Journal of
Mathematics, Statistics and Allied Fields
Volume 2, Issue 1, 2008

The Ricatti Equation in the M|G|∞ System Busy Cycle Study

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Abstract

The M|G|∞ queue system can be applied to model many social problems: sickness, unemployment, emigration, etc (see, for instance, Ferreira (1995, 1996, 2003a and 2003b)). In these situations it is very important to study the busy cycle length distribution of that queue system. It is shown, in this work, that the problem solution may be in the resolution of a Ricatti equation, following the work of Ferreira (2003).

In the M|G|∞ queue there are infinite servers. But in general it is not imposed the physical presence of infinite servers. Instead, in practical applications, it is supposed that when a customer arrives at the system it finds immediately an available server or that there is no difference between the customer and its server. This is the key idea to use the M|G|∞ queue to model sickness and unemployment problems, for example.

In the M|G|∞ system operation, as in any other queue system, there is a sequence of idle and busy periods. An idle period followed by a busy period is what is called a busy cycle. In sickness applications, an idle period is one when nobody is sick. A busy period is one that corresponds to an epidemic. Thus, a busy cycle is a sickness cycle. In unemployment applications, an idle period is a full employment period. A busy period is an unemployment period. So a busy cycle is an employment cycle.

Therefore it is important to study the M|G|∞ busy cycle length distribution and not only the busy period length distribution. The M|G|∞ idle period length is exponential with parameter λ, as it happens with any queue system with Poisson process arrivals at rate λ.

Calling the busy period and the busy cycle lengths Laplace transforms $\overline{B}(s)$ and $\overline{Z}(s)$, respectively, as the idle period and the busy period are independents (according to Takács (1962)), it follows:

$$\overline{Z}(s)=\frac{\lambda}{\lambda+s}\overline{B}(s) \qquad (1).$$

A situation in which a friendly M|G|∞ busy cycle length distribution is obtained is presented forwards.

The monotony of the probability that the M|G|∞ system is empty, as time function, being the initial instant a one in the beginning of a busy period (at which a customer arrives at the system finding it empty) is regulated by the $\frac{g(t)}{1-G(t)}-\lambda G(t),\ t\geq 0$ sign, see Ferreira (1996), where $g(.)$ and $G(.)$ are, respectively, the service time probability density function and distribution function.

Defining $\beta(t)=\frac{g(t)}{1-G(t)}-\lambda G(t)$ it follows

$$\frac{dG(t)}{dt}=-\lambda G^2(t)-(\beta(t)-\lambda)G(t)+\beta(t) \qquad (2),$$

a Ricatti equation in $G(.)$.

Detecting that $G(t)=1, t\geq 0$ is a solution, it is obtained the following result

$$G(t)=1-\frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t-\int_0^t \beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw-\left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw},$$

$$t\geq 0, -\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^\rho-1} \qquad (3).$$

Consider now the M|G|∞ busy period length Laplace transform, see Stadje (1985):

$$\overline{B}(s)=1+\lambda^{-1}\left(s-\frac{1}{\int_0^\infty e^{-st-\lambda\int_0^t[1-G(v)]dv}dt}\right) \qquad (4).$$

Substituting (3) in (4) it turns in

$$\overline{B}(s)=\frac{1-(s+\lambda)(1-G(0))L\left[e^{-\lambda t-\int_0^t \beta(u)du}\right]}{1-\lambda(1-G(0))L\left[e^{-\lambda t-\int_0^t \beta(u)du}\right]}, -\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^\rho-1} \qquad (5).$$

The capital letter L is a symbol for Laplace transform and

$$G(0)=\frac{\lambda\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw+e^{-\rho}-1}{\lambda\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw} \qquad (6).$$

After (1) and (5) it is possible to compute $\frac{1}{s}\overline{Z}(s)$ which inversion gives, for the M|G|∞ busy cycle distribution function, being $*$ the convolution operator,

$$Z(t)=\left(\lambda e^{-\lambda t}\right)*\left(1-(1-G(0))\left(e^{-\lambda t-\int_0^t \beta(u)du}+\lambda\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw\right)\right)*$$

$$*\sum_{n=0}^{\infty}\lambda^n(1-G(0))^n\left(e^{-\lambda t-\int_0^t \beta(u)du}\right)^{*n},-\lambda\le\frac{\int_0^t \beta(u)du}{t}\le\frac{\lambda}{e^{\rho}-1} \qquad (7).$$

If $\beta(t)=\beta$ (constant), see Ferreira (1998),

$$G^{\beta}(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{-(\lambda+\beta)t}-1\right)},t\ge 0,-\lambda\le\beta\le\frac{\lambda}{e^{\rho}-1} \qquad (8).$$

And

$$Z^{\beta}(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-e^{-\rho}(\lambda+\beta)t}+\frac{\beta}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-\lambda t},$$

$$t\ge 0,-\lambda\le\beta\le\frac{\lambda}{e^{\rho}-1} \qquad (9).$$

Consequently, if the service time distribution function is given by (8), the M|G|∞ busy cycle distribution function is the mixture of two exponential distributions.

Finally note that if $\beta=\frac{\lambda}{e^{\rho}-1}, Z^{\beta}(t)=1-\frac{\left(e^{\rho}-1\right)e^{-\frac{\lambda}{e^{\rho}-1}t}-e^{-\lambda t}}{e^{\rho}-2}, t\ge 0$.

And $Z(t)$, given by (7), satisfies

$$Z(t)\ge 1-\frac{\left(e^{\rho}-1\right)e^{-\frac{\lambda}{e^{\rho}-1}t}-e^{-\lambda t}}{e^{\rho}-2},t\ge 0,-\lambda\le\frac{\int_0^t \beta(u)du}{t}\le\frac{\lambda}{e^{\rho}-1} \qquad (10).$$

Note that (10) is coherent even for $\rho=\log 2$ since that

$$\lim_{\rho \to \log 2} \left(1 - \frac{\left(e^{\rho} - 1\right) e^{-\frac{\lambda}{e^{\rho}-1}t} - e^{-\lambda t}}{e^{\rho} - 2} \right) =$$

$$= \lim_{\rho \to \log 2} \frac{e^{\rho} - 2 - \left(e^{\rho} - 1\right) e^{-\frac{\lambda}{e^{\rho}-1}t} + e^{-\lambda t}}{e^{\rho} - 2} = 1 - (1 + \lambda t)\, e^{-\lambda t}.$$

Besides, it is possible to have

$$Z(t) \le 1 - e^{-\lambda t}, t \ge 0, -\lambda \le \frac{\int_0^t \beta(u)du}{t} \le \frac{\lambda}{e^{\rho} - 1} \qquad (11).$$

Acknowledgement

The authors are members of UNIDE/ISCTE Research Center and thank its support in this investigation.

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