ii)>=0

        teste_valores(jj)=ii;

        jj=jj+1;

    end

end

for tt=1:length(teste_valores)

    I_opt(tt)=I(teste_valores(tt))*f_Int_Case_A(xx(teste_valores(tt)));

end

Integral_Spline=spline(xx(teste_valores),[0,I_opt,0]);

Val_Integral=ppint(Integral_Spline, xx(min(teste_valores)), xx(max(teste_valores)));

V_option=(1/(2*pi))*exp(-rd*T)*Val_Integral;

rr=rr+1;

Resultado(rr,bb)=real(V_option);

end

bb=bb+1;
```

rr=0;

end

toc

Case 1

% function [Price,time, erro] = f_Heston_Barrier_MonteCarlo_DOC(Data)

v0=Data(1); % initial variance

k=Data(2); % rate of mean inversion

theta=Data(3); % long term variance

ro=Data(4); % Corelation between two processes Z1 e Z2

T=Data(5); % time to maturity (T)

rd=Data(6); % domestic rate

rf=Data(7); % foreign rate

LIVRE=Data(8);

sigma=Data(9); % volatility of the variance

K=Data(10); % strike price

L=Data(11); % lower barrrier

U=Data(12); % upper barrrier

C=Data(13); % cap

```
S0=Data(14); % spot
rr=0;
bb=1;
for Strike=80:5:100
for Barrier=105:5:145
K=Strike;
U=Barrier;
r=rd-rf;
Num_Intervals=300;
Num_tests=10000;
dt=T/Num_Intervals;
Pr=zeros(1,Num_tests);
teste=zeros(1,Num_tests);
teste(1:Num_tests) = 1;
tic
for n_test=1:Num_tests
    Z1=randn(1,Num_Intervals);
    Z2=randn(1,Num_Intervals);
```

```
Zv=Z1;

Zs=Z2;

S=zeros(1,Num_Intervals+1);

S(1)=S0;

v=zeros(1,Num_Intervals+1);

v(1)=v0;

for n=1:Num_Intervals

    v_plus=max(v(n),0);

    S(n+1)=S(n)*(1+(v_plus^0.5)*Zs(n)*dt^0.5);

    v(n+1)=v(n)+k*dt*(theta-v_plus)+sigma*v_plus^0.5*Zv(n)*dt^0.5;

    if S(n+1)>=U

        teste(n_test)=0;

        break

    end

end

if teste(n_test)==1

    Pr(n_test)=max(S(n+1)-K,0);

else
```

```
        Pr(n_test)=0;
    end
end
Price=mean(Pr); %sum(Pr(:))/Num_tests;
V_Opt_MC=exp(-rd*T)*Price;
rr=rr+1;
Resultado_MC(rr,bb)=real(V_Opt_MC);
end
bb=bb+1;
rr=0;
end
time=toc;
```

Double trapezoidal rule

```
function output = Double_Int_Trapez(x,y,z)
m=length(x); % num_int_x
n=length(y); % num_int_y
h=(x(1)-x(m))/(m-1);
k=(y(1)-y(n))/(n-1);
```

```
S1=z(1,1)+z(1,n)+z(m,1)+z(m,n);

sum_z_x=sum(z,1);

sum_z_y=sum(z,2);

S2=2*(sum_z_x(1)+sum_z_x(n)+sum_z_y(1)+sum_z_y(m)-2*S1);

z_apoio=z(2:m-1,2:n-1);

S3=4*sum(z_apoio(:));

output=1/4*h*k*(S1+S2+S3);
```

Case 3

```
syms x y u

v0=Data(1); % initial variance

k=Data(2); % rate of mean inversion

theta=Data(3); % long term variance

ro=Data(4); % Corelation between two processes Z1 e Z2

T=Data(5); % time to maturity (T)

rd=Data(6); % domestic rate

rf=Data(7); % foreign rate

V_Option_price=Data(8); % Resultado que é suposto dar

sigma=Data(9); % volatility of the variance
```

```
K=Data(10); % strike price

L=Data(11); % lower barrrier

U=Data(12); % upper barrrier

C=Data(13); % cap

S0=Data(14); % spot

B=U;

ro2=sqrt(1-ro^2);

LL=k*theta/sigma^2;

d=sqrt(k^2-2*sigma^2*1i*u); % u esta substituido por 1^1

e_plus=1+exp(-d*T);

e_minus=1-exp(-d*T);

% BE=besseli(2*LL-1,argum_B);

q=@(x,y)(rd-rf)*T-x/2+(ro/sigma)*(y-v0-k*theta*T+k*x);

I_1_1=@(x,y) exp(0.5*ro2^2.*x+q(x,y))*(normcdf((log(S0/K)+q(x,y)+ro2^2*x)./(ro2*(x.^1/2)))-normcdf((log(S0/B)+q(x,y)+ro2^2*x)./(ro2*(x.^1/2))));

I_1_0=@(x,y) normcdf((log(S0/K)+q(x,y))./(ro2*x.^1/2))-normcdf((log(S0/B)+q(x,y))./(ro2*x.^1/2

I_2_1_exp=@(x,y) exp(0.5*ro2^2.*x+q(x,y));

I_2_1_BS=@(x,y) (B/S0)^(2*q(x,y)./(ro2^2*x)+2);
```

```
I_2_1_Norm=@(x,y) normcdf((log(B^2/(S0*K))+q(x,y)+ro2^2*x)./(ro2*x.^1/2))-normcdf((log(B/S0)+q(x,y)+ro2^2*x)./(ro2*x.^1/2));

I_2_1=@(x,y) I_2_1_exp(x,y).*I_2_1_BS(x,y).*I_2_1_Norm(x,y);

I_2_0_BS=@(x,y) (B/S0)^(2*q(x,y)./(ro2^2.*x));

I_2_0_Norm=@(x,y) normcdf((log(B^2/S0/K)+q(x,y))./(ro2*x.^1/2))-normcdf((log(B/S0)+q(x,y))./(ro2*x.^1/2));

I_2_0=@(x,y) I_2_0_BS(x,y).*I_2_0_Norm(x,y);

f_Int_Case_C=@(x,y) S0*I_1_1(x,y)-K*I_1_0(x,y)-S0*I_2_1(x,y)+K*I_2_0(x,y);

N=2^8; % 16384=2^14

d_step=2^(4);

lambda_step=2*pi/(N*d_step);

mm=0:1:N-1;

pp=0:1:N-1;

uu=(mm-N/2)*d_step;

uu_max=(N-N/2)*d_step;

uu_min=(0-N/2)*d_step;

uu_0=uu(N/2+1);

xx=(pp-N/2)*lambda_step;

xx0=N/2+1;
```

```
d_vT_v2T_out=0.5/pi*exp(LL*k*T+(v0-y)*k/sigma^2);

den_coef=2*d/(sigma^2/e_minus);

den_exp=exp(-LL*d*T-(v0+y)*d*e_plus/(sigma^2*e_minus));

den_pot=(y*exp(d*T)/v0)^(LL-0.5);

den_argum_B=(4*d/(sigma^2*e_minus))*(y*v0*exp(-d*T))^0.5;

den_BE=besseli(2*LL-1,den_argum_B);

den_fun=den_coef*den_pot*den_BE*den_exp; % RealPart: Since "1" goses to 0 at 60+. Near 0 needs "u" around 1000. At 0 gives error. Max never at u=0.

y_min=0.001;

y_max=0.25+y_min;

y_step=0.001;

y_N=(y_max-y_min)/y_step;

yy=zeros(1,y_N+1);

d_vT_v2T_out_yy=zeros(1,y_N+1);

I=zeros(N,y_N+1);

I_opt=zeros(N/2,y_N+1);

for ii=1:y_N+1

    yy(ii)=y_min+y_step*(ii-1);
```

```
d_vT_v2T_out_yy(ii)=subs(d_vT_v2T_out,y,yy(ii));
if ii==100
    d_vT_v2T_out_yy_100=d_vT_v2T_out_yy(ii);
end
den_fun_fixed_y=subs(den_fun,y,yy(ii));
f_f=matlabFunction(den_fun_fixed_y); % All that depends only on (y and u)
if ii==100
    f_f_100=f_f;
end
phi_uu=f_f(uu);
if ii==100
    phi_uu_100=phi_uu;
end
four=zeros(1,N);
for m=1:N
    four(m)=((-1)^(m-1))*phi_uu(m); %((-1)^(m-1))*
end
```

```
if ii==100

    four_100=four;

end

GAMMA=fft(four);

if ii==100

    GAMMA_100=GAMMA;

end

for p=1:N

    I(p,ii)=GAMMA(p)*d_step*((-1)^(p-1)); %*((-1)^(p-1))

    if ii==100

        I_100(p)=I(p,ii);

    end

end

for tt=1:N/2

    I_opt(tt,ii)=I(N/2+tt,ii)*f_Int_Case_C(xx(N/2+tt),yy(ii))*d_vT_v2T_out_yy(ii);

    if isnan(I_opt(tt,ii))

        I_opt(tt,ii)=0;

    end
```

```
            if ii==100
                f_Int_Case_C_100=f_Int_Case_C(xx(N/2+tt),yy(ii));
                I_opt_100(tt)=I_opt(tt,ii);
            end
        end
    end
    I_opt_Real=I_opt;
    tr_xxx=xx(xx0:N);
    tr_yyy=yy;
    tr_zzz=I_opt_Real;
    Int_Opt=Double_Int_Trapez(tr_xxx,tr_yyy,tr_zzz);
    V_Opt_FFT=exp(-rd*T)*Int_Opt;
    yt=20;
    den_fun_fixed_yt=subs(den_fun,y,yy(yt));
    f_f_teste=matlabFunction(den_fun_fixed_yt);
    Teste_Zero1=quadgk(@(u) f_f_teste(u).*exp(-1i*xx(xx0+5)*u),uu_min,uu_max);
    Teste_Zero2=I(xx0+5,20);
    toc
```

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